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Some basic solutions in strain gradient elasticity theory of an arbitrary order

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THE LINEAR theory of elastic materials based on a differential equation of an arbitrary order is discussed. In the case of an isotropic material, the general form of solutions, as well as some special solutions are given in an explicit analytic form. The latter include the three-dimensional and two-dimensional fundamental solutions, the isotropic dilatation centre, and the straight dislocation line. For anisotropic materials, the one-dimensional fundamental solution is given which, after integrating over all directions, yields an integral representation of the three-dimensional fundamental solution. In the case of classical anisotropic elasticity, the corresponding range of integration reduces to a unit circle. The regularity and the asymptotic properties of the solutions are investigated.

W pracy rozpatruje się liniową teorię materiałów sprężystych, w której równanie podstawowe jest równaniem różniczkowym dowolnego rzędu. Dla materiałów izotropowych podano w jawnej postaci analitycznej ogólną formę rozwiązań, jak również niektóre rozwiązania specjalne. Rozwiązania specjalne obejmują trój- i dwuwymiarowe rozwiązania podstawowe, izotropowe centrum dylatacji oraz dyslokację prostoliniową. Dla materiałów anizotropowych podano jednowymiarowe rozwiązanie podstawowe, które, po scałkowaniu względem kierunków, daje całkowe przedstawienie trójwymiarowego rozwiązania podstawowego. W przypadku klasycznego materiału anizotropowego przedstawienie to sprowadza się do całki po okręgu jednostkowym. Przedyskutowano stopień regularności oraz asymptotyczne własności uzyskanych rozwiązań.

В работе рассмотрена линейная теория упругих материалов, в которой основное уравнение является дифференциальным уравнением произвольного порядка. Для изотропных материалов дано в явном аналитическом виде общее решение, а также некоторые специальные решения. Эти специальные решения содержат трех- и двумерные основные решения, изотропный центр дилатации и прямолинейную дислокацию. Для анизотропных материалов дано одномерное основное решение, которое после интегрирования по направлениям дает интегральное представление трехмерного основного решения. В случае классического анизотропного материала данное представление сводится к интегралу по единичной окружности. Обсуждена степень регулярности и асимптотические свойства полученных решений.

Introduction

IN RECENT years, a number of modified continuum theories have been developed with a view to improving the results of classical elasticity, mainly in regions of concentrated stresses. One of the simplest ways of improving the classical theory consists in introducing higher order derivatives into the equations of elastic equilibrium. In that way the classical equations of elasticity are replaced by more general differential equations of higher order. This approach will be referred to here as the strain gradient theory. A thorough formulation of the basic principles of this theory was given by TOUPIN (1962) and MINDLIN & TIERSTEN (1962). The strain gradient theory can be especially useful in considering crystal lattice defects in the framework of the theory of continuous media. Classical elasticity, being widely used in such cases, frequently gives results which are unsatisfactory in many respects.

Better results can be obtained by applying the strain gradient theory, even in the lowest order approximation, which consists in considering the elastic energy dependent on the first strain gradient in addition to its classical dependence on the strain itself. Then, instead of Lamé equation, we have to deal with an equation corrected by a fourth order term. However, by an appropriate choice of a particular form of strain gradient theory with higher order terms, we can obtain a more suitable modelling of the mechanical properties of a crystal including some non-local effects.

In the present paper, the general form of the medium is considered. We are concerned here mainly with estimating general possibilities of the theory rather than with calculations based on any particular form of it.

The fundamental equation of an arbitrary order is discussed and some basic solutions to it are given.

1. The fundamental equation

The general linear fundamental equation of the strain gradient theory can be written in the form:

$$P_{ij}(\partial)u_j = f_i, \tag{1.1}$$

where $P_{ij}(\partial)$ represents a tensor-operator which is a polynomial in the partial derivative operators $\partial = (\partial_1, \partial_2, \partial_3)$; u_i is the displacement field and f_i an external force field.

The usual tensor notation is not convenient in dealing with quantities of unspecified order. It can be simplified by the following convention.

Consider an arbitrary tensor quantity of an arbitrary order which is symmetric in a certain group of s indices:

$$a_{\ldots i_1 i_2 \ldots i_s \ldots}$$

Instead of specifying the value of every index in the group, it suffices to state how many indices take the values 1, 2 and 3, respectively. Thus, an arbitrarily large group of symmetric tensor indices can be replaced by three non-negative integers μ_1, μ_2, μ_3. Such a triple will be denoted by a single letter, e.g. $\mu = (\mu_1, \mu_2, \mu_3)$. The quantity

$$|\mu| \overset{\mathrm{df}}{=} \mu_1 + \mu_2 + \mu_3 \tag{1.2}$$

equals the number of tensor indices which correspond to the multi-index μ. For gradient operators of arbitrary orders the above convention allows us to write:

$$\partial^\mu \overset{\mathrm{df}}{=} \partial_1^{\mu_1} \partial_2^{\mu_2} \partial_3^{\mu_3}. \tag{1.3}$$

We can consider quantities with arbitrary numbers of multi-indices and/or usual tensor indices.

The operator $P_{ij}(\partial)$ in Eq. (1.1) can be conveniently written as:

$$P_{ij}(\boldsymbol{\partial}) = \sum_{0<|\mu|\leqslant r} a_{ij\mu}\partial^{\mu}, \tag{1.4}$$

where the coefficient $a_{ij\mu}$ has two tensor indices i, j and one multi-index μ. For any specified value of $|\mu|$, the quantity $a_{ij\mu}$ is equivalent to a tensor of order $2+|\mu|$ symmetric in the last $|\mu|$ indices.

Being evidently local, the differential Eqs. (1.1) might, at first sight, seem to have little relevance to any non-local physical effects. Let us, however, note the following features of the classical theory of elasticity:

a) it is governed by differential equations,

b) it involves no material constants which would have the dimension of length.

In fact, the elastic properties of a material are completely determined by the tensor of elastic constants c_{ijkl}, which has the dimension of stress. Even in the dynamical theory where one may combine the elastic constants with the mass density it is impossible to form a material constant of the dimension of length. In other words, the classical elastic medium has no intrinsic scale of length.

The locality property expressed by (a) is here considerably strengthened by the property (b). It is (a) and (b) together which render the classical theory of elasticity unable to describe any non-local physical effects. This is no longer true for a theory corrected by higher order terms in the equations of equilibrium. Being local in the weaker sense (a), such a theory introduces intrinsic scales of length in a material medium. The appropriate parameters can be expressed e.g., as ratios of coefficients of derivatives of different orders. Those intrinsic length parameters create a possibility for the strain gradient theory to describe certain effects resulting from the non-zero range of the real interatomic forces.

2. Energy

As in the classical theory of elasticity, we assume that the energy of a material body is entirely determined by its displacement field. The exact thermodynamic nature of this "energy", not necessarily the same in various cases, is irrelevant to our theme.

The above energy assumption imposes an important restriction on possible forms of the operator $P_{ij}(\boldsymbol{\partial})$. To obtain this restriction let us discuss the case of an unbounded medium. In this case the work done in an elementary process

$$\mathbf{u}(\mathbf{x}) \to \mathbf{u}(\mathbf{x}) + \delta\mathbf{u}(\mathbf{x}) \tag{2.1}$$

is given by the formula

$$\partial W = \int d_3x\,\delta u_i f_i. \tag{2.2}$$

Making use of the linearity of the relation between forces and displacements given by Eq. (1.1), we can easily integrate this formula. For the energy corresponding to a displacement field $\mathbf{u}(\mathbf{x})$ produced by the forces $\mathrm{f}(\mathbf{x})$, we obtain

$$W = \frac{1}{2}\int d_3x\,\mathbf{u}\cdot\mathbf{f} = \frac{1}{2}\int d_3x\,u_i P_{ij}(\partial)u_j. \tag{2.3}$$

Now let us consider a cyclic process:

$$0 \to \mathbf{u}^{(1)}(\mathbf{x}) \to \mathbf{u}^{(2)}(\mathbf{x}) \to 0. \tag{2.4}$$

According to the assumption, the work done in this process has to be zero. On the other hand, this work can be expressed as

$$0 = W_{01}+W_{12}+W_{20} = \frac{1}{2}\int d_3x\mathbf{u}^{(2)}\mathbf{f}^{(1)}+\frac{1}{2}\int dx(\mathbf{u}^{(2)}-\mathbf{u}^{(1)})(\mathbf{f}^{(2)}+\mathbf{f}^{(1)}) - \frac{1}{2}\int d_3x\mathbf{u}^{(2)}\mathbf{f}^{(2)} = \frac{1}{2}\int d_3x(\mathbf{u}^{(2)}\mathbf{f}^{(1)}-\mathbf{u}^{(1)}\mathbf{f}^{(2)}). \tag{2.5}$$

Making use of Eq. (1.1), we obtain

$$\int d_3x(u_i^{(2)}P_{ij}(\partial)u_j^{(1)}-u_i^{(1)}P_{ij}(\partial)u_j^{(2)}) = 0 \tag{2.6}$$

which holds for arbitrary fields $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$. After some integration by parts, we obtain from it:

$$P_{ij}(\partial) = P_{ij}^{+}(\partial), \tag{2.7}$$

where $P_{ij}^{+}(\partial)$ is the operator conjugate to $P_{ij}(\partial)$:

$$P_{ij}^{+}(\partial) = \sum_{0<|\mu|\leqslant r}(-)^{|\mu|}\partial^{\mu}a^{ji\mu}. \tag{2.8}$$

Thus, in order not to contradict the energy principle, the operator $P_{ij}(\partial)$ has to be self-conjugate.

As equation (2.3) shows, the global energy of an infinite medium for a given displacement field is uniquely determined by the form of the operator $P_{ij}(\partial)$. It does not, however, allow us to determine the energy density uniquely. In fact, introducing an energy density w so that

$$W = \int d_3xw, \tag{2.9}$$

we obtain an expression equivalent to (2.3), provided that the equation for δW resulting from (2.9) agrees with Eq. (2.2). The necessary and sufficient condition is (2.10)

$$P_{ij}(\partial)u_j = \frac{\delta W}{\delta u_j}, \tag{2.10}$$

where the last symbol denotes the functional derivative

$$\frac{\delta w}{\delta u_j} = \sum_{\mu}(-)^{\mu}\partial^{\mu}\frac{\partial w}{\partial u_{i,\mu}}. \tag{2.11}$$

In this case, the energy density w differs from $\frac{1}{2}\mathbf{u}\cdot\mathbf{f}$ in (2.3) by a divergence-type terms which does not affect the global energy.

The energy density w can be subjected to further requirements, such as invariance with respect to rigid translations and rotations of the medium or, for homogeneous deformations, correspondence to the classical theory of elasticity. With no initial stress present, the invariance requirement eliminates dependence w on u_i and ω_{ij} so that the energy density depends only on ε_{ij} and the derivatives of u_i of second and, possibly, higher orders:

(2.12) $$w = w(\varepsilon_{ij}, u_{i,\mu}), \qquad |\mu| \geqslant 2;$$

the symbols ω_{ij} and ε_{ij} have the standard meaning:

(2.13) $$\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,j}),$$

(2.14) $$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j\,i}).$$

The gradients of any non-zero order of the rotation tensor ω_{ij} can be entirely expressed by the gradients of the corresponding order of the strain tensor ε_{ij}. In fact, we can easily check that the equation

(2.15) $$\omega_{ij,k} = \varepsilon_{ik,j} - \varepsilon_{jk,i}$$

holds identically. Making use of this equation, we can write:

(2.16) $$u_{i,kj} = \varepsilon_{ij,k} + \omega_{ij,k} = \varepsilon_{ij,k} + \varepsilon_{ik,j} - \varepsilon_{jk,i},$$

which enables us to express the second order gradients of the displacement vector u_i by the first order gradient of the strain tensor ε_{ij}. By differentiating Eq. (2.16), the analogous equations for higher order gradients can be obtained. This leads us to the conclusion that energy density (2.12) can be expressed as a function of the strain tensor and its gradients of different orders:

(2.17) $$w = w(\varepsilon_{ij}, \varepsilon_{ij,\mu}), \qquad |\mu| \geqslant 1.$$

Given an expression for the energy density, the form of the operator $P_{ij}(\partial)$ is determined uniquely by Eq. (2.10). Therefore, in phenomenological formulation of the strain gradient theory, it is often convenient to begin with consideration of the energy density.

The energy of a finite body cannot be unambiguously determined without further investigation. This is because the surface energy may contain terms non-equivalent to any volume integrals. For our purposes, knowledge of the energy of an unbounded medium will suffice.

3. Conditions for the coefficients $a_{ij\mu}$

The coefficients $a_{ij\mu}$ are, in general, functions of the coordinates $\mathbf{x} = (x_1, x_2, x_3)$. We shall restrict ourselves to considering here a medium which is homogeneous,

(3.1) $$\partial_k a_{ij\mu} = 0$$

and centrosymmetric

(3.2) $$a_{ij\mu} = 0 \quad \text{for odd } |\mu|.$$

Othervise, the anisotropy considered is arbitrary. For any particular material, these coefficients are subject to the restrictions resulting from point symmetry.

According to (3.2), the operator $P_{ij}(\partial)$ contains only derivatives of even orders and, in particular, its order is even, $r = 2n$.

In the case of a homogeneous and centrosymmetric medium, the condition (2.7) can be written as

$$P_{ij}(\partial) = P_{ji}(\partial). \tag{3.3}$$

Thus the tensor $P_{ij}(\partial)$ has to be symmetric. This symmetry is equivalent to the symmetry of all $a_{ij\mu}$' s:

$$a_{ij\mu} = a_{ij\mu}. \tag{3.4}$$

Further important conditions for the $a_{ij\mu}$'s follow from stability considerations. For the medium to be stable, the energy (2.3) must be positive for non-vanishing fields $\mathbf{u}(\mathbf{x})$. In fact, a stronger condition is needed—namely, the energy density must be everywhere positive. According to KUNIN (1968), this is equivalent to the condition that the characteristic equation

$$\det(P_{ij}(ik) - \omega^2\delta_{ij}) = 0 \tag{3.5}$$

has only positive roots:

$$\omega_1^2(\mathbf{k}) > 0, \quad \omega_1^2(\mathbf{k}) > 0, \quad \omega_3^2(\mathbf{k}) > 0, \tag{3.6}$$

for any real $\mathbf{k} \neq 0$. The matrix $P_{ij}(i\mathbf{k})$ is defined as

$$P_{ij}(i\mathbf{k}) = \sum_{2 \leqslant |\mu| \leqslant 2n} (-)^{\frac{1}{2}} a_{ij\mu} k^{\mu}, \tag{3.7}$$

and according to the above condition must be positive-definite.

Making use of the Fourier transform

$$\hat{\mathbf{u}}(\mathbf{k}) = \int d_3 x e^{-i\mathbf{k}\mathbf{x}} \mathbf{u}(\mathbf{x}), \tag{3.8}$$

we can express the energy as

$$W = \frac{1}{2} \frac{1}{(2\pi)^3} \int d_3 k \, \hat{u}_i^*(\mathbf{k}) P_{ij}(i\mathbf{k}) \hat{u}_j(\mathbf{k}). \tag{3.9}$$

4. The operator $P_{ij}(\partial)$ and length parameters for an isotropic medium

Now we shall investigate the question as to what effect the higher order terms have on solutions to Eqs. (1.1). We shall begin with an isotropic medium.

In that case, the most general form of the operator $P_{ij}(\partial)$ is

$$P_{ij}(\partial) = -a(\Delta)(\Delta\delta_{ij} - \partial_i\partial_j) - b(\Delta)\partial_i\partial_j, \tag{4.1}$$

where $a(\Delta)$ and $b(\Delta)$ are polynomials in the Laplace operator Δ. Equation (1.1) takes the form

$$-a(\Delta)\Delta\mathbf{u} - [b(\Delta) - a(\Delta)]\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} = \mathbf{f}. \tag{4.2}$$

From correspondence with the classical theory, we have

$$a(0) = \mu, \quad b(0) = \lambda + 2\mu, \tag{4.3}$$

where μ and λ represent Lamé constants.

Let p and q be the orders of the polynomials $a(\Delta)$ and $b(\Delta)$ respectively, and

$$\begin{gathered} \beta_2^2, \beta_2^2, \ldots, \beta_p^2, \\ \beta_1'^2, \beta_2'^2, \ldots, \beta_q'^2 \end{gathered} \tag{4.4}$$

their roots, each taken according to its multiplicity. Then we can write:

$$\begin{aligned} a(\Delta) &= \text{const}\cdot(\beta_1^2 - \Delta)(\beta_2^2 - \Delta) \ldots (\beta_p^2, -\Delta), \\ b(\Delta) &= \text{const}\cdot(\beta_1'^2 - \Delta)(\beta_2'^2 - \Delta) \ldots (\beta_q'^2 - \Delta). \end{aligned} \tag{4.5}$$

The roots (4.4) are, in general, complex. They obey, however, the following restrictions:

(a) none of them is equal to zero;

(b) none of $\beta_r'^2$ or β_r^2 is a real negative number;

(c) non-real roots of $a(\Delta)$, as well as those of $b(\Delta)$, occur in mutually conjugate pairs.

The restriction (a) follows from correspondence with the classical theory of elasticity. In fact, from (4.5) we have

$$\begin{aligned} a(0) &= \text{const}\cdot\beta_1^2\beta_2^2 \ldots \beta_p^2 \\ b(0) &= \text{const}\cdot\beta_1'^2\beta_2'^2 \ldots \beta_q'^2. \end{aligned} \tag{4.6}$$

If one or more of the roots were zero, then, according to (4.3), one or both Lamé constants would be zero; this contradicts the classical theory.

The restriction (b) is a consequence of the stability condition. According to Eq. (4.1), the matrix (3.7) for an isotropic medium can be written as

$$P_{ij}(i\mathbf{k}) = a(-k^2)(k^2\delta_{ij} - k_i k_j) + b(-k^2) k_i k_j. \tag{4.7}$$

Thus, this matrix is positive-definite only when

$$a(-k^2) > 0 \quad \text{and} \quad b(-k^2) > 0. \tag{4.8}$$

If a real negative root of any of the two polynomials existed, it would contradict the inequalities (4.8) for certain real wave vectors **k**.

The conditions (a) and (b) imply that β's themselves can be so chosen that

$$\operatorname{Re}\beta_r > 0, \quad \operatorname{Re}\beta_s' > 0 \tag{4.9}$$

for any r and s.

The condition (c) is an immediate consequence of the fact that the coefficients of the polynomials $a(\Delta)$ and $b(\Delta)$ are real.

All the β's have the dimension of inverse length and therefore can be used as a convenient set of length parameters. The number of independent real length parameters equals the total number of the roots β_i and β_j': a real root determines one, and a pair of mutually conjugate complex roots determines two such parameters.

The classical elastic constants and the roots (4.4) determine completely the polynomials $a(\Delta)$ and $b(\Delta)$, and, consequently, the detailed form of Eq. (4.2).

In fact, according to Eqs. (4.5) and (4.3), we have the following representations

(4.10)
$$a(\Delta) = \frac{\mu}{\beta_1^2\beta_2^2 \dots \beta_p^2} (\beta_1^2-\Delta)(\beta_2^2-\Delta) \dots (\beta_p^2-\Delta)$$
$$= \mu(1-\beta_1^{-2}\Delta)(1-\beta_2^{-2}\Delta) \dots (1-\beta_2^{-2}\Delta),$$
$$b(\Delta) = \frac{\lambda+2\mu}{\beta_1'^2\beta_2'^2 \dots \beta_q'^2} (\beta_1'^2-\Delta)(\beta_2'^2-\Delta) \dots (\beta_q'^2-\Delta)$$
$$= (\lambda+2\mu)(1-\beta_1'^{-2}\Delta)(1-\beta_2'^{-2}\Delta) \dots (1-\beta_2'^{-2}\Delta).$$

5. The general form of solutions to homogeneous equations for isotropic media

Let us consider Eq. (4.2) in a region where $\mathbf{f} = 0$:

(5.1)
$$a(\Delta)\Delta\mathbf{u} + \big(b(\Delta) - \alpha(\Delta)\big) \operatorname{grad} \operatorname{div} \mathbf{u} = 0.$$

The displacement field can be decomposed into two parts

(5.2)
$$\mathbf{u} = \mathbf{v} + \mathbf{w},$$

so that

(5.3)
$$\operatorname{div} \mathbf{v} = 0 \quad \text{and} \quad \operatorname{curl} \mathbf{w} = 0.$$

Then the fields $\mathbf{v}$ and $\mathbf{w}$ have to satisfy the equations

(5.4)
$$a(\Delta)\Delta\mathbf{v} = 0 \quad \text{and} \quad b(\Delta)\Delta\mathbf{w} = 0.$$

In the classical theory of elasticity, where $a(\Delta)$ and $b(\Delta)$ are simply constants, the fields $\mathbf{v}$ and $\mathbf{w}$ (and $\mathbf{u}$, in consequence) have to be harmonic:

(5.5)
$$\Delta\mathbf{v}^{\text{class}} = 0 \quad \text{and} \quad \Delta\mathbf{w}^{\text{class}} = 0.$$

Thus, according to Eqs. (5.4), any classical solution is acceptable as a particular solution to the equations of the strain gradient theory. The non-classical solutions in which we are interested can be found by considering the equation

(5.6)
$$\Delta\mathbf{u} = \beta^2\mathbf{u},$$

where β^2 is a constant. Taking into account Eqs. (5.3) and (5.4), we see that if either

(5.7)
$$a(\beta^2) = 0 \quad \text{and} \quad \operatorname{div} \mathbf{u} = 0,$$

or

(5.8)
$$b(\beta^2) = 0 \quad \text{and} \quad \operatorname{curl} \mathbf{u} = 0,$$

then the solution $\mathbf{u}$ to Eq. (5.6) satisfies Eq. (5.1). In that case, the constant β^2 in Eq. (5.6) must be equal to one of the roots (4.4). We may conclude that a sum of the form

(5.9)
$$\mathbf{u} = \mathbf{u}^{\text{class}} + \sum_s \mathbf{u}^{(s)},$$

where $\mathbf{u}_{\text{class}}$ is a classical solution and the $\mathbf{u}^{(s)}$'s satisfy either

(5.10)
$$\Delta\mathbf{u}^{(s)} = \beta_s^2\mathbf{u}^{(s)}, \quad \operatorname{div} \mathbf{u}^{(s)} = 0,$$

or

$$\Delta \mathbf{u}^{(s)} = \beta_s'^2 \mathbf{u}^{(s)}, \quad \operatorname{curl} \mathbf{u}^{(s)} = 0, \tag{5.11}$$

is always a solution to Eq. (5.1).

In the case in which the polynomials $a(\Delta)$ and $b(\Delta)$ have no multiple roots, the inverse statement is also true: any solution to Eq. (5.1) is of the form (5.9). This can be proved by induction with respect to the number of factors in the representations (4.10). Thus, in that case the formula (5.9) represents the most general form of solution to the equations of the strain gradient theory.

If any of the polynomials $a(\Delta)$ and $b(\Delta)$ has a multiple root, the solution given by the formula (5.9) is not of the most general form. In this case, for any multiple root we have a family of particular solutions to Eqs. (5.1)

$$\mathbf{u}, \frac{\partial}{\partial \beta}\mathbf{u}, \ldots, \frac{\partial^{m-1}}{\partial \beta^{m-1}}\mathbf{u},$$

where $\mathbf{u}$ satisfies Eqs. (5.10) or (5.11), and m represents the multiplicity of the root β. This follows from the observation that if $\mathbf{u}$ is a solution to Eq. (5.6), then

$$(\beta^2 - \Delta)^{l+1} \frac{\partial^l}{\partial \beta^l}\mathbf{u} = 0 \tag{5.13}$$

for an arbitrary l. The particular solutions of the form (5.12) for all multiple roots should be taken into account.

In further paragraphs, we shall discuss briefly a few special solutions which are of some interest. Throughout this discussion we assume that none of the polynomials $a(\Delta)$ and $b(\Delta)$ in Eq. (4.2) has multiple roots. From the formulae directly valid under the above assumption, we can obtain expressions valid in the case of multiple roots by applying an appropriate limiting procedure.

6. The three-dimensional fundamental solution for an isotropic medium

We consider first the fundamental solution $G_{ij}(x)$ defined as the solution of the equation:

$$-a(\Delta)G_{ij,kk} - \big(b(\Delta) - a(\Delta)\big)G_{kj,ki} = \delta^{(3)}(\mathbf{x})\delta_{ij} \tag{6.1}$$

submitted to the condition of vanishing at infinity. The symbols $\delta^{(3)}(\mathbf{x})$ and δ_{ij} denote the three-dimensional Dirac delta and Krönecker delta, respectively. This solution can easily be found by making use of the Fourier transformation. It can be represented by the following Fourier integral

$$G_{ij}(\mathbf{x}) = \frac{1}{(2\pi)^2}\int d_3k\left[\frac{k^2\delta_{ij} - k_ik_j}{a(-k^2)} + \frac{k_ik_j}{b(-k^2)}\right]\frac{e^{i\mathbf{kx}}}{k^4}, \tag{6.2}$$

where k^{-4} is to be understood in the following sense

$$\frac{1}{k^4} = \frac{1}{2}\left[\frac{1}{(k+i0)^4} + \frac{1}{(k-i0)^4}\right]. \tag{6.3}$$

After performing angular integrations, we obtain the expression

$$G_{ij} = \frac{1}{(2\pi)^2 i} \int\limits_{-\infty}^{\infty} dk \left[\frac{k^2\delta_{ij} + \partial_i\partial_j}{a(-k^2)} - \frac{\partial_i\partial_j}{b(-k^2)} \right] \frac{e^{ikr}}{k^3 r}. \tag{6.4}$$

This integral can be computed conveniently by passing to the complex k-plane. Because $r > 0$, the integration contour can be closed in the upper half-plane. Then the contributions to the integral (6.2) arise from:

1) the pole at $k = 0$; according to the formula (6.3) this contribution is given by only a half of the residuum;

2) the poles at the roots of the polynomials $a(-k^2)$ and $b(-k^2)$ with $\operatorname{Im} k > 0$. These poles are at the points

$$k = i\beta_r, \quad r = 1, 2, \ldots, p, \tag{6.5}$$

and

$$k = i\beta'_s, \quad s = 1, 2, \ldots, q \tag{6.6}$$

with β''s restricted by inequalities (4.9).

The above procedure is equivalent to evaluating the integral (6.4) as

$$\frac{1}{2} \int\limits_{C_0} dk \ldots + \int\limits_{C_1} dk \ldots, \tag{6.7}$$

where the contours C_0 and C_1 are chosen as in Fig. 1. The contour C_1 encloses all the

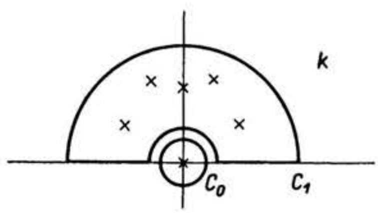

FIG. 1.

poles (6.5) and (6.6) and the contour C_0 encircles only the pole at $k = 0$. Summing up the above-mentioned contributions, we obtain

$$G_{ij} = \frac{1}{4\mu}\frac{1}{\mu}\delta_{ij}\left(\frac{1}{r} - \sum_s \alpha_s \frac{e^{-\beta_s r}}{r}\right) - \frac{1}{4\mu}\frac{1}{\mu}\partial_i\partial_j\left(\frac{r}{2} + \sum_s \frac{1}{\beta_s^2}\left(\frac{1}{r} - \alpha_s\frac{e^{-\beta_s r}}{r}\right)\right)$$
$$+ \frac{1}{4\pi}\frac{1}{\lambda+2\mu}\partial_i\partial_j\left(\frac{r}{2} + \sum_s \frac{1}{\beta_s'^2}\left(\frac{1}{r} - \alpha'_s\frac{e^{-\beta'_s r}}{r}\right)\right), \tag{6.8}$$

where the symbols α_s and α'_s are defined as

$$\alpha_s = \prod_{r \neq s} {}_r \frac{\beta_r^2}{\beta_r^2 - \beta_s^2}, \qquad \alpha'_s = \prod_{r \neq s} {}_r \frac{\beta_r'^2}{\beta_r'^2 - \beta_s'^2}. \tag{6.8}$$

By making use of the identities

$$\partial_i\partial_j r = \frac{\delta_{ij}}{r} - \frac{x_i x_j}{r^3},$$

(6.10) $$\partial_i\partial_j \frac{1}{r} = -\frac{4\pi}{3}\delta^{(3)}(\mathbf{x})\delta_{ij} - \frac{\delta_{ij}}{r^3} + \frac{3x_i x_j}{r^5},$$

$$\partial_i\partial_j \frac{e^{-\beta r}}{r} = -\frac{4\pi}{3}\delta^{(3)}(\mathbf{x})\delta_{ij} - \frac{e^{-\beta r}}{r}\left(\frac{1}{r}+\frac{\beta}{r}\right)\delta_{ij} + \frac{e^{-\beta r}}{r}\left(\frac{3}{r^2}+\frac{3\beta}{r}+\beta^2\right)\frac{x_i x_j}{r^2},$$

the angular dependence of G_{ij} can be demonstrated. From the point of view of behaviour as $r \to \infty$, three types of terms in the fundamental solution (6.8) can be distinguished

1) the terms of order $1/r$; these terms form the classical fundamental solution:

(6.11) $$G_{ij}^{\text{class}} = \frac{1}{4\pi}\frac{1}{\mu}\frac{\delta_{ij}}{r} - \frac{1}{4\pi}\left(\frac{1}{\mu} - \frac{1}{\lambda+2\mu}\right)\partial_i\partial_j\frac{r}{2}$$
$$= \frac{1}{8\pi r}\left[\frac{1}{\mu}\left(\delta_{ij} + \frac{x_i x_j}{r^2}\right) + \frac{1}{\lambda+2\mu}\left(\delta_{ij} - \frac{x_i x_j}{2}\right)\right];$$

2) the terms of order $1/r^3$

(6.12) $$-\frac{1}{4\pi}\left(\frac{1}{\mu}\sum_s \frac{1}{\beta_s^2} - \frac{1}{\lambda+2\mu}\sum_s \frac{1}{\beta_s'^2}\right)\partial_i\partial_j\frac{1}{r}$$
$$= \frac{1}{4\pi}\left(\frac{1}{\mu}\sum_s \frac{1}{\beta_s^2} - \frac{1}{\lambda+2\mu}\sum_s \frac{1}{\beta'^2}\right)\left(\frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5}\right);$$

3) the exponential terms which, by inequalities (4.9), decrease at infinity.

Hence, we can write

(6.13) $$G_{ij} = G_{ij}^{\text{class}} + \frac{1}{4\pi r^3}\left(\frac{1}{\mu}\sum_s \frac{1}{\beta_s^2} + \frac{1}{\lambda+2\mu}\sum_s \frac{1}{\beta_s'^2}\right)\left(\delta_{ij} - \frac{3x_i x_j}{r^2}\right)$$

+a finite number of exponentially decreasing terms.

For $r \to 0$, the classical fundamental solution exhibits a singularity of order $1/r$. This no longer holds in the case of the strain gradient theory. Taking into account the identities

(6.14) $$\sum \frac{1}{\beta_s^2}(1-\alpha_s) = 0, \qquad \sum \frac{1}{\beta_s'^2}(1-\alpha_s') = 0$$

and

(6.15) $$\sum \alpha_s = 1, \qquad \sum \alpha_s' = 1,$$

it can be seen from Eq. (6.8) that the terms of order $1/r$ cancel each other. Thus, the fundamental solution in the strain gradient theory is continuous at $r = 0$.

The actual form of the singularity at $r = 0$ depends on p and q, the orders of the polynomials $a(\Delta)$ and $b(\Delta)$. If $p > 1$ and $q > 1$, the gradients of G_{ij} up to second order, $\partial_k G_{ij}$ and $\partial_l\partial_k G_{ij}$, are continuous. If $p > 2$ and $q > 2$, then the gradients up to fourth order are continuous, and so on. In fact, if $p > 1$ and $q > 1$, the following equations are identically valid:

$$(6.16)\qquad \begin{aligned} &\sum_s \beta_s^{2m}\alpha_s = 0 \quad \text{for} \quad m = 1, 2, \ldots, p-1,\\ &\sum_s \beta_s'^{2m}\alpha_s = 0 \quad \text{for} \quad m = 1, 2, \ldots, q-1, \end{aligned}$$

so that sufficiently low odd powers of r in the series representing the solution (6.8) are cancelled. Specifically, the following asymptotic expression is valid for small r:

$$(6.17)\qquad G_{ij} = \frac{1}{4\pi}\frac{1}{\mu}(\Delta\delta_{ik} - \partial_i\partial_k)\sum_s\left(\frac{\alpha_s}{\beta_s^2}\frac{\operatorname{sh}\beta_s r}{r} - \frac{\beta_s^{2p}d_s}{(2p+2)!}r^{2p+1}\right)$$
$$-\frac{1}{4\pi}\frac{1}{\lambda+2\mu}\partial_i\partial_k\sum_s\left(\frac{\alpha_s'}{\beta_s'^2}\frac{\operatorname{sh}\beta_s' r}{r} - \frac{\beta_s'^{2q}\alpha_s'}{(2q+2)!}r^{2q+1}\right) + 0(r^{2p+1}) + 0(r^{2q+1}).$$

It follows from the above expression that up to order

$$(6.18)\qquad 2\min(p, q) - 2,$$

the gradients of the fundamental solution G_{ij} are continuous.

7. A centre of dilatation in an isotropic medium

Now we consider a spherically symmetric field $\mathbf{u}$ which has a singularity at $r = 0$, satisfies Eq. (5.1) for $r \neq 0$, and vanishes at infinity. The spherical symmetry implies that, in spherical coordinates, such a field has only the radial component which does not depend on angular variables:

$$(7.1)\qquad u_r = u_r(r), \quad u_\theta = u_\varphi = 0.$$

Consequently, we have curl $\mathbf{u} = 0$. The classical solution of this problem [ESHELBY (1956)] is of the form:

$$(7.2)\qquad u_r^{\text{class}} = \frac{c}{4\pi r^2},$$

and is completely determined by specifying the value of the constant c. The meaning of this constant is as follows: it equals the increase of volume of the infinite medium caused by the centre under consideration.

To obtain non-classical particular solutions, we shall consider Eq. (5.11) which in the present case takes the form:

$$(7.3)\qquad \frac{\partial}{\partial r}\frac{1}{r^3}\frac{\partial}{\partial r}r^2 u_r = \beta_s'^2 u_r.$$

Hence, the particular solutions we are seeking can be written as

$$(7.4)\qquad \frac{\partial}{\partial r}\frac{e^{-\beta_s' r}}{r}, \quad s = 1, 2, \ldots, q.$$

Taking into account the classical term, we obtain the following general solution of our problem:

(7.5) $$u_r = \frac{c}{4\pi r^2} + \sum_s A_s \frac{\partial}{\partial r} \frac{e^{-\beta'_s r}}{r},$$

where c and A_s are arbitrary constants. This solution can also be represented as

(7.6) $$\mathbf{u} = -\operatorname{grad}\varphi$$

with

(7.7) $$\varphi = \frac{c}{4\pi r} - \sum_s A_s \frac{e^{-\beta'_s r}}{r}.$$

The non-classical terms in (7.5) decrease exponentially at infinity. Hence the constant c retains its classical meaning. But the coefficients of the exponential terms remaining undetermined, the total volume change no longer determines the complete solution.

In order to explain this indeterminacy, let us write the solution (7.5) in a somewhat different form. First of all, we construct a solution with singularity at $r = 0$ as weak as possible. Manipulating with q the constants A_s enables us to eliminate q singular terms

(7.8) $$\frac{1}{r}, r, r^3, \ldots . r^{2q-3}$$

in the expansion of the potential (7.7). According to Eqs. (6.15) and (6.16), this requires the constants A_s to have the values $c\alpha'_s/4\pi$, so that

(7.9) $$\mathbf{u} = \operatorname{grad} \frac{c}{4\pi}\left(-\frac{1}{r} + \sum_s \alpha'_s \frac{e^{-\beta'_s r}}{r}\right)$$

is the least singular solution.

Next, we consider more singular solutions. Note that, because all the roots $\beta_s'^2$ are different, the Vandermonde determinant

(7.10) $$\begin{bmatrix} 1 & \beta_1'^2 \ldots \beta_1'^{2(q-1)} \\ 1 & \beta_2'^2 \ldots \beta_2'^{2(q-1)} \\ \cdots & \cdots\cdots\cdots\cdots \\ 1 & \beta_q'^2 \ldots \beta_q'^{2(q-1)} \end{bmatrix}$$

differs from zero.

Thus, instead of $A_1, A_2, \ldots, A_q$, we can introduce a new set of arbitrary constants $c_1, c_2, \ldots, c_q$ such that

(7.11) $$A_s = \frac{c}{4\pi}\alpha'_s + \frac{\alpha'_s \beta_s'^2}{4\pi}(c_1 + c_2\beta_s'^2 + \ldots + c_q \beta'^{2(q-1)}).$$

Then we have

(7.12) $$\mathbf{u} = c\cdot\mathbf{u}^{(0)} + c_1\mathbf{u}^{(1)} + \ldots c_q\mathbf{u}^{(q)},$$

where

(7 13) $$\mathbf{u}^{(0)} = \operatorname{grad}\frac{1}{4\pi}\left(-\frac{1}{r} + \sum_s \alpha'_s \frac{e^{-\beta'_s r}}{r}\right)$$

and

$$\mathbf{u}^{(k)} = \operatorname{grad} \frac{1}{4\pi} \sum_s \alpha'_s \beta'^{2k}_s \frac{e^{-\beta'_s r}}{r}, \quad k = 1, 2, \ldots, q. \tag{7.14}$$

It follows from Eq. (6.16) that the greater is k the more singular is the solution $\mathbf{u}^{(k)}$. Moreover, we have

$$\mathbf{u}^{(k+1)} = \Delta \mathbf{u}^{(k)} \quad \text{for} \quad k = 0, 1 \ldots q-1 \tag{7.15}$$

and

$$u_i^{(0)} = -(\lambda+2\mu) G_{ij,j}, \tag{7.16}$$

where G is the fundamental solution (6.8). Thus, by Eq. (6.1), solution (7.12) corresponds to the following singular forces

$$\mathbf{f} = -(\lambda+2\mu)[c + c_1\Delta + c_2\Delta^2 + \ldots + c_q\Delta^q] \operatorname{grad} \delta^{(3)}(\mathbf{x}). \tag{7.17}$$

The classical term

$$-(\lambda+2\mu) c \operatorname{grad} \delta^{(3)}(\mathbf{x})$$

in (7.17) represents an isotropic distribution of forces of the type shown in Fig. 2. The remaining terms represent isotropic distributions of multipole forces up to order q. The case of dipole forces is illustrated in Fig. 3.

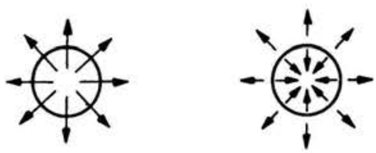

FIG. 2. FIG. 3.

In classical elasticity, multipole distributions of forces at the centre make no contribution to the solutions at $r \neq 0$.

8. The two-dimensional fundamental solution for an isotropic medium

As the next example, we shall discuss the plane fundamental solution which corresponds to the forces uniformly distributed along the z-axis. The appropriate equation is

$$-a(\Delta) F_{ij,kk} - \big(b(\Delta) - a(\Delta)\big) F_{kj,ki} = \delta^{(2)}(\mathbf{x}) \delta_{ik}, \tag{8.1}$$

where $\delta^{(2)}(\mathbf{x})$ is the two-dimensional Dirac delta $\delta(x)\delta(y)$. The solution F_{ij} can be calculated in a systematic way—e.g., by making use of the procedures of Sec. 6 or 7. However, we can obtain it more easily by analyzing the form of the fundamental solution (6.8). The factors responsible for G_{ij} being the solution of Eq. (6.1)—apart from algebraic properties of the coefficients—can be stated in the form of the equations

$$\Delta A = B, \quad \Delta B = \delta, \quad (\Delta - \beta^2) C = \delta, \tag{8.2}$$

where δ stands for $\delta^{(3)}(\mathbf{x})$ and

$$A = -\frac{r}{8\pi},$$

(8.3) $$B = -\frac{1}{4\pi r}, \qquad r = \sqrt{x^2+y^2+z^2},$$

$$C = -\frac{e^{-\beta r}}{4\pi r}.$$

Therefore, if we solve Eq. (8.2) with two-dimensional δ and put these solutions into the expression (6.8) in place of the terms (8.3), we obtain the solution F_{ij}. The appropriate solutions for A, B, C are

$$A = \frac{1}{8\pi} r^2(\log r - 1),$$

(8.4) $$B = \frac{1}{2\pi}\log r, \qquad r = \sqrt{x^2+y^2},$$

$$C = -\frac{1}{2\pi} K_0(\beta_s r),$$

where K_0 denotes the modified Hankel function. Therefore,

(8.5) $$F_{33} = -\frac{1}{2\pi\mu}\Big(\log r + \sum_s \alpha_s K_0(\beta_r r)\Big),$$

$$F_{13} = F_{23} = F_{31} = F_{32} = 0$$

and for $\alpha, \beta = 1, 2$,

$$F_{\alpha\beta} = -\frac{1}{2\pi\mu}\delta_{\alpha\beta}\Big(\log r + \sum_s \alpha_s K_0(\beta_s r)\Big) + \frac{1}{2\pi\mu}\partial_\alpha\partial_\beta\left(\frac{r^2}{4}\log r - \frac{r^2}{4} + \sum_s \frac{1}{\beta_s^2}\big(\log r + \alpha_s K_0(\beta_s r)\big)\right) - \frac{1}{2\pi(\lambda+2\mu)}\partial_\alpha\partial_\beta\left(\frac{r^2}{4}\log r - \frac{r^2}{2}\sum_s \frac{1}{\beta_s'^2}\big(\log r + \alpha_s' K_0(\beta_s' r)\big)\right).$$

The corresponding classical expressions are

(8.7) $$F_{33}^{\text{class}} = -\frac{1}{2\mu\pi}\log r$$

and

(8.8) $$F_{33}^{\text{class}} = -\frac{1}{4\pi}\left(\frac{1}{\mu} + \frac{1}{\lambda+2\mu}\right)\delta_{\alpha\beta}\log r + \frac{1}{4\pi}\left(\frac{1}{\mu} - \frac{1}{\lambda+2\mu}\right)\frac{x_\alpha x_\beta}{r^2},$$

possible constant terms being disregarded.

The modifications of F_{ij} introduced by the strain gradient theory are quite similar to these of G_{ij}. From the inequalities (4.9), we have

$$(8.9)\qquad |\arg \beta_s r| < \frac{\pi}{2}, \quad |\arg \beta'_s r| < \frac{\pi}{2},$$

so that the terms containing K_0 decrease exponentially at infinity. Therefore, for large values of r we have

$$(8.10)\qquad F_{33} = F_{33}^{\text{class}} + \text{a finite number of exponentially decreasing terms}$$

and

$$(8.13)\qquad F_{\alpha\beta} = F_{\alpha\beta}^{\text{class}} + \frac{1}{2\pi r^2}\left(\frac{1}{\mu}\sum_s \frac{1}{\beta_s^2} - \frac{1}{\lambda+2\mu}\sum_s \frac{1}{\beta_s'^2}\right)\left(\delta_{\alpha\beta} - 2\frac{x_\alpha x_\beta}{r}\right)$$

$$+\text{a finite number of exponentially decreasing terms.}$$

To investigate the behaviour of F_{ij} for $r \to 0$, we make use of the following expansion of K_0 (TRANTER, 1968):

$$(8.12)\qquad K_0(x) = -\left(\gamma + \log\frac{x}{2}\right) I_0(x) + \sum_{k=1}^{\infty} \frac{x^{2k}}{4^k (k!)^2}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k}\right),$$

where γ is Euler's constant and I_0 is the modified Bessel function of the first kind.

We obtain

$$(8.13)\qquad F_{33} = \frac{1}{2\pi\mu}\frac{1}{4^p (p!)^2}\sum_s \alpha_s \beta_s^{2p} r^{2p}\left(-\log r + 1 + \frac{1}{2} + \ldots + \frac{1}{p}\right)$$

$$-\frac{1}{2\pi\mu}\sum_s \alpha_s \left(\gamma + \log\frac{\beta_s}{2}\right) I_0(\beta_s r) + O(r^{2p+2}\log r)$$

and

$$(8.14)\qquad F_{\alpha\beta} = \frac{1}{2\pi\mu}(\partial_\alpha \partial_\beta - \left[\frac{1}{4^{p+1}[(p+1)!]^2}\sum_s \alpha_s \beta_s^{2p} r^{2p+2} \times\right.$$

$$\left.\times\left(-\log r + 1 + \frac{1}{2} + \ldots + \frac{1}{p+1}\right) - \sum_s \frac{\alpha_s}{\beta_s^2}\left(\gamma + \log\frac{\beta_s}{2}\right) I_0(\beta_s r)\right]$$

$$+\frac{1}{2\pi(\lambda+2\mu)}\partial_\alpha\partial_\beta\left[\frac{1}{4^{q+1}[(q+1)!]^2}\sum_s \alpha'_s \beta_s'^{2q} r^{2q+2}\left(-\log r + 1\right.\right.$$

$$\left.\left.+\frac{1}{2} + \ldots + \frac{1}{q+1}\right) - \sum_s \frac{\alpha'_s}{\beta_s'^2}\left(\gamma + \log\frac{\beta'_s}{2}\right) I_0(\beta'_s r)\right] + O(r^{2q+2}\log r).$$

Therefore, F_{33} is continuously differentiable up to order $2p-1$, and $F^{\alpha\beta}$—up to order $2\min(p, q) - 1$.

9. A dislocation line in an isotropic medium

As the last example, we consider an infinitely long straight dislocation line. We shall assume that the z axis coincides with the dislocation line and make use of polar coordinates in the x, y plane:

$$\theta = \operatorname{arc\,tg}\frac{y}{x}, \qquad r = \sqrt{x^2+y^2}. \tag{9.1}$$

A dislocation is defined by the following condition: for any complete circuit around the dislocation line, the displacements at the end and starting point, respectively, must differ from each other by the Burgers vector **b**. Thus, for the displacement field **u** of a dislocation, we can write

$$\mathbf{u} = -\mathbf{b}\frac{\theta}{2\pi} + \text{a single-valued displacement field.} \tag{9.2}$$

The single-valued part of the displacement field should be determined from the condition that the medium remains in static equilibrium with no external forces acting on it. Putting $\mathbf{f} = 0$, we can safely make use of Eq. (4.2) throughout the medium except the dislocation line. This, however, is not sufficient: considering the equation of equilibrium at $r \neq 0$ does not enable us to get rid of possible terms arising from distributions of forces concentrated at the dislocation line. Therefore, it is necessary to consider equations of equilibrium for arbitrary values of r, including $r = 0$. On the other hand, we cannot make direct use of Eq. (4.2) at $r = 0$. This equation is valid for single-valued displacement fields, because in deriving it we made use of the fact that the second order derivatives commute:

$$u_{i,jk} = u_{i,kj}. \tag{9.3}$$

For θ we have

$$\begin{aligned} \frac{\partial}{\partial y}\frac{\partial\theta}{\partial x} &= \frac{y^2-x^2}{r^4} + \pi\delta(x)\delta(y), \\ \frac{\partial}{\partial x}\frac{\partial\theta}{\partial y} &= \frac{y^2-x^2}{r^4} - \pi\delta(x)\delta(y), \end{aligned} \tag{9.4}$$

so that relation (9.3) is not valid at the dislocation line. In order to obtain the equation valid for multivalued dislocation fields, we shall write the expression for the symmetric stress tensor which, for single-valued displacement fields, should be compatible with Eq. (4.2):

$$\sigma_{ik} = a(\Delta)(u_{i,k}+u_{k,i}) + [b(\Delta)-2a(\Delta)]u_{l,l}\delta_{ik}. \tag{9.5}$$

Then, the equation $\sigma_{ik,k} = 0$ takes the form:

$$a(\Delta)u_{i,kk} + [b(\Delta)-a(\Delta)]u_{k,ki} + a(\Delta)(u_{k,ik}-u_{k,ki}) = 0. \tag{9.6}$$

Let the Burgers vector of the dislocation be $\mathbf{b} = (b_1, b_2, b_3)$ and the displacement vector, which does not depend on z, be $\mathbf{u} = (u, v, w)$. In place of Eq. (9.6) we can now write

$$a(\Delta)\Delta u+[b(\Delta)-a(\Delta)]\frac{\partial}{\partial x}\operatorname{div}\mathbf{u}+a(\Delta)\left(\frac{\partial^2}{\partial y\partial x}-\frac{\partial^2}{\partial x\partial y}\right)v = 0,$$

(9.7) $$a(\Delta)\Delta v+[b(\Delta)-a(\Delta)]\frac{\partial}{\partial y}\operatorname{div}\mathbf{u}+a(\Delta)\left(\frac{\partial^2}{\partial x\partial y}-\frac{\partial^2}{\partial y\partial x}\right)u = 0,$$

$$a(\Delta)\Delta w = 0,$$

with

(9.8) $$\Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}, \qquad \operatorname{div}\mathbf{u} = \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}.$$

Taking into account the fact that

(9.9) $$\Delta\theta = 0$$

in the entire x, y plane, we see that the classical solution for the screw dislocation

(9.10) $$w = -b_3\frac{\theta}{2\pi}, \qquad u = v = 0,$$

remains valid.

With the two-dimensional permutation symbol

(9.11) $$\varepsilon_{12} = -\varepsilon_{21} = 1, \qquad \varepsilon_{11} = \varepsilon_{22} = 0,$$

the first two Eqs. (9.7) can be written as

(9.12) $$a(\Delta)u_{\alpha,\beta\beta}+[b(\Delta)-a(\Delta)]u_{\beta,\beta\alpha}+a(\Delta)\left(\frac{\partial^2}{\partial y\partial y}-\frac{\partial^2}{\partial x\partial y}\right)\varepsilon_{\alpha\beta}u_\beta = 0.$$

For the edge dislocation it is convenient to put

(9.13) $$u_\alpha = \mathring{u}_\alpha+u'_\alpha,$$

where

(9.14) $$\mathring{u}_\alpha = \frac{1}{2\pi}(-b_\alpha\theta+\varepsilon_{\alpha\beta}b_\beta\log r).$$

Then, making use of the following equations

(9.15) $$\Delta\log r = -2\pi\delta^{(2)}(\mathbf{x}),\qquad \frac{\partial\theta}{\partial x} = -\frac{\partial\log r}{\partial y},\qquad \frac{\partial\theta}{\partial y} = \frac{\partial\log r}{\partial x},$$

together with Eqs. (9.4) and (9.9), we see that

(9.16) $$\mathring{u}_{\beta,\beta} = 0$$

and

(9.17) $$\Delta\mathring{u}_\alpha+\left(\frac{\partial^2}{\partial y\partial x}-\frac{\partial^2}{\partial x\partial\partial}\right)\varepsilon_{\alpha\beta}u_\beta = -2\varepsilon_{\alpha\gamma}b_\gamma\delta^{(2)}(\mathbf{x}).$$

Hence, putting the expression (9.13) into Eq. (9.12), we obtain the following equation for u'_α:

$$a(\Delta)u'_{\alpha,\beta\beta}+[b(\Delta)-a(\Delta)]u'_{\beta,\beta\alpha} = 2\varepsilon_{\alpha\gamma}b_\gamma a(\Delta)\delta^{(2)}(\mathbf{x}). \tag{9.18}$$

The solution of this equation can be represented as

$$u'_\alpha = -2\varepsilon_{\beta\gamma}b_\gamma a(\Delta)F_{\alpha\beta}, \tag{9.19}$$

where $F_{\alpha\beta}$ is the plane fundamental solution (8.6).

To calculate $a(\Delta)F_{\alpha\beta}$, let us note that for $r \neq 0$ the following equations hold:

$$\begin{gathered} a(\Delta)\log r = \mu\log r,\\ a(\Delta)\frac{r^2}{4}(\log r-1) = \mu\frac{r^2}{4}(\log r-1)-\mu\sum_s\frac{1}{\beta_s^2}\log r,\\ a(\Delta)K_0(\beta r) = a(\beta^2)K_0(\beta r). \end{gathered} \tag{9.20}$$

Then we obtain

$$a(\Delta)F_{\alpha\beta} = -\frac{1}{2\pi}\delta_{\alpha\beta}\log r+\frac{1}{2\pi}\frac{\lambda-\mu}{\lambda+2\mu}\partial_\alpha\partial_\beta\frac{r^2}{4}(\log r-1)-\frac{1}{2\pi}\frac{\mu}{\lambda+2\mu}\partial_\alpha\partial_\beta\psi, \tag{9.21}$$

where

$$\psi = \sum_s\frac{\alpha'_s}{\beta_s'^2}\frac{a(\beta_s'^2)}{\mu}K_0(\beta'_s r)+\left(\sum\frac{1}{\beta_s'^2}-\sum\frac{1}{\beta_s^2}\right)\log r. \tag{9.44}$$

Finally, we have

$$u_\alpha = u_\alpha^{\mathrm{class}}+\frac{1}{\pi}\frac{\mu}{\lambda+2\mu}\varepsilon_{\beta\gamma}b_\gamma\partial_\alpha\partial_\beta\psi, \tag{9.23}$$

where

$$u_\alpha^{\mathrm{class}} = -\frac{1}{2\pi}\left(b_\alpha\theta+\frac{\mu}{\lambda+2\mu}\varepsilon_{\alpha\beta}b_\gamma\log r-\frac{\lambda+\mu}{\lambda+2\mu}\varepsilon_{\beta\gamma}b_\gamma\frac{x_\alpha x_\beta}{r^2}\right)+\mathrm{const} \tag{9.24}$$

represents the classical solution for the edge dislocation.

Let us note that when the polynomials $a(\Delta)$ and $b(\Delta)$ have the same roots—i.e.,

$$\frac{1}{\mu}a(\Delta) = \frac{1}{\lambda+2\mu}b(\Delta), \tag{9.25}$$

then $\psi = 0$ and the whole solution for the edge dislocation becomes classical.

In general, the classical solution is modified. For large values of r, the general features of this modification are quite similar to those previously discussed. In this case, apart from the classical terms, we have the term proportional to

$$\partial_\alpha\partial_\beta\log r = \frac{1}{r^2}\left(\delta_{\alpha\beta}-\frac{2x_\alpha x_\beta}{r^2}\right), \tag{9.26}$$

which is $O(1/r^2)$, and a finite number of terms which decrease exponentially at infinity. Both of these terms can vanish identically in certain particular cases: the $O(1/r^2)$ term vanishes when

$$\sum\frac{1}{\beta_s'^2} = \sum\frac{1}{\beta_s^2}, \tag{9.27}$$

and the exponential terms vanish when all $\beta_s'^2$ are roots of $a(\Delta)$, i.e. when

$$a(\Delta) = b(\Delta)c(\Delta), \tag{9.28}$$

where $c(\Delta)$ is another polynomial in Δ.

In the previous sections we have seen that the solutions corresponding to given forces **f** are more regular than the classical ones. This no longer holds for dislocations: the solution for the screw dislocation, being entirely classical, retains its classical singularity at $r = 0$. The situation may be even worse for the edge dislocation.

The type of singularity depends mainly on the relation between p and q, the orders of the polynomials $a(\Delta)$ and $b(\Delta)$. Consider first the case $p < q$. Then, according to Eqs. (6.14), (6.15) and (6.16), the following equation holds:

$$\sum \frac{\alpha_s'}{\beta_s'^2} \frac{a(\beta_s'^2)}{\mu} = \sum \frac{1}{\beta_s'^2} - \sum \frac{1}{\beta_s^2}. \tag{9.29}$$

Putting expansion (8.12) into Eq. (9.22), we see that the log terms in ψ cancel each other, so that

$$\psi = -\frac{r^2}{4}(\log r - 1) + O(r^{2+2q-2p}\log r) + \text{a regular part}. \tag{9.30}$$

Thus,

$$\partial_\alpha \partial_\beta \psi = -\frac{1}{2}\left(\delta_{\alpha\beta}\log r + \frac{x_\alpha x_\beta)}{r^2}\right) + O(r^{2(q-p)}\log r) + \text{a regular part}, \tag{9.31}$$

$$u_\alpha = -\frac{1}{2\pi}\left(b_\alpha \theta + \frac{2\mu}{\lambda+\mu}\varepsilon_{\alpha\gamma} b_\gamma \log r - \frac{\lambda}{\lambda+2\mu}\varepsilon_{\beta\gamma} b_\gamma \frac{x_\alpha x}{r^2}\right) + O(r^{2(q-p)}\log r) + \text{a regular part}. \tag{9.32}$$

Hence, in the case of $p < q$, the singularity of the edge dislocation field at $r = 0$ is, with different numerical coefficients, of the classical type.

In the case of $p = q$, Eq. (9.29) is still valid, but instead of (9.30) we obtain

$$\psi = \left(\frac{\beta_1'^2 \beta_2'^2 \ldots \beta_p'^2}{\beta_1^2 \beta_2^2 \ldots \beta_p^2} - 1\right)\frac{r^2}{4}(\log r - 1) + O(r^4 \log r) + \text{a regular part}. \tag{9.33}$$

Thus, also in this case the singularity remains classical, the modification of the numerical coefficients being different from that in the case of $p < q$.

In the case in which $p > q$, Eq. (9.29) is not valid. Then,

$$\psi = -\sum \frac{\alpha_s'}{\beta_s'^2} \operatorname{cut} \frac{a(\beta_s'^2)}{\mu} \log r + \text{const} + O(r^2 \log r), \tag{9.34}$$

where cut $a(\beta_s'^2)$ denotes the part of the polynomial $a(\beta_s'^2)$ consisting of terms of orders higher than q. The ψ-term in the solution (9.23) now has a singularity of the type

$$\partial_\alpha \partial_\beta \psi = -\sum \frac{\alpha_s'}{\beta_s'^2} \operatorname{cut} \frac{a(\beta_s'^2)}{\mu} \frac{1}{r^2}\left(\delta_{\alpha\beta} - \frac{2x_\alpha x_\beta}{r^2}\right) + O(\log r), \tag{9.35}$$

which is $O(1/r^2)$, and thus stronger than the classical one.

10. The one dimensional fundamental solution

From the computational point of view, the case of anisotropic media is far more complicated than the isotropic. In particular, the analysis carried out in Secs. 4–9 does not apply here. Nevertheless, although in the general case of anisotropy we are not able to obtain corresponding solutions in an elementary analytic form, we shall discuss some of their important features.

We begin with the one-dimensional fundamental solution which, apart from being of considerable interest as regards certain problems of plane defects or plane boundaries, will be useful in further considerations of the three-dimentional case.

Let us consider the δ-type forces uniformly distributed on a certain plane with the normal unit vector $\mathbf{v}$. Let $\Gamma^{\mathbf{v}}_{ij}(\xi)$ be the solution of the equation

(10.1) $$P_{ij}(\partial)\Gamma^{\nu}_{jk}(\xi) = \delta(\xi)\delta_{ik}$$

subjected to the condition that the corresponding deformations remain bounded at infinity. The symbol ξ denotes the coordinate perpendicular to the plane considered,

(10.2) $$\xi = \mathbf{v}\mathbf{x}.$$

According to the assumption (3.2), in place of $P_{ij}(\partial)$ we can write in the present case

(10.3) $$P_{ij}(\partial) = P_{ij}(\mathbf{v}, \Delta) = -\Delta a_{ij}(\mathbf{v}, \Delta),$$

where

(10.4) $$\Delta = \frac{d^2}{d\xi^2},$$

$P_{ij}(\mathbf{v}, \Delta)$ and $a_{ij}(\mathbf{v}, \Delta)$ being polynomials in Δ. From the correspondence with classical elasticity we have

(10.5) $$a_{ij}(\mathbf{v}, 0) = c_{ikjl}\nu_k\nu_l,$$

where c_{iklj} is the classical tensor of elastic moduli. According to the positive-definiteness condition (3.6), the inverse $P^{-1}_{ij}(i\mathbf{k})$ exists for any real $\mathbf{k} \neq 0$. Therefore the solution of Eq. (10.1) can be represented as the Fourier integral

(10.6) $$\Gamma^{\nu}_{ij}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, P^{-1}_{ij}(\mathbf{v}, -k^2)e^{ik\xi}$$

with the procedure at $k = 0$ described in Sec. 6.

Consider now the values of Δ at which the matrix $P_{ij}(\mathbf{v}, \Delta)$ becomes singular. With the exception of $\Delta = 0$ these values are the roots of the equation

(10.7) $$\det a_{ij}(\mathbf{v}, \Delta) = 0.$$

The matrix $a_{ij}(\mathbf{v}, \Delta)$ being polynomial, Eq. (10.7) has a finite number of roots:

(10.8) $$\beta_1^2(\mathbf{v}), \beta_2^2(\mathbf{v}), \ldots, \beta_r^2(\mathbf{v}),$$

which are algebraic functions of the direction $\mathbf{v}$. The same reasons as in the isotropic case enable us to accept inequalities (4.9) for $\beta_s(\mathbf{v})$ with arbitrary $\mathbf{v}$ and s.

For the sake of simplicity we introduce also an assumption concerning the multiplicity of the roots (10.8).

We define the rank r_s of a root $\beta_s^2(\nu)$ as

$$r_s = 3-\operatorname{rank} a_{ij}\big(\nu, \beta_s^2(\nu)\big). \tag{10.9}$$

The multiplicity of any root $\beta_s^2(\nu)$ cannot be smaller than its rank r_s and we assume that it exactly equals r_s (in the particular case of isotropy this assumption means that each of the polynomials $a(\Delta)$ and $b(\Delta)$ has single roots only).

Under the above assumption, introducing the following symbols:

$$\overset{s}{Q}_{ij} = -\operatorname{res} P_{ij}^{-1}(\nu, \Delta) \quad \text{at} \quad \Delta = \beta_s^2(\nu) \tag{10.10}$$

and

$$\overset{\circ}{Q}_{ij} = -\operatorname{res} P_{ij}^{-1}(\nu, \Delta) = (c_{ikjl}\nu_k\nu_l)^{-1} \quad \text{at} \quad \Delta = 0, \tag{10.11}$$

we can express the integral (10.6) as

$$\Gamma_{ij}^{\nu}(\xi) = -\frac{1}{2}\overset{\circ}{Q}_{ij}(\nu)|\xi| + \frac{1}{2}\sum_s \frac{\overset{s}{Q}_{ij}(\nu)}{\beta_s(\nu)} e^{-\beta_s(\nu)|\xi|}, \tag{10.12}$$

where the summation runs over all different roots $\beta(\nu)$. Thus, apart from the classical term, the one-dimensional fundamental solution contains only terms which decrease exponentially at infinity. To explain the behaviour of this solution for $\xi \to 0$, let us note that

$$\overset{s}{Q}_{ij}\beta_s^{2m}(\nu) = -\operatorname{res}[P_{ij}^{-1}(\nu, \Delta)\Delta^m] \quad \text{at} \quad \Delta = \beta_s^2(\nu). \tag{10.13}$$

Then, making use of the theorem that the sum of the residues of the matrix-function

$$P_{ij}^{-1}(\nu, z)z^m \tag{10.14}$$

equals zero, we obtain

$$\sum_s \overset{s}{Q}_{ij}(\nu)\beta_s^{2m}(\nu) = (\operatorname{res}_0 + \operatorname{res}_\infty)[P_{ij}(\nu, \Delta)\Delta^m]. \tag{10.15}$$

In order not to be involved in algebraic details, we assume here that the highest order term of the polynomial $P_{ij}(\nu, \Delta)$,

$$\sum_{|\alpha|=2n} a_{ij\mu}\nu^{\mu}, \tag{10.16}$$

is not only positive semi-definite, as follows from the stability condition, but also positive--definite. (In the case of an isotropic medium, the corresponding assumption is $p = q$.) Then we have

$$\sum_s \overset{s}{Q}_{ij}(\nu) = -\overset{\circ}{Q}(\nu) \qquad (n \geqslant 2),$$

$$\sum_s \overset{s}{Q}_{ij}(\nu)\beta_s^{2m}(\nu) = 0 \quad \text{for} \quad m = 1, 2, \ldots, n-2, \ (n \geqslant 3), \tag{10.17}$$

$$\sum_s \overset{s}{Q}_{ij}(\mathbf{v})\beta_s^{2m}(\mathbf{v}) = -\Big(\sum_{|\mu|=2n} a_{ij\mu}\nu_\mu\Big)^{-1} \overset{\text{df}}{=} -R_{ij}(\mathbf{v}) \quad \text{for} \quad m = n-1, (n \geqslant 3).$$

Hence we have for $\xi \to 0$

$$(10.18) \qquad I_{ij}^{\nu} = \frac{1}{2}\frac{1}{(2n-1)!}R_{ij}(\mathbf{v})\,|\xi|^{2n-1} + O(|\xi|^{2n+1}) + \text{a regular part}.$$

11. The three-dimensional fundamental solution for an anisotropic medium

The three-dimensional fundamental solution can be easily obtained from the one-dimensional solution by making use of the following equation:

$$(11.1) \qquad \delta^{(2)}(\mathbf{x}) = -\frac{1}{8\pi^2}\int da\,\delta''(\mathbf{v}\mathbf{x}),$$

where the integration runs over the unit sphere $(\mathbf{v})^2 = 1$. Then we have

$$(11.12) \qquad G_{ij}(\mathbf{x}) = -\frac{1}{8\pi^2}\int da\frac{d^2}{d\xi^2}I_{ij}^{\nu}(\xi),$$

with ξ given by Eq. (10.2).

In the classical case, when the solution (10.12) contains its first term only, expression (11.2) takes the form

$$(11.3) \qquad G_{ji}^{\text{class}}(\mathbf{x}) = \frac{1}{8\pi^2}\int da\,(c_{ikjl}\nu_k\nu_l)^{-1}\delta(\mathbf{v}\mathbf{x}).$$

Introducing a spherical system of coordinates such that

$$(11.4) \qquad \mathbf{v}\mathbf{x} = r\cos\theta, \qquad da = \sin\theta\,d\theta\,d\varphi,$$

we can write

$$(11.5) \qquad \delta(\mathbf{v}\mathbf{x}) = \frac{1}{r}\delta\left(\frac{\pi}{2}-\theta\right)$$

and

$$(11.6) \qquad G_{ij}^{\text{class}} = \frac{1}{8\pi^2 r}\int_C d\varphi\,(c_{ikjl}\nu_k\nu_l)^{-1},$$

where we integrate over the unit circle in the plane perpendicular to the vector $\mathbf{x}$ (Fig. 4). In the non-classical case, $(n \geqslant 2)$, we make use of the following equation:

$$(11.7) \qquad \frac{d^2}{d\xi^2}e^{-\beta|\xi|} = \beta^2 e^{-\beta|\xi|} - 2\beta\delta(\xi).$$

Then, by the first Eq. (10.17), the δ-terms cancel each other and we obtain

$$(11.8) \qquad G_{ij}(\mathbf{x}) = -\frac{1}{16\pi^2}\int da\sum_s \beta_s(\mathbf{v})\overset{s}{Q}_{ij}(\mathbf{v})e^{-\beta_s(\mathbf{v})|\mathbf{v}\mathbf{x}|}.$$

To obtain the asymptotic expansion of $G_{ij}(\mathbf{x})$ for $r \to 0$, we put the expansion (10.18) into the expression (11.2). Then we have

$$(11.9) \qquad G_{ij}(\mathbf{x}) = -\frac{1}{16\pi^2}\frac{1}{(2n-3)!}\int da\, R_{ij}(\mathbf{v})|\cos\theta|^{2n-3}r^{2n-3} + O(r^{2n-1}) + \text{a regular part},$$

where θ denotes the angle between $\mathbf{x}$ and $\mathbf{v}$.

Thus the most singular term of $G_{ij}(\mathbf{x})$ at $r = 0$ is of order $2n-4$, so that the gradients of $G_{ij}(\mathbf{x})$ up to order $2n-4$ are continuous.

Now let us briefly examine the behaviour of $G_{ij}(\mathbf{x})$ at infinity. Note that a slowly decreasing contribution to the integral (11.8) can arise only from an immediate neighbourhood of these points for which the exponent $\beta_s(\mathbf{v})|\mathbf{vx}|$ equals zero. The contribution from

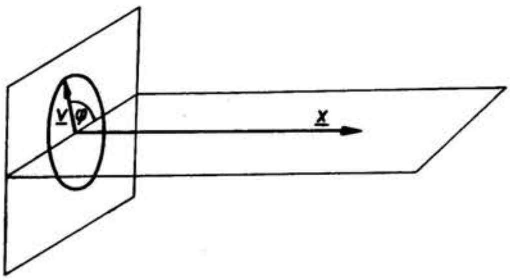

FIG. 4.

the remaining area decreases exponentially as $r \to \infty$. Since $\beta_s(\mathbf{v}) \neq 0$, the exponent equals zero only for $\mathbf{v} \perp \mathbf{x}$—i.e., on the circle C.

Making use of the fact that $\beta_s(\mathbf{v})$ and $Q_{ij}(\mathbf{v})$ are even functions of $\mathbf{v}$, the integral (11.8) can be rewritten as

$$(11.10) \qquad G_{ij}(\mathbf{x}) = -\frac{1}{8\pi^2}\int_0^{2\pi} d\varphi \int_0^1 dq \sum_s \beta_s(\mathbf{v}) \overset{s}{Q}_{ij}(\mathbf{v}) e^{-\beta_s(\mathbf{v})qr},$$

where

$$(11.11) \qquad q = \cos\theta.$$

Integrating this by parts, by means of the equation

$$(11.12) \qquad e^{-\beta_s(\mathbf{v})qr} = -\frac{B_s(\mathbf{v})}{r}\frac{d}{dq}e^{-\beta_s(\mathbf{v})qr},$$

where

$$(11.13) \qquad B_s(\mathbf{v}) = \frac{1}{\dfrac{d(\beta_s(\mathbf{v})q)}{dq}},$$

we obtain:

$$(11.14)\qquad G_{ij}(\mathbf{x}) = \frac{1}{8\pi^2 r}\int\limits_0^{2\pi} d\varphi \sum_s B_s(\mathbf{v})\beta_s(\mathbf{v})\overset{s}{Q}_{ij}(\mathbf{v})e^{-\beta_s(\mathbf{v})qr}\Big|_{q=0}^{q=1} - \frac{1}{8\pi^2 r}\int\limits_0^{2\pi} d\varphi \int\limits_0^1 dq\left(\frac{d}{dq}\sum_s B_s(\mathbf{v})\beta_s(\mathbf{v})\overset{s}{Q}_{ij}(\mathbf{v})\right)e^{-\beta_s(\mathbf{v})qr}.$$

The first term gives the classical fundamental solution at $q = 0$ and an exponential term at $q = 1$. The second term can again be integrated by parts. This time, the subintegral function being odd, it gives another exponential term at $q = 1$ but no slowly decreasing term at $q = 0$. Therefore,

$$(11.15)\qquad G_{ij}(\mathbf{x}) = G_{ij}^{\text{class}}(\mathbf{x}) + O\left(\frac{1}{r^3}\right),$$

where

$$(11.16)\qquad O\left(\frac{1}{r^3}\right) = = \int\limits_0^{2\pi} d\varphi \int\limits_0^1 dq\left(\sum_s \frac{d}{dq}B_s(\mathbf{v})\frac{d}{dq}B_s(\mathbf{v})\beta_s(\mathbf{v})\overset{s}{Q}_{ij}(\mathbf{v})\right)e^{-\beta_s(\mathbf{v})qr} + \text{exponential terms.}$$

The main contribution to $O(1/r^3)$ can be calculated by further integration by parts. In this way, we obtain

$$(11.17)\qquad G_{ij}(\mathbf{x}) = G_{ij}^{\text{class}}(\mathbf{x}) - \frac{1}{8\pi r^3}\int\limits_C d\varphi \frac{d^2}{dq^2}\sum_s \frac{\overset{s}{Q}_{ij}(\mathbf{v})}{\beta_s^2(\mathbf{v})} + O\left(\frac{1}{r^5}\right).$$

This procedure can be continued, yielding terms $O(1/r^m)$ with arbitrary odd m plus exponentially decreasing ones.

Conclusion

For isotropic materials of an arbitrary order, the basic solutions corresponding to point or line sources, including a dislocation line, are given in an analytic form. Apart from the classical terms, they contain a certain number of exponentially decreasing short-range terms and, usually, a long-range term which descends a little more rapidly than the classical one.

As the discussion of the three-dimensional fundamental solution shows, the asymptotic properties of solutions for anisotropic materials are similar, but instead of a single long-range non-classical term there occurs a series of terms of different order.

All the solutions corresponding to force-type sources are more regular than the classical ones.

Even in the case of the lowest order of the strain gradient theory, the fundamental solutions are continuous. For the higher order theories, these solutions become continuously differentiable an appropriate number of times. This fact, in particular, creates the possibility of obtaining non-singular interactions of point defects within the framework of the strain gradient theory. On the other hand, the solutions corresponding to dislocation-type sources retain at least classical singularity. In some particular cases, this singularity can even be stronger than the classical one.

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