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Dynamic boundary problem

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In this paper, the initial-boundary problem for hyperbolic equation by means of the solution of the elliptic problem with parameter is solved.

W pracy rozwiązuje się problem początkowo-brzegowy dla równania hiperbolicznego za pomocą rozwiązania eliptycznego problemu brzegowego z parametrem.

В работе приводится решение начально-краевой проблемы для гиперболического уравнения при помощи решения эллиптической задачи с параметром.

1. Introduction

The aim of this paper is to construct the solutions of the initial-boundary problem for a normal hyperbolic equation [1] of an arbitrary order, in a finite cylinder $\Omega' = [t_2, t_1]\times\Omega$, the differential operator having the form $A = E_t+L$. Here E_t is an elliptic operator dependent on a parameter. The system of the boundary conditions B_t^j is normal in the sense of [3] and the boundary problem $E_t u_t = \psi$ in Ω, $B_t^j u_t = g^j$ on Γ is a well posed elliptic boundary problem [2] dependent on a parameter t.

The solution is constructed under the assumption that we know the solution $u_t = u_t(\psi, g^j)$ of an elliptic boundary problem with a parameter as a function of arbitrary admissible ψ, g^j. By means of the relation $u_t = u_t(\psi)+u_t(g^j)$ we satisfy an arbitrary boundary condition in the hyperbolic problem, while ψ is chosen in such a way that the hyperbolic equation is satisfied in the weak sense

$$\big(u_t(\psi), \bar{A}\varphi\big)+\big(u_t(g^j), \bar{A}\varphi\big) = (f, \varphi),$$

where g^j and f are known, the initial condition being $u_t = 0$ for $t < 0$.

An equivalent form is the equation for ψ

$$(\psi, \bar{u}_t\bar{A}\varphi) = (f, \varphi)-\big(u_t(g^j), \bar{A}\varphi\big).$$

In the paper, we make an essential use of the theorem on regularity of the solutions of an elliptic equation with respect to the parameter, proved in [1].

We now proceed to the detailed statement of the problem.

Let $\Omega' = (t_2, t_1)\times\Omega$ be a cylindrical domain in the Cartesian space R_{xt}^{n+1} the elements of which are the sequences $x' = (x_1, x_2, \ldots, x_n, t)$ of real numbers, Ω is a bounded domain in R_x^n the boundary Γ of which is an $(n-1)$ -dimensional manifold of class C^∞, (t_2, t_1) is an open interval, bounded in R_x^1 and containing the origin.

Consider the differential operator

$$A = \sum_{|\alpha| \leq m} a_\alpha(x, t) D^\alpha_{xt}, \tag{1.1}$$

the coefficients of which belong to the space $C^\infty(G)$ where $\Omega' \Subset G \subset R^{n+1}_{xt}$, i.e., Ω' is contained in G together with the closure, $D^\alpha_{xt} = i^{-p} \frac{\partial}{\partial x_{\alpha_1}} \cdot \frac{\partial}{\partial x_{\alpha_2}} \cdot \ldots \frac{\partial}{\partial x_{\alpha_p}}$, $x = (x_1, \ldots, x_n)$, $x' = (x, t)$, α is a multi-index, i.e., a sequence $\alpha_1, \alpha_2, \ldots, \alpha_p$ of the indices α_ν each of the latter being an integer greater than 1 and smaller than $(n+1)$ and $x_{n+1} \equiv t$; $|\alpha|$ is the number of the indices α_ν in the sequence α and $|\alpha| = p$.

We assume that A is a strictly hyperbolic operator [1] with respect to every plane $t = \text{const}$, $(x, t) \in G$.

The operator A is associated with the system of boundary operators

$$\begin{gathered} B^j_t = \sum_{|\alpha| \leq m_j} b_{j\alpha}(x, t) D^\alpha_x, \\ b_{j\alpha}(x, t) \in C^\infty(\Gamma'), \qquad \Gamma' = \Gamma \times (t_2, t_1). \end{gathered} \tag{1.2}$$

We assume that A has the form

$$A = E_t + L, \tag{1.3}$$

where E_t is an elliptic operator dependent on the parameter t, such that for every fixed $t \in (t_2, t_1) \supset [t'_2, t'_1]$, the boundary problem

$$\begin{aligned} E_t u_t &= f_t \text{ in } \Omega, \\ B^j_t u_t &= g^j_t \text{ on } \Gamma, \quad j = 1, 2\text{-} \ldots, l \end{aligned} \tag{1.4}$$

is elliptic [2].

The system of boundary operators B^j_t is normal with respect to the operator E_t, i.e., there exist differential operators $\overline{B}^j_t$, S^j_t and $\overline{S}^j_t$ such that the Green formula [3]

$$\int_\Omega (E_t u\bar{v} - u\bar{E}_t \bar{v}) dx = \sum_{j=1}^{l} \int_\Gamma (B^j_t u \bar{S}^j_t \bar{v} - S^j_t u \bar{B}^j_t \bar{v}) d\Gamma \tag{1.5}$$

holds for all u and $v \in C^\infty(\bar{\Omega})$ with fixed $t \in (t'_2, t'_1)$. The example is the classical Green formula for the Laplace operator

$$\int_\Omega (\Delta u\bar{v} - u\Delta\bar{v}) dx = \int_\Gamma \left(\frac{\partial u}{\partial n} \bar{v} - u \frac{\partial \bar{v}}{\partial n} \right) d\Gamma.$$

The adjoint to the boundary problem (1.4) is the following:

$$\begin{aligned} \bar{E}_t v &= h_t \quad \text{in } \Omega, \\ \bar{B}^j_t v_t &= d_t \quad \text{on } \Gamma, \quad j = 1, 2, \ldots, l. \end{aligned} \tag{1.6}$$

The boundary problems (1.4) and (1.6) are connected by the [3].

THEOREM 1. *In order that the elliptic boundary problem* (1.4) *with a normal system of boundary operators B have a solution, it is necessary and sufficient that the relation*

$$\int_{\Omega} f_t \bar{\mathring{v}}_t dx - \sum_{j=1}^{l} \int_{\Gamma} g_t^j \bar{S}_t^j \bar{\mathring{v}}_t d\Gamma = 0 \tag{1.7}$$

holds for every solution $\mathring{v}_t$ of the homogeneous boundary problem

$$\begin{aligned} \bar{E}_t \mathring{v}_t &= 0 \quad \text{in } \Omega, \\ \bar{B}_t^j \mathring{v}_t &= 0 \quad \text{on } \Gamma, \quad j = 1, 2, \ldots, l. \end{aligned} \tag{1.8}$$

The theorem is also true when the operators E_t, B_t^j and $\bar{E}_t$, $\bar{B}_t^j$ are exchanged.

Moreover, we assume that the assumptions of the theorem on the regularity of an elliptic boundary problem with respect to the parameter t are satisfied. In particular, we assume that the dimension of the null subspace N_t, i.e. the set of solutions of the boundary problem

$$\begin{aligned} E_t \mathring{u}_t &= 0 \quad \text{in } \Omega, \\ B_t^j \mathring{u} &= 0 \quad \text{on } \Gamma, \quad j = 1, 2, \ldots, l \end{aligned} \tag{1.9}$$

is equal to σ and independent of t: $\sigma = \dim N_t$.

This theorem, proved in [1], implies that there exists in N_t the basis $\mathring{u}_t^1, \mathring{u}_t^2, \ldots, \mathring{u}_t^\sigma$, such that $\mathring{u}_t^r$ is of class C^∞ with respect to t.

The operator L in the formula (1.3) has the form

$$L = \sum_{\substack{|\alpha| < \eta \\ |\beta| \leqslant m}} a_\alpha(x, t) D_x^\alpha D_t^\beta, \tag{1.10}$$

where η is a number to be defined later.

The initial-boundary problem is formulated as follows: in an appropriate subspace of distributions $D'(\Omega')$ we seek u_t such that

$$\begin{aligned} Au_t &= f_t \quad \text{in } \Omega', \\ B_t^j u_t &= g_t^j \quad \text{on } \Gamma', \quad j = 1, \ldots, l, \quad u_t = 0 \quad \text{for} \quad t < 0 \end{aligned} \tag{1.11}$$

for given f_t, g_t^j.

2. Some function spaces

We shall now introduce certain function spaces in the language in which the construction of the solution of the problem (1.4) is carried out.

The Banach space of distributions $H^{r,s}(R_{xt}^{n+1})$ is the completion of the space $C^\infty(R_{xt}^{n+1})$ of infinitely differentiable functions f with compact supports in the norm

$$\|f; R_{xt}^{n+1}\|_{r,s} = \left[\int (1+|\xi'|^2)^r (1+|\xi|^2)^s |f(\xi')|^2 d\xi'\right]^{1/2}, \tag{2.1}$$

(ξ') is the Fourier transform of the function $f(x')$,

$$\xi' = (\xi_1, \xi_2, \ldots, \xi_{n+1}), \; |\xi'|^2 = \xi_1^2 + \xi_2^2 + \ldots + \xi_{n+1}^2,$$

$\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, r, s are arbitrary real numbers.

The space dual to $H^{r,s}(R_{xt}^{n+1})$ is the space $H^{-r,-s}(R_{xt}^{n+1})$. $H^{r,s}(\Omega')$ is the space of distributions f over Ω' such that f is the restriction to Ω' of a $f_1 \in H^{r,s}(R_{xt}^{n+1})$ and then we set

$$\|f; \Omega'\|_{r,s} = \inf \|f_1; R_{xt}^{n+1}\|_{r,s}, \tag{2,2}$$

where inf is taken over all extensions of f_1. The space $H^{r,s}(\Omega')$ is the completion in the norm (1.2) of the function space $C^\infty(\bar{\Omega}')$. The space dual to $H^{r,s}(\Omega')$ is $H_{\bar{\Omega}'}^{-r,-s}$ consisting of the elements $F \in H^{-r,-s}(R_{xt}^{n+1})$ the supports of which are contained in $\bar{\Omega}'$.

The space $H_{\bar{\Omega}'}^{-r,-s}$ is a completion in the norm $\|F; R_{xt}^{n+1}\|_{-r,-s}$ of the subspace $C_{\Omega'}^\infty$ of functions $F \in C_0^\infty(R_{xt}^{n+1})$ with supports contained in Ω'.

Remark. In the definition of the norm (2.2) the class of extensions f_1 of the function f can be restricted by assuming that the supports of f_1 are contained in an interval $(\tilde{t}_2, \tilde{t}_1)$, such that $[t_2, t_1] \subset (\tilde{t}_2, \tilde{t}_1) \Subset (t_2', t_1')$, (e.g. $[\tilde{t}_2, \tilde{t}_1] \subset (t_2', t_1')$); then we obtain an equivalent norm. In what follows we assume that f_1 do have this property.

Let $\sum_\nu \varphi_\nu(x) = 1$, $\varphi(x) \geqslant 0$, $\varphi_\nu \in C_0^\infty(R_x^n)$ be a finite partition of unity [1] in the neighbourhood of Ω. We assume that the surface Γ can be covered by a set of domains σ_ν, such that $\sigma_\nu \cap \Omega$ is mapped one-to-one by means of an infinitely differentiable function $\varkappa_\nu = \varkappa_\nu(\zeta)$, $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n)$; $\varkappa_\nu > 0$, into parts of the semi-space $\bar{\sigma}_\nu \subset R_\zeta^n$, in such a manner that $\Gamma_\nu = \sigma_\nu \subset \Gamma$ is a set of points with the coordinates $\varkappa_\nu(\zeta_1, \ldots, \zeta_{n-1}, 0)$, where $\zeta = (\zeta_1, \ldots, \zeta_{n-1}) \in \Gamma_\nu \subset R_\zeta^{n-1}$. Assume that ν takes values for which the support of φ_ν has common points with Γ under the additional assumption that the support of φ_ν is contained in σ_ν. Since $\Gamma' = [t_2, t_1] \times \Gamma$, the local coordinate system for Γ' is given by the mapping $(x_{n+1}, x) \to (\zeta_{n+1}, \varkappa(\zeta))$.

Denote by Γ'_∞ the Cartesian product $\Gamma'_\infty = R_1^t \times \Omega$. For $f \in C_0^\infty(\Gamma_\infty)$, there exists the norm

$$\|f; \Gamma'_\infty\|_{r,s} = \Big[\sum_\nu \|\varphi_\nu f; R_\zeta^n\|_{r,s}\Big]^{1/2}, \tag{2.3}$$

defined in the local coordinate systems.

The space $H^{r,s}(\Gamma'_\infty)$ is a completion in the norm (2.3) of the space $C_0^\infty(\Gamma'_\infty)$; the space $H^{-r,-s}(\Gamma'_\infty)$ is dual to $H^{r,s}(\Gamma'_\infty)$.

Furthermore, we denote by $H^{r,s}(\Gamma')$ the space of distributions f over Γ', such that f is a restriction to Γ' of a $f_1 \in H^{r,s}(\Gamma'_\infty)$ with the norm

$$\|f; \Gamma'\|_{r,s} = \inf \|f_1; \Gamma'_\infty\|_{r,s}, \tag{2.4}$$

inf being taken over all extensions f_1. In accordance with our remark we confine ourselves to the extensions f_1 the supports of which are contained in $(\tilde{t}_2, \tilde{t}_1)$.

The dual space $H_{\Gamma'}^{-r,-s}$ consists of elements $F \in H^{-r,-s}(\Gamma'_\infty)$ the supports of which are contained in (t_2, t_1).

By means of the above-introduced spaces we define the Cartesian product

$$(2.5) \qquad K^{r,s}(\Omega') = H^{r,s-m}(\Omega') \times H^{r,s-m_1-1/2}(\Gamma') \times \ldots \times H^{r,s-m_l-1/2}(\Gamma').$$

The norm of the element $v = (f_1, g_1, \ldots, g_l) \in K^{r,s}(\Omega')$ has the form

$$(2.6) \qquad |||v; \Omega'|||_{r,s} = \|f;\Omega'\|_{r,s-m} + \|g_1; \Gamma'\|_{r,s-m_1-1/2} + \ldots + \|g_l;\Gamma'\|_{r,s-m_l-1/2}.$$

The dual space is the following:

$$(2.7) \qquad K_{\Omega'}^{-r,-s} = H_{\Omega'}^{-r,-s+m} \times H_{\Gamma'}^{-r,-s+m_1+1/2} \times \ldots \times H_{\Gamma'}^{-r,-s+m_l+1/2},$$

the norm of the element $V = (F, G_1, \ldots, G_l) \in K_{\Omega'}^{-r,-s}$, being

$$(2.8) \qquad |||V; R_{xt}^{n+1}|||_{-r,-s} = \|F; R_{xt}^{n+1}\|_{-r,-s+m} + \|G_1; \Gamma'_\infty\|_{-r,-s+m_1+1/2} + \ldots + \|G_l; \Gamma'_\infty\|_{-r,-s+m_l+1/2}.$$

The functionals in the space $H^{r,s}(\Omega')$ have the form

$$(2.9) \qquad (u, U) = \int u(\xi')\bar{U}(\xi')\,d\xi', \qquad \text{where } U \in H_{\Omega'}^{-r,-s}.$$

The spaces $H^{r,s}(\Omega')$ and $H_{\Omega'}^{-r,-s}$ have the structure of a Hilbert space with the scalar product in $H^{r,s}(\Omega')$

$$(2.10) \qquad (u, w; \Omega')_{r,s} = \int (1+|\xi'|^2)^r(1+|\xi|)^s u_1(\xi')\overline{w_1(\xi')}\,d\xi',$$

where u_1, w_1 are extensions of u, w to a function belonging to $H^{r,s}(R_{xt}^{n+1})$, such that for an arbitrary $\varphi \in C_0^\infty(R_{xt}^{n+1}/\Omega')$,

$$(u_1, \varphi; R_{xt}^{n+1})_{r,s} = (w_1, \varphi; R_{xt}^{n+1})_{r,s} = 0.$$

The scalar product in $H_{\Omega'}^{-r,-s}$ is identical with that in $H^{-r,-s}(R_{xt}^{n+1})$; it has the form

$$(2.11) \qquad (U, W; R_{xt}^{n+1})^{-r,-s} = \int (1+|\xi'|^2)^{-r}(1+|\xi|^2)^{-s}U(\xi')\overline{W(\xi')}\,d\xi'.$$

The functionals in $K^{r,s}(\Omega')$ have the form

$$(2.12) \qquad (v, V) = (f, F) + (g^1, G^1) + \ldots + (g^l, G^l), \qquad V = (F, G^1, \ldots, G^l) \in K_{\Omega'}^{-r,-s}.$$

The scalar product is defined as follows:

$$(2.13) \qquad (v_1, v_2; \Omega')_{r,s} = (f_1, f_2; \Omega')_{r,s} + (g_1^1, g_2^1; \Gamma')_{r,s-m_l-1/2} + (g_1^l, g_2^l; \Gamma')_{r,s-m_l-1/2} \qquad v_i \in K^{r,s}(\Omega')$$

and similarly, we have for the scalar product in $K_{\Omega'}^{-r,-s}$

$$(2.14) \qquad (V_1, V_2; R_{xt}^{n+1})_{-r,-s} = (F_1, F_2; R_{xt}^{r+1}) + (G_1^1, G_2^1; \Gamma'_\infty)_{-r,-s+m_1+1/2} + \ldots + (G_1^l G_2^l; \Gamma'_\infty)_{-r,-s+m_l+1/2}.$$

Remark: The meaning of the symbol $(\cdot, \cdot)$ in the above definitions is different in different spaces; it becomes unique only when the elements on which it acts are indicated.

3. Elliptic boundary problems with a parameter

In this Section we present the basic results concerning the elliptic boundary problem [1, 2] and then we make certain generalizations to elliptic problems with a parameter in function spaces over the domain Ω'.

Let us replace in the definitions of the spaces $H^{r,s}(\Omega')$ and $K^{r,s}(\Omega')$, Ω', Γ' by the symbols Ω, Γ and set $r = 0$. The obtained spaces denote by $H^s(\Omega)$ and $K^s(\Omega)$.

The corresponding norms have the form

$$\|f;\Omega\|_s = \inf \int (1+|\xi|^2)^s |f_1(\xi)|^2 d\xi,$$

where inf is taken over all extensions f_1 of the function f,

$$|||v;\Omega||| = \|f;\Omega\|_{s-m} + \|g^1;\Gamma\|_{s-m_1-\frac{1}{2}} + \ldots + \|g^l;\Gamma\|_{s-m_l-\frac{1}{2}},$$

and the norms on Γ are in the local coordinate systems. As in the preceding Section, we introduce the scalar products, e.g. for $v_1, v_2 \in K^s(\Omega)$:

$$(v_1, v_2;\Omega)_s = (f_1, f_2;\Omega)_{s-m} + \sum_{i=1}^{l} (g_1^i, g_2^i;\Gamma)_{s-m_i-\frac{1}{2}}.$$

To obtain the dual spaces H_Ω^{-s} and K_Ω^{-s} we perform identical modifications of the definitions of the above-introduced spaces $H_{\Omega'}^{-r,-s}$ and $K_{\Omega'}^{-r,-s}$, i.e., we replace Ω', Γ' by Ω, Γ and set $r = 0$. This concerns also the corresponding definitions of the norms and scalar products in which, moreover, the integration with respect to the variable ξ' is replaced by integration with respect to ξ.

The boundary problem (1.4) is associated with the continuous mapping

$$T_t : u_t \to (E_t u_t, B_t^1 u_t, \ldots, B_t^l u_t) \tag{3.1}$$

of the space $H^s(\Omega)$ into the space $K^s(\Omega)$, where for every t the function $u_t \in H^s(\Omega)$, $s > s_0 = \max(m_1, \ldots, m_l) + \dfrac{1}{2}$.

The mapping T_t satisfies for a fixed t the fundamental inequality

$$\|u_t;\Omega\|_s \leqslant C_t(|||T_t u_t;\Omega|||_s + \|u_t;\Omega\|_{s-1}), \qquad s > s_0. \tag{3.2}$$

The mapping $\overline{T}_t$ dual to T_t

$$\overline{T}_t : (F_t, G_t^1, \ldots, G_t^l) \to \overline{E}_t F_t + \overline{B}_t^1 G_t^1 + \ldots + B_t^l G_t^l \tag{3.3}$$

of the space K_Ω^{-s} into the space H_Ω^{-s} is given by the relation

$$(T_t u_t, V_t) = (u_t, \overline{T}_t V_t) \tag{3.4}$$

for a fixed t and $u_t \in H^s(\Omega)$, $V_t \in K_\Omega^{-s}$. The functionals in both sides of (3.4) are, of course, defined in different Banach spaces.

For a fixed t, the mapping $\overline{T}_t$ satisfies also the fundamental inequality

$$|||V_t;R_x^n|||_{-s} \leqslant \overline{C}_t(\|\overline{T}_t V_t;R_x^n\|_{-s} + |||V_t;R_x^n|||_{-s+1}), \qquad s > s_0 \tag{3.5}$$

Now the boundary problem (1.4) takes the form

$$T_t u_t = v_t; \tag{3.6}$$

and the corresponding problem adjoint to (3.6) can be written as follows [2]:

$$\overline{T}_t V_t = U_t. \tag{3.7}$$

A closed theory of the above two problems was given in [2].

Denote now by N_t the null subspace of the mapping T_t, i.e. the set of elements $\mathring{u}_t$ satisfying the equation

$$T_t \mathring{u}_t = 0. \tag{3.8}$$

The null subspace of the mapping $\overline{T}_t$ consists of elements $\mathring{V}_t$ satisfying the equation

$$\overline{T}_t \mathring{V}_t = 0; \tag{3.9}$$

we denote it by $\overline{N}_t$. In what follows we assume that $\sigma = \dim N_t$ (i.e. the dimension of N_t) and $\overline{\sigma} = \dim N_t$ are independent of t. In view of the theorem on regularity of solutions of elliptic problems [1], there exist bases $\{\mathring{u}_t^\nu\}_{\nu=1}^{\sigma} \subset N_t$ and $\{\mathring{V}_t^\nu\}_{\nu=1}^{\overline{\sigma}} \subset \overline{N}_t$ regular with respect to t, i.e. the mappings

$$t \to \mathring{u}_t^\nu, \qquad t \to \mathring{V}_t^\nu \tag{3.10}$$

are of class C^∞.

One of the fundamental results of the general theory is that for every t the dimensions of N and $\overline{N}_t$ are finite.

Every element $\mathring{u}_t \in N_t$ can be represented in the form of a finite sum

$$\mathring{u}_t = \sum_{\nu=1}^{\sigma} c_\nu(t) \mathring{u}_t^\nu. \tag{3.11}$$

Similarly, if $V_t \in \overline{N}_t$, then

$$\mathring{V}_t = \sum_{\nu=1}^{\overline{\sigma}} \overline{c}_\nu(t) \mathring{V}_t^{\nu}. \tag{3.12}$$

The continuity of the mapping T_t for a fixed t is equivalent to the inequality

$$|||T_t u_t; \Omega|||_s \leqslant c\|u_t; \Omega\|_s, \tag{3.13}$$

where s is an arbitrary real; similarly, the continuity of T implies that

$$\|\overline{T}_t V_t; R_x^n\|_{-s} \leqslant \overline{c}|||.V_t; R_x^n|||_{-s}, \tag{3.14}$$

where the constants $c, \overline{c}$ are independent of $t \in [t_2', t_1']$ and s is an arbitrary real.

If u_t as a function of (x, t) belongs to $C^\infty(\overline{\Omega}')$, (3.13) implies that

$$|||T_t u_t; \Omega'|||_{r,s} \leqslant c\|u_t; \Omega'\|_{r,s}, \tag{3.15}$$

where r, s are arbitrary reals.

The inequality (3.15) suggests an extension of the mapping T_t to the space $H^{r,s}(\Omega')$.

DEFINITION 1. The extension by continuity of the mapping T_t to the space $H^{r,s}(\Omega')$ will be denoted by T.

The dual mapping is given by the formula

$$(Tu, V) = (u, \overline{T}V), \quad V \in K_{\Omega'}^{-r,-s}. \tag{3.16}$$

The inequality (3.15) implies an inequality for $\overline{T}$, namely

$$\|\overline{T}V; R_{xt}^{n+1}\|_{-r,-s} \leqslant \overline{c}|||V; R_{xt}^{n+1}|||_{-r,-s}, \tag{3.17}$$

where r, s are arbitrary reals.

We shall later need a certain subspace.

DEFINITION 2. By $M_{\tilde{\Omega}}^s$ we denote the subspace of mappings $t \xrightarrow{\varphi} \varphi_t$ of class C^∞ of the real R_1^t into the space $H^s(\Omega)$ with the support contained in $(\tilde{t}_2, \tilde{t}_1)$, where

$$s > s_0 = \max(m_1, \ldots, m_l) + \frac{1}{2},$$

$$[t_2, t_1] \subset (\tilde{t}_2, \tilde{t}_1) \Subset (t_2', t_1'),$$

$$\tilde{\Omega} = \Omega \times (\tilde{t}_2, \tilde{t}_1).$$

LEMMA 1. *For every $\varphi \in M_{\tilde{\Omega}}^s$, the inequality*

$$\|\varphi_t; \Omega\|_s \leqslant c(|||T_t\varphi_t; \Omega|||_s + \|\varphi_t; \Omega\|_{s-1} \tag{3.18}$$

holds, the constant C being independent of t.

P r o o f. For a fixed t the value φ_t of the mapping φ belongs to $H^s(\Omega)$. For every $t_\nu \in [\tilde{t}_2, \tilde{t}_1]$, there exists a neighbourhood $\sigma_\nu \subset (t_2', t_1')$ such that $t_\nu \in \sigma_\nu$ and in the neighbourhood σ_ν the inequality (3.2) holds with a constant C_ν independent of t. The Borel theorem implies that there exists a finite number of neighbourhoods $\sigma_\nu (\nu = 1, \ldots, \nu_0)$, such that $\sigma_\nu \subset (t_2', t_1')$ and $[\tilde{t_2'}, \tilde{t}_1] \subset \bigcup_\nu \sigma_\nu$. Taking $C = \max C_\nu, \nu = 1, \ldots, \nu_0$ we arrive at (3.18).

Lemma 1 yields

LEMMA 2. *For $u \in C^\infty(\overline{\Omega}')$, we have the inequality*

$$\|u; \Omega'\|_{0,s} \leqslant c(|||T_t u; \Omega'|||_{0,s} + \|u; \Omega'\|_{0,s-1}), \quad s > s_0. \tag{3.19}$$

P r o o f. We extend u to a function $\varphi \in M_{\tilde{\Omega}}^s$. Substituting into the formula (3.18) and integrating the squares of both sides with respect to t from $+\infty$ to $-\infty$, on the basis of the Parseval equation, taking inf over all extensions, we obtain (3.19).

Similarly to the general theory of elliptic problems, the inequality (3.19) can be modified.

LEMMA 3. *Let $t \to \mathring{u}_t^\nu$ be a mapping of class C^∞ of the real line R_t^1 into the set of elements of a finite basis $\{\mathring{v}_t^\nu\}_{\nu=1}^{\tau} \subset N_t$, such that for a fixed t*

$$(\mathring{u}_t^\nu, \mathring{u}_t^\mu; \Omega)_s = 0, \quad \text{if} \quad \mu \neq \nu \tag{3.20}$$

and in accordance with the above assumptions $\sigma = \dim N_t$ is independent of t; then, there *exists a decomposition of the space $M^s_{\tilde{\Omega}}$ into the orthogonal sum*

$$M^s_{\tilde{\Omega}} = \overset{1}{M}{}^s_{\tilde{\Omega}} \oplus \mathring{M}^s_{\tilde{\Omega}} \tag{3.21}$$

such that for a fixed t

$$(\overset{1}{\varphi}, \mathring{\varphi}; \Omega)_s = 0, \quad \text{if} \quad \overset{1}{\varphi} \in \overset{1}{M}{}^s_{\tilde{\Omega}}, \tag{3.22}$$

$\mathring{M}^s_{\tilde{\Omega}}$ being the space of mappings $\mathring{\varphi}$,

$$t \to \mathring{\varphi}_t = \sum_{\nu=1}^{\sigma} c^\nu(t) \mathring{u}^\nu_t, \quad c^\nu(t) \in C^\infty(R'_t). \tag{3.23}$$

and the support $c^\nu(t) \subset (\tilde{t}_2, \tilde{t}_1)$.

P r o o f. Let $\varphi \in M^s_{\tilde{\Omega}}$; we represent φ in the form of the sum $\varphi = \overset{1}{\varphi} + \mathring{\varphi}$, where $\mathring{\varphi}_t = \sum_{\nu=1}^{\sigma} c^\nu(t)\mathring{u}^\nu_t$, $c^\nu(t) = (\varphi_t, \mathring{u}; \Omega)_s \|\mathring{u}^\nu_t ;\Omega\|_s^{-2}$. Since the dimension of the basis σ is independent of t, this expression is meaningful; moreover, $c^\nu(t) \in C^\infty$. Assuming $\overset{1}{\varphi}_t = \varphi_t - \sum_{\nu=1}^{\sigma} c^\nu(t)\mathring{u}^\nu_t$ we complete the proof.

LEMMA 4. *If $\varphi_t \in \overset{1}{M}{}^s_{\tilde{\Omega}}$, then the inequality* (3.18) *takes the form*

$$\|\varphi_t ; \Omega\|_s \leqslant c|||T_t \varphi_t ; \Omega|||_s. \tag{3.24}$$

P r o o f. If $\varphi_t \in \overset{1}{M}{}^s_{\tilde{\Omega}}$, then for every t

$$(\varphi_t, \mathring{u}^\nu_t ; \Omega)_s = 0, \quad \nu = 1, \dots, \sigma.$$

Hence, for a fixed t, the inequality (3.24) follows from the general theory of elliptic problems.

Lemma 4 implies an important generalization of the inequality (3.24). To this end, we decompose the space $H^{0,s}(\Omega')$ into an orthogonal sum of two subspaces

$$H^{0,s}(\Omega') = \overset{1}{H}{}^{1,s}(\Omega') \oplus \mathring{H}^{0,s}(\Omega'), \tag{3.25}$$

where $\mathring{H}^{0,s}(\Omega')$ is the completion in the norm $\| \cdot ; \Omega'\|_{0,s}$ of the set of elements of the form $\mathring{u} = \sum_{\nu=1}^{\sigma} c^\nu(t)\mathring{u}^\nu_t$, $c^\nu(t) \in C^\infty$ and $(\overset{1}{u}, \mathring{u}, \Omega')_{0,s} = 0$ for $\overset{1}{u} \in \overset{1}{H}{}^{0,s}(\Omega')$.

LEMMA 5. *For $u \in \overset{1}{H}{}^{0,s}(\Omega')$, the following inequality holds:*

$$\|u ; \Omega'\|_{0,s} \leqslant c|||Tu ; \Omega'|||_{0,s}. \tag{3.26}$$

Here T is an extension by continuity of the mapping T_t to the space $H^{0,s}(\Omega')$.

P r o o f. The function space of $C^\infty(\Omega')$ is dense in $H^{0,s}(\Omega')$. On the other hand, (3.24) yields

$$\int \|\varphi ; \Omega\|_s^2 \, dt \leqslant C \int |||T_t \varphi_t ; \Omega|||_s^2 \, dt. \tag{3.27}$$

This formula is true for functions $\varphi \in C^\infty(\Omega')$ and vanishing in $R^{n+1}_{xt} - \Omega'$ such that

$$(\varphi, \mathring{u}; \Omega')_{0,s} = 0 \quad \text{for} \quad \mathring{u} \in \mathring{H}^{0,s}(\Omega').$$

On the basis of the definition of the norm in $H^{0,s}(\Omega')$, (3.27) implies that for such φ:

$$\|\varphi;\Omega'\|_{0,s} \leqslant c|||T_t\varphi;\Omega'|||_{0,s}, \tag{3.28}$$

and hence, in view of the continuity of T_t and the definition of the extension T, we arrive at the formula (3.26).

An analogous reasoning for the dual mapping $\overline{T}_t$ leads to

LEMMA 6. *Let V: $t \to V_t$ be a mapping of class C^∞ of the real line R_t^1 into the space K_Ω^{-s} with support contained in (t_2, t_1). Then*

$$|||V; R_{xt}^{n+1}|||_{0,-s} \leqslant \bar{c}(\|\overline{T}_t V; R_{xt}^{n+1}\|_{0,-s} + |||V; R_{xt}^{n+1}|||_{0,-s+1}). \tag{3.29}$$

Denote now by $\mathring{K}_{\Omega'}^{0,-s}$ the completion of the set of functions of the form $\mathring{V} = \sum_{\nu=1}^{\sigma} c^\nu(t)V_t^\nu$, $c^\nu(t) \in C^\infty[t_2, t_1]$ in the norm $|||\cdot; R_{xt}^{n+1}|||_{0,-s}$, where $\mathring{V}_t^\nu$ is the basis of the null subspace $\overline{N}_t$ of the mapping $\overline{T}$ of class C^∞ with respect to t; the elements of this basis satisfy for a fixed t the orthogonality condition

$$(\mathring{V}_t^\nu, \mathring{V}_t^\mu; R_x^n)_{-s} = 0, \quad \nu \neq \mu.$$

The existence of the above basis follows from the assumption that the dimension of the subspace $\overline{N}_t$ is independent of t. Let us represent the space $K_{\Omega'}^{0,-s}$ in the form of the orthogonal sum

$$K_{\Omega'}^{0,-s} = \overset{1}{K}{}_{\Omega'}^{0,-s} \oplus \mathring{K}_{\Omega'}^{0,-s}, \tag{3.30}$$

where $(\overset{1}{V}, \mathring{V}; R_{xt}^{n+1})_{0,-s} = 0$ for $\overset{i}{V} \in \overset{i}{K}{}_{\Omega'}^{0,-s}$.

LEMMA 7. *For $V \in \overset{1}{K}{}_{\Omega'}^{0,-s}$, we have the inequality*

$$|||V; R_{xt}^{n+1}|||_{0,-s} \leqslant \bar{c}\|\overline{T}V; R_{xt}^{n+1}\|_{0,-s}, \tag{3.31}$$

where $\overline{T}$ is the extension by continuity of the mapping to the space $K_{\Omega'}^{0,-s}$.

Proof. The Lemma constitutes a modification of the inequality (3.29) analogous to Lemma 5 for the inequality (3.19). The proof is also similar.

Lemmas 5 and 7 lead to generalizations of the theorems of existence of solutions of elliptic boundary problems to the case of elliptic boundary problems with a parameter.

THEOREM 2. *In order that the equation*

$$Tu = b, \quad b \ni K^{0,s}(\Omega') \tag{3.32}$$

has a solution $u \in H^{0,s}(\Omega')$ it is necessary and sufficient that $(b, \mathring{V}) = 0$ for an arbitrary $\mathring{V} \in \mathring{K}_{\Omega'}^{0,-s}$, i.e. on the set of solutions of the equation $\overline{T}\mathring{V} = 0$.

Proof. The theorem follows in the usual way from the theorem on extension of functionals and the inequality (3.26).

THEOREM 3. *In order that the equation*

$$\overline{T}V = U, \quad U \in H_{\Omega'}^{0,-s} \tag{3.33}$$

has a solution $V \in K_{\Omega}^{0;-s}$ *it is necessary and sufficient that*

$$(\mathring{u}, U) = 0 \tag{3.34}$$

for an arbitrary $\mathring{u} \in \mathring{H}^{0,s}(\Omega')$ *i.e., on the set of solutions of the equation* $T\mathring{u} = 0$.

P r o o f. The theorem follows from the theorem on extension of functionals and the inequality (3.31).

Theorems 2 and 3 are valid for general elliptic boundary problems. In our case when the system of the boundary operators $B_t^j (j = 1, \ldots, l)$ is normal, we obtain [3]

THEOREM 4. *Let* $U: t \to U_t = (F_t, G_t^1, \ldots, G_t^l)$ *be a mapping of class* C^∞ *of the real line* R_t^1 *into the space* $H^{s-m}(\Omega) \times H^{s-\bar{m}_1-1/2}(\Gamma) \times \ldots \times H^{s-\bar{m}_l-1/2}(\Gamma)$ *with support contained in* $[t_2, t_1]$, *where* $\bar{m}_j$ *is the order of the operator* $\bar{B}_t^j$. *The adjoint boundary problem*

$$\begin{aligned} \bar{E}_t w_t &= F_t \quad \text{in} \quad \Omega, \\ \bar{B}_t^j w_t &= G_t^j \quad \text{on} \quad \Gamma, \quad j = 1, 2, \ldots, l, \end{aligned} \tag{3.35}$$

has a solution $w_t \in M_{\Omega}^s$ *with support contained in* $[t_2, t_1]$, *if and only if, for an arbitrary* $\mathring{u}_t \in \mathring{H}^{0,s}(\Omega')$,

$$\int_{\Omega'} \mathring{u}_t \bar{F}_t \, d\Omega' - \int_{\Gamma'} \sum_{j=1}^{l} S_t^j \mathring{u} \bar{G}_t^j \, d\Gamma' = 0, \tag{3.36}$$

where $\Gamma' = \Gamma \times [t_2, t_1]$ *and the boundary operators* S_t^j, B_t^j *are defined by the formula* (1.5).

P r o o f. The space $\mathring{H}^{0,s}(\Omega')$ is the completion of a dense set of elements of the form $\sum_{\nu=1}^{l} c^\nu(t) \mathring{u}_t^\nu$, where $c^\nu(t)$ is an arbitrary function belonging to $C^\infty[t_2, t_1]$. The arbitrariness of $c^\nu(t)$ implies that (3.36) is equivalent to the relation

$$\int_{\Omega} \mathring{u}_t^\nu \bar{F}_t \, d\Omega - \int_{\Gamma} \sum_{j=1}^{l} S_t^j \mathring{u}_t^\nu \bar{G}_t^j \, d\Gamma = 0. \tag{3.37}$$

This is the usual condition of existence of a solution of the problem (3.35) for a fixed t. The regularity of the solution w_t with respect to t follows from the general theorem [1].

Theorem 4 leads to

LEMMA 8. If 1) $F \in M_{\tilde{\Omega}}^{-s+m-1}$, $\quad s > s_0 = \max(m_1, \ldots, m_l) + \frac{1}{2}$ *and the support of* F *is contained in* $[t_2, t_1]$,

2) $\bar{L}F \in M_{\tilde{\Omega}}^{-s+m-1-\eta}$, *where* L *is the operator defined by the formula* (1.3), η *is the order of the highest derivative with respect to the variables* $x_j (j = 1, \ldots, n)$ *appearing in the operator* $\bar{L}$,

3) *the inequality* $-s+m-1-\eta > s_0 - m$ *holds,*

4) $(L\mathring{u}_t, F) = 0$ *for every* $\mathring{u}_t \in \mathring{H}^{0,s}(\Omega')$,

then there exists a solution $w \in M_{\Omega}^{s_0}$ *of the boundary problem*

$$\begin{aligned} \bar{E}_t w &= \bar{L}F \quad \text{in } \Omega', \\ \bar{B}_t^j w &= 0 \quad \text{on } \Gamma', \end{aligned} \tag{3.38}$$

or every $t \in [\tilde{t_2}, \tilde{t_1}]$ *and the support* $w \subset [t_2, t_1]$.

4. Construction of the solution

Consider the general boundary problem with the parameter

$$(4.1)\qquad Tw = b, \quad b \in R(T) \subset K^{0,s}(\Omega'),$$

where T is the extension of the mapping T_t to the space $H^{0,s}(\Omega')$. For the problem (4.1), Theorem 2 on the existence of the solution is valid.

In general, the solution w is the sum

$$(4.2)\qquad w = u + \mathring{u},$$

where $u \in \overset{1}{H}{}^{0,s}(\Omega')$ and $\mathring{u} \in \mathring{H}^{0,s}(\Omega')$ i.e., $T\mathring{u} = 0$, $\mathring{u} = \sum_{\nu=1}^{l} c^{\nu}(t)\mathring{u}$.

With every $b \in R(T)$ (i.e. the set of values of the mapping T), we can associate one and only one $u \in \overset{1}{H}{}^{0,s}(\Omega')$ such that $Tu = b$. Thus, there exists a one-to-one mapping

$$u\colon \quad b \to u(b).$$

Let $b = (\psi, g_1, \ldots, g_l)$. We shall write $u(\psi, 0, \ldots, 0) = u(\psi)$, $u(0, g_t^1, \ldots, g_t^l) = u(g)$.

Thus, (4.2) has the form

$$(4.4)\qquad w = u(\psi) + u(g) + \mathring{u}.$$

The particular solutions $u(\psi)$, $u(g)$ satisfy the equations

$$(4.5)\qquad \begin{aligned} E_t u(\psi) &= \psi \quad \text{in } \Omega', \\ B_t^j u(\psi) &= 0 \quad \text{on } \Gamma' = \Gamma \times (t_2, t_1), \end{aligned}$$

$$(4.6)\qquad \begin{aligned} E_t u(g) &= 0 \quad \text{in } \Omega', \\ B_t^j u(g) &= g_t^j \quad \text{on } \Gamma'. \end{aligned}$$

R e m a r k. In what follows we assume that we know the solutions $u = u(\psi)$, $u = u(g)$ for arbitrary $\psi, g \in R(T) \subset K^{0,s}(\Omega')$ possessing the properties (4.5), (4.6) and a basis $u_t^{\nu}(\nu = 1, \ldots, l)$ of the subspace N_t regular with respect to t.

Observe that the inequality

$$(4.7)\qquad \|u(\psi); \Omega'\|_{0,s} \leqslant c\|\psi, \Omega'\|_{0,s-m},$$

constituting a particular case of (3.26) is true. It is equivalent to the continuity of the mapping u. Hence, the mapping $\bar{u}$ dual to u exists and is continuous:

$$(4.8)\qquad \big(u(\psi), U\big) = \big(\psi, \bar{u}(U)\big).$$

Furthermore, for an arbitrary $U \in H_{\Omega'}^{0,-s}$,

$$(4.9)\qquad \|\bar{u}(U); R_{xt}^{n+1}\|_{0,-s+m} \leqslant \bar{c}\|U; R_{xt}^{n+1}\|_{0,-s}.$$

We shall now perform a modification of the inequalities (3.15) and (3.17). A particular case of (3.15) is

$$4.10)\qquad \|E_t u_t; \Omega'\|_{r,s-m} \leqslant c\|u_t; \Omega'\|_{r,s},$$

where r, s are reals and $u \in \bar{C}^{\infty}(\Omega')$. The inequality (4.10) implies that

$$\|E_t u_t;\, \Omega'\|_{r-m,s} \leqslant c\|u;\, \Omega'\|_{r,s} \tag{4.11}$$

and the latter inequality leads to the relation

$$\|\bar{E}_t F;\, R^{n+1}_{xt}\|_{-r,-s} \leqslant \bar{c}\|F;\, R^{n+1}_{xt}\|_{-r+m,-s}, \tag{4.12}$$

where r, s are reals and $F \in C^{\infty}_{\Omega'}$.

The basis of the construction of the initial-boundary problem (1.11) is the following

LEMMA 9. *Let η be the order of the highest derivative with respect to the variables $x_i(i = 1, \ldots, n)$ in the operator L defined by the formula* (1.10). *If* 1) $F \in C^{\infty}_0(R^{n+1}_{xt})$, 2) *the support of F is contained in Ω',* 3) $(L\mathring{u}, F) = 0$ *for an arbitrary $\mathring{u} \in H^{0,s}(\Omega')$ (the space $H^{0,s}(\Omega')$ is defined by the formula* (3.25)), 4) $m-1-\eta \geqslant 0$, *then the inequality*(¹)

$$\|\bar{u}(\bar{A}F,);\, \Omega'\|_m \geqslant c\|F;\, R^{n+1}_{xt}\|_{m-1}, \tag{4.13}$$

is true; here $\bar{u}$ is the mapping dual to u defined by the formula (4.8) *and the constant C depends on Ω'.*

Proof. In the general theory of the Cauchy problem [1] we have the inequality

$$\|\bar{A}F;\, R^{n+1}_{xt}\|_0 \geqslant c\|F;\, R^{n+1}_{xt}\|_{m-1} \tag{4.14}$$

for an arbitrary $F \in C^{\infty}_{\Omega'}$. Lemma 8 implies that for F satisfying 1) 2), 3) there exists $v_t \in M^{m}_{\tilde{\Omega}}$ such that

$$\bar{E}_t v_t = \bar{L}F \quad \text{in } \Omega',$$
$$\bar{B}^j_t v_t = 0 \qquad \text{on } \Gamma' = [t_2, t_1]\times\Gamma.$$

In view of the continuity of the operator $\bar{E}_t$,

$$\|v_t;\, \Omega'\|_m \geqslant \|\bar{E}_t v_t;\, \Omega'\|_0,$$

where we have denoted $\|\cdot\|_{s,0} = \|\cdot\|_s$. Observe that for $\varphi \in C^{\infty}_0(\Omega')$ we have

$$\|\varphi;\, \Omega'\|_0 = \|\varphi;\, R^{n+1}_{xt}\|_0.$$

On the basis of the inequality (4.14) we find that the mapping $\bar{u}_t(\cdot)$ dual to $u_t(\cdot)$ satisfies the inequality

$$\|F+v_t;\, \Omega'\|_m = \|\bar{u}(\bar{E}_t F+\bar{E}_t v_t);\, \Omega'\|_m$$
$$= \|\bar{u}(\bar{E}_t F+\bar{L}F);\, \Omega'\|_m \geqslant c\|\bar{E}_t F+\bar{L}F;\, \Omega'\|_0 = c\|\bar{A}F; R^{n+1}_{xt}\|_0 \geqslant \|F;\, R^{n+1}_{xt}\|_{m-1},$$

Q.E.D.

The fundamental theorem follows directly from the last Lemma.

THEOREM 5. If 1) $b_t = f_t - Lu_t(g) \in M^{-m+1}_{\tilde{\Omega}}$; 2) $b_t = 0$ *for* $t < 0$, 3) $m-1-\eta \geqslant 0$, *where η is the order of the highest derivative with respect to the variables $x_j(j = 1, \ldots, n)$,* 4) *for $\nu = 1, \ldots, \sigma$ there exist solutions $\mathring{\varphi}^{\nu}_t$ of the equations $\bar{L}\mathring{\varphi}^{\nu}_t = \mathring{u}^{\nu}_t$, $\varphi_t \in H^{m-1}(\Omega')$, where $\mathring{u}^{\nu}_t$ is an l-dimensional orthogonal basis normed in $\mathring{H}^0(\Omega')$ with the scalar product $(\cdot, \cdot)$ and*

(¹) We denote hereafter $\|\cdot\|_0 = \|\cdot\|$

$E_t\mathring{u}_t^\nu = 0$, $B_t^j\mathring{u}_t^\nu = 0$, then there exists a) *$\psi_t \in H_{\Omega'}^{-m}$ such that $t \to \psi_t$ is a mapping of the real line R_t^1 into the space H_{Ω}^{-s} of class $C^\infty(R_t^1)$ with the support contained in $[0, T_2]$ and ψ_t is a solution of the equation $\big(\psi_t, \bar{u}(\bar{A}\varphi_t)\big) = (f_t, \varphi_t) - \big(Lu(g), \varphi_t\big)$; this equation holds for all $\varphi_t \in \overset{1}{H}{}_{\Omega'}^m$, i.e. orthogonal to all elements of the form $L\mathring{u}_t$, where $\mathring{u}_t = \sum_{\nu=1}^{\sigma} d^\nu(t)\mathring{u}_t$;* b) *the system of functions $c_\nu(t)$ defined by the formula in the scalar product in R^n,*

$$c_\nu(t) = (b, \mathring{\varphi}_t^\nu) - \big(\psi_t, \bar{u}(\bar{A}\mathring{\varphi}_t^\nu)\big) - \big(Lu_t(g), \mathring{\varphi}_t^\nu\big), \qquad \nu = 1, \dots, \sigma,$$

such that the equation

$$\Big(\sum_{\nu=1}^{\sigma} c_\nu(t)\,\mathring{u}_t^\nu + u_t(\psi_t), \bar{A}\varphi_t\Big) = \big(f_t - Au_t(g), \varphi_t\big)$$

holds, c) *the solution of the initial-boundary problem*

$$Aw_t = f_t \quad \text{in } \Omega', \qquad B_j^t u_t = g_t^j \quad \text{on } \Gamma'$$

is of the form

$$w_t = \sum_{\nu=1}^{\sigma} c_\nu(t)\mathring{u}_t^\nu + u_t(\psi_t) + u_t(g),$$

where $u_t(\psi_t)$ is the extension by continuity to the space $H_{\Omega'}^{-m}$ defined by the formula

$$\big(u_t(\psi_t), \bar{T}V_t\big) = \big(\psi_t, \bar{u}_t(\bar{E}P_t)\big);$$

d) *for all ψ_t satisfying the condition $(\psi_t, \mathring{\psi}_t)_{-m} = 0$, where $\mathring{\psi}_t$ is an arbitrary solution of the equation $\big(\mathring{\psi}_t, \mathring{\bar{u}}(\bar{A}\varphi_t)\big) = 0$, the inequality*

$$\|\psi_t; R_{xt}^{n+1}\|_{-m} \leqslant C\|Au(\psi_t); \Omega'\|_{-m+1}$$

holds. This inequality implies the regularity for ψ_t, i.e. if $b_t \in M_{\Omega'}^1$, then $\psi_t \in M_{\Omega'}^0$; then the boundary conditions in the ordinary sense are satisfied.

The function ψ_t is constructed in the usual manner by means of the Banach-Hahn theorem on the extension of functionals, and the inequality (4.13).

5. Mixed boundary problem

Consider now an application of our results to the solution of the mixed initial-boundary problem

$$(5.1) \qquad \begin{aligned} &Au = f \quad \text{in } \Omega', \qquad B_t^j u = g_t^j \quad \text{on } \Gamma_j', \qquad j = 1, 2,\\ &u = 0 \quad \text{for } t < 0, \end{aligned}$$

where the boundary $\bar{\Gamma}$ is the sum $\bar{\Gamma} = \bar{\Gamma}_1 \cup \bar{\Gamma}_2$, B_t^j are two systems of normal boundary operators, each satisfying the conditions stated at the beginning of this paper.

We shall base on the theorem [4] concerning elliptic mixed boundary problems.

THEOREM 6. *If the system of the boundary operators* $B_t^j = B_t^j(x, D)$ *is normal for* $j = 1, 2$ *(for a fixed t), then for* $f_t \in C^\infty(\bar{\Omega})$ *and* $(f_t, N^+) = 0$, *where* N^+ *is a subspace of* $H^s(\Omega)$ *of a finite dimension, then there exists a solution* $u \in L^2(\Omega)$ *of the mixed boundary problem*

$$E_t u_t = f_t \quad \text{in } \Omega, B_t^j u_t = 0 \quad \text{on } \Gamma_j, \quad j = 1, 2,$$

such that $u \in C^\infty(F)$ *for every closed set* $F \subset \bar{\Omega}\gamma$ *where* $\Gamma_1 \cap \Gamma_2 = \gamma$.

As before we assume that the solution of the mixed initial-boundary problem (4.1) has the form

$$\tilde{w}_t = u_t(\psi_t) + u_t(g_t) + \sum_{\nu=1}^{\sigma} c_\nu(t)\, \mathring{u}_t^\nu, \tag{5.2}$$

where $u_t(\psi_t)$ is the solution of the mixed elliptic problem following from Theorem 2,

$$\begin{aligned} &E_t u_t(\psi_t) = \psi_t, \\ &E_t u_t(g_t) = 0, \quad B_t^j u_t(g_t) = g_t^j \quad \text{on } \Gamma_j, \quad j = 1, 2, \end{aligned} \tag{5.3}$$

ψ_t is the unknown function, g_t^j the function obtained on the basis of the problem (5.1). The function ψ_t should be chosen in such a manner that the equation

$$(A\tilde{w}_t, \varphi) = 0 \tag{5.4}$$

is satisfied for any $\varphi \in C_0^\infty(\Omega')$.

Applying a procedure similar to that in the preceding Sections it can be proved that there holds a theorem basically the same as that proved above, provided the auxiliary function $u_t(\cdot)$ connected with the continuous boundary problem is replaced by the function defined in Theorem 2.

This follows also from the fact that every function given by the formula (5.2) can be represented by means of a function w_t defined by Theorem 1, in particular by c). The problem of establishing an effective realization of the functions ψ_t and $c_\nu(t)$ appearing in (5.2) can be solved by reducing the expression (5.4) to an infinite system of algebraic equations; the author considered this problem for elliptic equations.

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