The surface waveguide. Accurate solution

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THE PAPER presents the exact formulation and solution of the problem of the surface elastic waveguides. The solutions known so far either concerned certain particular forms of boundary conditions or were approximate in the asymptotic sense. The method proposed in the paper is general enough to be applied to a wide class of surface waveguides — particularly in semi-conductors — what is of a great practical importance in microwave acoustics.

W pracy podano ścisłe sformułowanie i rozwiązanie problemu powierzchniowego falowodu sprężystego. Dotąd istniejące rozwiązania dotyczyły szczególnych przypadków warunków brzegowych lub były przybliżone w sensie asymptotycznym. Metoda rozwiązania zaproponowana w pracy jest dostatecznie ogólna i może być zastosowana do szerokiej klasy problemów falowodowych fal powierzchniowych, w szczególności w piezopółprzewodnikach, co posiada zasadnicze znaczenie praktyczne w akustyce mikrofalowej.

В работе дана точная формулировка задачи о поверхностном упругом волноводе и найдено решение этой задачи. Решения, известные ранее в литературе, были получены либо для частных случаев краевых условий, либо для приближений асимптотического характера. Предложенный в данной работе метод решения является достаточно общими и может быть применен к широкому классу задач о поверхностных волноводах, в частности к пьезополупроводникам, имеющим основное практическое значение в микроволновой акустике.

1. Introduction

THE PRACTICAL importance of problems of surface waveguides, in the domain of microwave acoustics in particular, has led to considerable interest in those problems despite their mathematical difficulties involved. One of the fundamental methods for practical realization of surface waveguides is by applying to the surface of the medium in which the wave is transferred a thin layer of a lower velocity of transverse waves, than in the medium.

A survey of the literature of the problem was given in [1] and a number of application problems in microwave engineering were discussed in [4].

The existing solutions of the propagation and damping problem of guided elastic surface waves are either accurate qualitative solutions with particular boundary conditions [2] or approximate asymptotic solutions. A fundamental solution of such a type was obtained by TIERSTEN [3], and modified by ADKINS and HUGHES [4].

The aim of the present paper is to find an accurate solution to the problem under consideration. By accurate solution we mean a solution of the accurate boundary-value problem with no asymptotic simplification, the general solution of the problem being found numerically, by means of a computer, with any desired accuracy. The present results may be confronted with those of [3 and 4]. Thus, for instance, in the present paper, the attenuation curves of the guided surface wave follow from general considerations, while in [3 and 4] these curves were assumed beforehand.

The present method has been tested by applying it to the simpler problem of a surface waveguide in the case of the wave equation [5, 6].

In Ref. [1] was studied a practical problem of microwave acoustics — that of a semi-conductor waveguide on the surface of a piezo-electric body. This enables us to amplify or reduce the damping of the surface wave by means of a drifting electron stream. To the best of the author's knowledge this is the first accurate solution of such a problem to be published.

In the above references, the mass of the guiding layer was taken into account, its rigidity being disregarded. In further papers, this effect will also be taken into consideration.

In the present paper we shall confine ourselves to the study of a wave symmetric about the waveguide axis. On the basis of the intermediate calculations [1, 6] and those of the present paper, we express our conviction that the fact that a wave skew symmetric about the waveguide axis has not yet been produced experimentally is a result of the wave guiding layer being unable to guide such a wave.

In Sec. 2, we shall formulate the problem. Section 3 will be devoted to the solution method and Sec. 4 — to the computation results for a layer of gold on fused silica and conclusions.

2. The statement of the problem

We seek for a plane harmonic wave propagating in the direction of the axis of the layer and dying out with increasing distance from the free surface and the wave guiding layer. The set of axes is assumed as represented in Fig. 1.

The components of the displacement vector are expressed by a scalar and vector potential:

$$(2.1) \qquad\qquad U_i = \Phi_{,i} + e_{ijk}\Psi_{k,l},$$

where e_{ijk} is an absolutely asymmetric quantity of the third valence. From the equations
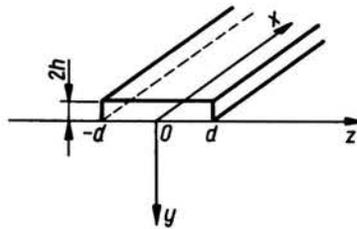
FIG. 1.

of the theory of elasticity it follows that

$$(2.2) \qquad\qquad \ddot{\Phi} - a_1^2 \Phi_{,kk} = 0, \quad \ddot{\Psi}_i - a_2^2 \Psi_{i,kk} = 0.$$

The compatibility condition is:

$$(2.3) \qquad\qquad \Psi_{i,i} = 0.$$

The following conditions must be satisfied on the free surface $x_2 = 0$:

$$(2.4) \qquad \sigma_{zj} = \begin{cases} 0 & |x_3| > d \\ 2h\bar{\varrho}\ddot{u}_j & |x_3| < d \end{cases} \qquad j = 1, 2, 3,$$

where $\bar{\varrho}$ is the density of the material of the layer. The inertia forces of the thin layer have been taken into consideration, its rigidity being disregarded. The stresses are connected with the components of the displacement vector by the relations:

$$(2.5) \qquad \delta_{ik} = \varrho a_2^2 \left[2\varepsilon_{ik} + \delta_{ik} \left(\frac{a_1^2}{a_2^2} - 2 \right) \varepsilon_{jj} \right],$$

$$\varepsilon_{ij} = u_{(i,k)} = \frac{1}{2}(u_{i,k} + u_{k,i}).$$

We seek for solutions bounded for $x_2 = 0$, $x_3 \to \pm d$ and dying out for $y \to \infty$ and $|z| \to \infty$. The set of equations (2.2), (2.3) with the boundary conditions constitute the general formulation of the problem.

3. The general solution

A solution satisfying the conditions formulated in Sec. 2 is sought for in the form:

$$(3.1) \qquad \begin{aligned} \Phi(x, y, z, t) &= \mathrm{Re}\{\Phi(p, z)e^{ik(vt-x)}\}, \\ \Psi_l(x, y, z, t) &= \mathrm{Re}\{\Psi_l(y, z)e^{ik(vt-x)}\}. \end{aligned}$$

Use will be made of the Fourier complex transformation as defined by the equations:

$$(3.2) \qquad \begin{aligned} \bar{\Phi}(y, \alpha) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(y, z)e^{i\alpha z} dz, \\ \bar{\Psi}_l(y, \alpha) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi_l(y, z)e^{i\alpha z} dz. \end{aligned}$$

Since the equations are satisfied and the solution dies out with increasing distance from the free surface, it follows that

$$(3.3) \qquad \begin{aligned} \bar{\Phi}(y, \alpha) &= \bar{C}(\alpha)e^{-\beta_1(\alpha, k, v)y}, \\ \bar{\Psi}_l(y, \alpha) &= D_l(\alpha)e^{-\beta_2(\alpha, k, v)y}, \end{aligned}$$

where

$$(3.4) \qquad \beta_j(\alpha, k, v) = \sqrt{\alpha^2 + k^2 \left(1 - \frac{v^2}{a_i^2} \right)}, \qquad \mathrm{Re}\,\beta_j > 0, \qquad j = 1, 2.$$

From the condition (2.3) it follows that

$$(3.5) \qquad \bar{D}_2 = \bar{D}_1 \left(-\frac{ik}{\beta_2} \right) + \bar{D}_3 \left(-\frac{i\alpha}{\beta_2} \right).$$

On introducing the abbreviated notations

(3.6) $$\bar{C} = A_1, \quad \bar{D}_1 = A_2, \quad \bar{D}_3 = A_3,$$

we have:

(3.7) $$\bar{u}_j = \sum_{j=1}^{3} M_{jl} A_l e^{ik(vt-x)},$$

where

$$M_{11} = -ike^{-\beta_1 y}, \quad M_{12} = \frac{k\alpha}{\beta_2} e^{-\beta_2 y}, \quad\quad\quad M_{13} = -k^2\left(1 - \frac{v^2}{a_2^2}\right)\beta_2^{-1} e^{-\beta_2 y},$$

$$M_{21} = -\beta_1 e^{-\beta_1 y}, \quad M_{22} = -i\alpha e^{-\beta_2 y}, \quad\quad\quad M_{23} = ike^{-\beta_2 y},$$

$$M_{31} = -i\alpha e^{-\beta_1 y}, \quad M_{32} = \left(\alpha^2 - k^2\frac{v^2}{a_2^2}\right)\beta_2^{-1} e^{-\beta_2 y}, \quad M_{33} = -\frac{k\alpha}{\beta_2} e^{-\beta_2 y}.$$

From the boundary conditions (2.4) we have, after transformation,

(3.8) $$\sum_{l=1}^{3} A_l N_{jl} = -\frac{2h\bar{\varrho}k^2 v^2}{\varrho a_2^2 \sqrt{2\pi}} \int_{-d}^{d} U_j(0, z) e^{i\alpha z} dz,$$

where

(3.9) $$U_j(y, z) = U_j(x, y, z, t) e^{-ik(vt-x)}, \quad s_2 = \left(\frac{a_1}{a_2}\right)^2, \quad s_1 = s_2 - 2,$$

$$N_{11} = 2ik\beta_1, \quad N_{12} = -2\alpha k, \quad N_{12} = k^2\left(2 - \frac{v^2}{a_2^2}\right),$$

$$N_{21} = 2\beta_2^2 + k^2\frac{v^2}{a_2^2}, \quad N_{22} = 2i\alpha\beta_2, \quad N_{23} = -2ik\beta_2,$$

$$N_{31} = 2i\alpha\beta_1, \quad N_{32} = -2\alpha^2 + k^2\frac{v^2}{a_2^2}, \quad N_{33} = 2k\alpha.$$

On solving (3.8), we obtain

(3.10) $$A_l = -\frac{2h\bar{\varrho}k^2 v^2}{\varrho a_2^2 \sqrt{2\pi}} \sum_{j=1}^{3} W_{lj} \int_{-d}^{d} U_j(0, \zeta) e^{i\alpha\zeta} d\zeta$$

and $\{W_{ij}\}$ is the inverse matrix of $\{N_{il}\}$. From the definition of the inverse transformation, (3.7) and (3.8), we find

(3.11) $$U_j(0, z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{l=1}^{3} M_{jl} A_l e^{-i\alpha z} d\alpha, \quad j = 1, 2, 3.$$

Taking into consideration (3.10), and reverting the order of operations, we obtain a set of three homogeneous integral equations

(3.12) $$U_i(0, z) + G\sum_{j=1}^{3} \int_{-d}^{d} U_j(0, \zeta) J_{ij}(z - \zeta) d\zeta = 0,$$

where

$$(3.13) \qquad J_{ij}(z) = \frac{1}{2} \int\limits_{-\infty}^{\infty} \left(\sum_{l=1}^{3} M_{il} W_{lj} \right) e^{-i\alpha z} d\alpha.$$

By writing out (3.13), we find:

$$J_{11}(z) = \int\limits_{0}^{\infty} \frac{\cos\alpha z}{M(\alpha)} \frac{1}{\beta_2} \left[2k^2\beta_2^2 + 4\alpha^2\beta_1\beta_2 - \left(2\beta_2^2 + k^2 \frac{v^2}{a_2^2} \right)(\alpha^2 + \beta_2^2) \right] d\alpha,$$

$$J_{12}(z) = ik \int\limits_{0}^{\infty} \frac{\cos\alpha z}{M(\alpha)} \left[2\beta_1\beta_2 - \left(2\beta_2^2 + k^2 \frac{v^2}{a_2^2} \right) \right] d\alpha,$$

$$J_{13}(z) = ik(-1) \int\limits_{0}^{\infty} \frac{\alpha \sin\alpha z}{\beta_2 M(\alpha)} \left[2\beta_2^2 + \left(2\beta_2^2 + k^2 \frac{v^2}{a_2^2} \right) - 4\beta_1\beta_2 \right] d\alpha,$$

$$J_{21}(z) = -J_{12}(z),$$

$$(3.14)$$

$$J_{22}(z) = k^2 \frac{v^2}{a_2^2} \int\limits_{0}^{\infty} \frac{\cos\alpha z}{M(\alpha)} \beta_1 \, d\alpha,$$

$$J_{23}(z) = \int\limits_{0}^{\infty} \frac{\sin\alpha z}{M(\alpha)} \alpha \left[\left(2\beta_2^2 + k^2 \frac{v^2}{a_2^2} \right) - 2\beta_1\beta_2 \right] d\alpha,$$

$$J_{31}(z) = J_{13}(z),$$

$$J_{32}(z) = -J_{23}(z),$$

$$J_{33}(z) = k^2 \int\limits_{0}^{\infty} \frac{\cos\alpha z}{M(\alpha)} \frac{1}{\beta_2} \left[4(\beta_1\beta_2 - \beta_2^2) + \frac{v^2}{a_2^2}(\beta_2^2 - k^2) \right] d\alpha,$$

where

$$M(\alpha) = \left(2\beta_2^2 + k^2 \frac{v^2}{a_2^2} \right)^2 - 4\beta_1\beta_2(k^2 + \alpha^2).$$

The integrals representing the kernels J_{ik} are weakly singular for $i = k$ and absolutely convergent for $i \neq k$. The set of integral equations obtained will be solved by the method described in [1, 6]. To this end, we shall construct the solution over the interval $(-d/2 < z < d/2)$ and generalize by means of Eqs. (3.10) and (3.7) over the entire interval of the variables y and z.

Bearing in mind the symmetry about the axis of the waveguide, the solution is sought for in the form of the series:

$$U_1(0, z) = \sum_{n=0}^{N} a_n \cos n\pi \frac{z}{d},$$

$$U_2(0, z) = i \sum_{n=0}^{N} b_n \cos n\pi \frac{z}{d},$$

(3.15)

$$U_3(0, z) = i \sum_{n=0}^{N} c_n \sin n\pi \frac{z}{d}.$$

The solution (3.15) should satisfy the set of Eqs. (3.12) in the mean with the respective weights:

$$\cos m\pi \frac{z}{d}, \quad -i \cos m\pi \frac{z}{d}, \quad -i \sin m\pi \frac{z}{d}, \quad m = 0, 1, ..., N.$$

We obtain the following homogeneous set of equations:

(3.16) $\quad \sum_{n=0}^{N} (a_n \alpha_{nm}^{(\delta)} + b_n \beta_{nm}^{(\delta)} + c_n \gamma_{nm}^{(\delta)}) = 0, \quad \delta = 0, 1, 2 \quad m = 0, 1, 2, ..., N,$

where

$$\alpha_{nm}^{(0)} = \delta_{nm} + \frac{G}{(1+\delta_{m0})d} \int\limits_{-d}^{d} \int\limits_{-d}^{d} \cos n\pi \frac{\zeta}{d} \cos m\pi \frac{z}{d} J_{11}(z-\zeta) dz\, d\zeta,$$

$$\beta_{nm}^{(0)} = -\frac{G}{d(1+\delta_{m0})} \int\limits_{-d}^{d} \int\limits_{-d}^{d} \cos n\pi \frac{\zeta}{d} \cos m\pi \frac{z}{d} \operatorname{Im}[J_{12}(z-\zeta)] d\zeta\, dz,$$

$$\gamma_{nm}^{(0)} = \frac{-G}{d(1+\delta_{m0})} \int\limits_{-d}^{d} \int\limits_{-d}^{d} \sin n\pi \frac{\zeta}{d} \cos m\pi \frac{z}{d} \operatorname{Im}[J_{13}(z-\zeta)] d\zeta\, dz,$$

$$\alpha_{nm}^{(1)} = \beta_{nm}^{(0)}(1+\delta_{m0}),$$

(3.17)

$$\beta_{nm}^{(1)} = \delta_{nm} + \frac{G}{d(1+\delta_{m0})} \int\limits_{-d}^{d} \int\limits_{-d}^{d} \cos n\pi \frac{\zeta}{d} \cos m\pi \frac{z}{d} J_{22}(z-\zeta) dz\, d\zeta,$$

$$\gamma_{nm}^{(1)} = \frac{G}{d(1+\delta_{m0})} \int\limits_{-d}^{d} \int\limits_{-d}^{d} \sin n\pi \frac{\zeta}{d} \cos m\pi \frac{z}{d} J_{23}(z-\zeta) dz\, d\zeta,$$

$$\alpha_{nm}^{(2)} = \gamma_{nm}^{(0)}(1+\delta_{m0}),$$

$$\beta_{nm}^{(2)} = \gamma_{mn}^{(1)}(1+\delta_{m0}),$$

$$\gamma_{nm}^{(2)} = \delta_{nm} + \frac{G}{d(1+d_{m0})} \int\limits_{-d}^{d} \int\limits_{-d}^{d} \sin n\pi \frac{\zeta}{d} \sin m\pi \frac{z}{d} J_{33}(z-\zeta) d\zeta\, dz.$$

From the existence of non-trivial solutions of (3.16),

(3.18) $\det\{(3.16)\} = 0,$

we find the dispersion curve or curves

$$(3.19) \qquad \frac{v}{a_2} = f(k),$$

expressing at the same time all the quantities a_n, b_n and c_n by a_0, for instance,

$$(3.20) \qquad \alpha_n = \frac{a_n}{a_0}, \qquad \beta_n = \frac{b_n}{a_0}, \qquad \gamma_n = \frac{c_n}{a_0}.$$

Taking into consideration (3.19) and (3.20), we obtain from (3.10)

$$(3.21) \qquad \frac{A_l}{a_0} = -g\sqrt{2\pi}\sin(\alpha d) \sum_{n=0}^{N} \frac{(-1)^n}{(\alpha d)^2 - (n\pi)^2}[(\alpha d)\alpha_n W_{11}$$

$$+ \beta_n i W_{12}(\alpha d) + \gamma_n(-W_{13})n\pi],$$

where

$$g = (2h/d)(\bar{\varrho}/\varrho)\varkappa^2 V^2 \pi^{-1}, \qquad \varkappa = kd, \qquad V = v/a_2.$$

Then, from (3.7) and (3.2), we find:

$$(3.22) \qquad \frac{u_j(x, y, z, t)}{a_0} = \mathrm{Re}\left\{\frac{1}{\sqrt{2\pi}} e^{ik(vt-x)} \int_{-\infty}^{\infty} \sum_{l=1}^{3} M_{jl} A_l e^{-i\alpha z} dz\right\},$$

and the computation formulae:

$$(3.23) \qquad \frac{u_1(x, y, z, t)}{a_0 \cos k(vt-x)} = -2g \int_0^{\infty} \frac{\cos\left(u\frac{z}{d}\right)\sin u}{m(u)} u \sum_{n=0}^{N} \frac{(-1)^n}{u^2 - (n\pi)^2} \times$$

$$\times \left\{ e^{-\beta_1(u)\frac{y}{d}}[\alpha_n 2\varkappa^2 \beta_2(u) + \beta_n \varkappa(2\beta_2^2(u) + \varkappa^2 V^2) - \gamma_n n\pi 2\varkappa\beta^2(u)]\right.$$

$$+ e^{-\beta_2(u)\frac{y}{d}}\left[\alpha_n \frac{1}{\beta_2(u)}[4u^2\beta_1(u)\beta_2(u) - (2\beta_2^2(u) + \varkappa^2 V^2)(u^2 + \beta_2^2(u))]\right.$$

$$\left.\left. + \beta_n[-2\varkappa\beta_1(u)\beta_2(u)] + \gamma_n(-1)n\pi\frac{\varkappa}{\beta_2(u)}[(2\beta_2^2(u) + \varkappa^2 V^2) - 4\beta_1(u)\beta_2(u)]\right]\right\}du,$$

$$\frac{u_2(x, y, z, t)}{a_0 \sin k(vt-x)} = 2g \int_0^{\infty} \frac{\sin u \gos\left(u\frac{z}{d}\right)u}{m(u)} \sum_{n=0}^{N} \frac{(-1)^n}{u^2 - (n\pi)^2} \times$$

$$\times \left[e^{-\beta_1(u)\frac{y}{d}}\{\alpha_n[-2\varkappa\beta_1(u)\beta_2(u)] + \beta_n(-\beta_1(u))[2\beta_2^2(u) + \varkappa^2 V^2]\right.$$

$$+ \gamma_n n\pi 2\beta_1(u)\beta_2(u)\} + e^{-\beta_2(u)\frac{y}{d}}\{\alpha_n\varkappa[2\beta_2^2(u) + \varkappa^2 V^2]$$

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$$+\beta_n 2\beta_1(u)(\varkappa^2+u^2)+\gamma_n n\pi(-1)[2\beta_2^2(u)+\varkappa^2 V^2]\}\bigg]du\,,$$

$$\frac{u_3(x,y,z,t)}{a_0\sin k(vt-x)}=-2g\int_0^\infty\frac{\sin u\sin\left(u\dfrac{z}{d}\right)}{m(u)}\sum_{n=0}^N\frac{(-1)^n}{u^2-(n\pi)^2}\times$$

$$\times\bigg[e^{-\beta_1(u)\frac{y}{d}}\{\alpha_n 2\varkappa u^2\beta_2(u)+\beta_n u^2[2\beta_2^2(u)+\varkappa^2 V^2]+\gamma_n(-1)n\pi 2u^2\beta_2(u)\}$$

$$+e^{-\beta_2(u)\frac{y}{d}}\bigg\{\alpha_n\frac{\varkappa u^2}{\beta_2(u)}[(2\beta_2^2(u)+\varkappa^2 V^2)-4\beta_1(u)\beta_2(u)]+\beta_n[-2u^2\beta_1(u)\beta_2(u)]$$

$$+\gamma_n\frac{n\pi}{\beta_2(u)}[-4\varkappa^2\beta_1(u)\beta_2(u)+(2\beta_2^2(u)+\varkappa^2 V^2)(\beta_2^2(u)+\varkappa^2)]\bigg\}\bigg]du\,,$$

where

(3.24)

$$\beta_1(u)=\sqrt{u^2+\varkappa^2(1-V^2/s_2)}\,,$$

$$\beta_2(u)=\sqrt{u^2+\varkappa^2(1-V^2)}\,,$$

$$m(u)=[2\beta_2^2(u)+\varkappa^2 V^2]^2-4\beta_1(u)\beta_2(u)[\varkappa^2+u^2]\,.$$

The equations discussed in the present section represent the general method for solving the problem by means of a computer. An example will be considered in the next section. Although the procedure concerns particular boundary conditions, it is general and may be applied with success to other boundary-value problems.

Let us proceed now to solve an example and to a discussion of the application of a computer.

4. Computation results and applications

The numerical computations were performed by means of the EMC ZAM-41 computer using the SAKO code. Integration over infinite intervals was replaced by integration over very large finite intervals, the influence of the upper limit on the accuracy of the procedure being tested. The integrals over finite intervals were calculated by dividing them into from 4 to 8 parts depending on the type of the integrand. Integration in each subinterval was performed by 16-point approximation, using the Gauss-Legendre method. The influence of the number N of terms of the series (3.15) on the accuracy of the results constitutes a very interesting subject to study. It is illustrated by Table 1, from which it is seen that the rate of convergence is determined by the ratio of the thickness of the plate to the wavelength. This ratio can be expressed by, for instance, the parameter $\Psi=2\pi\dfrac{2h}{\lambda}=2hk$ used by THIERSTEN. As a result of the known features of the approximate computation of eigenvalues [8], the computation of the phase velocity entails an error by more than

one order of magnitude smaller than that of computation of the form of the propagatin
wave.

Table 1

$$y = 0$$
$$y = 0, z = 0, z = 10\lambda$$

$\psi = 2hk$	N	v/a_2	$u_1/a_0\cos k(vt-x)$	$u_2/a_0\sin k(vt-x)$	$u_3/a_0\sin k(vt-x)$
0.02	0	0.90400027	1.00629	-1.37889	$1.01755 \cdot 10^{-2}$
	1	0.90400027	1.00697	-1.37989	$1.00242 \cdot 10^{-2}$
	2	0.90400027	1.00696	-1.37988	$1.00336 \cdot 10^{-2}$
0.2	0	0.21816	1.11204	-1.11960	$-8.6764 \cdot 10^{-3}$
	1	0.23823	1.74734	-2.62282	$-3.2394 \cdot 10^{-2}$
	2	0.24755	3.29550	-6.12884	$-7.7029 \cdot 10^{-2}$
	3	0.24971	3.90330	-8.03962	$-1.1138 \cdot 10^{-1}$

For very thin plates $\Psi = 0.02$, the difference of phase velocities for $N = 0, 1, 2, 3$
is undetectable.

For thick plates — $\Psi = 2$ for instance, which is a value beyond the range of physical
sense, the results imply that for $N = 3$, the error is of order one or two per cent. Similarly,
although the computed value of the difference of displacements between $N = 0$ and $N = 1$
reaches, for $\Psi = 0.02$, the fourth decimal figure, and between $N = 1$ and $N = 2$ no more
than the sixth decimal figure, the error for $\Psi = 2$, between $N = 2$ and $N = 3$, is of order
40%. These results explain also the success of asymptotic methods based on prior assump-
tion of the variability of the wave with increasing distance from the surface of the body
or the waveguide layer [3, 4].

The analysis was performed for a layer of gold on fused silica the properties of which
are described in [7]. The dispersion curve is represented in Fig. 2.
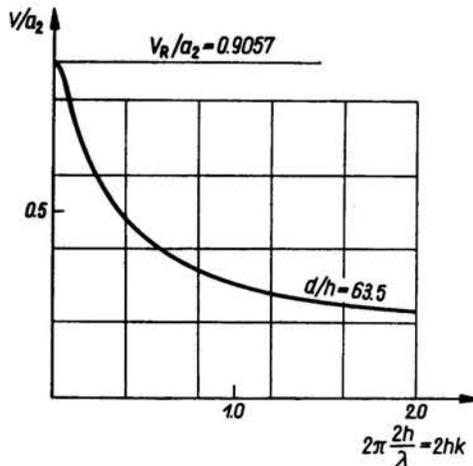
FIG. 2.

The propagating wave obtained by computation for a particular instant of time is represented in Fig. 3, which is a spatial representation. It shows the phase shift between the displacements and the symmetry about the waveguide axis. It is very interesting to study
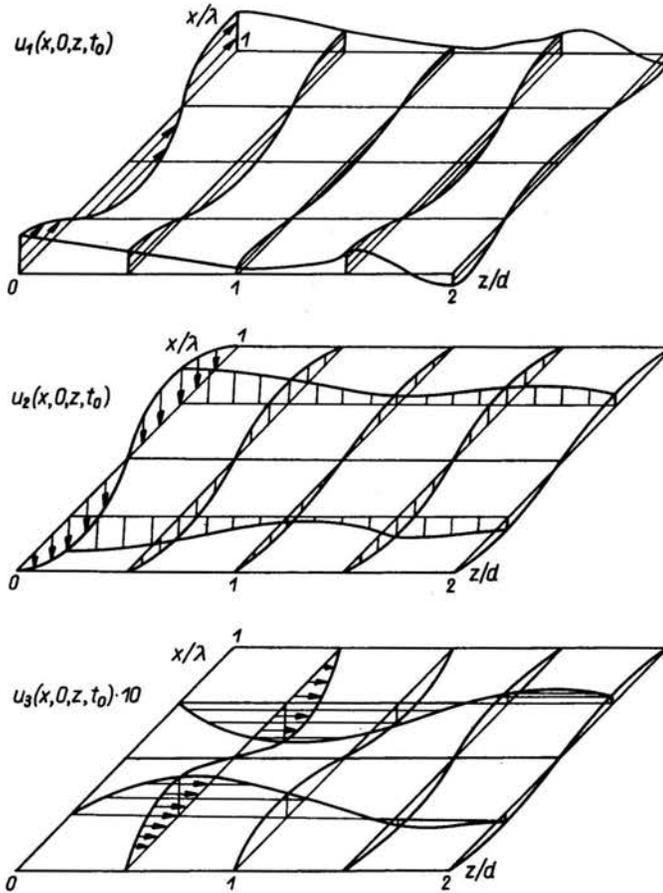
FIG. 3.

the process of dying out of the wave with increasing distance from the surface. This is represented in Fig. 4 up to a depth of 1,5 wavelengths. Further attenuation is monotonic, which is illustrated by Table 2.

The analysis shows that there are in the surface layer very sharp gradients of the perturbed field. The attenuation is non-oscillatory but there is a single change of sign of u_1 and u_3.

By way of example, the variability type of the wave at the surface and at a depth of one wavelength is shown in Figs. 5 and 6, respectively, where $\lambda = 6.28319 \cdot 10^{-5}$ m, $d/h = 63.5$, $v = 0.509 \, a_2 \; i$ denote depth in wavelengths, and k displacement component divided by $\cos k(vt-x)$ or the function $a_0 \sin k(vt-x)$, respectively.

From the analysis it follows that the displacement dies out in an oscillatory manner with increasing distance from the middle of the waveguide similarly to the sine integral function for a value of the argument much above zero.
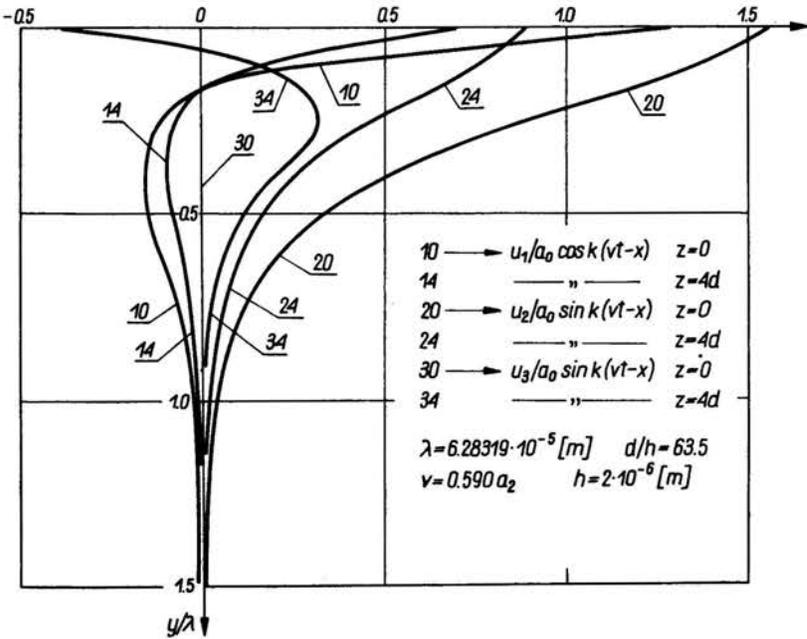
FIG. 4.

Table 2

y/λ	$u_1/u_0 \cos k(vt-x)$	$u_2/a_0 \sin k(vt-x)$	$u_3/a_0 \sin k(vt-x)$
0	0.74500	-0.89205	$-3.9456 \cdot 10^{-3}$
1	$-0.94367 \cdot 10^{-2}$	$-1.53436 \cdot 10^{-2}$	$-3.9530 \cdot 10^{-5}$
1.5	$-8.40146 \cdot 10^{-4}$	$-1.2164 \cdot 10^{-3}$	$-1.5623 \cdot 10^{-5}$
2	$-6.6043 \cdot 10^{-5}$	$-9.0546 \cdot 10^{-5}$	$-2.0224 \cdot 10^{-6}$
2.5	$-4.9089 \cdot 10^{-6}$	$-6.5296 \cdot 10^{-6}$	$-2.0776 \cdot 10^{-7}$
3.5	$-1.0803 \cdot 10^{-8}$	$-1\,3907 \cdot 10^{-8}$	$-7.9041 \cdot 10^{-10}$

$$z \approx 4\lambda$$

It follows also that the displacement in the direction of the z-axis is by five to ten times as small as the displacements in the directions of the remaining axes. It appears that the approximation consisting in the rejection of this displacement in the equations and the rejection of the relevant equation of dynamic equilibrium gives a sufficiently accurate dispersion curve and a correct image of the process of dying out of the signal.

The present results are not directly comparable to those of Thiersten which were obtained by taking into consideration the rigidity of the waveguide layer, approximate methods

being used. The results are in general, however, identical for very thin layers, asymptotic methods being used, the Tiersten results are correct only in a limited variability range of the parameters — on the other hand, Tiersten was obliged to asssume beforehand the manner in which guided surface waves die out.

There are no such expressions in the present paper and the manner in which waves die out results from the solution. It is worth while to observe that, by contrast with Tiersten's conclusions, the dying out process is non-monotonic although there were no significant
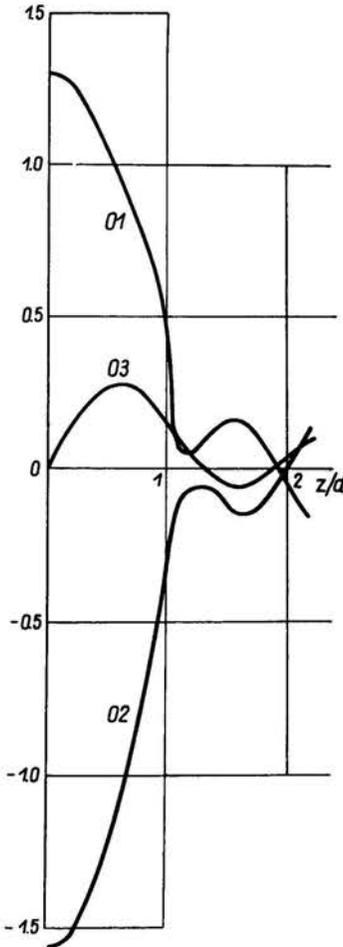
FIG. 5.
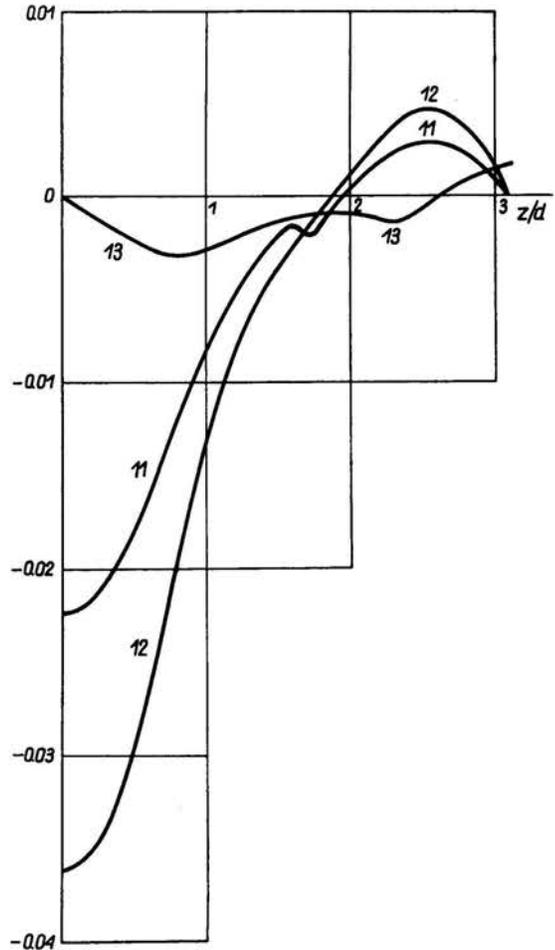
FIG. 6.

quantitative differences. In addition, the present solution is universal, because the method used can be applied to any set of boundary conditions and a body of any type (cf. [1]).

This method will be used to analyse the influence of the rigidity of the waveguide layer and other problems of surface waveguides in piezo-semiconducting bodies (a semi-conducting layer on a piezo-electric body etc.), as well as curvature effects of the waveguide.

References

1. S. Kaliski, *Guided and amplified surface waves in piezo-semiconductors,* Proc. Vibr. Probl., **13**, 2, 1972.
2. S. Kaliski, *Waveguide of elastic surface waves,* Bull. Acad. Polon. Sci., Série Sci. Techn., **17**, 10, 1969.
3. H. F. Tiersten, *Elastic surface waves guided by thin films,* J. Appl. Phys., **40**, 2, 1969.
4. L. R. Adkins, A. J. Hughies, *Elastic surface waves guided by thin films: Gold on Fused Quartz,* IEEE Tran. Mi. T.T. **17**, 11, 1969.
5. S. Kaliski, *About the surface guide effect for the wave equation,* Bull. Acad. Polon. Sci., Série Sci. Techn. [in print].
6. Kaliski, L. Solarz, *Surface wavequide for a wave medium,* Proc. Vibr. Probl., **13**, 1, 1972.
7. M. P. Mason, *Physical acoustics and the properties of solid,* Princeton, New Jersey 1958.
8. Z. Dżygadło, S. Kaliski, L. Solarz, E. Włodarczyk, *Drgania i fale,* Red. nauk. S. Kaliski, PWN, 1967.

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