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Defects as initial deformations

E. KOSSECKA (WARSZAWA)

THE INITIAL (plastic) deformations corresponding to defects are considered. Singular fields of initial (plastic) distortion $\overset{\circ}{\beta}$, having the character of a delta function and corresponding to velocity field $\overset{\circ}{v}$, are constructed. Next, the following equations are derived: the equation which enables us to derive the total (discontinuous) displacement field for the medium with a defect, when $\overset{\circ}{\beta}$ and $\overset{\circ}{v}$ are given, and the equations from which elastic distortion and velocity field corresponding to it are to be derived when $\overset{\circ}{\beta}$ and $\overset{\circ}{v}$ are given. The particular solutions of these equations are derived.

W pracy rozważa się deformacje wstępne (plastyczne) odpowiadające defektom. Konstruuje się osobliwe — mające charakter funkcji delta — pole wstępnej (plastycznej) dystorsji $\overset{\circ}{\beta}$ i odpowiadające mu osobliwe pole prędkości $\overset{\circ}{v}$. Następnie wyprowadza się kolejno równanie pozwalające wyznaczyć całkowite (nieciągłe) pole przemieszczeń dla ośrodka z defektem przy danych $\overset{\circ}{\beta}$ i $\overset{\circ}{v}$ oraz równania pozwalające wyznaczyć pole dystorsji sprężystej i odpowiadające mu pole prędkości przy danych $\overset{\circ}{\beta}$ i $\overset{\circ}{v}$. Podane są rozwiązania szczególne tych równań.

В работе рассмотрены начальные деформации (остаточные пластические деформации), соответствующие дефектам. Дается конструкция поля предварительной (пластической) дисторсии $\overset{\circ}{\beta}$, имеющей особенность типа функции дельта, и соответствующего ей поля скоростей $\overset{\circ}{v}$, с особенностью того же типа. Далее, выводятся уравнения, дающие возможность определения полного (разрывного) поля перемещений в среде с дефектом, при заданных $\overset{\circ}{\beta}$ и $\overset{\circ}{v}$, а также уравнения, описывающие поле упругой дисторсии и соответствующее ей поле скоростей для заданных $\overset{\circ}{\beta}$ и $\overset{\circ}{v}$. Даются частные решения этих уравнений.

Introduction

IN PAPERS devoted to the defects theory, we are faced with different methods of treatment of the basic problem — the construction of the solution describing a surface defect. Some of these methods are quite general, others — e.g., the methods applied in the two-dimensional crack theory — are applicable only to special problems.

In the most general case, one has to construct the mathematical model of a dynamic surface defect (crack or dislocation), which has the surface of an arbitrary shape depending eventually on time.

If we want to describe a medium with a defect by means of the displacement field $\mathbf{u}$, a defect is modelled by the displacement field having the jump discontinuity $\mathbf{U}$ on the defect surface S and describing outside this surface the free elastic medium. The problem of construction of such a field can be solved by means of the Somigliana formula, which is the consequence of the Green formula in the theory of elasticity. This method — called also the potentials method and introduced to the theory of elasticity by W. D. KUPRADSE (see [2]) — generalized to the case of moving surfaces, was applied by H. ZORSKI in [1]. It is also discussed in detail in [3, 4, 5, 6]. The method applied in the crack theory, on the basis of the properties of surface distributions of double and single forces, is equivalent to the potentials method, as was discussed in detail in [5].

The problem of a defect may, however, be treated in two ways, different from the above-mentioned. We make use of the fact that the singularities of the displacenent field describing a medium with a defect are concentrated on the defect surface S. We, thus, represent the expressions for distortion and velocity fields as the sum of the regular and singular parts. The regular part of the distortion $\boldsymbol{\beta}$ is called the elastic distortion, the singular part $\mathring{\boldsymbol{\beta}}$ is called the plastic or initial distortion (the same concerns the velocities $\mathbf{v}$ and $\mathring{\mathbf{v}}$). The expressions for the singular parts of the distortion field — $\mathring{\boldsymbol{\beta}}$ and the velocity field — $\mathring{\mathbf{v}}$ can easily be constructed explicitly on the basis of the jump condition for the $\mathbf{u}$ field. They have the form of generalized functions of the Dirac delta function type, concentrated on the defect surface.

The equation of motion of the medium is the consequence of the assumption that every piece of the medium is in a state of dynamic equilibrium—e.g., does not act with a resultant force. In this equation, the elastic distortion field $\boldsymbol{\beta}$ appears and the velocity field $\mathbf{v}$ corresponding to it. The relations between the derivatives of the displacement field, the elastic distortion and the velocity field corresponding to it, play the role of subsidiary conditions. Now, we can obtain solutions of the problem in two different ways. First, we can insert the subsidiary conditions directly into equilibrium equation; we then obtain the inhomogenous equation for the displacement field $\mathbf{u}$, the inhomogeneity being given in terms of initial distortions and velocities. This method we shall term the initial deformation method. Secondly, we can eliminate from the subsidiary conditions the derivatives of the displacement field $\mathbf{u}$ and obtain these conditions as the relations between the fields $\boldsymbol{\beta}$ $\mathbf{v}$ and $\mathring{\boldsymbol{\beta}}$, $\mathring{\mathbf{v}}$; relations of this kind are frequently called constraint equations for the fields $\boldsymbol{\beta}$ and $\mathbf{v}$. From the indicated conditions together with the equation of motion, we then obtain two equations of the type of the Lamé equation for the fields $\boldsymbol{\beta}$ and $\mathbf{v}$. This method is applicable in the dislocations theory and is a natural approach to the dislocation problem, since in this case only the elastic distortions have a physical interpretation (see [3, 7, 8, 9]). It can easily be generalized to the case of continuous distribution of dislocations. However, it can be applicable also to any surface defect.

1. Singularities of the distortion field on the defect surface

In the general case, a defect is described by the displacement field $\mathbf{u}$, having on the defect surface S, represented by the radius vector $\boldsymbol{\zeta}$, the jump discontinuity $\mathbf{U}$. We assume that $\mathbf{U}$ is a continuous and differentiable (at least twice) function of the vector $\boldsymbol{\zeta}$. We assume, moreover, that the displacement field $\mathbf{u}$ is everywhere finite, the distortion field $u_{i,k}$ and the velocity field $\mathring{\mathbf{u}}$ being at the same time continuous everywhere outside the defect surface. $\mathbf{U}$ may also be a function of time.

We have then:

$$\begin{gathered}\lim_{\varepsilon\to 0}\{\mathbf{u}(\boldsymbol{\zeta}+\varepsilon\mathbf{k})-\mathbf{u}(\boldsymbol{\zeta}-\varepsilon\mathbf{k})\} = \mathbf{U}, \\ \boldsymbol{\zeta}\in S, \quad \varepsilon \geqslant 0, \quad |\mathbf{k}| = 1, \quad \mathbf{k}\cdot\mathbf{n} \geqslant 0.\end{gathered} \tag{1.1}$$

The above limit does not depend on the direction of the unit vector $\mathbf{k}$. Symbolically, we write (1.1) in the following form:

$$[\![\mathbf{u}(\boldsymbol{\zeta})]\!] = \mathbf{U}(\boldsymbol{\zeta}), \quad \boldsymbol{\zeta} \in S. \tag{1.2}$$

The double brackets denote here the discontinuity of the function $\mathbf{u}(\mathbf{x})$ at the point $\boldsymbol{\zeta}$ of the surface S.

At the same time, the following identity holds:

$$u_i(\boldsymbol{\zeta}+\varepsilon\mathbf{k}) - u_i(\boldsymbol{\zeta}-\varepsilon\mathbf{k}) = \oint_{\lambda} d\lambda_k u_{i,k}, \quad \zeta \in S, \tag{1.3}$$

where λ is any curve joining the point $\boldsymbol{\zeta}-\varepsilon\mathbf{k}$ and $\boldsymbol{\zeta}+\varepsilon\mathbf{k}$. Without loss of generality, in what follows we shall assume that λ is a segment of the straight line having the direction of $\mathbf{k}$ (see Fig. 1):

$$\mathbf{x}(l) \in \lambda : \mathbf{x}(l) = \boldsymbol{\zeta} + l\mathbf{k}, \quad -\varepsilon \leqslant l \leqslant +\varepsilon.$$

Thus, we write (1.3) in the form:

$$\lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} dl k_k u_{i,k}(\mathbf{x}(l)) = \mathbf{U}_i. \tag{1.4}$$

Consequently, the value of the integral remains finite, the lenght of the integration path tending simultaneously to zero (with the assumption of continuity of the function $u_{i,k}$ in the neighbourhood of the surface S). It follows, therefore, that the function $u_{i,k}$ has sin-

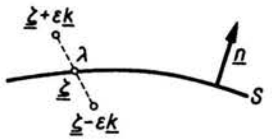

FIG. 1

gularity of the type of a Dirac delta function on the surface S. This singular part of the distortion field $u_{i,k}$ is related to the discontinuity of the displacement field $\mathbf{u}$; in what follows, we shall term it initial (or plastic) distortion and shall denote as $\overset{\circ}{\beta}_{lk}$. Therefore,

$$u_{l,k} = \beta_{lk} + \overset{\circ}{\beta}_{lk}. \tag{1.5}$$

The field β_{ik} is taken to be the elastic distortion; we assume that β_{ik} does not have strong singularities on S except at least on its boundary.

We shall prove that β_{ik} is a tensor generalized function having the form:

$$\overset{\circ}{\beta}_{ik} = \int_S ds_k U_i \delta_3(\mathbf{x}-\boldsymbol{\zeta}), \quad \boldsymbol{\zeta} \in S, \tag{1.6}$$

where $ds_k = dsn_k$, $\mathbf{n}$ being the normal vector of the surface S. Let us consider the case in which the surface S is the part of the xy-plane described as follows:

$$\begin{gathered} \mathbf{n} = [0,0,1], \quad \boldsymbol{\zeta} = [\zeta_1, \zeta_2, 0], \\ a_1 \leqslant \zeta_1 \leqslant a_2, \\ \zeta_2'(\zeta_1) \leqslant \zeta_2 \leqslant \zeta_2''(\zeta_1), \end{gathered} \tag{1.7}$$

The components of the tensor $\overset{\circ}{\beta}_{ik}$ have the form:

$$\overset{\circ}{\beta}_{i1} = 0, \quad \overset{\circ}{\beta}_{i2} = 0, \tag{1.8}$$

$$\overset{\circ}{\beta}_{i3} = \int_S ds U_i(\zeta_1, \zeta_2)\delta(x-\zeta_1)\delta(y-\zeta_2)\delta(z)$$

$$= \int_{a_1}^{a_2} d\zeta_1 \int_{\zeta_2'(\zeta_1)}^{\zeta_2''(\zeta_1)} d\zeta_2 U_i(\zeta_1, \zeta_2)\delta(x-\zeta_1)\delta(y-\zeta_2)\delta(z)$$

$$= \int_{a_1}^{a_2} d\zeta_1 U_i(\zeta, y)[\eta(y-\zeta_2'(\zeta_1))-\eta(y-\zeta_2''(\zeta_1))]\delta(x-\zeta_1)\delta(z)$$

$$= U_i(x, y)[\eta(x-a_1)-\eta(x-a_2)][\eta(y-\zeta_2'(x))-\eta(y-\zeta_2''(x))]\delta(z),$$

$$\eta(x) = \frac{1}{2}\operatorname{sgn}(x).$$

Let us denote:

$$[\eta(x-a_1)-\eta(x-a_2)][\eta(y-\zeta_2'(x))-\eta(y-\zeta_2''(x))] = \underset{S}{\theta}(x, y). \tag{1.9}$$

The function $\underset{S}{\theta}(x, y)$ is equal to unity inside the cylinder outlined by the straight line sliding along the boundary of the surface S, and is equal to zero outside this cylinder. We write $(1.8)_3$ in the form:

$$\overset{\circ}{\beta}_{i3} = \underset{S}{\theta}(x, y)\delta(z)U_i(x, y). \tag{1.10}$$

Let us now calculate the expression (1.4) integrating along λ, which is a segment of the straight line intersecting the xy-plane at the point $\overset{\circ}{\boldsymbol{\zeta}} = [\overset{\circ}{\zeta}_1, \overset{\circ}{\zeta}_2, 0)$.

$$\int_\lambda d\lambda_r U_{i,r} = \int_\lambda dlk_r \beta_{ir} + \int dlk_r \overset{\circ}{\beta}_{ir}. \tag{1.11}$$

From the assumption concerning regularity of the function $\boldsymbol{\beta}$, it follows that the first integral appearing on the right-hand side of the above formula tends to zero simultaneously with the integration path tending to zero. In the second integral, we interchange the integration with respect to 1 with the integration with respect to the coordinate z:

$$dlk_3 = dz; \quad -\varepsilon k_3 \leqslant z \leqslant \varepsilon k_3,$$

$$\mathbf{x}(z) = \left[\overset{\circ}{\zeta} + \frac{k_1}{k_3}z, \overset{\circ}{\zeta}_2 + \frac{k_2}{k_3}z, z\right].$$

$$\int_\lambda dlk_r \overset{\circ}{\beta}_{ir}(\mathbf{x}) = \int_\lambda dlk_3 \overset{\circ}{\beta}_{i3}(\mathbf{x}) = \int_{-\varepsilon k_3}^{\varepsilon k_3} dz \overset{\circ}{\beta}_{i3}(\mathbf{x}(z)) \tag{1.12}$$

$$\approx \int_{-\varepsilon k_3}^{\varepsilon k_3} dz \underset{S}{\theta}[x(z), y(z)]\delta(z)U_i(x(z), y(z)) = \underset{S}{\theta}[x(0), y(0)]U_i(x(0), y(0))$$

$$\approx \underset{S}{\theta}(\overset{\circ}{\zeta}_1, \overset{\circ}{\zeta}_2)U_i(\overset{\circ}{\zeta}_1, \overset{\circ}{\zeta}_2).$$

Consequently, the integral (1.12) is equal to **U** if the line intersects the surface S; otherwise, it is equal to zero. Thereby the condition (1.4) is fulfilled. The generalization of the above proof to the case of a surface of arbitrary shape is not difficult.

2. Singularities of the velocity field on the defect's surface

The particular case constitutes a defect the surface of which is moving about in the medium; obviously, we assume that the motion $\boldsymbol{\zeta}(t)$ is regular function of time. The assumption that the defect surface — that is, the surface of discontinuity of the field $\mathbf{u}(\mathbf{x}, t)$ — is in motion does not affect the form of the conditions (1.1), (1.5), (1.6). However, the set of singular points of the function $\mathbf{u}(\mathbf{x}, t)$ considered as a function of four variables, is altered. The formula (1.1) states that at a fixed instant of time the increment of the displacement field between two points which are situated no matter how close to each other but on the opposite faces of the surface S, is finite. Notice, however, that in the case of a moving surface, the fixed point of the medium at two instants of time arbitrarily near each other can be found on two different faces of the surface S — if only at some instant of time t it coincides with some point of the surface $\boldsymbol{\zeta}(t)$; moreover, $\dot{\boldsymbol{\zeta}} \cdot n \neq 0$.

Let us consider the following expression:

$$(2.1) \qquad \mathbf{u}(\boldsymbol{\zeta}(t), t+\varepsilon) - \mathbf{u}(\boldsymbol{\zeta}(t), t-\varepsilon) = \mathbf{u}(\boldsymbol{\zeta}(t), t+\varepsilon) - \mathbf{u}(\boldsymbol{\zeta}(t) + 2\dot{\boldsymbol{\zeta}}\varepsilon, t+\varepsilon)$$

$$+\mathbf{u}(\boldsymbol{\zeta}(t) + 2\dot{\boldsymbol{\zeta}}\varepsilon, t+\varepsilon) - \mathbf{u}(\boldsymbol{\zeta}(t), t-\varepsilon) \approx -[\mathbf{u}(\boldsymbol{\zeta}(t+\varepsilon) + \dot{\boldsymbol{\zeta}}\varepsilon, t+\varepsilon) - \mathbf{u}(\boldsymbol{\zeta}(t+\varepsilon) - \dot{\boldsymbol{\zeta}}\varepsilon, t+\varepsilon)]$$

$$+[\mathbf{u}(\boldsymbol{\zeta}t+\varepsilon) + \dot{\boldsymbol{\zeta}}\varepsilon, t+\varepsilon) - \mathbf{u}(\boldsymbol{\zeta}(t-\varepsilon) + \dot{\boldsymbol{\zeta}}\varepsilon, t-\varepsilon)].$$

The first term of the above formula represents the increment of the displacement vector **u** at the moment $t+\varepsilon$ between two points at the space-like distance $2\dot{\zeta}\varepsilon$ and situated on two different faces of the surface S; it tends to $-\mathbf{U}$. The second term represents the increment of **u** between two points at the space-like distance $2\dot{\zeta}\varepsilon$ and the time-like distance 2ε, but situated on the same face of the surface S (at distance $(\dot{\zeta}n)\varepsilon$ from the surface); it tends to zero. We thus obtain the analog of the formula (1.1):

$$(2.2) \qquad \lim_{\varepsilon \to 0} \{\mathbf{u}(\boldsymbol{\zeta}(t), t+\varepsilon) - \mathbf{u}(\boldsymbol{\zeta}(t), t-\varepsilon)\} = -\mathbf{U}, \qquad \boldsymbol{\zeta}(t) \in S.$$

Because at the same time

$$(2.3) \qquad \mathbf{u}(\boldsymbol{\zeta}(t), t+\varepsilon) - \mathbf{u}(\boldsymbol{\zeta}(t), t-\varepsilon) = \int_{t-\varepsilon}^{t+\varepsilon} dt' \dot{\mathbf{u}}(\boldsymbol{\zeta}(t), t'),$$

we obtain the result:

$$(2.4) \qquad \lim_{\varepsilon \to 0} \int_{t-\varepsilon}^{t+\varepsilon} dt' \dot{\mathbf{u}}(\mathbf{x}, t') = -\mathbf{U},$$

$$\text{if} \qquad \mathbf{x} = \boldsymbol{\zeta}(t), \qquad t-\varepsilon \leqslant t' \leqslant t+\varepsilon.$$

It can be seen from the above that the velocity field $\dot{\mathbf{u}}$ has at points $[\boldsymbol{\zeta}(t), t]$ singularity of the type of a delta function, which makes the result of integration (2.4) finite. The velocity field we shall thus represent as the sum of the regular part $\mathbf{v}$ — corresponding to the elastic distortion field, and the singular part $\mathring{\mathbf{v}}$ — corresponding to the initial (plastic) distortion:

$$\dot{u}_i = v_i + \mathring{v}_i. \tag{2.5}$$

We shall prove that $\mathring{v}_i$ is a generalized function having the form:

$$\mathring{v}_i = -\int_S ds_k U_i \dot{\zeta}_k \delta_3(\mathbf{x}-\boldsymbol{\zeta}). \tag{2.6}$$

Let us consider the surface S described by the formulae (1.7), with this change only that $\zeta_3 = \zeta_3(t)$. In this case:

$$\mathring{v}_i = -\int_S ds_3 U_i \dot{\zeta}_3 \delta_3(\mathbf{x}-\boldsymbol{\zeta}) = -U_i(x,y)\dot{\zeta}_3(t)\theta(x,y)\delta\big(z-\zeta_3(t)\big). \tag{2.7}$$

Let us now calculate the following integral:

$$\int_{t-\varepsilon}^{t+\varepsilon} dt' \dot{u}_i(\mathbf{x},t') = \int_{t-\varepsilon}^{t+\varepsilon} dt' v_i(\mathbf{x},t') + \int_{t-\varepsilon}^{t+\varepsilon} dt' \mathring{v}_i(\mathbf{x},t'). \tag{2.8}$$

The first of the integrals appearing on the right-hand side of the above formula tends to zero. When calculating the second one, we shall put for $\mathring{v}_i$ the expression (2.7) and interchange the integration with respect to t with the integration with respect to ζ_3:

$$\begin{aligned} \int_{t-\varepsilon}^{t+\varepsilon} dt' \mathring{v}_i(\mathbf{x},t') &= -\int_{t-\varepsilon}^{t+\varepsilon} dt' U_i \dot{\zeta}_3(t') \underset{S}{\theta}(x,y)\delta\big(z-\zeta_3(t)\big) \\ &= -\underset{S}{\theta}(x,y) \int_{\zeta_3(t-\varepsilon)}^{\zeta_3(t+\varepsilon)} d\zeta_3 U_i \delta\big(z-\zeta_3(t)\big) \\ &= -U_i(x,y)\underset{S}{\theta}(x,y)\big[\eta\big(z-\zeta_3(t-\varepsilon)\big)-\eta\big(z-\zeta_3(t+\varepsilon)\big)\big]. \end{aligned} \tag{2.9}$$

Consequently, the integral (2.9) is equal to $-\mathbf{U}$ if in the vicinity of the moment t the point $\mathbf{x}$ coincides with the point of the surface $\boldsymbol{\zeta}(t')$, $t-\varepsilon \leqslant t' \leqslant t+\varepsilon$ (at the limit $\mathbf{x} = \boldsymbol{\zeta}(t)$); in the opposite case it is equal to zero. Thereby, the condition (2.4) is fulfilled. Now it should be mentioned that the formulae (1.6), (2.6) can be proved in an elegant manner by means of four-dimensional formalism.

3. The equations of equilibrium of a medium with defects

3.1. The solution for the field u

The basis of calculations of the distorsion and velocity fields in a medium with defects is the basic equation of dynamics. It is stated, namely, that the elastic distortion and the corresponding velocity field describe the medium, every part of which can be treated as a part of the free medium, not acting with the resultant force. In other words, we are dealing with a free medium in a certain state of deformation, described by the distortion $\boldsymbol{\beta}$, which needs not be integrable (integrable is the sum $\boldsymbol{\beta}$ and $\mathring{\boldsymbol{\beta}}$ only).

The equation of motion has the form:

$$(3.1)\qquad \varrho\frac{d}{dt}v_i - c_{iklm}\nabla_k\beta_{lm} = 0.$$

Further, we have the subsidiary conditions:

$$(3.2)\qquad \begin{aligned} u_{l,k} &= \beta_{lk}+\mathring{\beta}_{lk} \Rightarrow \beta_{lk} = u_{l,k}-\mathring{\beta}_{lk}.\\ \dot{u}_l &= v_l+\mathring{v}_l \Rightarrow v_l = \dot{u}_l-\mathring{v}_l. \end{aligned}$$

Inserting the above expressions for $\boldsymbol{\beta}$ and $\mathbf{v}$ into Eq. (3.1), we obtain the equation of motion as an equation from which the $\mathbf{u}$ field is to be determined, $\mathring{\boldsymbol{\beta}}$ and $\mathring{\mathbf{v}}$ being given:

$$(3.3)\qquad \left[\varrho\delta_{il}\frac{d^2}{dt^2} - c_{iklm}\nabla_k\nabla_m\right]u_l = \varrho\frac{d}{dt}\mathring{v}_i - c_{iklm}\nabla_k\mathring{\beta}_{lm}.$$

The right-hand side here represents the field of forces acting on the ideal continuum and producing the distortion and velocity the continuous parts of which are the deformation and velocity of the medium with defects.

Taking into account the formulae (1.6), (2.6), we write (3.3) in the form:

$$(3.4)\qquad \begin{gathered} \left[\varrho\delta_{il}\frac{d^2}{dt^2} - c_{iklm}\nabla_k\nabla_m\right]u_l = X_i,\\ X_i = -\varrho\frac{d}{dt}\int\limits_S ds_k U_i\dot{\zeta}_k\delta_3(\mathbf{x}-\boldsymbol{\zeta}) - c_{iklm}\nabla_k\int\limits_S ds_m U_l\delta_3(\mathbf{x}-\boldsymbol{\zeta}). \end{gathered}$$

The particular solution of Eqs. (3.4), being the convolution of the inhomogeneity $\mathbf{X}$ and the dynamic Green tensor $\mathbf{G}$ of the Lamé equation (see [11, 12])

$$u_i = G_{ik} * X_k,$$

has the form:

$$(3.5)\qquad u_i = -\int\limits_{-\infty}^{\infty} dt'\int\limits_S ds_b U_n\left[\varrho\dot{\zeta}_b\frac{\partial}{\partial t}G_{in}(\mathbf{x}-\boldsymbol{\zeta}, t-t') + c_{nbrs}\nabla_s G_{ir}(\mathbf{x}-\boldsymbol{\zeta}, t-t')\right].$$

The properties of the force field (3.4) and of the solution (3.5) for the static case are discussed in detail in [6].

3.2. The solutions for the fields $\boldsymbol{\beta}$ and $\mathbf{v}$

If we are to derive the equations for the fields $\boldsymbol{\beta}$ and $\mathbf{v}$, and to find directly the expressions for these fields, we proceed as follows. From the system of Eqs. (3.2), we eliminate the field $\mathbf{u}$. Acting on the first equation with the rotation operation, we obtain:

$$(3.6)\qquad \mathscr{E}_{klm}\nabla_l\beta_{im} = -\mathscr{E}_{klm}\nabla_l\mathring{\beta}_{im}.$$

Acting now on the first equation with the operation d/dt and on the second with ∇_k, and subtracting one from the other, we obtain:

$$(3.7)\qquad \frac{d}{dt}\beta_{lk} - v_{l,k} = -\frac{d}{dt}\mathring{\beta}_{lk} + \mathring{v}_{l,k}.$$

The conditions (3.6), (3.7) are the subsidiary conditions — or the constraint conditions for the fields $\boldsymbol{\beta}$ and $\mathbf{v}$. Note that the generalized functions appearing on the right-hand sides of Eqs. (3.6), (3.7) have a complicated structure and, in general, it is difficult to construct them directly. For the case of a dislocation only, these expressions have comparatively simple form; they are given as the line integrals of the delta function (see [3]).

When we set out to solve the system of Eqs. (3.1), (3.6), (3.7), we proceed as follows: we act on (3.1) with the operation ∇ and on (3.7) with the operation $\varrho(d/dt)$, and add them together; then we act on (3.1) with d/dt and on (3.7) with $c_{iklm}\nabla_k$ and add them together. We obtain the following set of equations:

$$\varrho\frac{d^2}{dt^2}\beta_{ls}-c_{iklm}\beta_{im,ks} = \varrho\frac{d}{dt}\mathring{v}_{l,s}-\varrho\frac{d^2}{dt^2}\mathring{\beta}_{ls},$$

$$(3.8)\qquad \varrho\frac{d^2}{dt^2}v_i-c_{iklm}v_{l,mk} = -c_{iklm}\nabla_k\left[\frac{d}{dt}\mathring{\beta}_{lm}-\mathring{v}_{l,m}\right],$$

$$\mathscr{E}_{klm}\beta_{im,l} = -\mathscr{E}_{klm}\mathring{\beta}_{im,l}.$$

Multiplying Eq. $(3.8)_3$ by the tensor $\mathscr{E}_{abk}$, we obtain:

$$(3.9)\qquad \beta_{ia,b}-\beta_{ib,a} = -[\mathring{\beta}_{ia,b}-\mathring{\beta}_{ib,a}].$$

Inserting the above into Eq. $(3.8)_1$, we obtain for the field $\boldsymbol{\beta}$ an equation of the type of the Lamé equation.

Finally, the set of equations for the fields $\boldsymbol{\beta}$ and $\mathbf{v}$ is:

$$\left[\varrho\delta_{il}\frac{d^2}{dt^2}-c_{iklm}\nabla_k\nabla_m\right]v_l = -c_{iklm}\nabla_k\left[\frac{d}{dt}\mathring{\beta}_{lm}-\mathring{v}_{l,m}\right],$$

$$(3.10)\qquad \left[\varrho\delta_{il}\frac{d^2}{dt^2}-c_{iklm}\nabla_k\nabla_m\right]\beta_{ls} = -c_{iklm}\nabla_k[\mathring{\beta}_{lm,s}-\mathring{\beta}_{ls,m}]-\varrho\frac{d}{dt}\left[\frac{d}{dt}\mathring{\beta}_{is}-\mathring{v}_{i,s}\right].$$

These are equations of the Lamé type; particular solutions of them are given in the form of convolutions of the inhomogeneities appearing on the right-hand sides of the equations and the dynamic Green tensor of the Lamé equation:

$$v_i = G_{ij} * X_j,$$

$$v_i = -c_{jklm}\int_{-\infty}^{\infty}dt'\left[\int_S ds_m U_l\frac{\partial}{\partial t}G_{ij,k}-\int_S ds_n\dot{\zeta}_n U_l G_{ij,km}\right],$$

$$(3.11)\qquad \beta_{ir} = G_{ij} * X_{jr},$$

$$\beta_{ir} = -\mathscr{E}_{mra}\mathscr{E}_{abc}c_{jklm}\int_{-\infty}^{\infty}dt'\int_S ds_b U_l G_{ij,kc}-\varrho\int_{-\infty}^{\infty}dt'\left[\int_S ds_r U_j\frac{\partial^2}{\partial t^2}G_{ij}-\int_S ds_n\dot{\zeta}_n U_j\frac{\partial}{\partial t}G_{ij,r}\right],$$

$$G_{ij} = G_{ij}(\mathbf{x}-\boldsymbol{\zeta}, t-t').$$

As may be seen from the above, in the general case the solutions have the form of surface integrals. Only for the case of a dislocation (U = const) can they be transformed into line integrals [see (10.3)].

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