Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics. Volume I: Foundations

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Dedicated to Professor Beloslav Riečan on his 75th anniversary

A convergence theorem on effect algebras

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Abstract

There is a method of the local representation of a sequence of observables by a sequence of random variables. It was successfully used in the probability theory on MV-algebras. In the paper the method is extended from MV-algebras to effect algebras.

Keywords: ergodic, effect algebra, observable.

1 Introduction

The classical Kolmogorov probability theory works with a probability space $(\Omega, \mathcal{S}, p)$ and sequences $(\xi_n)_{n=1}^{\infty}$ of random variables $\xi_n : \Omega \to R$. In quantum structures ([3]) some other approach is convenient: sequences $(x_n)_{n=1}^{\infty}$ of σ-homomorphisms from the Borel family $\mathcal{B}(R)$ to the quantum structure are considered. Very interesting example important in multi-valued logic as well as in quantum structures is an MV-algebra ([1],[8], for the probability theory on MV-algebras see [12]). One of the effective methods of the probability theory on MV-algebras is the local representation of a sequence $(x_n)_{n=1}^{\infty}$ of observables by a sequence $(\xi_n)_{n=1}^{\infty}$ of random variables in an appropriate probability space. This method was used for the first time in [9] (see also [13]. It was successfully used in probability theory on MV-algebras ([7], [5], [10], [14], for a review see [12], [3]), in ergodic theory [6], in topology ([11]), etc. From the a.e. convergence of the corresponding se- quence in the Kolmogorov space the a.e. convergence of

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the original sequence follows. In the paper we present a variant of the local representation method in effect algebras ([4], see also [3], an equivalent structure is a D-poset ([2])). It seems that by the method many results of the classical probability theory could be transpose to the quantum probability theory even in the effect algebras case.

Definition 1 *Let E be a nonempty set, $\oplus$ is a partially binary operation on E and $0_E, 1_E$ are elements of E. When the following conditions are fulfilled:*

1. *if $a \oplus b$ exists, then $b \oplus a$ exist and $a \oplus b = b \oplus a$*
2. *if $(a \oplus b) \oplus c$ exists on E, then $a \oplus (b \oplus c)$ exists and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ for all $a, b, c \in E$*
3. *if $(a \oplus 1)$ is defined then $a = 0_E$*
4. *for every element a in E exists element b in E: $a \oplus b = 1_E$*

then the system $(E, \oplus, 0_E, 1_E)$ is called effect algebra.

In the following text we will use the notation:

$$\mathcal{S} = \{(-\infty, t)\} \cup \{R\} \cup \{\emptyset\}\,; t \in R$$
$$\mathcal{B}(R) = \text{ Borel set}$$

We want to introduce probability on effect algebras. We need to define two basic mappings on E for this aim. The first is called observable and the second usefull mapping is state on m.

Definition 2 *The mapping $x : \mathcal{S} \to E$ is called observable, if the following statements are fulfilled:*

1. $x(R) = 1_E; x(\emptyset) = 0_E$,
2. $A \subset B \Rightarrow x(A) \leq x(B)$ *for all* $A, B \in \mathcal{S}$,
3. $A_n \nearrow A \Rightarrow x(A_n) \nearrow x(A)$ *for all* $A_n, A \in \mathcal{S}$,
4. $B_n \nearrow \emptyset \Rightarrow x(B_n) \searrow 0$.

Definition 3 *The mapping $m : E \to \langle 0, 1 \rangle$ is a state if the following rules holds*

1. $m(1_E) = 1, m(0_E) = 0$

2. $s \leq t, m(s) \leq m(t)$*; for all* $s, t \in R$

3. $a_n \nearrow a \Rightarrow m(a_n) \nearrow m(a)$
$a_n \searrow a \Rightarrow m(a_n) \searrow m(a)$ *for every* $a_n, a \in R$

Definition 4 *The mapping* F *will be defined as a composite mapping of state and observable*

$$F : R \to \langle 0, 1\rangle : F(t) = m(x(-\infty, t))$$

Theorem 1 *The mapping* F *introduced in previous definition is a distribution function.*

Proof: We will prove the four properties of distribution function

1. let a, b be element of R and $a \leq b$ then by using properties of observable and state holds:

$$F(a) = m(x(-\infty, a)) \leq m(x(-\infty, b)) = F(b)$$

   So the function F is nondecreasing

2. $a \in R : \lim\limits_{x \to a^-} F(x) = F(a)$ This property results from the third properties of states and observables.

3. $\lim\limits_{a \to \infty} F(a) = \lim\limits_{a \to \infty} m(x(-\infty, a)) = m(x(R)) = 1$

4. $\lim\limits_{a \to -\infty} F(a) = \lim\limits_{a \to -\infty} m(x(-\infty, a)) = m(x(\emptyset)) = 0$

Definition 5 *The sequence of the observables* $(x_n)_{n=1}^{\infty}$ *is called regular, if* $\forall n \in N$ *there exists an observable* $h_n : \mathcal{B}(R^n) \to E$ *with the following property:*

$$h_n(A_1 \times A_2 \times ... \times A_n) = x_1(A_1) \wedge x_2(A_2) \wedge ... \wedge x_n(A_n).$$

The sequence of mappings h_n *is called regulating sequence.*

It is easy to see that it holds:

$$m(h_{n+1}(A \times R)) = m(h_n(A)).$$

2 Kolmogorov construction

Let $(x_n)_{n=1}^{\infty}$ be a regular sequence of the observables in E and for all $n \in N$ the mapping h_n be the observable from the previous definition.

Let $C = \left\{\pi_n^{-1}(K), K \in \mathcal{B}(R^n), n \in N\right\}$ be a set of all cylinders, where the function $\pi_n : R^N \to R^n$ defined by $\pi_n\left((u_i)_1^{\infty}\right) = (u_1, ..., u_n)$ is called the n-th coordinate random vector.

Then there exists a mapping such that $P : \sigma(C) \to \langle 0, 1\rangle$

$$P \circ \pi_n^{-1} = m \circ h_n : \mathcal{B}(R^n) \to \langle 0, 1\rangle .$$

for every $n \in N$.

Let for every $n \in N$ the function $g_n : R^n \to R$ be Borel measurable function.

For each $n \in N$ the function $\xi_n : R^N \to R$ is given by

$$\xi_n\left((u_i)_{i=1}^{\infty}\right) = u_n.$$

We define the mapping η_n by this way:

$$\eta_n = g_n(\xi_1, \xi_2, ..., \xi_n) = g_n \circ \pi_n$$

for all $n \in N$ and for every Borel function g_n. So we can write for the mapping $\eta_n^{-1} : \mathcal{B}(R) \to \mathcal{B}(R^n)$ the following formula:

$$\eta_n^{-1}(A) = \pi_n^{-1}\left(g_n^{-1}(A)\right)$$

for all $n \in N, \forall A \in \mathcal{B}(R)$.

By y_n we denote the mapping from $\mathcal{B}(R)$ to the effect algebra E given by the equality:

$$y_n = h_n \circ g_n^{-1}.$$

Then for the mapping $m \circ y_n : \mathcal{B}(R) \to \langle 0, 1\rangle$ there hold the equalities:

$$P \circ \eta_n^{-1} = P \circ \pi_n^{-1} \circ g_n^{-1} = m \circ h_n \circ g_n^{-1} = m \circ y_n$$

for every natural number n.

3 Upper and lower limits

In the next step we modify the almost everywhere convergence with help of lim sup and lim inf and the construction of translations formulas for these limits. This modification gives us the possibility to proved the convergence theorem. You can see that:

$$\limsup_{n\to\infty} \eta_n(\omega) < t \Leftrightarrow \omega \in \bigcup_{p=1}^{\infty}\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} \eta_n^{-1}\left(\left(-\infty, t-\frac{1}{p}\right)\right).$$

This relation justifies the following definition.

Definition 6 *The sequence $(x_n)_{n=1}^{\infty}$ of observables on σ-complete effect algebra E has $\limsup\limits_{n\to\infty}$ if there exists a mapping $\overline{x} : (S) \to E$ such that*

$$\overline{x}\left((-\infty, t)\right) = \bigvee_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} x_n\left(\left(-\infty, t-\frac{1}{p}\right)\right)$$

for every $t \in R$. If such observable exists, we write $\overline{x} = \limsup\limits_{n\to\infty} x_n$.

Definition 7 *The sequence $(x_n)_{n=1}^{\infty}$ of observables on σ-complete effect algebra E has $\liminf\limits_{n\to\infty}$ if there exists a mapping $\underline{x}$ such that*

$$\underline{x}\left((-\infty, t)\right) = \bigvee_{p=1}^{\infty}\bigwedge_{k=1}^{\infty}\bigvee_{n=k}^{\infty} x_n\left(\left(-\infty, t-\frac{1}{p}\right)\right)$$

for every $t \in R$. If such observable exists, we write $\underline{x} = \liminf\limits_{n\to\infty} x_n$.

Proposition 1 *If a sequence $(x_n)_{n=1}^{\infty}$ of the observables on an effect algebras E has $\limsup\limits_{n\to\infty} x_n$ and $\liminf\limits_{n\to\infty} x_n$, then the following inequality holds for every $t \in R$:*

$$\limsup_{n\to\infty} x_n\left((-\infty, t)\right) \le \liminf_{n\to\infty} x_n\left((-\infty, t)\right)$$

Proof: Let $\omega \in \Omega$ and $k \ge 1, k \in N$.

Evidently the following inequalities hold:

$$\bigwedge_{n=k}^{\infty} x_n\left(\left(-\infty, t-\frac{1}{p}\right)\right) \le \bigwedge_{n=k+1}^{\infty} x_n\left(\left(-\infty, t-\frac{1}{p}\right)\right)$$
$$\bigvee_{n=k+1}^{\infty} x_n\left(\left(-\infty, t-\frac{1}{p}\right)\right) \le \bigvee_{n=k}^{\infty} x_n\left(\left(-\infty, t-\frac{1}{p}\right)\right).$$

It is clear then for every $k \in N$:

$$\bigwedge_{n=k+1}^{\infty} x_n\left(\left(-\infty, t-\frac{1}{p}\right)\right) \leq \bigvee_{n=k+1}^{\infty} x_n\left(\left(-\infty, t-\frac{1}{p}\right)\right)$$

So we have for all $p \in N$

$$\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} x_n\left(\left(-\infty, t-\frac{1}{p}\right)\right)) \leq \bigwedge_{k=1}^{\infty}\bigvee_{n=k}^{\infty} x_n\left(\left(-\infty, t-\frac{1}{p}\right)\right).$$

Hence:

$$\bigvee_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} x_n\left(\left(-\infty, t-\frac{1}{p}\right)\right) \leq \bigvee_{p=1}^{\infty}\bigwedge_{k=1}^{\infty}\bigvee_{n=k}^{\infty} x_n\left(\left(-\infty, t-\frac{1}{p}\right)\right)$$

and that is what we need.

Theorem 2 *Let E be an effect algebra. Let $(x_n)_{n=1}^{\infty}$ be a regular sequence of the observables in effect algebra E with state m, $(\xi_n)_{n=1}^{\infty}$ be a sequence of corresponding projections, $g_n : R^N \to R$ be Borel measurable function for every natural number n. Then the following inequalities holds:*

$$P\left(\overline{\eta}^{-1}\left((-\infty, t)\right)\right) \leq m\left(\overline{y}\left((-\infty, t)\right)\right) \text{ for all } t \in R$$
$$P\left(\underline{\eta}^{-1}\left((-\infty, t)\right)\right) \geq m\left(\underline{y}\left((-\infty, t)\right)\right) \text{ for all } t \in R.$$

Proof:

From the equality:

$$P\left(\overline{\eta}^{-1}\left((-\infty, t)\right)\right) = \lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} P\left(\bigcap_{n=k}^{k+i}\eta_n^{-1}\left(\left(-\infty, t-\frac{1}{p}\right)\right)\right)$$

and by using previous results we have:

$$P\left(\bigcap_{n=k}^{k+i}\eta_n^{-1}\left(\left(-\infty, t-\frac{1}{p}\right)\right)\right) = P\left(\pi_{k+i}^{-1}\left(\bigcap_{n=k}^{k+i} g_n^{-1}\left(\left(-\infty, t-\frac{1}{p}\right)\right)\right)\right) =$$
$$m\left(h_{k+i}\left(\bigcap_{n=k}^{k+i}\left\{(t_1, .., t_{k+i}); g_n(t_1, .., t_n) < t-\frac{1}{p}\right\}\right)\right) \leq$$
$$m\left(\bigwedge_{n=k}^{k+i} h_n\left(g_n^{-1}\left(\left(-\infty, t-\frac{1}{p}\right)\right)\right)\right) = m\left(\bigwedge_{n=k}^{k+i} y_n\left(\left(\left(-\infty, t-\frac{1}{p}\right)\right)\right)\right)$$

As you can see the first part of the theorem is proved.

From the equality:

$$P\left(\underline{\eta}_n^{-1}\left((-\infty,t)\right)\right)=\lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty}P\left(\bigcup_{n=k}^{k+i}\eta_n^{-1}\left(\left(-\infty,t-\frac{1}{p}\right)\right)\right)$$

and by using previous results we have:

$$P\left(\bigcup_{n=k}^{k+i}\eta_n^{-1}\left(\left(-\infty,t-\frac{1}{p}\right)\right)\right)=P\left(\pi_{k+i}^{-1}\left(\bigcup_{n=k}^{k+i}g_n^{-1}\left(\left(-\infty,t-\frac{1}{p}\right)\right)\right)\right)=$$
$$m\left(h_{k+i}\left(\bigcup_{n=k}^{k+i}\left\{(t_1,..,t_{k+i});g_n(t_1,..,t_n)<t-\frac{1}{p}\right\}\right)\right)\geq$$
$$m\left(\bigvee_{n=k}^{k+i}h_n\left(g_n^{-1}\left(\left(-\infty,t-\frac{1}{p}\right)\right)\right)\right)=m\left(\bigvee_{n=k}^{k+i}y_n\left(\left(\left(-\infty,t-\frac{1}{p}\right)\right)\right)\right)$$

4 The convergence theorem

Now we extend the convergence m-almost everywhere by the following way.

Definition 8 *The sequence $(y_n)_{n=1}^{\infty}$ of observables on effect algebra converges m-almost everywhere, if*

1. $\forall t\in R: m\left(\overline{y}\left((-\infty,t)\right)\right)=m\left(\underline{y}\left((-\infty,t)\right)\right)$
2. *The function $F(t)=m\left(\overline{y}\left((-\infty,t)\right)\right)$ is a distribution function.*

Theorem 3 *Let $(x_n)_{n=1}^{\infty}$ be a regular sequence of observables $x_n:\mathcal{S}\to E$, h_n be a regulating sequence and for all $n\in N$ $g_n:R^n\to R$ be a Borel function. We define a sequence $(\eta_n)_{n=1}^{\infty}$ for every natural number n by this way: $\eta_n=g_n(\xi_1,\xi_2,...,\xi_n)$, where $\xi_n:R^N\to R$ is n-th coordinate random vector. If the sequence $(\eta_n)_{n=1}^{\infty}$ converges P-almost everywhere to η, then the sequence $y_n=h_n\circ g_n^{-1}$ converges m-almost everywhere and*
$P\left(\eta^{-1}\left((-\infty,t)\right)\right)=m\left(\overline{y}\left((-\infty,t)\right)\right)$, for any $t\in R$.

Proof:
We have seen that the following equalities hold

$$P\left(\overline{\eta}^{-1}\left((-\infty,t)\right)\right)\leq m\left(\overline{y}\left((-\infty,t)\right)\right)\leq m\left(\underline{y}\left((-\infty,t)\right)\right)\leq$$
$$P\left(\underline{\eta}^{-1}\left((-\infty,t)\right)\right)\text{ for all }t\in R.$$

therefore the equality for the limes superior and limes inferior is satisfied.

So we have the required equality: $P\left(\eta^{-1}\left((-\infty, t)\right)\right) = m\left(\overline{y}\left((-\infty, t)\right)\right)$, for any $t \in R$.

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References

[1] Chang C. C. (1958). Algebraic analysis of many- valued logics: Trans. Amer. ath. Soc., 88, 467– 490.

[2] Chovanec F., Kôpka P. (1992). On a representation of observables in D-posets on fuzzy sets: Tatra Mth. Math. Publ., 1, 15–18.

[3] Dvurečenskij A., Pulmannová A. (2000). New trends in quantum structures. Kluwer, Dordrecht.

[4] Foulis D.J., Bennett M. K. (1994). The difference poset of monotone functions: Found Phys., 24, 1325–1346.

[5] Jurečková M. (2001). On the conditional expectation on probability MV-algebras with product: Soft comput., 5, 381–385.

[6] Jurečková M. (2000). A note on the individual ergodic theorem on product MV-algebras: Internat. J. Theoret. Phys., 39, 737–760.

[7] Jurečková M., Riečan B. (1997). Weak Law of large numbers for weak observables in MV- algebras: Tatra Mth. Math. Publ., 12, 221–228.

[8] Mundici D. (1986). Interpretation of AFC*- algebras in Lukasiewicz sentential calculus: J. Funct. Anal., 65, 15–63.

[9] Riečan B. (1996). On the almost everywhere convergence of observables in some algebraic structures: Atti Sem. Mat. Fis. Univ. Modena 44, 95–104.

[10] Riečan B. (2001). Almost everywhere convergence in probability MV-algebras with product: Soft Comput., 5, 396–399.

[11] Riečan B. (2000). On the LP space of observables: Internat. J. Theoret. Phys., 39, 847–854.

[12] Riečan B., Mundici D. (2002). Probability on MV-algebras: Handbook of Measure Theory, Elsevier, Amsterdam, 869–909.

[13] Riečan B., Neubrunn T. (1997). Integral, measure and ordering. Kluwer, Dordrecht.

[14] Vrábelová M. (2000). On the conditional probability in product MV-algebras: Soft comput., 4, 58–61.

[15] Atanassov K. (1999) Intuitionistic Fuzzy Sets: Theory and Applications. Springer-Verlag.

[16] Zadeh L.A. (1965) Fuzzy sets. Information and Control, 8, 338–353.

The papers presented in this Volume 1 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.
It may be viewed as a result of fruitful discussions held during the Tenth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2011) organized in Warsaw on September 30, 2011 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT - Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bystrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, and the University of Westminster, Harrow, UK:

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The consecutive International Workshops on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGNs) have been meant to provide a forum for the presentation of new results and for scientific discussion on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems have also been included. The Tenth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2011) is a continuation of this undertaking, and provides many new ideas and results in the areas concerned.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

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