Developments in Fuzzy Sets,
Intuitionistic Fuzzy Sets,
Generalized Nets and Related Topics.
Volume I: Foundations

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Dedicated to Professor Beloslav Riečan on his 75th anniversary

Systems of intuitionistic fuzzy equations

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Abstract

The idea of intuitionistic fuzzy sets is a generalization of Zadeh's concept of fuzzy sets. This extension of fuzzy set was introduced by Atanassov in 1986 [2]. In the next years intuitionistic fuzzy sets was considered and discussed. Intuitionistic fuzzy relations was started by Atanassov in 1995. Especially Bustince and Burillo published few articles about these relations and gave the most general definition of relations composition.

Based on previous results the problem of solving of intuitionistic fuzzy relational equations has been discussed by many authors e.g. Peeva and Kyosev [14] or Zhou and Bao [17].

In this article, we consider systems of intuitionistic fuzzy equations as a generalization of systems of equations with $\max -T$ and $\min -S$ products, where T is triangular norm and S is triangular conorm. The goal of this paper is the complete description of the family of all solutions. Moreover, we investigate correlation between systems of intuitionistic fuzzy equations and systems of fuzzy equations.

Keywords: fuzzy equations, intuitionistic fuzzy equations, systems of fuzzy equations, systems of intuitionistic fuzzy equations.

1 Introduction

Intuitionistic fuzzy sets introduced by Atanassov in [1] are the extension of fuzzy sets introduced by Zadeh in [16].

Definition 1 ([1])**.** *Let $X \neq \emptyset$ be a given set. An intuitionistic fuzzy set A in X is*

$$A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\},$$

where

$$\mu_A : X \to [0,1], \quad \nu_A : X \to [0,1]$$

with the condition $0 \leqslant \mu_A(x) + \nu_A(x) \leqslant 1$ for all $x \in X$. The set of all intuitionistic fuzzy sets in X is denoted by $IFS(X)$.

The numbers $\mu(x)$ and $\nu(x)$ denote respectively the degree of membership and the degree of non-membership of element x in set A. Obviously, when

$$\nu(x) = 1 - \mu(x)$$

for all x in X, the set A is a fuzzy set.

Definition 2 ([4], Definition 1)**.** *Let $X, Y \neq \emptyset$ be finite crisp sets. An intuitionistic fuzzy relation R between sets X and Y is an intuitionistic fuzzy set R of $X \times Y$, i.e.*

$$R = \{((x,y), \mu_R(x,y), \nu_R(x,y)) : (x,y) \in X \times Y\},$$

where

$$\mu_R : X \times Y \to [0,1], \quad \nu_R : X \times Y \to [0,1]$$

with the condition $0 \leqslant \mu_R(x,y) + \nu_R(x,y) \leqslant 1$ for all $(x,y) \in X \times Y$. $IFS(X \times Y)$ denotes the set of all intuitionistic fuzzy relations between sets X and Y.

Let us denote $M = \{1, \ldots, m\}$, $N = \{1, \ldots, n\}$.

For $X = \{x_1, \ldots, x_m\}$, $Y = \{y_1, \ldots, y_n\}$, matrix representation of intuitionistic fuzzy relation $R \in IFS(X \times Y)$

$$(r_{ij}^{\mu}, r_{ij}^{\nu}) = R(x_i, y_j) = (\mu_R(x_i, y_j), \nu_R(x_i, y_j)), \; i \in M, j \in N$$

we call an intuitionistic fuzzy membership matrix (see [13], [14]).

Corollary 1. *Obviously, each intuitionistic fuzzy membership matrix of intuitionistic fuzzy relation R can be divided on two matrices $R^{\mu} = (r_{ij}^{\mu})$ and $R^{\nu} = (r_{ij}^{\nu})$ as follows:*

$$r_{ij}^{\mu} = \mu_R(x_i, y_j), \; r_{ij}^{\nu} = \nu_R(x_i, y_j), \; i \in M, j \in N.$$

Moreover, we have

$$r_{ij}^{\mu} + r_{ij}^{\nu} \leqslant 1 \quad \text{for } i \in M, \; j \in N. \tag{1}$$

Example 1. The intuitionistic fuzzy membership matrix

$$A = \begin{bmatrix} (0,0.9) & (0.3,0.4) & (0.5,0.3) \\ (0.8,0.2) & (0.3,0.6) & (0.6,0.1) \\ (1,0) & (0.2,0.7) & (0.1,0.5) \end{bmatrix},$$

can be rewritten as matrices

$$A^{\mu} = \begin{bmatrix} 0 & 0.3 & 0.5 \\ 0.8 & 0.3 & 0.6 \\ 1 & 0.2 & 0.1 \end{bmatrix}, A^{\nu} = \begin{bmatrix} 0.9 & 0.4 & 0.3 \\ 0.2 & 0.6 & 0.1 \\ 0 & 0.7 & 0.5 \end{bmatrix}.$$

The lattice operations $\vee$ and $\wedge$ are used for numbers:

$$a \vee b = \max(a,b), \quad a \wedge b = \min(a,b), \quad a,b \in [0,1],$$

$$\bigvee_{i=1}^{n} a_i = \max_{i \in N} a_i, \quad \bigwedge_{i=1}^{n} a_i = \min_{i \in N} a_i, \quad a_i \in [0,1],\ i \in N.$$

Definition 3 ([4])**.** *Let $P, R \in IFS(X \times Y)$. We define*

- *inclusion*
  $P \subset R \Leftrightarrow \underset{(x,y)\in X\times Y}{\forall} (\mu_P(x,y) \leqslant \mu_R(x,y)$ *and* $\nu_P(x,y) \geqslant \nu_R(x,y))$,
- *disjunction*
  $P \cup R = \{((x,y), \mu_P(x,y) \vee \mu_R(x,y), \nu_P(x,y) \wedge \nu_R(x,y))\}$,
- *conjunction*
  $P \cap R = \{((x,y), \mu_P(x,y) \wedge \mu_R(x,y), \nu_P(x,y) \vee \nu_R(x,y))\}$.

An intuitionistic fuzzy membership matrices A, B and vectors a, b of intuitionistic fuzzy relations are ordered by

$$(A \preceq B) \Leftrightarrow (a_{ij}^{\mu} \leqslant b_{ij}^{\mu} \text{ and } a_{ij}^{\nu} \geqslant b_{ij}^{\nu}),$$

$$(a \preceq b) \Leftrightarrow (a_j^{\mu} \leqslant b_j^{\mu} \text{ and } a_j^{\nu} \geqslant b_j^{\nu}), \quad i \in M, j \in N.$$

Example 2. Let us consider intuitionistic fuzzy membership matrix A from Example 1 and matrix B

$$B = \begin{bmatrix} (0.7,0.9) & (0.5,0.5) & (0.6,0.3) \\ (0.9,0.2) & (0.3,0.6) & (1,0.1) \\ (1,0.7) & (0.5,0.7) & (0.7,0.9) \end{bmatrix},$$

We obtain matrices

$$B^{\mu} = \begin{bmatrix} 0.7 & 0.5 & 0.6 \\ 0.9 & 0.3 & 1 \\ 1 & 0.5 & 0.7 \end{bmatrix}, B^{\nu} = \begin{bmatrix} 0.9 & 0.5 & 0.3 \\ 0.2 & 0.6 & 0.1 \\ 0.7 & 0.7 & 0.9 \end{bmatrix}.$$

We see that $A^{\mu} \leqslant B^{\mu}$, $A^{\nu} \leqslant B^{\nu}$ and intuitionistic fuzzy membership matrices A and B are incomparable.

An intuitionistic fuzzy relations and their properties have been investigated by many authors, especially by Bustince and Burillo ([4], [5], [6]).

2 A binary operations

In this section, we recall some kind of binary operation in $[0, 1]$.

Definition 4 ([9], Definition 1.1)**.** *A triangular norm (t-norm) is a binary operation* $T : [0, 1]^2 \to [0, 1]$*, which is commutative, associative, increasing one and it has neutral element* $e = 1$*.*

Example 3 (cf. [9], Examples 1.2, 1.12)**.** The triangular norms:

$$T_M(x, y) = x \wedge y \quad \text{(minimum)}\,, \tag{2}$$

$$T_P(x, y) = x \cdot y \quad \text{(product)}\,, \tag{3}$$

$$T_L(x, y) = 0 \vee (x + y - 1) \quad \text{(Łukasiewicz } t\text{−norm)}\,, \tag{4}$$

$$T_D(x, y) = \begin{cases} 0, & \text{if } x, y \in (0, 1) \\ x \wedge y, & \text{otherwise} \end{cases} \quad \text{(drastic product)}\,, \tag{5}$$

$$T_{FD}(x, y) = \begin{cases} 0, & \text{if } x + y \leqslant 1 \\ x \wedge y, & \text{otherwise} \end{cases} \quad \text{(Fodor } t\text{−norm)}\,. \tag{6}$$

Definition 5 ([9], Definitions 1.13)**.** *A triangular conorm (t-conorm) is a binary operation* $S : [0, 1]^2 \to [0, 1]$*, which is commutative, associative, increasing one and it has neutral element* $e = 0$*.*

Example 4. [cf. [9], Example 1.14] The triangular norms:

$$S_M(x, y) = x \wedge y \quad \text{(maximum)}\,, \tag{7}$$

$$S_P(x, y) = x + y - x \cdot y \quad \text{(probabilistic sum)}\,, \tag{8}$$

$$S_L(x, y) = 1 \wedge (x + y) \quad \text{(Łukasiewicz } t\text{−conorm)}\,, \tag{9}$$

$$S_D(x,y) = \begin{cases} 1, & \text{if } x,y \in (0,1] \\ x \vee y, & \text{otherwise} \end{cases} \quad \text{(drastic product)}\ , \tag{10}$$

$$S_{FD}(x,y) = \begin{cases} 1, & \text{if } x+y \geqslant 1 \\ x \vee y, & \text{otherwise} \end{cases} \quad \text{(Fodor } t\text{−conorm)}\ . \tag{11}$$

Remark 1. *Let $p \in \mathbb{N}_2$ and α be a triangular norm or conorm. We use the following notation*

$$\underset{i=1}{\overset{2}{\alpha}}(a_i) = \alpha(a_1,a_2), \quad \underset{i=1}{\overset{p}{\alpha}}(a_i) = \alpha(\underset{i=1}{\overset{p-1}{\alpha}}(a_i), a_p) \tag{12}$$

Definition 6. *Let $*, *' : [0,1]^2 \to [0,1]$. Operations $*$, $*'$ are dual operations, when they are fulfilling condition*

$$a *' b = 1 - (1-a) * (1-b) \tag{13}$$

as generalized de Morgan laws.

Lemma 1 ([9], Proposition 1.15)**.** *A function $S : [0,1]^2 \to [0,1]$ is a t-conorm if and only if exists t-norm T such that for all $(x,y) \in [0,1]$*

$$S(x,y) = 1 - T(1-x, 1-y). \tag{14}$$

Example 5. From Lemma 1 we know that the following pairs:

$$(T_M, S_M),\ (T_P, S_P),\ (T_L, S_L),\ (T_D, S_D),\ (T_{FD}, S_{FD})$$

are de Morgan pairs.

Lemma 2. *Let λ, $*$ be a triangular norms or comorms and $P \subset \mathbb{N}$. If $(*, *')$ and (λ, λ') are de Morgan pairs then*

$$\underset{j}{\lambda'}((1-a_j) *' (1-b_j)) = 1 - \underset{j \in P}{\lambda}(a_j * b_j), \quad j \in P,\ a_j, b_j \in [0,1]. \tag{15}$$

Proof. From (14) we get

$$\underset{j}{\lambda'}((1-a_j) *' (1-b_j)) = 1 - \underset{j \in P}{\lambda}(1-(1-a_j) *' (1-b_j)) = 1 - \underset{j \in P}{\lambda}(a_j * b_j).$$

□

Definition 7 ([7], Definition 1)**.** *Let $* : [0,1]^2 \to [0,1]$. The binary operation $\overset{*}{\to}$ and $\overset{*}{\leftarrow}$ in $[0,1]$ are called induced implication and dual induced implication by the operation $*$, where*

$$a \overset{*}{\to} b = \max\{t \in [0,1] : a * t \leqslant b\}, \tag{16}$$

$$a \overset{*}{\leftarrow} b = \min\{t \in [0,1] : a * t \geqslant b\}, \tag{17}$$

if they exist for $a, b \in [0,1]$.

Example 6 ([3], Table 1.3; [7], Example 3)**.** Triangular norms from Example 3 induce the following implications:

$$a \overset{T_M}{\to} b = \begin{cases} 1, & a \leqslant b \\ b, & a > b \end{cases}, a \overset{T_P}{\to} b = \begin{cases} 1, & a \leqslant b \\ \frac{b}{a}, & a > b \end{cases},$$

$$a \overset{T_L}{\to} b = (1-a+b) \wedge 1, \quad a \overset{T_{FD}}{\to} b = \begin{cases} 1, & a \leqslant b \\ (1-a) \vee b) & a > b \end{cases} \text{ for } a, b \in [0,1].$$

but the operation T_D has the implication $a \overset{T_D}{\to} b$ not for all $a, b \in [0,1]$, i.e.

$$a \overset{T_D}{\to} b = \begin{cases} 1, & a \leqslant b \\ b, & a = 1,\ b \in [0,1) \end{cases}.$$

However, for $b < a < 1$ we get $\{t \in [0,1] : T_D(a,t) \leqslant b\} = [0,1)$ (cf. (5)) and the greatest element in (16) does not exist. For triangular conorms (from Example 4), we often obtain induced implication only for $a \leqslant b$. For example:

$$a \overset{S_M}{\to} b = b, \quad a \overset{S_P}{\to} b = \frac{b-a}{1-a}, \quad a \overset{S_L}{\to} b = b - a \quad \text{for } \ a \leqslant b.$$

But these triangular conorms induces the following dual implications:

$$a \overset{S_M}{\leftarrow} b = \begin{cases} 0, & a \geqslant b \\ b, & a < b \end{cases}, \quad a \overset{S_P}{\leftarrow} b = \begin{cases} 0, & a \geqslant b \\ \frac{b-a}{1-a}, & a < b \end{cases},$$

$$a \overset{S_L}{\leftarrow} b = \begin{cases} 0, & a \geqslant b \\ b-a & a < b \end{cases}.$$

From [7] Theorem 3 (and knowing [8], Corollary 2 and [9], Proposition 1.22) we get

Theorem 1. *If $*$ is left continuous triangular norm, then it has the induced implication $\overset{*}{\to}$.*

Theorem 2. *If $\diamond$ is right continuous triangular conorm, then it has the dual induced implication $\overset{\diamond}{\leftarrow}$.*

Theorem 3 ([7], Theorem 2)**.** *Let $* : [0,1]^2 \to [0,1]$ be an increasing operation and $a, b \in [0,1]$. If the induced implication (16) exists, then it has the following properties*

$$a * (a \overset{*}{\to} b) \leqslant b \leqslant a \overset{*}{\to} (a * b), \tag{18}$$

$$(a * c \leqslant b) \Leftrightarrow (c \leqslant a \overset{*}{\to} b). \tag{19}$$

Theorem 4 ([10], Theorem 3)**.** *Let $\diamond : [0,1]^2 \to [0,1]$ be an increasing operation and $a, b \in [0,1]$. If the dual induced implication (17) exists, then it has the following properties*

$$a \overset{\diamond}{\leftarrow} (a \diamond b) \leq b \leq a \diamond (a \overset{\diamond}{\leftarrow} b), \tag{20}$$

$$(c \diamond a) \geq b \Leftrightarrow c \geq a \overset{\diamond}{\leftarrow} b, \tag{21}$$

3 The product of intuitionistic fuzzy membership matrices

Let us denote $L^* = \{(x,y) \in [0,1]^2 : x + y \leqslant 1\}$, $card\, X = m$, $card\, Y = n$, $card\, Z = p$.

Elements in L^* are ordered by

$$(x \preceq y) \Leftrightarrow (x_1 \leqslant y_1 \text{ and } x_2 \geqslant y_2), \tag{22}$$

where $x = (x_1, x_2)$, $y = (y_1, y_2) \in L^*$.

Moreover the lattice operations $\vee$ and $\wedge$ in L^* we use for:

$$x \vee y = (\max(x_1, y_1), \min(x_2, y_2)), \qquad x \wedge y = (\min(x_1, y_1), \max(x_2, y_2)),$$

where $x = (x_1, x_2)$,$y = (y_1, y_2)$, $x, y \in L^*$.

Corollary 2. *We have in L^* the greatest element $\mathbf{1} = (1,0)$ and the lowest $\mathbf{0} = (0,1)$.*

Definition 8 ([4], Definition 3)**.** *Let $\alpha, \lambda, *, \diamond$ be triangular norms or triangular conorms, $R \in IFS(X \times Y)$, $P \in IFS(Y \times Z)$. The relation $P \overset{\alpha,*}{\underset{\lambda,\diamond}{\circ}} R \in IFS(X \times Z)$ is the composition of relations P and R, where*

$$P \overset{\alpha,*}{\underset{\lambda,\diamond}{\circ}} R = \{((x,z), \mu_{P \overset{\alpha,*}{\underset{\lambda,\diamond}{\circ}} R}(x,z), \nu_{P \overset{\alpha,*}{\underset{\lambda,\diamond}{\circ}} R}(x,z) : (x,y) \in X \times Z\}$$

and

$$\mu_{P \overset{\alpha,*}{\underset{\lambda,\lambda}{\circ}} R}(x,z) = \underset{y}{\alpha}\{\mu_R(x,y) * \mu_P(y,z)\},$$

$$\nu_{P \overset{\alpha,*}{\underset{\lambda,\diamond}{\circ}} R}(x,z) = \underset{y}{\lambda}\{\nu_R(x,y) \diamond \nu_P(y,z)\},$$

whenever

$$\mu_{P \overset{\alpha,*}{\underset{\lambda,\diamond}{\circ}} R}(x,z) + \mu_{P \overset{\alpha,*}{\underset{\lambda,\diamond}{\circ}} R}(x,z) \leqslant 1.$$

Based on Definition 8, we introduce $\overset{\alpha,*}{\underset{\lambda,\diamond}{\circ}}$ product of intuitionistic fuzzy membership matrices.

Definition 9. *Let $\alpha, \lambda, *, \diamond$ be triangular norms or conorms, $A \in (L^*)^{m\times n}$, $B \in (L^*)^{n\times p}$. By $\overset{\alpha,*}{\underset{\lambda,\diamond}{\circ}}$ product of intuitionistic fuzzy membership matrices A and B we call the matrix $A \overset{\alpha,*}{\underset{\lambda,\diamond}{\circ}} B$, where*

$$(A \overset{\alpha,*}{\underset{\lambda,\diamond}{\circ}} B)_{ik} = (\underset{j\in N}{\alpha}(a^{\mu}_{ij} * b^{\mu}_{jk}), \underset{j\in N}{\lambda}(a^{\nu}_{ij} \diamond b^{\nu}_{jk}))$$

and $(A \overset{\alpha,*}{\underset{\lambda,\diamond}{\circ}} B)^{\mu}_{ik} + (A \overset{\alpha,*}{\underset{\lambda,\diamond}{\circ}} B)^{\nu}_{ik} \leqslant 1$ for $i \in M,\ 1 \leqslant k \leqslant p$.

Corollary 3. *Let $\alpha = \vee, \lambda = \wedge$ and $*, \diamond$ be triangular norms or conorms and $A \in (L^*)^{m\times n}$, $c \in (L^*)^n$. The product $A \overset{\vee,*}{\underset{\wedge,\diamond}{\circ}} c$ has the following form:*

$$(A \overset{\vee,*}{\underset{\wedge,\diamond}{\circ}} c)_i = (\bigvee_{j=1}^{n}(a^{\mu}_{ij} * c^{\mu}_j), \quad \bigwedge_{j=1}^{n}(a^{\nu}_{ij} \diamond c^{\nu}_j)), \tag{23}$$

where

$$(A \overset{\vee,*}{\underset{\wedge,\diamond}{\circ}} c)^{\mu}_i + (A \overset{\vee,*}{\underset{\wedge,\diamond}{\circ}} c)^{\nu}_i \leqslant 1 \text{ and } i \in M. \tag{24}$$

In this paper we pay attention on product, which is presented as (23). This kind of product has been researched by few authors. Especially the fulfilling of the condition (24) has been investigated. Based on results of Burillo, Bustince from 1995 we have

Theorem 5 (cf. [4], Proposition 1)**.** *Let $\lambda, \alpha,\ *,\ \diamond$ be a triangular norms or conorms and (λ, λ') and $(\diamond, \diamond')$ be de Morgan pairs. If $\alpha \leqslant \lambda'$ and $* \leqslant \diamond'$, then the condition*

$$(A \overset{\lambda,*}{\underset{\alpha,\diamond}{\circ}} c)^{\mu}_i + (A \overset{\lambda,*}{\underset{\alpha,\diamond}{\circ}} c)^{\nu}_i \leqslant 1$$

is true for all $i \in M$.

Proof. From (1) and assumptions we get

$$(a^{\mu}_{ij} * c^{\mu}_j) \leqslant ((1 - a^{\nu}_{ij}) * (1 - c^{\nu}_j)) \leqslant ((1 - a^{\nu}_{ij}) \diamond' (1 - c^{\nu}_j)).$$

Therefore, using (14) and (15) we have

$$\underset{j\in N}{\alpha}(a^{\mu}_{ij} * c^{\mu}_j) \leqslant \underset{j\in N}{\alpha}((1 - a^{\nu}_{ij}) \diamond' (1 - c^{\nu}_j)) \leqslant$$

$$\underset{j\in N}{\lambda'}((1-a_{ij}^{\nu})\diamond'(1-c_j^{\nu}))=1-\underset{j\in N}{\lambda}(a_{ij}^{\nu}\diamond c_j^{\nu}),$$

then

$$\underset{j\in N}{\alpha}(a_{ij}^{\mu}*c_j^{\mu})+\underset{j\in N}{\lambda}(a_{ij}^{\nu}\diamond c_j^{\nu})\leqslant 1.$$

$\square$

Corollary 4. *Let $*$, $\diamond$ be a triangular norms or conorms. If $(\diamond,\diamond')$ is de Morgan pair and $*\leqslant\diamond'$, then the condition*

$$(A\overset{\vee,*}{\underset{\wedge,\diamond}{\circ}}c)_i^{\mu}+(A\overset{\vee,*}{\underset{\wedge,\diamond}{\circ}}c)_i^{\nu}\leqslant 1$$

is true for all $i\in M$.

Theorem 6 (cf. [4], Theorem 3)**.** *If $*,\diamond$ are triangular norms or conorms and $\diamond'$ be a dual operation of $\diamond$, we have*

$$(A\overset{\vee,*}{\underset{\wedge,\diamond}{\circ}}c)_i^{\mu}+(A\overset{\vee,*}{\underset{\wedge,\diamond}{\circ}}c)_i^{\nu}\leqslant 1\Leftrightarrow\underset{j\in N}{\forall}\ \underset{k\in N}{\exists}(a_{ij}^{\mu}*c_j^{\mu})\leqslant(1-a_{ik}^{\nu}\diamond' 1-c_k^{\nu})$$

for $i\in M$.

Proof. Since $(A\overset{\vee,*}{\underset{\wedge,\diamond}{\circ}}c)_i^{\mu}+(A\overset{\vee,*}{\underset{\wedge,\diamond}{\circ}}c)_i^{\nu}\leqslant 1$, we have

$$(A\overset{\vee,*}{\underset{\wedge,\diamond}{\circ}}c)_i^{\mu}\leqslant 1-(A\overset{\vee,*}{\underset{\wedge,\diamond}{\circ}}c)_i^{\nu}.$$

It means that

$$\bigvee_{j\in N}(a_{ij}^{\mu}*c_j^{\mu})\leqslant 1-\bigwedge_{j\in N}(a_{ij}^{\nu}\diamond c_j^{\nu})=\bigvee_{j\in N}(1-(a_{ij}^{\nu}\diamond c_j^{\nu}))=$$

$$\bigvee_{j\in N}((1-a_{ij}^{\nu})\diamond'(1-c_j^{\nu})).$$

Then for all $i\in M$ and $j\in N$ exists $k\in N$ for which we have

$$(a_{ij}^{\mu}*c_j^{\mu})\leqslant((1-a_{ik}^{\nu})\diamond'(1-c_k^{\nu})).$$

Now, we assume that for all $j\in N$ exists $k\in N$ such that $(a_{ij}^{\mu}*c_j^{\mu})\leqslant(1-a_{ik}^{\nu}\diamond' 1-c_k^{\nu})$. It means that

$$\bigvee_{j\in N}(a_{ij}^{\mu}*c_j^{\mu})\leqslant\bigvee_{k\in N}((1-a_{ik}^{\nu})\diamond'(1-c_k^{\nu}))=\bigvee_{j\in N}[(1-(a_{ij}^{\nu}\diamond c_j^{\nu})]=$$

$$1 - \bigwedge_{j\in N} (a^{\nu}_{ij} \diamond c^{\nu}_{j}).$$

Therefore

$$\bigvee_{j\in N} (a^{\mu}_{ij} * c^{\mu}_{j}) + \bigwedge_{j\in N} (a^{\nu}_{ij} \diamond c^{\nu}_{j}) \leqslant 1.$$

So we get

$$(A \overset{\vee,*}{\underset{\wedge,\diamond}{\circ}} c)^{\mu}_{i} + (A \overset{\vee,*}{\underset{\wedge,\diamond}{\circ}} c)^{\nu}_{i} \leqslant 1$$

for all $i \in M$. □

Example 7. Let $\alpha = \vee, \lambda = \wedge, * = T_{FD}, \diamond = \cdot$ and

$$A = \begin{bmatrix} (0, 0.9) & (0.3, 0.4) & (0.5, 0.3) \\ (0.8, 0.2) & (0.3, 0.6) & (0.6, 0.1) \\ (1, 0) & (0.2, 0.7) & (0.1, 0.5) \end{bmatrix}, c = \begin{bmatrix} (0.3, 0.2) \\ (0.2, 0.8) \\ (0.2, 0.4) \end{bmatrix}.$$

We have

$$A^{\mu} = \begin{bmatrix} 0 & 0.3 & 0.5 \\ 0.8 & 0.3 & 0.6 \\ 1 & 0.2 & 0.1 \end{bmatrix}, A^{\nu} = \begin{bmatrix} 0.9 & 0.4 & 0.3 \\ 0.2 & 0.6 & 0.1 \\ 0 & 0.7 & 0.5 \end{bmatrix},$$

$$c^{\mu} = \begin{bmatrix} 0.3 \\ 0.2 \\ 0.2 \end{bmatrix}, c^{\nu} = \begin{bmatrix} 0.2 \\ 0.8 \\ 0.4 \end{bmatrix}$$

and from $(A \overset{\vee,*}{\underset{\wedge,\diamond}{\circ}} c)_i = (\bigvee_{j=1}^{n} (a_{ij} * c_j),\ \bigwedge_{j=1}^{n} (a_{ij} \diamond c_j))$ we get

$$(A \overset{\vee,*}{\underset{\wedge,\diamond}{\circ}} c)^{\mu} = \begin{bmatrix} 0 \\ 0.3 \\ 0.3 \end{bmatrix}, (A \overset{\vee,*}{\underset{\wedge,\diamond}{\circ}} c)^{\nu} = \begin{bmatrix} 0.12 \\ 0.04 \\ 0 \end{bmatrix}, A \overset{\vee,*}{\underset{\wedge,\diamond}{\circ}} c = \begin{bmatrix} (0, 0.12) \\ (0.3, 0.04) \\ (0.3, 0) \end{bmatrix}.$$

4 The system of intuitionistic fuzzy equations

System of intuitionistic fuzzy equations has the form

$$A \overset{\alpha,*}{\underset{\lambda,\diamond}{\circ}} x = b,$$

where $A \in (L^*)^{m\times n}$, $b \in (L^*)^m$ are intuitionistic fuzzy membership matrix and vector and $x \in (L^*)^n$ is the vector of unknowns.

We consider special case of system of intuitionistic fuzzy equations.

Corollary 5. *Let $\alpha = \vee, \lambda = \wedge$ and $*, \diamond$ be triangular norms or conorms and $A \in (L^*)^{m\times n}$, $b \in (L^*)^m$. The system of intuitionistic fuzzy equations has the following form $A \overset{\vee,*}{\underset{\wedge,\diamond}{\circ}} x = b$, i.e.:*

$$(\bigvee_{j=1}^{n}(a_{ij}^{\mu} * x_j^{\mu}),\ \bigwedge_{j=1}^{n}(a_{ij}^{\nu} \diamond x_j^{\nu})) = (b_i^{\mu}, b_i^{\nu}), \quad i \in M.$$

The system of intuitionistic fuzzy equations can be considered as two separate systems of fuzzy equations. This kind of method of solving system of intuitionistic fuzzy equations was introduced by Peeva (see [13], [14]) for $\alpha = \diamond = \vee$ and $* = \lambda = \wedge$.

Corollary 6. *Let $A \in (L^*)^{m\times n}$, $b \in (L^*)^m$ and $\alpha = \vee, \lambda = \wedge$ and $*, \diamond$ be triangular norms or conorms. The system of intuitionistic fuzzy equations $A \overset{\vee,*}{\underset{\wedge,\diamond}{\circ}} x = b$ can be divided to system of equations $A^{\mu} \circ x^{\mu} = b^{\mu}$ with* $\max -*$ *product, i.e.:*

$$\bigvee_{j=1}^{n}(a_{ij}^{\mu} * x_j^{\mu}) = b_i^{\mu}$$

and system of equations $A^{\nu} \bullet x^{\nu} = b^{\nu}$ with $\min -\diamond$ *product, i.e.:*

$$\bigwedge_{j=1}^{n}(a_{ij}^{\nu} \diamond x_j^{\nu}) = b_i^{\nu}$$

for $i \in M$.

The family of all solutions of $A \overset{\vee,*}{\underset{\wedge,\diamond}{\circ}} = b$ we denote by $S(A, b, *, \diamond)$, i.e.

$$S(A, b, *, \diamond) = \{x \in L^* : A \overset{\vee,*}{\underset{\wedge,\diamond}{\circ}} = b\}.$$

It means that

$$S(A, b, *, \diamond) = \{x = (x_1, x_2) \in L^* : A \circ x_1 = b \text{ and } A \bullet x_2 = b\}.$$

Definition 10. *By minimal solutions of system $A \overset{\vee,*}{\underset{\wedge,\diamond}{\circ}} = b$ with $\overset{\vee,*}{\underset{\wedge,\diamond}{\circ}}$ product we call minimal elements in $S(A, b, *, \diamond)$ with respect to the partial order (22) (if they exist). The set of all minimal solution we denote by $S^0(A, b, *, \diamond)$.*

Based on [7] (Theorems 4, 6) we have

Theorem 7. *Let $*$ be a left-continuous triangular norm. If the vector $u \in [0,1]^n$ is solution of $A \circ x = b$, where*

$$u_j = \bigwedge_{i=1}^{m} (a_{ij} \overset{*}{\to} b_i) \quad \textit{for } j \in N,$$

then u is the greatest solution of $A \circ x = b$.

Proof. From Theorem 1 we know that the implications $a_{ij} \overset{*}{\to} b_i$ exist for any $i \in M$, $j \in N$. Using (18) we obtain

$$(A \circ u)_i = \bigvee_{j=1}^{n} (a_{ij} * \bigwedge_{l=1}^{m} (a_{lj} \overset{*}{\to} b_l)) \leqslant \bigvee_{j=1}^{n} (a_{ij} * (a_{ij} \overset{*}{\to} b_i)) \leqslant \bigvee_{j=1}^{n} b_i = b_i.$$

It give us that $A \circ u \leqslant b$.

Let now y is solution of $A \circ x = b$. Using (19) we get

$$a_{ij} * y_j \leqslant \bigvee_{k=1}^{m} a_{ik} * y_k \leqslant b_i, \quad (a_{ij} * y_j \leqslant b_i) \Leftrightarrow (y_j \leqslant (a_{ij} \overset{*}{\to} b_i))$$

for any $i \in M$. After adding infimmum, we have

$$x_j \leqslant \bigwedge_{i=1}^{m} (a_{ij} \overset{*}{\to} b_i) = u_j.$$

Then u is the greatest solution of $A \circ x = b$. $\square$

From [10] we have

Theorem 8. *If $\diamond$ is a right-continuous triangular conorm and the vector $z \in [0,1]^n$ is a solution of $A \bullet x = b$, where*

$$z_j = \bigvee_{i=1}^{m} (a_{ij} \overset{\diamond}{\leftarrow} b_i) \quad \textit{for } j \in N,$$

then z is the lowest solution of $A \bullet x = b$.

Proof. From Theorem 2 we know that the dual implications $a_{ij} \overset{\diamond}{\leftarrow} b_i$ exist for any $i \in M$, $j \in N$ Using (21)

$$\bigwedge_{j=1}^{n} (a_{ij} \diamond z_j) = \bigwedge_{j=1}^{n} (a_{ij} \diamond \bigvee_{k=1}^{m} (a_{kj} \overset{\diamond}{\leftarrow} b_k)) \geq \bigwedge_{j=1}^{n} (a_{ij} \diamond (a_{ij} \overset{\diamond}{\leftarrow} b_i) \geq b_i)$$

we get that z is a solution of inequality $A \circ x \geq b$. Let y be a solution of $A \circ x = b$. Using (21), (20), we get

$$a_{ij} * y_j \geqslant \bigwedge_{k=1}^{m} a_{ik} \diamond y_k \geqslant b_i, \quad (a_{ij} \diamond y_j = y_j \diamond a_{ij} \geqslant b_i) \Leftrightarrow (y_j \geqslant (a_{ij} \overset{\diamond}{\leftarrow} b_i))$$

for any $i \in M$. So we have

$$(x_j \geq \bigvee_{k=1}^{m} (a_{kj} \overset{\diamond}{\leftarrow} b_k)) = z_j.$$

Then z is the lowest solution of $A \bullet x = b$. □

From Theorems 7, 8 we get

Theorem 9. *Let $*$, $\diamond$ are left-continuous triangular norm and right-conti-nuous triangular conorm, respectively. If the vector $g \in (L^*)^n$ is solution of $A \underset{\wedge,\diamond}{\overset{\vee,*}{\circ}} x = b$, where*

$$g_j^{\mu} = \bigwedge_{i=1}^{m} (a_{ij}^{\mu} \overset{*}{\rightarrow} b_i^{\mu}),$$

$$g_j^{\nu} = \bigvee_{i=1}^{m} (a_{ij}^{\nu} \overset{\diamond}{\leftarrow} b_i^{\nu}) \quad \textit{for } j \in N,$$

then g is the greatest solution of $A \underset{\wedge,\diamond}{\overset{\vee,*}{\circ}} x = b$.

In the papers [15], [11] has been considered the problem of existence and determination of minimal solution of $A \circ x = b$. As a common result of them we have

Theorem 10. *If $*$ a is right-continuous triangular norm and the set of all solution of $A \circ x = b$ is not empty, then each vector $x \in [0,1]^n$, which is solution of $A \circ x = b$, is bounded from below by some minimal solution from the set of all solutions of $A \circ x = b$.*

In a dual way and based on [12] we get

Theorem 11. *If $\diamond$ is a left-continuous triangular conorm and the set of all solutions of $A \bullet x = b$ is not empty, then each vector $x \in [0,1]^n$, which is solution of $A \bullet x = b$, is bounded from above by some maximal solution from the set of all solutions of $A \bullet x = b$.*

From Theorems 10, 11 we obtain

Theorem 12. *If $*$ is a right-continuous triangular norm and $\diamond$ is left-conti-nuous triangular conorm and the set of all solutions of $A \circ x = b$ is not empty, then each vector $x \in (L^*)^n$, which is solution of $A \underset{\wedge,\diamond}{\overset{\vee,*}{\circ}} x = b$, is bounded from below by some minimal solution v from the set of all solutions of $A \underset{\wedge,\diamond}{\overset{\vee,*}{\circ}} = b$.*

From theorems 9 and 12 we have

Theorem 13. *If $*$ is continuous triangular norm and $\diamond$ is continuous triangular conorm and $S(A, b, *, \diamond) \neq \emptyset$, then*

$$S(A, b, *, \diamond) = \bigcup_{v \in S^0(A,b,*,\diamond)} [v, g]$$

and $g = \max S(A, b, *, \diamond)$.

Example 8. Let $\alpha = \vee, * = \wedge, \lambda = \wedge, \diamond = \vee$ and

$$A = \begin{bmatrix} (0.5, 0.4) & (0.3, 0.2) & (0.7, 0.3) \\ (0.5, 0.6) & (0.6, 0.6) & (0.8, 0.8) \\ (0.2, 0.2) & (0.8, 0.9) & (0.2, 0.5) \end{bmatrix}, b = \begin{bmatrix} (0.5, 0.2) \\ (0.5, 0.6) \\ (0.2, 0.5) \end{bmatrix}.$$

From that we have

$$A^\mu = \begin{bmatrix} 0.5 & 0.3 & 0.7 \\ 0.5 & 0.6 & 0.8 \\ 0.2 & 0.8 & 0.2 \end{bmatrix}, A^\nu = \begin{bmatrix} 0.4 & 0.2 & 0.3 \\ 0.6 & 0.6 & 0.8 \\ 0.2 & 0.9 & 0.5 \end{bmatrix},$$

$$b^\mu = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.2 \end{bmatrix}, b^\nu = \begin{bmatrix} 0.2 \\ 0.6 \\ 0.5 \end{bmatrix}.$$

We need to solve two systems of equations $A^\mu \circ x^\mu = b^\mu$ and $A^\nu \bullet x^\nu = b^\nu$. Because the greatest solution of $A^\mu \circ x^\mu = b^\mu$ is u^μ and the lowest solution of $A^\nu \bullet x^\nu = b^\nu$ is z^ν (in $([0, 1]^n, \leqslant)$), where

$$u^\mu = \begin{bmatrix} 1 \\ 0.2 \\ 0.5 \end{bmatrix}, z^\nu = \begin{bmatrix} 0.5 \\ 0 \\ 0 \end{bmatrix},$$

then the greatest solution of $A \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} x = b$ is g (in $((L^*)^n, \preceq)$), where

$$g = \begin{bmatrix} (1, 0.5) \\ (0.5, 0) \\ (0.5, 0) \end{bmatrix}.$$

Example 9. Let $\alpha = \vee, * = \wedge, \lambda = \wedge, \diamond = \vee$ and

$$A = \begin{bmatrix} (0.5, 0.4) & (0.3, 0.2) & (0.7, 0.3) \\ (0.5, 0.6) & (0.6, 0.6) & (0.8, 0.8) \\ (0.2, 0.2) & (0.8, 0.9) & (0.2, 0.5) \end{bmatrix}, b = \begin{bmatrix} (0.5, 0.2) \\ (0.5, 0.6) \\ (0.2, 0.5) \end{bmatrix}.$$

Because minimal solutions of $A^\mu \circ x^\mu = b^\mu$ are w^1, w^2 and maximum solutions of $A^\nu \bullet x^\nu = b^\nu$ are w^3, w^4 (in $([0,1]^n, \leqslant)$), where

$$w^1 = \begin{bmatrix} 0.5 \\ 0 \\ 0 \end{bmatrix}, w^2 = \begin{bmatrix} 0 \\ 0 \\ 0.5 \end{bmatrix}, w^3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, w^4 = \begin{bmatrix} 0.5 \\ 0 \\ 1 \end{bmatrix},$$

then minimal solutions of $A \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} x = b$ are v^1, v^2, v^3, v^4 (in $((L^*)^n, \preceq)$), where

$$v^1 = \begin{bmatrix} (0.5, 1) \\ (0, 0) \\ (0, 0) \end{bmatrix}, v^2 = \begin{bmatrix} (0.5, 0.5) \\ (0, 0) \\ (0, 1) \end{bmatrix},$$

$$v^3 = \begin{bmatrix} (0, 1) \\ (0, 0) \\ (0.5, 0) \end{bmatrix}, v^4 = \begin{bmatrix} (0, 0.5) \\ (0, 0) \\ (0.5, 1) \end{bmatrix}.$$

Therefore, we obtain the set of all solution of this system of intuitionistic fuzzy equations $[v^1, g] \cup [v^2, g] \cup [v^3, g] \cup [v^4, g]$, where g is calculated in Example 8.

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The papers presented in this Volume 1 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.
It may be viewed as a result of fruitful discussions held during the Tenth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2011) organized in Warsaw on September 30, 2011 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT - Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bystrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, and the University of Westminster, Harrow, UK:

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The consecutive International Workshops on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGNs) have been meant to provide a forum for the presentation of new results and for scientific discussion on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems have also been included. The Tenth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2011) is a continuation of this undertaking, and provides many new ideas and results in the areas concerned.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

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