Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics. Volume I: Foundations

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Properties of interval-valued fuzzy relations and Atanassov operators

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Abstract

The goal of this paper is to consider of properties of interval-valued fuzzy relations which were introduced by L.A. Zadeh in 1975. Fuzzy set theory turned out to be a useful tool to describe situations in which the data are imprecise or vague. Interval-valued fuzzy set theory is a generalization of fuzzy set theory which was introduced also by Zadeh in 1965. This paper is continuation of examinations by Pękala (2009) on the interval-valued fuzzy relations. We study standard properties of interval-valued fuzzy relations in the context of Atanassov operators.

Keywords: fuzzy relations, interval-valued fuzzy relations, Atanassov operators.

1 Introduction

The idea of a fuzzy relation was defined in [27]. An extension of fuzzy set theory is interval-valued fuzzy set theory. Any interval-valued fuzzy set is defined by an interval-valued membership function: a mapping from the given universe to the set of all closed subintervals of [0,1] (it means that information is incomplete). In this work we study preservation of properties of interval-valued fuzzy relations

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by powers of this relations, by lattice operations and Atanassov operators. Consideration of diverse properties of the composition is interesting not only from a theoretical point of view but also for the applications, since the composition of interval-valued fuzzy relations has proved to be useful in several fields, see for example, [19] (performance evaluation), [24] (genetic algorithm), [18] (approximate reasoning) or in other (see [1, 15]). Moreover, it is interesting to use Atanassov operators as parameters in Intuitionistic Fuzzy Systems. Furthermore, interval-valued fuzzy relations are applied in classification and in decision making [5]. Interval-valued fuzzy relations (sets) are equivalent to some other extensions of fuzzy relations (sets) (see [13]). Among others, interval-valued fuzzy relations are isomorphic to Atanassov's intuitionistic fuzzy relations. This fact was noticed by several authors [3, 12, 13]. An Atanassov's intuitionistic fuzzy relation is a pair of fuzzy relations, namely a membership and a nonmembership function, which represent positive and negative aspects of the given information. This objects introduced by Atanassov and originally called intuitionistic fuzzy relations were recently suggested to be called Atanassov's intuitionistic fuzzy relations or just bipolar fuzzy relations [14]. An Atanassov's intuitionistic fuzzy set theory is also widely applied in solving real-life problems. An example of such application is the optimization in Atanassov's intuitionistic fuzzy environment (an extension of fuzzy optimization and an application of Atanassov's intuitionistic fuzzy sets) where by applying this concept it is possible to reformulate the optimization problem by using degrees of rejection of constraints and values of the objective which are non-admissible. This concept allows one to define a degree of rejection which cannot be simply a complement of the degree of acceptance. Atanassov's intuitionistic fuzzy optimization problem may be applied to nutrient medium optimization in bioprocess engineering and to decision making in biomedical engineering (cf. [2]). The idea of a positive and negative information was confirmed by psychological investigations [10]. Moreover, multiattribute decision making using Atanassov's intuitionistic fuzzy sets is possible (see [21, 22]). If it comes to Atanassov's intuitionistic fuzzy relations, the effective approach to deal with decision making in medical diagnosis was proposed with the use of composition of such objects (cf. [11]).

In this work we recall some concepts and results useful in our further considerations. Next, we study properties of powers of interval-valued fuzzy relations. Finally, we consider standard properties of interval-valued fuzzy relations and we study connections of its properties with lattice operations and some Atanassov operators, so we consider preservation of some properties of interval-valued fuzzy relations by lattice operations and some Atanassov operators.

2 Basic definitions

First we recall the notion of the lattice operations and the order in the family of interval-valued fuzzy relations. Let X, Y, Z be non-empty sets.

Definition 1 (cf. [26], [25]). *Let $Int([0,1])$ be the set of all closed subintervals of $[0,1]$. An interval-valued fuzzy relation R between universes X, Y is a mapping $R : X \times Y \to Int([0,1])$ such that*

$$R(x,y) = [\underline{R}(x,y), \overline{R}(x,y)] \in Int([0,1]), \tag{1}$$

for all pairs $(x,y) \in (X \times Y)$.
The class of all interval-valued fuzzy relations between universes X, Y will be denoted by IVFR$(X \times Y)$ or IVFR(X) for $X = Y$.

Interval-valued fuzzy relations reflect the idea that membership grades are often not precise and the intervals represent such uncertainty.

The boundary elements in *IVFR*$(X \times Y)$ are $\mathbf{1} = [1,1]$ and $\mathbf{0} = [0,0]$.

Let $S, R \in$ *IVFR*$(X \times Y)$. Then for every $(x,y) \in (X \times Y)$ we can define

$$S(x,y) \leq R(x,y) \Leftrightarrow \underline{S}(x,y) \leq \underline{R}(x,y), \overline{S}(x,y) \leq \overline{R}(x,y), \tag{2}$$

$$(S \vee R)(x,y) = [\max(\underline{S}(x,y), \underline{R}(x,y)), \max(\overline{S}(x,y), \overline{R}(x,y))], \tag{3}$$

$$(S \wedge R)(x,y) = [\min(\underline{S}(x,y), \underline{R}(x,y)), \min(\overline{S}(x,y), \overline{R}(x,y))], \tag{4}$$

where operations $\vee$ and $\wedge$ are the supremum and the infimum in *IVFR*$(X \times Y)$, respectively. Similarly, for arbitrary set $T \neq \emptyset$

$$(\bigvee_{t \in T} R_t)(x,y) = [\bigvee_{t \in T} \underline{R_t}(x,y), \bigvee_{t \in T} \overline{R_t}(x,y)], \tag{5}$$

$$(\bigwedge_{t \in T} R_t)(x,y) = [\bigwedge_{t \in T} \underline{R_t}(x,y), \bigwedge_{t \in T} \overline{R_t}(x,y)]. \tag{6}$$

The pair $(IVFR(X \times Y), \leqslant)$ is a partially ordered set. Operations $\vee, \wedge$ are the supremum and the infimum in $IVFR(X \times Y)$, respectively. As a result, the family $(IVFR(X \times Y), \vee, \wedge)$ is a lattice (for the notion of a lattice and other related concepts see [6]) which is a consequence of the fact that $([0,1], \max, \min)$ is a lattice. The lattice $IVFR(X \times Y)$ is complete. This fact follows from the notion of the supremum $\bigvee$ and infimum $\bigwedge$ and from the fact that the values of fuzzy relations are from the interval $[0,1]$ which, with the operations maximum and minimum, forms a complete lattice. As a result (*IVFR*$(X \times Y), \vee, \wedge)$ is a complete, distributive lattice. It was mentioned in the Introduction that interval-valued fuzzy relations (sets) are equivalent to some other extensions of fuzzy relations (sets), i.e. Atanassov's intuitionistic fuzzy relation.

Definition 2 (cf. [4]). *Let* $X \neq \emptyset$, $\Re$, $\Re^d : X \times Y \to [0,1]$ *be fuzzy relations fulfilling the condition*

$$\Re(x,y) + \Re^d(x,y) \leq 1, \ \ (x,y) \in (X \times Y).$$

A pair $\rho = (\Re, \Re^d)$ *is called an Atanassov's intuitionistic fuzzy relation. The family of all Atanassov's intuitionistic fuzzy relations described in the given sets* X, Y *is denoted by* $AIFR(X \times Y)$.

Basic operations and relations for $\rho = (\Re, \Re^d)$, $\sigma = (\Im, \Im^d)$ are the union, the intersection.

$$\rho \cup \sigma = (\max(\Re, \Im), \min(\Re^d, \Im^d)), \ \rho \cap \sigma = (\min(\Re, \Im), \max(\Re^d, \Im^d)). \quad (7)$$

Moreover, the order is defined by

$$\rho \leq \sigma \ \Leftrightarrow \ (\Re \leq \Im, \ \Im^d \leq \Re^d). \quad (8)$$

The isomorphism which proves the equivalence between Atanassov's intuitionistic fuzzy relations and interval-valued fuzzy relations is the following

Theorem 1 (cf. [12]). *The mapping* ψ : *IVFR*$(X \times Y) \to AIFR(X \times Y)$ *is an isomorphism between the lattices* (*IVFR*$(X \times Y), \vee, \wedge)$ *and* $(AIFR(X \times Y), \cup, \cap)$, *where* $R(x,y) = [\underline{R}(x,y), \overline{R}(x,y)], R \in$ *IVFR*$(X \times Y)$ *and* $\psi(R(x,y)) = (\underline{R}(x,y), 1 - \overline{R}(x,y))$, $(x,y) \in (X \times Y)$.

The presented above isomorphism is due to the appropriate choice of operations in both theories, however the initial ideas behind each theory are different (see [14]).

Now, we will give the notions of some Atanassov's operators for interval-valued fuzzy relations. This notions follow the ones introduced by Atanassov for intuitionistic fuzzy relations [5].

Definition 3. *Let* $R \in IVFR(X \times Y)$, $\alpha, \ \beta \in [0,1], \ \alpha + \beta \leq 1$, *we define the operators* $F_{\alpha,\beta}, P_{\alpha,\beta}, Q_{\alpha,\beta}, \boxplus_\alpha, \boxtimes_\alpha : Int([0,1]) \to Int[0,1]$ *such that*

$$F_{\alpha,\beta}(R(x,y)) =$$

$$[\underline{R}(x,y) + \alpha(\overline{R}(x,y) - \underline{R}(x,y)), \overline{R}(x,y) - \beta(\overline{R}(x,y) - \underline{R}(x,y))], \quad (9)$$

$$P_{\alpha,\beta}(R(x,y)) = [\max(\alpha, \underline{R}(x,y)), \max(1-\beta, \overline{R}(x,y))], \quad (10)$$

$$Q_{\alpha,\beta}R((x,y)) = [\min(\alpha, \underline{R}(x,y)), \min(1-\beta, \overline{R}(x,y))], \quad (11)$$

$$\boxplus_\alpha(R(x,y)) = [\alpha\underline{R}(x,y)), \alpha\overline{R}(x,y))], \quad (12)$$

$$\boxtimes_\alpha(R(x,y)) = [1 + \alpha(\underline{R}(x,y) - 1), 1 + \alpha(\overline{R}(x,y) - 1)]. \quad (13)$$

In particular, if $F_{\alpha,1-\alpha}(R) = D_\alpha(R)$, then $D_\alpha(R) = \underline{R}(x,y) + \alpha(\overline{R}(x,y) - \underline{R}(x,y))$ and $F_{0,1}(R) = D_0(R) = \underline{R}$, $F_{1,0}(R) = D_1(R) = \overline{R}$.

3 Properties of interval-valued fuzzy relations

For our further considerations we need the following properties

Definition 4. *Let* $* : [0,1]^2 \to [0,1]$. *Operation* $*$ *is infinitely sup-distributive, if*

$$\bigvee_{t \in T} (x_t * y) = (\bigvee_{t \in T} x_t) * y, \quad \bigvee_{t \in T} (y * x_t) = y * (\bigvee_{t \in T} x_t). \tag{14}$$

3.1 Properties of powers

Let us recall the notion of the composition and powers in IVFR.

Definition 5 (cf. [17], [9])**.** *Let* $* : [0,1]^2 \to [0,1]$,
$S \in$ *IVFR*$(X \times Y)$, $R \in$ *IVFR*$(Y \times Z)$. *By the* $\sup -*$ *composition of relations* S *and* R *we call the relation* $S \circ R \in$ *IVFR*$(X \times Z)$,

$$(S \circ R)(x,z) = [(\underline{S} \circ \underline{R})(x,z), (\overline{S} \circ \overline{R})(x,z)],$$

where

$$(\underline{S} \circ \underline{R})(x,z) = \bigvee_{y \in Y} (\underline{S}(x,y) * \underline{R}(y,z)), \quad (\overline{S} \circ \overline{R})(x,z) = \bigvee_{y \in Y} (\overline{S}(x,y) * \overline{R}(y,z))$$

and $(\underline{S} \circ \underline{R})(x,z) \leq (\overline{S} \circ \overline{R})(x,z)$.

As a consequence of results of [23] we obtain:
If $*$ is an isotonic operation, then we may prove

$$\bigvee_{t \in T} (S_t \circ R) \leq (\bigvee_{t \in T} S_t) \circ R, \quad \bigwedge_{t \in T} (S_t \circ R) \geq (\bigwedge_{t \in T} S_t) \circ R. \tag{15}$$

Moreover, if an operation $*$ is infinitely sup-distributive, then $\sup -*$ composition is also infinitely sup-distributive.

Proposition 1 (cf. [23])**.** *If* $*$ *is associative, infinitely sup-distributive operation with a zero element z=0 and a neutral element e=1, then*
(*IVFR*$(X), \circ$) *is an ordered semigroup with the identity* $\boldsymbol{I} = [I, I]$.

In the sequel we consider arbitrary binary operations $* : [0,1]^2 \to [0,1]$ which are associative and infinitely sup-distributive (these conditions imply that $*$ is isotonic). As a result a special case of $*$ may be a left-continous triangular norm or conorm. If $*$ is associative and infinitely sup-distributive, then in a semigroup (*IVFR*$(X), \circ$) we can consider the powers of its elements, i.e. relations R^n for $R \in$ *IVFR*(X), $n \in \mathbb{N}$. Analogously to [20] we define

Definition 6. *By the powers of a relation* $R \in$ *IVFR*(X) *we call interval-valued fuzzy relations*

$$R^1 = R,\ R^{m+1} = R^m \circ R,\ \text{where } m = 1, 2, \ldots\ . \tag{16}$$

By the upper closure $R^\vee$ *and the lower closure* $R^\wedge$ *of the relation* R *we call*

$$R^\vee = \bigvee_{k=1}^{\infty} R^k, \qquad R^\wedge = \bigwedge_{k=1}^{\infty} R^k, \tag{17}$$

where $R^k = [\underline{R}^k, \overline{R}^k]$.

Theorem 2. *If* $R \in IVFR(X)$*,* $*$ *is associative and infinitely sup - distributive, then*

$$R^n \circ R^\vee = R^\vee \circ R^n, \tag{18}$$

$$(R^\vee)^n \geq (R^n)^\vee, \tag{19}$$

$$(R^\wedge)^{n+1} \leq (R^{n+1})^\wedge,\ n = 1, 2, ... \tag{20}$$

Proof. By infinite sup-distributivity of composition $\circ$ we obtain

$$R \circ R^\vee = R \circ (\bigvee_{k=1}^{\infty} R^k) = \bigvee_{k=1}^{\infty} R^{k+1} = \bigvee_{k=2}^{\infty} R^k,$$

$$R^\vee \circ R = (\bigvee_{k=1}^{\infty} R^k) \circ R = \bigvee_{k=1}^{\infty} (R^k \circ R) = \bigvee_{k=2}^{\infty} R^k,$$

which proves that R and $R^\vee$ are commuting. Then by mathematical induction we obtain commutativity of their powers and we get (18). Inequalities (19), (20) are obtained by mathematical induction using also infinite sup-distributivity, isotonicity of composition $\circ$ (see [23], Lemma 1) and sub-distributivity from (15).

□

From the above conditions we obtain the following dependence between upper and lower closers

Corollary 1. *If* $R \in IVFR(X)$*,* $*$ *is associative and infinitely sup - distributive, then*

$$(R^\vee)^\vee = R^\vee,\ (R^\wedge)^\wedge \leq R^\wedge,\ (R^\wedge)^\vee \leq (R^\vee)^\wedge. \tag{21}$$

3.2 Reflexivity

Now we examine some standard properties. We apply the following ones which are modifications of the properties applied in [8] and [16]. We study preservation of these properties by powers, lattice operations and Atanassov operators. First, we give definition of reflexivity.

Definition 7. *Let $R \in IVFR(X)$. A relation R is called reflexive if*

$$R(x, x) = \mathbf{1} \ \text{for all} \ \ x \in X. \tag{22}$$

Proposition 2. *Let $R \in IVFR(X)$ and an operation $*$ be associative and infinitely sup - distributive having the neutral element 1. If R is reflexive, then*

$$R^n, \ R^{\vee}, \ R^{\wedge} \tag{23}$$

are reflexive.

Proof. Let R be reflexive, so $[I, I] \leq [\underline{R}, \overline{R}]$. By infinite sup - distributivity we have isotonicity of the composition, so $\mathbf{I} \leq R \leq R^2$, $R \leq R^n = [\underline{R}^n, \overline{R}^n]$, as a result R^n, $n \in \mathbb{N}$ are reflexive. Moreover, by the property of supremum and infimum we obtain $\mathbf{I} \leq R^{\vee}$ and $\mathbf{I} \leq R^{\wedge}$, so $R^{\vee}$ and $R^{\wedge}$ are reflexive. □

Similarly, lattice operations and only some Atanassov operators preserve reflexivity.

Proposition 3. *Let $R, S \in IVFR(X)$, $\alpha, \beta \in [0, 1]$, $\alpha + \beta \leq 1$. If relations R, S are reflexive, then*

$$R \wedge S, \quad R \vee S \tag{24}$$

$$F_{\alpha,\beta}(R), \quad P_{\alpha,\beta}(R), \quad \boxtimes_{\alpha}(R) \tag{25}$$

are reflexive.

Proof. If relations R, S are reflexive, then $(R \wedge S)(x, x) = R(x, x) \wedge S(x, x) = [\underline{R}(x, x) \wedge \underline{S}(x, x), \overline{R}(x, x) \wedge \overline{S}(x, x)] = [1, 1]$. Similarly, for one reflexive relation we obtain reflexivity of $R \vee S$.
We prove for example reflexivity of $F_{\alpha,\beta}(R) : F_{\alpha,\beta}(R)(x, x) =$
$[\underline{R}(x, x) + \alpha\overline{R}(x, x) - \alpha\underline{R}(x, x), \overline{R}(x, x) - \beta\overline{R}(x, x) + \beta\underline{R}(x, x)] =$
$[1 + \alpha - \alpha, 1 - \beta + \beta] = [1, 1]$. □

3.3 Irreflexivity

Now we consider irreflexivity.

Definition 8. *Let $R \in IVFR(X)$. A relation R is called irreflexive if*

$$R(x,x) = \mathbf{0} \text{ for all } x \in X. \tag{26}$$

Directly from definition of the lower closure we obtain preservation irreflexivity.

Proposition 4. *Let $R \in IVFR(X)$ and an operation $*$ be associative and infinitely sup - distributive. If R is irreflexive, then $R^{\wedge}$ is irreflexive.*

Moreover, for lattice operations and some Atanassov operators we get.

Proposition 5. *Let $R, S \in IVFR(X)$, $\alpha, \beta \in [0,1]$, $\alpha + \beta \leq 1$. If relations R, S are irreflexive, then*

$$R \wedge S, \quad R \vee S \tag{27}$$

$$F_{\alpha,\beta}(R), \quad Q_{\alpha,\beta}(R), \quad \boxplus_{\alpha}(R) \tag{28}$$

are irreflexive.

Proof. We prove for example irreflexivity of $F_{\alpha,\beta}(R)$. If relations R is irreflexive, then $F_{\alpha,\beta}(R)(x,x) =$
$[\underline{R}(x,x) + \alpha\overline{R}(x,x) - \alpha\underline{R}(x,x), \overline{R}(x,x) - \beta\overline{R}(x,x) + \beta\underline{R}(x,x)] = [0,0]$.
By the properties of minimum, maximum and operation product we have irreflexivity of $Q_{\alpha,\beta}(R)$ and $\boxplus_{\alpha}(R)$. □

3.4 Symmetry

Definition 9. *Let $R \in IVFR(X)$. A relation R is called symmetric if*

$$R(x,y) = R(y,x) \text{ for all } x, y \in X. \tag{29}$$

Proposition 6. *Let $R \in IVFR(X)$, an operation $*$ be is associative, infinitely sup - distributive and commutative. If R is symmetric, then*

$$R^n, \quad R^{\vee}, \quad R^{\wedge} \quad n \in \mathbb{N}, \tag{30}$$

are symmetric.

Proof. Let R be symmetric and $*$ be is associative, infinitely sup - distributive and commutative. Then for $x, z \in X$ we see that
$R^2(x,z) = [\bigvee_{y\in X}(\underline{R}(x,y) * \underline{R}(y,z)), \bigvee_{y\in X}(\overline{R}(x,y) * \overline{R}(y,z))] =$
$[\bigvee_{y\in X}(\underline{R}(y,z) * \underline{R}(x,y)), \bigvee_{y\in X}(\overline{R}(y,z) * \overline{R}(x,y))] =$
$[\bigvee_{y\in X}(\underline{R}(z,y) * \underline{R}(y,x)), \bigvee_{y\in X}(\overline{R}(z,y) * \overline{R}(y,x))] = R^2(z,x)$
and by mathematical induction we obtain symmetry of R^n. Moreover, by the properties of infimum and supremum we complete the proof. □

Directly by definition of Atanassov operators and lattice operations we see that

Proposition 7. *Let $R, S \in$ IVFR(X), $\alpha, \beta \in [0,1]$, $\alpha + \beta \le 1$. If relations R, S are symmetric, then*

$$R \wedge S, \quad R \vee S \tag{31}$$

$$F_{\alpha,\beta}(R), \quad P_{\alpha,\beta}(R), \quad Q_{\alpha,\beta}(R), \quad \boxtimes_\alpha(R), \quad \boxplus_\alpha(R) \tag{32}$$

are symmetric.

3.5 Asymmetry

Definition 10 (cf. [7]). *Let $R \in$ IVFR(X). A relation R is called asymmetric if*

$$R(x,y) \neq R(y,x), \ \overline{R}(x,y) - \underline{R}(x,y) = \overline{R}(y,x) - \underline{R}(y,x), \ x \neq y, \ x, y \in X. \tag{33}$$

Proposition 8. *Let $R \in$ IVFR(X), $\alpha, \beta \in [0,1]$, $\alpha + \beta \le 1$. If relation R is asymmetric, then*

1. *$F_{\alpha,\beta}(R)$ is asymmetric.*

2. *$\boxtimes_\alpha(R)$, $\boxplus_\alpha(R)$ are asymmetric for $\alpha, \beta \in (0,1]$, $\alpha + \beta \le 1$.*

3. *If $\underline{R} \ge \alpha$, $\overline{R} \ge 1 - \beta$, then $P_{\alpha,\beta}(R)$ is asymmetric.*

4. *If $\underline{R} \le \alpha$, $\overline{R} \le 1 - \beta$, then $Q_{\alpha,\beta}(R)$ is asymmetric.*

Proof. We prove for example the first and the third condition. Let $x, y \in X$ and $R \in$ IVFR(X) be asymmetric, i.e.
$\overline{R}(x,y) - \underline{R}(x,y) = \overline{R}(y,x) - \underline{R}(y,x)$ and $\underline{R}(x,y) \neq \underline{R}(y,x)$,
$\overline{R}(x,y) \neq \overline{R}(y,x)$, then
$\alpha(\overline{R}(x,y) - \underline{R}(x,y)) = \alpha(\overline{R}(y,x) - \underline{R}(y,x))$ and

$\underline{R}(x,y) + \alpha(\overline{R}(x,y) - \underline{R}(x,y)) \neq \underline{R}(y,x) + \alpha(\overline{R}(y,x) - \underline{R}(y,x))$,
$\overline{R}(x,y) - \beta(\overline{R}(x,y) - \underline{R}(x,y)) \neq \overline{R}(y,x) - \beta(\overline{R}(y,x) - \underline{R}(y,x))$, so
$F_{\alpha,\beta}(R)(x,y) \neq F_{\alpha,\beta}(R)(y,x)$.
$\overline{R}(x,y) - \beta(\overline{R}(x,y) - \underline{R}(x,y)) - (\underline{R}(x,y) + \alpha(\overline{R}(x,y) - \underline{R}(x,y))) =$
$\overline{R}(y,x) - \beta(\overline{R}(y,x) - \underline{R}(y,x)) - (\underline{R}(y,x) + \alpha(\overline{R}(y,x) - \underline{R}(y,x)))$.
Thus $F_{\alpha,\beta}(R)$ is asymmetric. Moreover, if $\underline{R} \geq \alpha$ and $\overline{R} \geq 1-\beta$, then
$\max(\alpha, \underline{R}(x,y)) = \underline{R}(x,y) \neq \underline{R}(y,x) = \max(\alpha, \underline{R}(y,x))$,
$\max(1-\beta, \overline{R}(x,y)) = \overline{R}(x,y) \neq \overline{R}(y,x) = \max(1-\beta, \overline{R}(y,x))$,
so $P_{\alpha,\beta}(R)(x,y) \neq P_{\alpha,\beta}(R)(y,x)$.
$\max(1-\beta, \overline{R}(x,y)) - \max(\alpha, \underline{R}(x,y)) = \overline{R}(x,y) - \underline{R}(x,y) =$
$\overline{R}(y,x) - \underline{R}(y,x) = \max(1-\beta, \overline{R}(y,x)) - \max(\alpha, \underline{R}(y,x))$.
Thus $P_{\alpha,\beta}(R)$ is asymmetric. The other conditions we may prove in a similar way. □

3.6 Transitivity

In this section we continue the study from [23] and we examine transitivity. Similarly to definitions of properties of fuzzy relations considered by Kaufmann we have

Definition 11. *Let $R \in IVFR(X)$. A relation R is called subidempotent (transitive) if $R^2 \leq R$.*

We know that

Theorem 3 (cf. [23])**.** *Let $*$ be an associative, infinitely sup - distributive operation and $R \in IVFR(X)$. $R^{\vee}$ is the least subidempotent relation greater than or equal to R. Moreover, the relation R is subidempotent if and only if $R = R^{\vee}$.*

Corollary 2. *Let $R \in IVFR(X)$, an operation $*$ be associative and infinitely sup - distributive. If R is symmetric and reflexive, then $R^{\vee}$ is symmetric, reflexive and transitive.*

In [7], the authors introduced for Atanassov's intuitionistic fuzzy relations the concept of a "partially included relation". The justification for consideration of partially included relations is connected with the fact that Atanassov's operators $D_{\alpha}(R)$, where $\alpha \in [0,1]$, do not generally keep the transitivity property of a Atanassov's intuitionistic fuzzy relation. However, for Atanassov's intuitionistic fuzzy relations which are partially included, transitivity of a Atanassov's intuitionistic fuzzy relation ρ is equivalent to transitivity of operator $D_{\alpha}(R)$ for each $\alpha \in [0,1]$. We consider a more general form of the concept of a partially included Atanassov's intuitionistic fuzzy relation. Namely

Definition 12 (cf. [7]). *A relation R ∈IVFR(X) is called partially included if*

$$sgn(\underline{R}(x,z) - \underline{R}(z,y)) = sgn(\overline{R}(x,z) - \overline{R}(z,y)), \;\; x,y,z \in X. \qquad (34)$$

Proposition 9. *Let $R \in$ IVFR(X), $\alpha, \beta \in [0,1]$, $\alpha + \beta \leq 1$ and $* = \wedge$. If R is partially included and transitive, then $F_{\alpha,\beta}(R)$ is transitive.*

Proof. Let $R^2 \leq R$ and R be partially included, $x, y \in X$. From (34) we obtain
$((1-\alpha)\underline{R}(x,z) + \alpha\overline{R}(x,z)) \wedge ((1-\alpha)\underline{R}(z,y) + \alpha\overline{R}(z,y)) =$
$(1-\alpha)(\underline{R}(x,z) \wedge \underline{R}(z,y)) + \alpha(\overline{R}(x,z) \wedge \overline{R}(z,y))$.
Then $F^2_{\alpha,\beta}(R)(x,y) =$
$[(\underline{R}(x,y) + \alpha(\overline{R}(x,y) - \underline{R}(x,y)))^2, (\overline{R}(x,y) - \beta(\overline{R}(x,y) - \underline{R}(x,y)))^2] =$
$[((1-\alpha)\underline{R}(x,y) + \alpha\overline{R}(x,y))^2, ((1-\beta)\overline{R}(x,y) + \beta\underline{R}(x,y))^2] =$
$[\bigvee_{z\in X}((1-\alpha)\underline{R}(x,z) + \alpha\overline{R}(x,z)) \wedge ((1-\alpha)\underline{R}(z,y) + \alpha\overline{R}(z,y)),$
$\bigvee_{z\in X}((1-\beta)\overline{R}(x,z) + \beta\underline{R}(x,z)) \wedge ((1-\beta)\overline{R}(z,y) + \beta\underline{R}(z,y))]$.
From the above considerations we have
$[\bigvee_{z\in X}((1-\alpha)(\underline{R}(x,z) \wedge \underline{R}(z,y)) + \alpha(\overline{R}(x,z) \wedge \overline{R}(z,y))),$
$\bigvee_{z\in X}((1-\beta)(\overline{R}(x,z) \wedge \overline{R}(z,y)) + \beta(\underline{R}(x,z) \wedge \underline{R}(z,y)))] \leq$
$[\bigvee_{z\in X}(1-\alpha)(\underline{R}(x,z) \wedge \underline{R}(z,y)) + \bigvee_{z\in X}\alpha(\overline{R}(x,z) \wedge \overline{R}(z,y)),$
$\bigvee_{z\in X}(1-\beta)(\overline{R}(x,z) \wedge \overline{R}(z,y)) + \bigvee_{z\in X}\beta(\underline{R}(x,z) \wedge \underline{R}(z,y))] =$
$F_{\alpha,\beta}(R^2)(x,y)$, so by the isotonicity $F_{\alpha,\beta}$ we obtain
$F^2_{\alpha,\beta}(R)(x,y) \leq F_{\alpha,\beta}(R)(x,y)$. □

Moreover, the following Atanassov operators preserve transitivity, what we may prove from the Definition 3 and by distributivity of some operations

Proposition 10. *Let $R \in$ IVFR(X), $* = \wedge$, $\alpha, \beta \in [0,1]$, $\alpha + \beta \leq 1$.*
If R is transitive, then

$$P_{\alpha,\beta}(R), \;\; Q_{\alpha,\beta}(R), \;\; \boxplus_\alpha(R), \;\; \boxtimes_\alpha(R) \qquad (35)$$

are transitive.

4 Conclusion

In this work we consider preservation of properties of interval-valued fuzzy relations (reflexivity, irreflexivity, symmetry, asymmetry and transitivity) by closures, powers, lattice operations and Atanassov operators. We can also consider other Atanassov operators and the dual composition to the one defined in Definition 5. This is the $\inf -*'$ composition with the dual binary operation $*'$, where

$$x *' y = 1 - (1-x) * (1-y) \;\text{ for }\; x, y \in [0,1].$$

Thus we may obtain similar properties to presented in this paper theorems but for other Atanassov operators.

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The papers presented in this Volume 1 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.
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The Eighth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2009) has been meant to commence a new series of scientific events primarily focused on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Moreover, other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems are discussed.

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