Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics. Volume I: Foundations

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Systems Research Institute
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Eigen fuzzy vectors for $\min - *$ product

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Abstract

Eigen fuzzy vectors and values were introduced by E. Sanchez [11] and K.H. Kim, F.W. Roush [8].

Some articles (cf. [11], [12], [3]) consider eigen fuzzy vectors for $\max - \min$ product with discussion of the greatest solution and algorithms for reach it. In [15] author consider properties of eigen fuzzy vectors for $\max - \min$ product with some assumption about given matrix. Moreover, M. Wagenknecht, K. Hartmann (cf. [13]) consider fuzzy eigen vectors from given intervals.

Later, those vectors were generalized for $\max -$prod (cf. [2]) and $\min - T$ (cf. [6]) product, where T denotes triangular norms.

Our considerations concern eigen fuzzy vectors for $\min - *$ product, where $*$ is a monotonic operation with additional properties.

Keywords: eigen fuzzy vectors, eigenvectors, fuzzy relation equations.

1 Introduction

In this paper we present properties of eigen fuzzy vectors for $\min - *$ product, where $*$ is a monotonic operation with additional properties.

In this study we formulate and solve two problems. The first one is about properties of eigen fuzzy vectors for given matrix and operation. The second

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part is about properties of set of all matrices with common eigen fuzzy vector. Moreover we present methods to obtain the least eigen fuzzy vector.

For short, operations $\vee$ and $\wedge$ are used for numbers

$$a \vee b = \max\{a, b\},\ a \wedge b = \min\{a, b\},\ a, b \in [0, 1] \tag{1}$$

and for matrices with elements from $[0, 1]$:

$$(A \vee B)_{ij} = \max\{a_{ij}, b_{ij}\}, (A \wedge B)_{ij} = \min\{a_{ij}, b_{ij}\}, \tag{2}$$

and similarly

$$(A \leqslant B) \Leftrightarrow \underset{i,j}{\forall} (a_{ij} \leqslant b_{ij}). \tag{3}$$

In particular

$$(a \leqslant b) \Leftrightarrow \underset{k}{\forall} (a_k \leqslant b_k) \tag{4}$$

for $a, b \in [0, 1]^n$.

Moreover, for short we denote $\mathbf{0} = [0, 0, \ldots, 0]^T$ and $\mathbf{1} = [1, 1, \ldots, 1]^T$.

2 Properties of monotonic operations

We use a binary operation $* : [0, 1]^2 \to [0, 1]$ with some additional assumptions.

Lemma 1 (cf. [5], Lemma 5)**.** *Let* $* : [0, 1]^2 \to [0, 1]$. *The following three conditions are equivalent:*

$$\underset{a,b,c \in [0,1]}{\forall} (b \leqslant c) \Rightarrow (a * b \leqslant a * c, \quad b * a \leqslant b * c), \tag{5}$$

$$\underset{a,b,c \in [0,1]}{\forall} a * (b \wedge c) = (a * b) \wedge (a * c), \quad (b \wedge c) * a = (b * a) \wedge (c * a), \tag{6}$$

$$\underset{a,b,c \in [0,1]}{\forall} a * (b \vee c) = (a * b) \vee (a * c), \quad (b \vee c) * a = (b * a) \vee (c * a). \tag{7}$$

Example 1 ([9])**.** *Typical binary operations* $*$ *on interval* $[0, 1]$ *are triangular norms and triangular conorms, especially*

$$T_M(x, y) = x \wedge y, \quad S_M(x, y) = x \vee y, \tag{8}$$

$$T_P(x, y) = x \cdot y, \quad S_P(x, y) = x + y - x \cdot y, \tag{9}$$

$$T_L(x, y) = (x + y - 1) \vee 0, \quad S_L(x, y) = (x + y) \wedge 1, \tag{10}$$

$$T_D(x, y) = \begin{cases} x \wedge y, & x \vee y = 1 \\ 0 & \textit{for other} \end{cases}, \ S_D(x, y) = \begin{cases} x \vee y, & x \wedge y = 0 \\ 1 & \textit{for other} \end{cases}, \tag{11}$$

$$T_{FD}(x,y) = \begin{cases} 0, & x+y \leqslant 0,5 \\ x \wedge y & \textit{for other} \end{cases}, S_{FD}(x,y) = \begin{cases} 1, & x+y > 0,5 \\ x \vee y & \textit{for other} \end{cases} \tag{12}$$

for $x, y \in [0, 1]$.

Lemma 2 ([9], Remark 1.5, 1.16)**.** *Operations from Example 1 have the following ordering*

$$T_D \leqslant T \leqslant T_M \leqslant S_M \leqslant S \leqslant S_D, \tag{13}$$

where T is a triangular norm and S is a triangular conorm.

Definition 1 (cf. [1], p. 327)**.** *An operation $*$ is infinitely* $\inf -$*distributive (distributive over arbitrary infimum) if*

$$\underset{a,b_t \in [0,1]}{\forall} \; a * (\bigwedge_{t\in T} b_t) = \bigwedge_{t \in T}(a * b_t),\; (\bigwedge_{t\in T} b_t) * a = \bigwedge_{t\in T}(b_t * a) \tag{14}$$

for arbitrary set $T \neq \emptyset$ of indexes. The set of all increasing operations $*$ with the neutral element $e = 0$ will be denoted by D.

Definition 2 (cf. [4], Definition 1)**.** *Let $a, b \in [0, 1]$. The following binary operation in $[0, 1]$ is called the dual implication induced by the operation $*$*

$$a \overset{*}{\leftarrow} b = \min\{t \in [0,1] : a * t \geqslant b\} \quad \textit{for } a, b \in [0,1], \tag{15}$$

if they exist.

Corollary 1. *If binary operation $* : [0, 1]^2 \to [0, 1]$ is increasing with neutral element $e = 0$, then we have*

$$a \overset{*}{\leftarrow} a = 0 \quad \textit{for } a \in [0,1]. \tag{16}$$

Lemma 3 (cf. [4], Lemma 2)**.** *If $* : [0, 1]^2 \to [0, 1]$ is an increasing operation, then the dual implication induced by the operation $*$ is decreasing with respect to the first variable and increasing with respect to the second variable.*

Theorem 1 (cf. [10], Theorem 3, p. 186)**.** *Let $* : [0, 1]^2 \to [0, 1]$ be increasing operation and $a, b, c, d \in [0, 1]$. If dual implication exists for suitable arguments then it has the following properties*

$$a \overset{*}{\leftarrow} (a * b) \leqslant b \leqslant a * (a \overset{*}{\leftarrow} b), \tag{17}$$

$$(a * c \geqslant b) \Leftrightarrow (c \geqslant a \overset{*}{\leftarrow} b), \tag{18}$$

$$(a \overset{*}{\leftarrow} b) \vee (a \overset{*}{\leftarrow} d) \geqslant a \overset{*}{\leftarrow} (b \vee d). \tag{19}$$

Proof. From (15) we obtain $b \leqslant a * (a \overset{*}{\leftarrow} b)$ and

$$a \overset{*}{\leftarrow} (a * b) = \min\{c \in [0,1] : a * c \geqslant a * b\} \leqslant b,$$

thus we have (17). By monotonicity we obtain

$$c \geqslant a \overset{*}{\leftarrow} b \Rightarrow a * c \geqslant b.$$

According to the formula (17) and Lemma 3 we obtain

$$a * c \geqslant b \Rightarrow c \geqslant a \overset{*}{\leftarrow} (a * c) \geqslant a \overset{*}{\leftarrow} b,$$

which proves (18). Let $b, d \in L$ where $b \leqslant d$. From this we get

$$a \overset{*}{\leftarrow} (b \vee d) = a \overset{*}{\leftarrow} d \leqslant (a \overset{*}{\leftarrow} b) \vee (a \overset{*}{\leftarrow} d),$$

what gives (19). □

Definition 3 (cf. [14]). *Let* $A \in [0,1]^{m \times n}$, $B \in [0,1]^{n \times p}$ *and operation* $* : [0,1]^2 \to [0,1]$. *The matrix* $A \circ B$ *is called* $\min - *$ *product of matrices* A *and* B *if*

$$(A \circ B)_{ik} = \bigwedge_{j=1}^{n} (a_{ij} * b_{jk}), \quad i = 1, 2, \ldots, m, \ k = 1, 2, \ldots, p. \tag{20}$$

Lemma 4. *If an operation* $*$ *is infinitely distributive and associative, then product (20) has the following properties*

$$(A \circ B) \circ C = A \circ (B \circ C), \tag{21}$$

$$\alpha * (A \circ B) = (\alpha * A) \circ B. \tag{22}$$

Proof. From (20) and assumptions we obtain
$((A \circ B) \circ C)_{il} = \bigwedge\limits_{k=1}^{n} \bigwedge\limits_{j=1}^{n} ((a_{ij} * b_{jk}) * c_{kl}) = \bigwedge\limits_{j=1}^{n} \bigwedge\limits_{k=1}^{n} (a_{ij} * (b_{jk} * c_{kl})) = (A \circ (B \circ C))_{il}$,
which gives (21). Moreover we get
$\alpha * (A \circ B)_{ik} = \alpha * \bigwedge\limits_{j=1}^{n} (a_{ij} * b_{jk}) = \bigwedge\limits_{j=1}^{n} \alpha * (a_{ij} * b_{jk}) = \bigwedge\limits_{j=1}^{n} ((\alpha * a_{ij}) * b_{jk}) = ((\alpha * A) \circ B)_{ik}$
what proves (22). □

Lemma 5. *If an operation* $*$ *is increasing, then* $\min - *$ *product is also increasing, it means*

$$(A \leqslant B) \Leftarrow (A \circ C \leqslant B \circ C, \ C \circ A \leqslant C \circ B). \tag{23}$$

Proof. Since $*$ increasing operation then by (3) we get
$(A \leqslant B) \Leftrightarrow \underset{i,j}{\forall} (a_{ij} \leqslant b_{ij}) \Rightarrow \underset{i,j,l}{\forall} (a_{ij} * c_{jl} \leqslant b_{ij} * c_{jl}) \Rightarrow \underset{i,l}{\forall} (\bigwedge_{j=1}^{n} (a_{ij} * c_{jl}) \leqslant \bigwedge_{j=1}^{n} (b_{ij} * c_{jl})) \Leftrightarrow (A \circ D \leqslant B \circ D)$,
and similarly we get the second part of (23). □

Corollary 2. *Let $A, B \in [0,1]^{n\times n}$, $b, c \in [0,1]^n$. If an operation $*$ is increasing, then*

$$(A \leqslant B) \Leftarrow (A \circ c \leqslant B \circ c), \tag{24}$$

$$(c \leqslant d) \Leftarrow (A \circ c \leqslant A \circ d). \tag{25}$$

3 Eigen fuzzy vectors

Let $n \in \mathbb{N}$, $A \in [0,1]^{n\times n}$.

Definition 4 ([8], Definition 4.2)**.** *A column (row) eigen fuzzy vector of matrix A is a vector x such that $A \circ x = \lambda * x$ (or $x \circ A = \lambda * x$) for some $\lambda \in [0,1]$. Such number λ is called eigen fuzzy value. The family of all such solutions is denoted by $E_\lambda(A,*)$ for some λ.*

We look for solutions $x \in [0,1]^n$, $x = (x_1, x_2, \ldots, x_n)^T$ of $\min -*$ system $A \circ x = \lambda * x$, i.e.

$$\bigwedge_{j=1}^{n} (a_{ij} * x_j) = \lambda * x_i,\ i = 1, \ldots, n. \tag{26}$$

Example 2. *Let $\circ$, $\circledast$ mean $\min - S_M$ and $\min - S_L$ composition, respectively. We have*

$$A = \begin{bmatrix} 0.5 & 0.7 \\ 1 & 0.9 \end{bmatrix}, v = \begin{bmatrix} 0.5 \\ 0.9 \end{bmatrix} A \circ v = \begin{bmatrix} 0.5 \\ 0.9 \end{bmatrix} = \lambda * v, A \circledast v = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$v \in E_\lambda(A, S_M)$ for $\lambda \in [0, 0.5]$, but $v \notin E_\mu(A, S_L)$ for $\mu \in [0,1]$.

Lemma 6. *Let $*$ be an operation with neutral element $e = 0$.*

$$\mathbf{0} \in E_0(A,*) \Leftrightarrow \bigwedge_{j=1}^{n} (a_{ij}) = 0, \quad i = 1, \ldots, n. \tag{27}$$

Proof. Knowing that $e = 0$ we get

$$\bigwedge_{j=1}^{n} (a_{ij} * 0) = \bigwedge_{j=1}^{n} a_{ij} = 0 \quad \text{for } i = 1, \ldots, n.$$

It proves (27). $\square$

Example 3. *Let $*$ be an operation with zero element $z = 0$. We have $0 \in E_\lambda(A, *)$ for any $\lambda \in [0, 1]$.*

Lemma 7. *Let $*$ be arbitrary t-conorm. If $\lambda > 0$ then $0 \notin E_\lambda(A, *)$ for any matrix A.*

Proof. Suppose that $0 \in E_\lambda(A, *)$ for $\lambda > 0$. Using Lemma 2 we have

$$A \circ 0 = \lambda * 0 \geqslant \lambda \vee 0 > 0.$$

It gives contradiction with supposition $0 \in E_\lambda(A, *)$. $\square$

Lemma 8. *If $*$ is increasing with $e = 0$ then $\mathbf{1} \in E_\lambda(A, *)$ for $\lambda \in [0, 1]$.*

Proof. Using increasing property we get

$$\bigwedge_{j=1}^{n} (a_{ij} * 1) \geqslant \bigwedge_{j=1}^{n} (0 * 1) = 1, \quad \text{for } i = 1, \ldots, n.$$

Moreover $1 = 0 * 1 \leqslant \lambda * 1 \leqslant 1$. Thus we obtain $\mathbf{1} \in E_\lambda(A, *)$ for any $\lambda \in [0, 1]$. $\square$

Example 4. *Let $* = T_L$, $n = 3$ and*

$$A = \begin{bmatrix} 0.2 & 0.9 & 0.5 \\ 0.6 & 0.7 & 0.3 \\ 1 & 0.5 & 0.9 \end{bmatrix}, \; A \circledast 1 = \begin{bmatrix} 0.2 & 0.9 & 0.5 \\ 0.6 & 0.7 & 0.3 \\ 1 & 0.5 & 0.9 \end{bmatrix} \circledast \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix}.$$

$T_L \notin D$, because T_L is increasing operation with $e = 1$. We can observe $\mathbf{1} \notin E_\lambda(A, *)$, where $\lambda \in [0, 1]$.

We introduce the following vectors $v^1 \in [0, 1]^n$, $v^0 \in [0, 1]^n$ and numbers α, δ:

$$v_i^1 = \bigwedge_{j=1}^{n} a_{ij}, \quad \alpha = \bigvee_{i=1}^{n} v_i^1, \quad v_{\lambda i}^0 = \lambda * \alpha, \quad \delta = \bigwedge_{j=1}^{n} v_j^1 \quad \text{for } i = 1, 2, \ldots n. \tag{28}$$

Lemma 9. *Let $\beta \in [\alpha, 1]$. If $x = [\beta, \dots, \beta]^T$, then $x \in E_\lambda(A, \vee)$ for $\lambda \leqslant \beta$.*

Proof. Using (28) we have $\beta = \alpha \vee \beta \geqslant v_i^1 \vee \beta \geqslant 0 \vee \beta = \beta$. From this it follows that

$$\bigwedge_{j=1}^{n} (a_{ij} \vee \beta) = (\bigwedge_{j=1}^{n} a_{ij}) \vee \beta = v_i^1 \vee \beta = \beta.$$

Moreover $\lambda \vee x_i = \lambda \vee \beta = \beta$ for $\lambda \leqslant \beta$. Thus $x \in E_\lambda(A, \vee)$ for $\lambda \leqslant \beta$. □

Corollary 3. *Let v_λ^0 as in (28). $v_\lambda^0 \in E_\lambda(A, \vee)$ for $\lambda \leqslant \delta$.*

Theorem 2. *If $v^1 \in E_\lambda(A, \vee)$, then $v^1 = \min E_\lambda(A, \vee)$ for $\lambda \leqslant \delta$.*

Proof. Let $x \in E_\lambda(A, \vee)$, thus we have

$$\lambda \vee x_i = \bigwedge_{j \in I} (a_{ij} \vee x_j^1) \geqslant \bigwedge_{j \in I} a_{ij} = v_i = \delta \vee v_i.$$

It means that $v^1 = \min E_\lambda(A, \vee)$ for $\lambda \leqslant \delta$. □

Example 5. *Let $* = \vee$, $\lambda = 0$ and matrix A be from Example 4.Using (28) we obtain v^0 and v^1.*

$$v^1 = \begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix}, \quad v^0 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}.$$

Based on Theorem 2 we get that $v^1 = \min E_0(A, \vee)$ and using Corollary 3 we know that $v_0^0 \in E(A, \vee)$. But exists vectors $x \in [v^1, v^0]$, such that $x \notin E_0(A, \vee)$. Particularly,

$$x = \begin{bmatrix} 0.3 \\ 0.4 \\ 0.5 \end{bmatrix} \notin E_0(A, \wedge), \quad \textit{because } A \circ x = \begin{bmatrix} 0.4 \\ 0.4 \\ 0.5 \end{bmatrix}$$

$x \notin E_0(A, \vee)$.

Lemma 10. *Let $*$ be an increasing, $\inf -$distributive operation with $e = 0$, $\beta \in [0, 1]$. If*

$$\underset{i \in I}{\forall} v_i^1 = \delta, \tag{29}$$

then $x = [\beta, \beta, \dots, \beta]^T$ is the eigen fuzzy vector and $\lambda = \delta$. Moreover if $* = \vee$ and $\beta \geqslant \delta$, then constant vectors are eigen fuzzy vectors x for $\lambda \leqslant \beta$ $\lambda \in [0, \beta]$.

Proof. Based on (28) and using assumption (29) we get

$$\bigwedge_{j=1}^{n}(a_{ij} * \beta) = (\bigwedge_{j=1}^{n}(a_{ij})) * \beta = v_i^1 * \beta = \delta * \beta = \lambda * \beta \quad \text{for } i \in \{1, 2, \ldots, n\}.$$

Especially for $* = \vee$ and $\beta \geqslant \delta$ we obtain

$$\beta = \beta \vee \beta \geqslant \delta \vee \beta \geqslant 0 \wedge \beta = \beta,$$

and thus $\lambda \in [0, \beta]$. □

Example 6. *Let $* = S_L$ and a matrix A be from Example 4. Matrix A does not fulfill (29). Based on (10) we can observe that for $\beta < 1$ constant vectors x are not eigen fuzzy vectors. Only vector $\mathbf{1}$ is an eigen fuzzy vector for $\lambda \in [0, 1]$.*

Corollary 4. *Under assumptions of Lemma 10, vector v^0 is an eigen fuzzy vector and vector $\mathbf{0}$ is the least eigen fuzzy vector for $\lambda = \delta$.*

Lemma 11. *Let $* \in D$. If exists $l \in \{1, 2, \ldots, n\}$ for which*

$$\forall_{i \in \{1,2,\ldots,n\}} \; v_i^1 = a_{il} \quad \textit{and} \quad a_{ll} = 0, \tag{30}$$

then $v^1 = \min E_0(A, *)$.

Proof. Based on assumptions we get

$$\bigwedge_{t=1}^{n}(a_{it} * \bigwedge_{j=1}^{n} a_{tj}) = \bigwedge_{t=1}^{n}(a_{it} * a_{tl}) = a_{il} * a_{ll} = a_{il} * 0 = a_{il} = 0 * a_{il}.$$

Thus $v^1 \in E_0(A, *)$. Moreover if exists another $y \in E_0(A, *)$ we obtain

$$y_i = \bigwedge_{j=1}^{n}(a_{ij} * y_j) \geqslant \bigwedge_{j=1}^{n}(a_{ij} * 0) = \bigwedge_{j=1}^{n} a_{ij} = a_{il} = v_i^1 \quad \text{for } i \in \{1, 2, \ldots, n\}.$$

Hence we have $v^1 = \min E_0(A, *)$. □

Example 7. *Let $* = T_L \notin D$. A matrix A fulfills condition (30)*

$$A = \begin{bmatrix} 0.7 & 0.3 & 0.4 \\ 0.5 & 0 & 0.8 \\ 1 & 0.9 & 0.9 \end{bmatrix}, x = \begin{bmatrix} 0.3 \\ 0 \\ 0.9 \end{bmatrix}, A \circ x = \lambda * \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We can see that $x \notin E_\lambda(A, *)$ for $\lambda \in [0, 1]$.

Theorem 3. *Let* $* \in D$. *If* A *is an idempotent matrix, then* $\min E_0(A, *) = v^1$.

Proof. From assumption, that A is an idempotent matrix we have

$$a_{ij} = \bigwedge_{t=1}^{n} (a_{it} * a_{tj}).$$

Using inf-distributivity we obtain
$\bigwedge_{j=1}^{n} (a_{ij} * v_j^1) = \bigwedge_{j=1}^{n} (a_{ij} * \bigwedge_{t=1}^{n} a_{jt}) = \bigwedge_{j=1}^{n} \bigwedge_{t=1}^{n} (a_{ij} * a_{jt}) = \bigwedge_{j=1}^{n} \bigwedge_{t=1}^{n} (a_{it}) = \bigwedge_{t=1}^{n} (a_{it}) =$
$v_j^1 = 0 * v_j^1$.
for each $i \in \{1, 2, \ldots, n\}$. It means that $v^1 \in E(A, *)$. Moreover if exists another $y \in E_0(A, *)$ we obtain

$$y_i = \bigwedge_{j=1}^{n} (a_{ij} * y_j) \geqslant \bigwedge_{j=1}^{n} (a_{ij} * 0) = \bigwedge_{j=1}^{n} a_{ij} = a_{il} = v_i^1 \quad \text{for } i \in \{1, 2, \ldots, n\}.$$

Since we have $v^1 = \min E_0(A, *)$. □

In the article [12] Sanchez gave the algorithm for $\min E_1(A, \wedge)$. In the same way we get algorithm for calculating $\min E_0(A, \vee)$.

1. Calculate $x := v^1$.
2. While $A \circ x \neq x$, we place $x := A \circ x$.
3. If $A \circ x = x$, then we have $\min E(A, \vee) := x$.

Example 8. *Let* $* = \wedge$ *and*

$$A = \begin{bmatrix} 1 & 0.4 & 0.5 \\ 0.5 & 0.8 & 1 \\ 1 & 0.6 & 0.5 \end{bmatrix}, v^1 = \begin{bmatrix} 0.4 \\ 0.5 \\ 0.6 \end{bmatrix}, v^2 = A \circ v^1 = \begin{bmatrix} 0.4 \\ 0.4 \\ 0.4 \end{bmatrix}.$$

We can observe that $v^2 \in E(A, *)$, *but* $0 = \min E(A, *)$.

4 Properties of the set of matrices with the same eigen fuzzy vector

The set of all matrices for which v is an eigen fuzzy vector with an eigen fuzzy value λ is denoted by $EF_\lambda(v, *)$.

Theorem 4. *Let $v \in [0,1)^n$ and $* \in D$ be a commutative, infinitely* $\inf -$*distributive operation.If matrix $W = (w_{ij})$ has the following form*

$$w_{ij} = v_j \overset{*}{\leftarrow} v_i, \quad \textit{for } i,j \in \{1,2,\ldots,n\}, \tag{31}$$

then $W = \min EF_0(v,*)$.

Proof. According to the formula (17) and by using commutativity of operation $*$ we obtain

$$\bigwedge_{j=1}^{n}(w_{ij} * v_j) = \bigwedge_{j=1}^{n}((v_j \overset{*}{\leftarrow} v_i) * v_j) = \bigwedge_{j=1}^{n}(v_j * (v_j \overset{*}{\leftarrow} v_i)) \geqslant \bigwedge_{j=1}^{n} v_i = v_i.$$

From Corollary 1 we have

$$\bigwedge_{j=1}^{n}(w_{ij} * v_j) = \bigwedge_{j=1}^{n}((v_j \overset{*}{\leftarrow} v_i) * v_j) = \bigwedge_{j=1}^{n}(0 * v_j) \leqslant \bigwedge_{j=1}^{n} v_j \leqslant v_i = 0 * v_i.$$

It means that v is an eigen fuzzy vector of W with an eigen fuzzy value $\lambda = 0$.

Suppose that $A \in EF_0(v,*)$. Based on (18) we have

$$(a_{ij} * v_j \geqslant v_i) \Leftrightarrow (a_{ij} \geqslant v_j \overset{*}{\leftarrow} v_i) \quad \text{for } i,j \in \{1,2,\ldots,n\}.$$

It proves that $W \leqslant A$.

Finally, we obtain $W = \min EF(v,*)$. □

Theorem 5. *Let $v \in [0,1)^n$ and $*$ be an increasing operation.*
If $A \in EF_\lambda(v,*)$ and $B \in EF_\mu(v,*)$, then

$$A \wedge B \in EF_{\lambda\wedge\mu}(v,*). \tag{32}$$

Proof. For given $A \in E_\lambda(v,*)$ and $B \in E_\mu(v,*)$, based on the Lemma 1, we obtain

$$(A \wedge B) \circ v = (A \circ v) \wedge (B \circ v) = (\lambda * v) \wedge (\mu * v) = (\lambda \wedge \mu) * v.$$

It means that $A \wedge B \in EF_{\lambda\wedge\mu}(v,*)$. □

Corollary 5. *Let $v \in [0,1)^n$ and $*$ be an increasing operation.*
If $A, B \in EF_\lambda(v,*)$, then $A \wedge B \in EF_\lambda(v,*)$.

Theorem 6. *Let $v \in [0,1)^n$ and $*$ be a triangular a triangular conorm. If $A \in EF_\lambda(v,*)$ and $B \in EF_\mu(v,*)$, then*

$$A \circ B \in EF_{\lambda*\mu}(v,*). \tag{33}$$

Proof. According to (21), (22) and using commutativity of operation $*$ we obtain

$$(A \circ B) \circ x = A \circ (B \circ x) = A \circ (\mu * x) = (A \circ x) * \mu = \lambda * x * \mu = \lambda * \mu * x.$$

In consequence we get (33). □

Theorem 7. *Let $v \in [0,1)^n$ and $*$ be an increasing operation with $e = 0$. If $A \leqslant B$ and $A, B \in EF_\lambda(v,*)$, then $[A,B] \in EF_\lambda(v,*)$.*

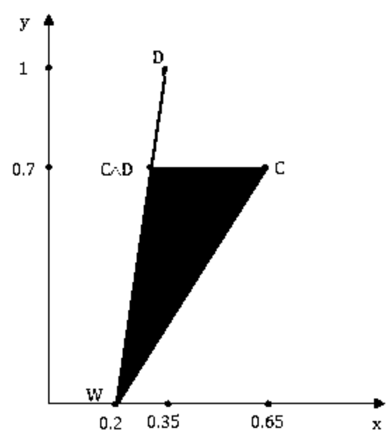
Proof. Let $A \leqslant C \leqslant B$. Using (23) we have

$$\lambda * x = A \circ x \leqslant C \circ x \leqslant B \circ x = \lambda * x.$$

It means that for arbitrary matrix $C \in [A,B]$, $C \in EF_\lambda(v,*)$. In consequence we obtain $[A,B] \in EF_\lambda(v,*)$. □

Corollary 6. *Let $v \in [0,1)^n$ and $*$ be an increasing operation with $e = 0$. If $A \leqslant B$ and $A, B \in EF_\lambda(v,*)$ then $C \vee D \in EF_\lambda(v,*)$.*

Example 9. *Let $\lambda \in [0, 0.3)$, $n = 2$, $* = \vee$ and*
$s = \begin{bmatrix} 0.3 \\ 0.4 \end{bmatrix}, W = \begin{bmatrix} 0 & 0 \\ 0.4 & 0 \end{bmatrix}, C = \begin{bmatrix} 0.3 & 1 \\ 1 & 0.4 \end{bmatrix}, D = \begin{bmatrix} 0.3 & 1 \\ 0.4 & 1 \end{bmatrix},$
We have $EF_\lambda(s,\vee) = [W,C] \cup [W,D]$.
$(C \wedge D) \circ s = \begin{bmatrix} 0.3 & 1 \\ 0.4 & 0.4 \end{bmatrix} \circ \begin{bmatrix} 0.3 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.4 \end{bmatrix},$
$(C \vee D) \circ s = \begin{bmatrix} 0.3 & 1 \\ 1 & 1 \end{bmatrix} \circ \begin{bmatrix} 0.3 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.3 \\ 1 \end{bmatrix} \neq s.$
Moreover, we can observe $C \wedge D \in [W,C]$ and $C \wedge D \in [W,D]$, but $C \vee D \notin EF_\lambda(v,\vee)$. We use $x = \frac{a_{11}+a_{11}}{2}$, $y = \frac{a_{21}+a_{22}}{2}$ to illustrate it.



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The papers presented in this Volume 1 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.
It may be viewed as a result of fruitful discussions held during the Eighth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2009) organized in Warsaw on October 16, 2009 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Centre for Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT – Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bistrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, and the University of Westminster, Harrow, UK:

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The Eighth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2009) has been meant to commence a new series of scientific events primarily focused on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Moreover, other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems are discussed.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

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