Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics. Volume I: Foundations

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Systems Research Institute
Polish Academy of Sciences
Newelska 6, 01-447 Warsaw, Poland
www.ibspan.waw.pl

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On fuzzy implications with ordered fuzzy numbers

W. Kosiński and D. Wilczyńska-Sztyma
Department of Computer Science
Polish-Japanese Institute of Information Technology
ul. Koszykowa 86, 02-008 Warsaw, Poland
and
Kazimierz Wielki University in Bydgoszcz
Faculty of Mathematics, Physics and Technology
ul. Chodkiewicza 30, 85-064 Bydgoszcz

wkos@pjwstk(ukw).edu.pl, dorotaws@ukw.edu.pl

Abstract

Ordered fuzzy numbers (OFN) invented by the first author and his two coworkers in 2002 make possible to utilize the fuzzy arithmetic and to construct the Abelian group of fuzzy numbers and then an ordered ring. This new model of fuzzy numbers overcomes known drawbacks of classical convex fuzzy numbers and at the same time has the algebra of crisp (non-fuzzy) numbers inside. The definition of OFN uses the extension of the parametric representation of convex fuzzy numbers. Operation of addition and multiplication by a positive scalar defined on OFNs give the same result as addition and scalar multiplication of their corresponding convex fuzzy numbers, if the numbers have the same orientation. However, subtraction, multiplication and division give quit different results for interval arithmetic and presented new algebra of OFNs. Fuzzy implications are discussed. With the help of a perceptron that realizes the classical binary implication a new fuzzy implication with ordered fuzzy numbers is proposed.

Keywords: convex fuzzy numbers, ordered fuzzy numbers, perceptron, fuzzy implications, defuzzification functionals.

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1 Introduction

A fuzzy implication (FI), commonly defined as a two-place operation on the unit interval, is an extension of the classical binary implication. It plays important roles in both mathematical and applied sides of fuzzy set theory. The importance of fuzzy implications in applications of fuzzy logic (FL) to approximate reasoning (AR), decision support systems (DSS), fuzzy control (FC), etc., is hard to exaggerate. Many different fuzzy implication operators have been proposed; most of them fit into one of the two classes: implication operations that are based on an explicit representation of implication $A \Longrightarrow B$ in terms of alternative, conjunction and negation and R-implications that are based on an implicit representation of implication $A \Longrightarrow B$ as the weakest C for which $C \wedge B$ implies A. However, some fuzzy implication operations cannot be naturally represented in this form [1].

For example, to the first class belong the Kleene–Dienes operation, called a binary implication, which is a fuzzy counterpart of the RHS of the binary logic tautology $a \Longrightarrow b \equiv b \vee \neg a$. To have it one has to invent the negation operator $\neg$ to the membership function μ_A of a fuzzy set A and to define a membership function of $\neg A$ as $\mu_{\neg A} := 1 - \mu_A$, and the alternative of two fuzzy sets $C = A \vee B$, and its membership function μ_C as $\mu_C := \max\{\mu_A, \mu_B\}$. Then the Kleene–Dienes implication $A \Longrightarrow B$ will be $\max\{1 - \mu_A, \mu_B\}$. The simple generalization of the last implication is the so-called S-implication $I_s(A, B)$ defined by the formula

$$I_s(A, B) = S(1 - \mu_A, \mu_B)\,, \tag{1}$$

where S is any S-norm [3]. This generalization is obvious in view of the fact that any S-norm is a generalization of the sum (alternative) of two fuzzy sets. Implication invented by Łukasiewicz [13] which takes the form $\min\{1, 1 - \mu_A + \mu_B\}$ together with the Reichenbach and Fodor implications serve as examples of $I_s(A, B)$ implication [1, 18].

The general class of the so-called Q-implication $I_q(A, B)$ can be derived from the formula

$$I_q(A, B) = S(1 - \mu_A, T(\mu_A, \mu_B))\,, \tag{2}$$

where S and T are general S-norm and T-norm, respectively. The example of Q–implication is the Zadeh implication [19] defined by

$$\max\{\min\{\mu_A, \mu_B\}, 1 - \mu_A\}\,.$$

The R–implication $I_r(A, B)$ is defined with the help of the formula

$$I_r(A, B) = \sup\{z \in [0, 1] | T(\mu_A, z) \leq \mu_B\}\,. \tag{3}$$

Examples of $I_r(A, B)$ are the Gougen and Gödel implications [1].

It is no difficult to check that all above listed implications are consistent with the classical binary logic implication (cf. the second table in Fig. 4). There are, however, the so-called 'engineering' implications $M(A, B)$ invented by Mamdani, in which any T-norm appears, i.e.

$$M(A, B) = T(\mu_A, \mu_B)\,. \tag{4}$$

In the classical case [14] Mamdani put $T = \min$, then Larsen [12] proposed $T = \text{prod}$, where prod represents the algebraic product; of course non them is consistent with the classical binary implication.

What is unpleasant with all those implications: they do not lead to convex fuzzy numbers, they have, in general, unbounded supports [16].

2 Ordered fuzzy numbers

Proposed recently by the first author and his two coworkers: P. Prokopowicz and D. Ślęzak [7, 8, 9, 10, 11] an extended model of convex fuzzy numbers [15] (CFN), called ordered fuzzy numbers (OFN), does not require any existence of membership functions. In this model an ordered fuzzy number is a pair of continuous functions defined on the interval $[0, 1]$ with values in **R**. To see OFN as an extension of CFN - model, take a look on a parametric representation know since 1986, [4]. of convex fuzzy numbers.

Then four algebraic operations have been proposed between fuzzy numbers and crisp (real) numbers, in which componentwise operations are present. In particular if $A = (f_A, g_A), B = (f_B, g_B)$ and $C = (f_C, g_C)$ are mathematical objects called ordered fuzzy numbers, then the sum $C = A + B$, product $C = A \cdot B$, division $C = A \div B$ and scalar multiplication by real $r \in \mathbf{R}$, are defined in natural way:

$$r \cdot A = (rf_A, rg_A),$$

and

$$f_C(y) = f_A(y) \star f_B(y) \qquad g_A(y) \star g_B(y)\,, \tag{5}$$

where "$\star$" works for "$+$", "$\cdot$", and "$\div$", respectively, and where $A \div B$ is defined, if the functions $|f_B|$ and $|g_B|$ are bigger than zero. Notice that the subtraction of B is the same as the addition of the opposite of B, i.e. the number $(-1) \cdot B$, and consequently $B - B = 0$. From this follows that any fuzzy algebraic equation $A + X = C$ with given A and C as OFN possesses a solution, that is OFN, as well. Moreover, to any convex and continuous[1] fuzzy number correspond two

[1] However, the recent extension presented in [6] includes all convex fuzzy numbers.

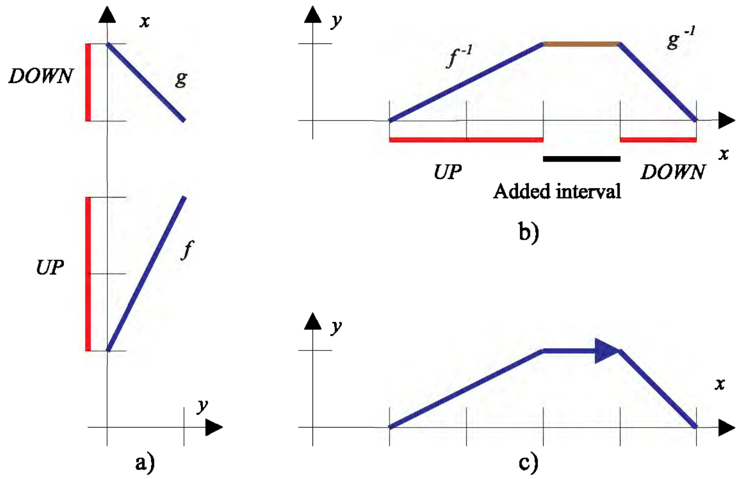

Figure 1: a) Example of an ordered fuzzy number; b) construction of the membership function; c) the arrow denotes the orientation and the order of inverted functions: first f and then g.

OFNs, they differ by the orientation: one has positive, say (f, g), another (g, f) has negative.

A relation of partial ordering in the space of all OFN, denoted by $\mathcal{R}$, can be introduced by defining the subset of 'positive' ordered fuzzy numbers: a number $A = (f, g)$ is not less than zero, and by writing

$$A \geq 0 \quad \text{iff} \quad f \geq 0, \, g \geq 0 \, . \tag{6}$$

In this way the set $\mathcal{R}$ becomes a partially ordered ring. Notice that in the definition of OFN it is not required that two continuous functions f and g are (partial) inverses of some membership function. Moreover, it may happen that the membership function corresponding to A does not exist; such numbers are called improper.

In any case for $A = (f, g)$ we call f - the up-part and g - the down-part of the fuzzy number A. To be in agreement with further and classical denotations of fuzzy sets (numbers), the independent variable of the both functions f and g is denoted by y (or some times by s), and their values by x. The continuity of both parts implies their images are bounded intervals, say UP and $DOWN$, respectively (Fig. 2), where boundaries for $UP = [l_A, 1_A^-]$, for $DOWN = [1_A^+, p_A]$ and for the $CONST = [1_A^-, 1_A^+]$ are used on Fig. 1.

In this new framework new fuzzy implications have been invented. One of the most promising implications is that suggested by P. Prokopowicz in his Ph.D.

thesis and called implication with multiplication, in which the algebraic structure of operations on OFN has been used. Before its formula will be given we have to repeat after its author [17] the '*corresponding*' membership function $\tilde{\mu}_A$, which can be defined for any proper[2] and improper ordered fuzzy number $A = (f, g)$ by the formulae:

$$\tilde{\mu}_A(x) = \max \arg\{f(s) = x, g(y) = x\} \text{ , if } x \in \text{Range}(f) \cup \text{Range}(g) \text{ ,}$$
$$\tilde{\mu}_A(x) = 1 \text{ , if } x \in [f(1), g(1)] \cup [g(1), f(1)] \text{ ,} \quad (7)$$
$$\text{and } \tilde{\mu}_A(x) = 0 \text{ , otherwise , where } s \in [0, 1] \text{ ,}$$

where one of the intervals $[f(1), g(1)]$ or $[g(1), f(1)]$ may be empty, depending on the sign of $f(1) - g(1)$, (i.e., if the sign is -1 then the second interval is empty). Then the implication with multiplication $P(A, B)$ is given by

$$P(A, B) = \tilde{\mu}_A B \,. \quad (8)$$

The result of this implication is an ordered fuzzy number and is close to the engineering implication of Mamdani type, rather.

3 New fuzzy implication

Let us recall the perceptron, the first model of a natural neuron. It is characterized by number n of possible inputs, a vector of connection weights $\boldsymbol{w} = [w_1, w_2, ..., w_n]$ and a bias (threshold) value θ corresponding to a nonlinearity of relationship between inputs and outputs. This threshold value characterizes the activation function of the perceptron which is a 1D jump function. To calculate the output y of the perceptron for the given input states $x_1, x_2, ..., x_n$ we calculate the weighted sum $z := w_1x_1 + w_2x_2 + ...w_nx_n$. If z is greater or equal to the threshold value θ, then the output is set as $y = 1$, otherwise $y = 0$. Perceptron's scheme of working is shown on Fig. 2. The perceptron's mathematical characteristics can be expressed as the equation of the straight line (generally - hyperplane in n-dimensional space) (Fig. 3) which is a graph of a decision function (cf. Fig. 3). Points above or on the line are classified as 1, points below as 0. The above-mentioned equation stems straightforwardly from the perceptron's formula: values of weights form components of the vector $\boldsymbol{w}$ perpendicular to the line (generally – to the hyperplane), and the threshold value gives the lines location. Here this vector is $\boldsymbol{w} = [-w, 1]$. Let us see how we can simulate simple

[2] If f is strictly increasing and g decreasing, and $f(s) \leq g(s)$, for $s \in [0, 1]$, then the classical membership function for A exists, and A is called proper; it represents a CFN.

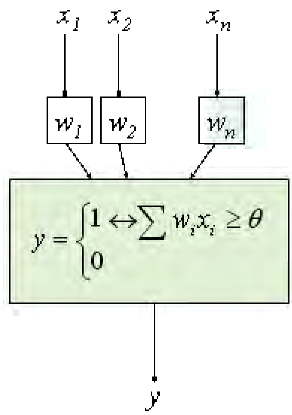

Figure 2: Scheme of action of perceptron

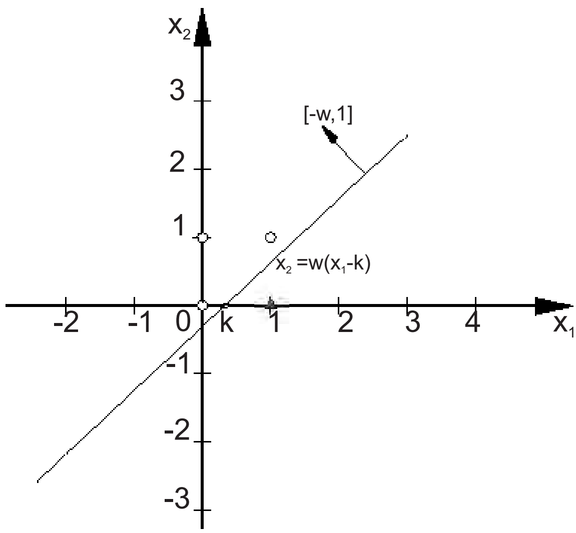

Figure 3: Decision function for implication

logical gates: the conjunction ”x_1 and x_2” and the implication ”$x_1 \implies x_2$”: the weight values are visible by corresponding inputs, while the thresholds are denoted as usual by θ. We can see in Fig. 4 that in the second case we left the weight w and the constant k unspecified, however, in order to satisfy the true values of the implication listed in the second table on the left, the both constants have to satisfy the constraint

$$0 \leq k < 1 \text{ and } w \in (0, \frac{1}{1-k})\,. \tag{9}$$

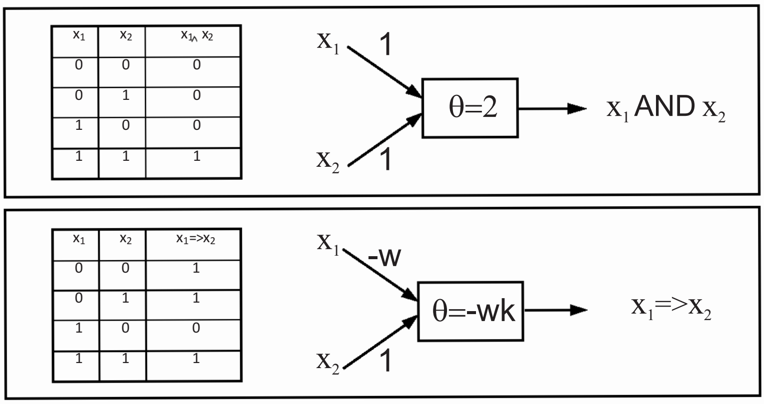

Figure 4: Two perceptrons simulate two logical gates

Concluding this part we can see that the action of the perceptron simulating the logical gate of implication "$x_1 \Longrightarrow x_2$" is based on the following inequality:

$$-wx_1 + x_2 \geq -wk\,, \tag{10}$$

which classifies points (x_1, x_2) from the plane $\mathbf{R}^2$ into two classes: those above the line $-wx_1 + x_2 = -wk$ to the class 1, and those below it - to the class 0.

Referring to OFN we remember that linear and nonlinear functionals on $\mathcal{R}$ give defuzzification functionals which are of a big importance in fuzzy reasoning. In our opinion not every functional can play a role of a defuzzification functional. Linear and continuous functional can be represented by a sum of two integrals with respect to a pair of Radon measures[3] on all Boreal subsets of the interval [0, 1], if we restrict the representation of OFN to pairs of continuous functions continuous.

In our understanding a most general class of defuzzification functionals should satisfied three conditions:

1. $\phi(c) = c\,,$
2. $\phi(A + c) = \phi(A) + c\,,$
3. $\phi(cA) = c\phi(A)\,,$ for any $c \in \mathbf{R}$ and $A \in \mathcal{R}\,.$

Here ϕ a general functional defined on $\mathcal{R}$ - the space of OFN, and by writing $\phi(c)$ we understand the action of ϕ on the crisp number c from $\mathbf{R}$, which regarded as

[3]Equivalently they are represented by a pair of functions of bounded variations, cf. [5].

OFN is represented by a pair of constant functions $(c^\dagger, c^\dagger)$, with $c^\dagger(s) = c\,, s \in [0,1]$. Notice that the condition 2. is a restricted additivity, since the second component in the argument of ϕ is a crisp number. However, if we require that $\phi(A+B) = \phi(A) + \phi(B)$, for any $B \in \mathcal{R}$, then we get the standard additivity of ϕ and, the full linearity, in view of 3.

Now we can pass to a new definition of fuzzy implication defined on order fuzzy numbers. First we notice that in view of (9) the following inequality

$$-w1 + 1 \geq -wk \tag{11}$$

holds. We want to pass to fuzzy 'points' from $\mathcal{R} \times \mathcal{R}$ and classify them in the similar way as it was done by the inequality (10) for crisp points (x_1, x_2). We should, however, translate those points to the neigbourhood of the origin (or better to say - of the point $(1,1)$) in such a way that after defuzzification the tautology (11) follows.

Definition. *Let A and $B \neq 0$ be two ordered fuzzy numbers. We say that A implies B, and write $A \Longrightarrow_{w,k} B$ if there exit two positive constants w and k satisfying (11), (or equivalently (9)), such that the following inequality*

$$-w(A - \phi(A) + 1) + (B - \phi(B) + 1) \geq -wk \tag{12}$$

regarded, in view of (6), as the inequality between elements of partially ordered ring $\mathcal{R}$ holds, for an arbitrary defuzzification functional ϕ.

Notice that if we apply the defuzzification functional ϕ to the LHS of (12) we get $\phi(-w(A - \phi(A) + 1) + (B - \phi(B) + 1)) = -w1 + 1$, in view of the above condition 2. This leads to the previous inequality (11). In this way we show that our definition is proper. We should add that if such pair of constants (w, k) does not exist then we say that the implication $A \Longrightarrow B$ is false.

In the next paper consequences of (12) will discussed and some examples will be given.

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The papers presented in this Volume 1 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.
It may be viewed as a result of fruitful discussions held during the Eighth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2009) organized in Warsaw on October 16, 2009 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Centre for Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT – Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bistrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, and the University of Westminster, Harrow, UK:

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The Eighth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2009) has been meant to commence a new series of scientific events primarily focused on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Moreover, other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems are discussed.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

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