Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics. Volume I: Foundations

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Hudetz entropy on IF-events

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Abstract

In this paper we study dynamical systems based on IF-events. We define a special type of the notion of the entropy on this systems and its Hudetz modification.

Keywords: IF-events, IF-dynamical system, IF-partition, Hudetz entropy.

1 Introduction

We start with classical dynamical systems $(\Omega, \mathcal{S}, P, T)$, where $(\Omega, \mathcal{S}, P)$ is a probability space and $T : \Omega \to \Omega$ is a measure preserving map, i.e. $T^{-1}(A) \in \mathcal{S}$ and $P(T^{-1}(A)) = P(A)$ for any $A \in \mathcal{S}$. The entropy of the dynamical system is defined as follows (see [8], [9]). Consider measurable partition $\mathcal{A} = \{A_1, ..., A_k\}$, where $A_i \in \mathcal{S}; i = 1, ..., k, A_i \cap A_j = \emptyset; i \neq j, \bigcup_{i=1}^{k} A_i = \Omega$. Its entropy is the number

$$H(\mathcal{A}) = \sum_{i=1}^{k} \varphi(P(A_i)),$$

where $\varphi(x) = -x \log x$, if $x > 0$, and $\varphi(0) = 0$. If $\mathcal{A} = \{A_1, ..., A_k\}$ and $\mathcal{B} = \{B_1, ..., B_l\}$ are two measurable partitions, then $T^{-1}(\mathcal{A}) = \{T^{-1}(A_1), ..., T^{-1}(A_k)\}$ and $\mathcal{A} \vee \mathcal{B} = \{A \cap B; A \in \mathcal{A}, B \in \mathcal{B}\}$ are measurable partitions, too. It can be proved that there exists

$$h(\mathcal{A}, T) = \lim_{n \to \infty} H\left(\bigvee_{i=0}^{n-1} T^{-i}(\mathcal{A})\right).$$

The entropy $h(T)$ of $(\Omega, \mathcal{S}, P, T)$ is defined as the supremum

$$h(T) = \sup\{h(\mathcal{A}, T); \mathcal{A} \text{ is a measurable partition}\}.$$

The aim of the Kolmogorov-Sinaj entropy was to distinguish non-isomorphic dynamical systems. Two dynamical systems with different entropies cannot be isomorphic.

The notion of the entropy has been extended using fuzzy partitions instead of set partitions (see [8], [9]). Let $\mathcal{T}$ be a tribe of fuzzy sets on Ω. Fuzzy partition is a set of functions $\mathcal{A} = \{f_1, ..., f_k\} \subset \mathcal{T}$ such that $\sum_{i=1}^{k} f_i = 1$. Then we define its entropy

$$H(\mathcal{A}) = \sum_{i=1}^{k} \varphi(m(f_i)). \tag{1}$$

Further

$$h(\mathcal{A}, \tau) = \lim_{n \to \infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} \tau^{i}(\mathcal{A})\right),$$

and, if $G \subset \mathcal{T}$ is an arbitrary non-empty set, then

$$h_G(\tau) = \sup\{h(\mathcal{A}, \tau); \mathcal{A} \text{ is a fuzzy partition}, \mathcal{A} \subset G\}.$$

We want to extend the notion of the entropy to dynamical systems based on IF-events. An IF-event is a pair $A = (\mu_A, \nu_A)$ of $\mathcal{S}$-measurable function $\mu_A, \nu_A : \Omega \to [0, 1]$ such that

$$\mu_A + \nu_A \leq 1.$$

If $f_A : \Omega \to [0, 1]$ is a fuzzy set, then the pair $(f_A, 1 - f_A)$ is an IF-event, of course IF-events present a larger family. Denote by $\mathcal{F}$ the family of all IF-events. On $\mathcal{F}$ we define partial binary operation $+$ and binary operation $\cdot$ by the formulas

A+B=$(\mu_A, \nu_A) + (\mu_B, \nu_B) = (\mu_A + \mu_B, \nu_A + \nu_B - 1)$, whenever $\mu_A + \mu_B \leq 1$ and $0 \leq \nu_A + \nu_B - 1 \leq 1$, and

$$A \cdot B = (\mu_A, \nu_A) \cdot (\mu_B, \nu_B) = (\mu_A \mu_B, \nu_A + \nu_B - \nu_A \nu_B).$$

Further

$$A_n \nearrow A \Longleftrightarrow \mu_{A_n} \nearrow \mu_A, \nu_{A_n} \searrow \nu_A,$$

where $A = (\mu_A, \nu_A), A_n = (\mu_{A_n}, \nu_{A_n}) \in \mathcal{F}$ $(n = 1, 2, ...)$ and

$$A \leq B \Longleftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B,$$

where $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \mathcal{F}$.

2 IF-dynamical system

Definition 1 *By a state on the family of all IF-events $\mathcal{F}$ we mean a mapping $m : \mathcal{F} \to [0, 1]$ satisfying the following conditions:*

(i) $m((1, 0)) = 1$*;*

(ii) If $A, B, C \in \mathcal{F}$ and $A + B = C$, then $m(A) + m(B) = m(C)$;

(iii) If $A_n \in \mathcal{F}(n = 1, 2, ...), A_n \nearrow A$, then $m(A_n) \nearrow m(A)$.

Definition 2 *Let $m : \mathcal{F} \to [0, 1]$ be a state on the family of all IF-events $\mathcal{F}$ and $\tau : \mathcal{F} \to \mathcal{F}$ be a mapping satisfying the following conditions:*

(I) If $A \in \mathcal{F}$, then $\tau(A) \in \mathcal{F}$ and $m(A) = m(\tau(A))$.

(II) If $A, B \in \mathcal{F}$ and there exists $A + B$, then $\tau(A + B) = \tau(A) + \tau(B)$.

Then a triplet $(\mathcal{F}, m, \tau)$ is an IF-dynamical system.

To any state on $\mathcal{F}$ there exists $\alpha \in [0, 1]$ such that

$$m_\alpha(A) = m(A) = m((\mu_A, \nu_A)) = (1 - \alpha) \int_\Omega \mu_A dP + \alpha \int_\Omega (1 - \nu_A) dP. \quad (2)$$

See [5]. Following this result it is reasonable to consider the family $\mathcal{F}$ and a mapping (state) $m_\alpha : \mathcal{F} \to [0, 1]$ defined by (2). Finally, let a mapping $\tau : \mathcal{F} \to \mathcal{F}$ be defined by $\tau(A) = \tau((\mu_A, \nu_A)) = (\mu_A \circ T, \nu_A \circ T) = A \circ T$. Since

$$m_\alpha(\tau((\mu_A, \nu_A))) = (1 - \alpha) \int_\Omega \mu_A \circ T dP + \alpha \int_\Omega (1 - \nu_A \circ T) dP =$$

$$= (1 - \alpha) \int_\Omega \mu_A dP \circ T^{-1} + \alpha(1 - \int_\Omega \nu_A dP \circ T^{-1}) =$$

$$= (1-\alpha)\int_\Omega \mu_A dP + \alpha\int_\Omega (1-\nu_A)dP = m_\alpha((\mu_A,\nu_A))$$

and

$$\tau((\mu_A,\nu_A)+(\mu_B,\nu_B)) = \tau((\mu_A+\mu_B,\nu_A+\nu_B-1)) =$$

$$= ((\mu_A+\mu_B)\circ T,(\nu_A+\nu_B-1)\circ T) = (\mu_A\circ T+\mu_B\circ T,\nu_A\circ T+\nu_B\circ T-1\circ T) =$$

$$= (\mu_A\circ T,\nu_A\circ T)+(\mu_B\circ T,\nu_B\circ T) = \tau((\mu_A,\nu_A))+\tau((\mu_B,\nu_B)),$$

then $(\mathcal{F},m_\alpha,\tau)$ is an IF-dynamical system.

3 Entropy of IF-partition

We want to define the entropy of the dynamical system $(\mathcal{F},m_\alpha,\tau)$. The crucial point in the definition is the notion of an IF-partition. We shall consider a family of all couples of fuzzy sets

$$\mathcal{M} = \{(f,g):\Omega\to[0,1]; f,g \text{ are } \mathcal{S}\text{-measurable}\}.$$

On $\mathcal{M}$ we define partial binary operation $\oplus$ and binary operation $\odot$ by the formulas

$(f,g)\oplus(h,k) = (f+h,g+k-1)$, whenever $f+h\le 1$ and $0\le g+k-1\le 1$,

and

$$(f,g)\odot(h,k) = (fh,g+k-gk).$$

See [6]. Of course, if $(f,g),(h,k)\in\mathcal{F}$ are IF-events, i.e. $f+g\le 1, h+k\le 1$, then

$$(f,g)\oplus(h,k) = (f,g)+(h,k) \quad\text{and}\quad (f,g)\odot(h,k) = (f,g)\cdot(h,k).$$

Recall that

$$\bigoplus_{i=1}^{k}(\mu_{A_i},\nu_{A_i}) = \left(\sum_{i=1}^{k}\mu_{A_i},\left(\sum_{i=1}^{k}\nu_{A_i}\right)-(n-1)\right)$$

and operations $\oplus,\odot$ fulfill the commutative, associative and distributive law.

Definition 3 *An IF-partition is any set $\mathcal{A} = \{(\mu_{A_1},\nu_{A_1}),...,(\mu_{A_k},\nu_{A_k})\}\subset\mathcal{M}$ such that*

$$(\mu_{A_1},\nu_{A_1})\oplus(\mu_{A_2},\nu_{A_2})\oplus...\oplus(\mu_{A_k},\nu_{A_k}) = (1,0).$$

If $\mathcal{A} = \{(\mu_{A_1},\nu_{A_1}),...,(\mu_{A_k},\nu_{A_k})\}$ and $\mathcal{B} = \{(\mu_{B_1},\nu_{B_1}),...,(\mu_{B_l},\nu_{B_l})\}$ are two IF-partitions, then we define

$$\mathcal{A}\vee\mathcal{B} = \{(\mu_{A_i},\nu_{A_i})\odot(\mu_{B_j},\nu_{B_j}); i=1,...,k, j=1,...,l\}.$$

Proposition 1 *If* $\mathcal{A} = \{(\mu_{A_1}, \nu_{A_1}), ..., (\mu_{A_k}, \nu_{A_k})\}$ *and* $\mathcal{B} = \{(\mu_{B_1}, \nu_{B_1}), ...$ $..., (\mu_{B_l}, \nu_{B_l})\}$ *are two IF-partitions, then* $\tau(\mathcal{A}) = \{\tau((\mu_{A_1}, \nu_{A_1})), ...$ $..., \tau((\mu_{A_k}, \nu_{A_k}))\}$ *and* $\mathcal{A} \vee \mathcal{B}$ *are IF-partitions, too.*

Proof.

$$\tau(\mathcal{A}) = \{\tau((\mu_{A_1}, \nu_{A_1})), ..., \tau((\mu_{A_k}, \nu_{A_k}))\} =$$
$$= \{(\mu_{A_1} \circ T, \nu_{A_1} \circ T), ..., (\mu_{A_k} \circ T, \nu_{A_k} \circ T)\}$$

Then

$$\bigoplus_{i=1}^{k} \tau((\mu_{A_i}, \nu_{A_i})) = \bigoplus_{i=1}^{k} (\mu_{A_i} \circ T, \nu_{A_i} \circ T) =$$
$$= \left(\left(\sum_{i=1}^{k} \mu_{A_i}\right) \circ T, \left(\sum_{i=1}^{k} \nu_{A_i} - (k-1)\right) \circ T\right) = (1 \circ T, 0 \circ T) = (1, 0).$$

Further, $\mathcal{A} \vee \mathcal{B} = \{(\mu_{A_i}, \nu_{A_i}) \odot (\mu_{B_j}, \nu_{B_j}); i = 1, ..., k, j = 1, ..., l\}$. Therefore

$$\bigoplus_{i=1}^{k} \bigoplus_{j=1}^{l} (\mu_{A_i}, \nu_{A_i}) \odot (\mu_{B_j}, \nu_{B_j}) = \bigoplus_{i=1}^{k} \bigoplus_{j=1}^{l} (\mu_{A_i}\mu_{B_j}, \nu_{A_i} + \nu_{B_j} - \nu_{A_i}\nu_{B_j}) =$$

$$= \left(\sum_{i=1}^{k}\sum_{j=1}^{l} \mu_{A_i}\mu_{B_j}, \sum_{i=1}^{k}\sum_{j=1}^{l} \nu_{A_i} + \sum_{i=1}^{k}\sum_{j=1}^{l} \nu_{B_j} - \sum_{i=1}^{k}\sum_{j=1}^{l} \nu_{A_i}\nu_{B_j} - (kl-1)\right) =$$

$$= \left(\left(\sum_{i=1}^{k} \mu_{A_i}\right)\left(\sum_{j=1}^{l} \mu_{B_j}\right), \sum_{j=1}^{l}\left(\sum_{i=1}^{k} \nu_{A_i}\right) + \sum_{i=1}^{k}\left(\sum_{j=1}^{l} \nu_{B_j}\right) - \right.$$
$$\left. - \left(\sum_{i=1}^{k} \nu_{A_i}\right)\left(\sum_{j=1}^{l} \nu_{B_j}\right) - (kl-1)\right) =$$
$$= (1.1, k(l-1) + l(k-1) - (l-1)(k-1) - (kl-1)) = (1, 0).$$

Proposition 2 *If* $\mathcal{A} = \{(\mu_{A_1}, \nu_{A_1}), ..., (\mu_{A_k}, \nu_{A_k})\}$ *is an IF-partition, then* $\mathcal{A}^\flat = \{\mu_{A_1}, ..., \mu_{A_k}\}$ *and* $\mathcal{A}^\sharp = \{1 - \nu_{A_1}, ..., 1 - \nu_{A_k}\}$ *are fuzzy partitions.*

Proof. Let $\mathcal{A} = \{(\mu_{A_1}, \nu_{A_1}), ..., (\mu_{A_k}, \nu_{A_k})\}$ be an IF-partition. Since

$$\bigoplus_{i=1}^{k} (\mu_{A_i}, \nu_{A_i}) = \left(\sum_{i=1}^{k} \mu_{A_i}, \sum_{i=1}^{k} \nu_{A_i} - (k-1)\right) = (1, 0),$$

we have

$$\sum_{i=1}^{k} \mu_{A_i} = 1$$

and

$$\sum_{i=1}^{k}(1-\nu_{A_i}) = k - \sum_{i=1}^{k}\nu_{A_i} = k-(k-1) = 1.$$

Indeed, $\mathcal{A}^\flat = \{\mu_{A_1}, ..., \mu_{A_k}\}$ and $\mathcal{A}^\sharp = \{1-\nu_{A_1}, ..., 1-\nu_{A_k}\}$ are fuzzy partitions.

Proposition 3 *If $\mathcal{A} = \{(\mu_{A_1}, \nu_{A_1}), ..., (\mu_{A_k}, \nu_{A_k})\}$ and $\mathcal{B} = \{(\mu_{B_1}, \nu_{B_1}), ..., (\mu_{B_l}, \nu_{B_l})\}$ are two IF-partitions, then*

$$(\mathcal{A} \vee \mathcal{B})^\flat = \mathcal{A}^\flat \vee \mathcal{B}^\flat$$

and

$$(\mathcal{A} \vee \mathcal{B})^\sharp = \mathcal{A}^\sharp \vee \mathcal{B}^\sharp.$$

Proof. Since $\mathcal{A} = \{(\mu_{A_1}, \nu_{A_1}), ..., (\mu_{A_k}, \nu_{A_k})\}$ and $\mathcal{B} = \{(\mu_{B_1}, \nu_{B_1}), ..., (\mu_{B_l}, \nu_{B_l})\}$, then we have

$$\mathcal{A} \vee \mathcal{B} = \{(\mu_{A_i}, \nu_{A_i}) \odot (\mu_{B_j}, \nu_{B_j}); i = 1, ..., k, j = 1, ..., l\} =$$
$$= \{(\mu_{A_i}\mu_{B_j}, \nu_{A_i} + \nu_{B_j} - \nu_{A_i}\nu_{B_j}); i = 1, ..., k, j = 1, ..., l\}.$$

By [9]

$$(\mathcal{A} \vee \mathcal{B})^\flat = \{\mu_{A_i}\mu_{B_j}; i = 1, ..., k, j = 1, ..., l\} =$$
$$= \{\mu_{A_i}; i = 1, ..., k\} \vee \{\mu_{B_j}; j = 1, ..., l\} = \mathcal{A}^\flat \vee \mathcal{B}^\flat$$

and

$$(\mathcal{A} \vee \mathcal{B})^\sharp = \{1 - \nu_{A_i} - \nu_{B_j} + \nu_{A_i}\nu_{B_j}; i = 1, ..., k, j = 1, ..., l\} =$$
$$= \{(1-\nu_{A_i})(1-\nu_{B_j}); i = 1, ..., k, j = 1, ..., l\} =$$
$$= \{1-\nu_{A_i}; i = 1, ..., k\} \vee \{1-\nu_{B_j}; j = 1, ..., l\} = \mathcal{A}^\sharp \vee \mathcal{B}^\sharp.$$

Theorem 1 *Let $m : \mathcal{F} \to [0,1]$ be a state, $\overline{m} : \mathcal{M} \to [0,1]$ defined by the equality $\overline{m}((\mu_A, \nu_A)) = m((\mu_A, 0)) + m((0, \nu_A)) - m((0,0))$. Then $\overline{m}$ is state and $\overline{m}|\mathcal{F} = m$. If $s : \mathcal{M} \to [0,1]$ is another extension of m, then $s = \overline{m}$.*

Proof. See [3].

Definition 4 *If $\mathcal{A} = \{(\mu_{A_1}, \nu_{A_1}), ..., (\mu_{A_k}, \nu_{A_k})\}$ is an IF-partition, then we define its entropy (with respect to a given state m_α)*

$$H_\alpha(\mathcal{A}) = (1-\alpha)H(\mathcal{A}^\flat) + \alpha H(\mathcal{A}^\sharp),$$

where H is the entropy of fuzzy partition (see equation (1)).

4 Entropy on IF-dynamical system

Proposition 4 *For any IF-partition $\mathcal{A}$ there exists*

$$\lim_{n\to\infty}\frac{1}{n}H_\alpha\left(\bigvee_{i=0}^{n-1}\tau^i(\mathcal{A})\right).$$

Proof. Let $\mathcal{A}=\{(\mu_{A_1},\nu_{A_1}),...,(\mu_{A_k},\nu_{A_k})\}$ be an IF-partition. By [9], there exist limits

$$\lim_{n\to\infty}\tfrac{1}{n}H\left(\bigvee_{i=0}^{n-1}\tau^i(\mathcal{A}^\flat)\right) \quad \text{and} \quad \lim_{n\to\infty}\tfrac{1}{n}H\left(\bigvee_{i=0}^{n-1}\tau^i(\mathcal{A}^\sharp)\right).$$

Hence there exists

$$\lim_{n\to\infty}\frac{1}{n}H_\alpha\left(\bigvee_{i=0}^{n-1}\tau^i(\mathcal{A})\right)=$$

$$=(1-\alpha)\lim_{n\to\infty}\frac{1}{n}H\left(\bigvee_{i=0}^{n-1}\tau^i(\mathcal{A}^\flat)\right)+\alpha\lim_{n\to\infty}\frac{1}{n}H\left(\bigvee_{i=0}^{n-1}\tau^i(\mathcal{A}^\sharp)\right).$$

Definition 5 *For every IF-partition $\mathcal{A}$ we define*

$$h_\alpha(\mathcal{A},\tau)=\lim_{n\to\infty}\frac{1}{n}H_\alpha\left(\bigvee_{i=0}^{n-1}\tau^i(\mathcal{A})\right)$$

and, if $G\subset\mathcal{M}$ is an arbitrary set, then the entropy of IF-dynamical system $(\mathcal{F},m_\alpha,\tau)$ is

$${}_Gh_\alpha(\tau)=\sup\{h_\alpha(\mathcal{A},\tau);\mathcal{A}\text{ is an IF-partition, }\mathcal{A}\subset G\}.$$

Since

$$h(\mathcal{A}^\flat,\tau)=\lim_{n\to\infty}\tfrac{1}{n}H\left(\bigvee_{i=0}^{n-1}\tau^i(\mathcal{A}^\flat)\right) \quad \text{and}$$
$$h(\mathcal{A}^\sharp,\tau)=\lim_{n\to\infty}\tfrac{1}{n}H\left(\bigvee_{i=0}^{n-1}\tau^i(\mathcal{A}^\sharp)\right),$$

then we have

$$h_\alpha(\mathcal{A},\tau)=(1-\alpha)h(\mathcal{A}^\flat,\tau)+\alpha h(\mathcal{A}^\sharp,\tau).$$

Theorem 2 *Let* $\mathcal{C} = \{C_1, ..., C_t\}$ *be a measurable partition of* Ω *being a generator, i.e.* $\sigma(\bigcup_{i=0}^{\infty} \tau^i(\mathcal{C})) = \mathcal{S}$. *Then for every IF-partition* $\mathcal{A} = \{(\mu_{A_1}, \nu_{A_1}), ..., (\mu_{A_k}, \nu_{A_k})\}$ *there holds*

$$h_\alpha(\mathcal{A}, \tau) \leq h_\alpha(\mathcal{C}, \tau) + \int_\Omega \left(\sum_{i=1}^{k} (1-\alpha)\varphi(\mu_{A_i}) + \alpha\varphi(1-\nu_{A_i}) \right) dP.$$

Proof. Put $\mathcal{C} = \{(\chi_{C_1}, 1-\chi_{C_1}), ..., (\chi_{C_t}, 1-\chi_{C_t})\}$ instead of $\mathcal{C} = \{C_1, ..., C_t\}$. By Theorem 10.3.4 in [9]

$$h(\mathcal{A}^\flat, \tau) \leq h(\mathcal{C}^\flat, \tau) + \int_\Omega \left(\sum_{i=1}^{k} \varphi(\mu_{A_i}) \right) dP$$

and

$$h(\mathcal{A}^\sharp, \tau) \leq h(\mathcal{C}^\sharp, \tau) + \int_\Omega \left(\sum_{i=1}^{k} \varphi(1-\nu_{A_i}) \right) dP.$$

This implies

$$h_\alpha(\mathcal{A}, \tau) \leq h_\alpha(\mathcal{C}, \tau) + \int_\Omega \left(\sum_{i=1}^{k} (1-\alpha)\varphi(\mu_{A_i}) + \alpha\varphi(1-\nu_{A_i}) \right) dP.$$

Of course this IF-entropy has the following defect.

Proposition 5 *If* $G = \{(\mu, 1-\mu); \mu \in [0,1]\}$, *then*

$${}_G h_\alpha(\tau) = \infty.$$

Proof. Put $\mathcal{A} = \{(\frac{1}{k}, 1-\frac{1}{k}), ..., (\frac{1}{k}, 1-\frac{1}{k})\}$, where $k \in \mathbf{N}$. Then $\mathcal{A}^\flat = \mathcal{A}^\sharp = \{1/k, ..., 1/k\}$ and

$$\mathcal{A}^\flat \vee \tau(\mathcal{A}^\flat) = \mathcal{A}^\sharp \vee \tau(\mathcal{A}^\sharp) = \{1/k^2, ..., 1/k^2\},$$

hence

$$H(\mathcal{A}^\flat \vee \tau(\mathcal{A}^\flat)) = H(\mathcal{A}^\sharp \vee \tau(\mathcal{A}^\sharp)) = -\sum_{i=1}^{k^2} \frac{1}{k^2} \log \frac{1}{k^2} = 2\log k,$$

and

$$H_\alpha(\mathcal{A} \vee \tau(\mathcal{A})) = 2\log k.$$

Similarly

$$H_\alpha\left(\bigvee_{i=0}^{n-1}\tau^i(\mathcal{A})\right) = n\log k,$$

hence

$$h_\alpha(\mathcal{A},\tau) = \log k.$$

Since $k \in \mathbf{N}$ was arbitrary, we obtain ${}_G h_\alpha(\tau) \geq \log k$ for every k. Therefore ${}_G h_\alpha(\tau) = \infty$.

To eliminate this defect we used the Hudetz modification of the notion of entropy (see [9]).

5 Hudetz entropy on IF-dynamical system

If we want to define the Hudetz entropy on an IF-dynamical system $(\mathcal{F}, m_\alpha, \tau)$, we must define the value $\varphi(A)$ for any $A \in \mathcal{M}$, where $\varphi(x) = -x\log x$, if $x > 0$, and $\varphi(0) = 0$.

Definition 6 *For any $A = (\mu_A, \nu_A) \in \mathcal{M}$ we define*

$$\varphi((\mu_A,\nu_A)) = (\varphi(\mu_A), 1-\varphi(1-\nu_A)) = (-\mu_A\log\mu_A, 1+(1-\nu_A)\log(1-\nu_A)).$$

Definition 7 *If $\mathcal{A} = \{A_1, ..., A_k\}$ is an IF-partition, then we define its Hudetz entropy (with respect to a given state m_α) by the formula*

$$\widehat{H_\alpha}(\mathcal{A}) = H_\alpha(\mathcal{A}) - \sum_{i=1}^{k}\overline{m}_\alpha(\varphi(A_i)),$$

where $\overline{m}_\alpha$ is the extension of state m_α.

Proposition 6 *If $\mathcal{A} = \{A_1, ..., A_k\}$ is an IF-partition, then*

$$\widehat{H_\alpha}\left(\bigvee_{i=0}^{n-1}\tau^i(\mathcal{A})\right) = H_\alpha\left(\bigvee_{i=0}^{n-1}\tau^i(\mathcal{A})\right) - n.\overline{m}_\alpha\left(\bigoplus_{i=1}^{k}\varphi(A_i)\right),$$

hence there exists

$$\lim_{n\to\infty}\frac{1}{n}\widehat{H_\alpha}\left(\bigvee_{i=0}^{n-1}\tau^i(\mathcal{A})\right) = h_\alpha(\mathcal{A},\tau) - \overline{m}_\alpha\left(\bigoplus_{i=1}^{k}\varphi(A_i)\right).$$

Proof. Let $\mathcal{A} = \{A_1, ..., A_k\}$ be an IF-partition. Since

$$\mathcal{A} \vee \tau(\mathcal{A}) = \{A_i \odot \tau(A_j); i, j = 1, ..., k\},$$

we have

$$\widehat{H_\alpha}(\mathcal{A} \vee \tau(\mathcal{A})) = H_\alpha(\mathcal{A} \vee \tau(\mathcal{A})) - \sum_{i=1}^{k}\sum_{j=1}^{k} \overline{m}_\alpha\Big(\varphi(A_i \odot \tau(A_j))\Big).$$

Since $A_i = (\mu_{A_i}, \nu_{A_i}); i = 1, ..., k$, we have

$$\sum_{i=1}^{k}\sum_{j=1}^{k} \overline{m}_\alpha\Big(\varphi(A_i \odot \tau(A_j))\Big) = \sum_{i=1}^{k}\sum_{j=1}^{k} \overline{m}_\alpha\Big(\varphi((\mu_{A_i}, \nu_{A_i}) \odot (\mu_{A_i} \circ T, \nu_{A_i} \circ T))\Big) =$$

$$= \sum_{i=1}^{k}\sum_{j=1}^{k} \overline{m}_\alpha\Big(\varphi((\mu_{A_i}(\mu_{A_j} \circ T), \nu_{A_i} + (\nu_{A_j} \circ T) - \nu_{A_i}(\nu_{A_j} \circ T)))\Big) =$$

$$= \overline{m}_\alpha\Bigg(\bigoplus_{i=1}^{k}\bigoplus_{j=1}^{k}\Big(-\mu_{A_i}(\mu_{A_j} \circ T)\log(\mu_{A_i}(\mu_{A_j} \circ T)),$$

$$1 + (1 - \nu_{A_i} - (\nu_{A_j} \circ T) + \nu_{A_i}(\nu_{A_j} \circ T))\log(1 - \nu_{A_i} - \nu_{A_j} \circ T + \nu_{A_i}(\nu_{A_j} \circ T))\Big)\Bigg) =$$

$$= \overline{m}_\alpha\Bigg(\Bigg(\sum_{i=1}^{k}\sum_{j=1}^{k}\Big(-\mu_{A_i}(\mu_{A_j} \circ T)(\log \mu_{A_i} + \log(\mu_{A_j} \circ T))\Big),$$

$$\sum_{i=1}^{k}\sum_{j=1}^{k}\Big(1 + (1 - \nu_{A_i})(1 - \nu_{A_j} \circ T)(\log(1 - \nu_{A_i}) + \log(1 - \nu_{A_j} \circ T))\Big) - (k^2 - 1)\Bigg)\Bigg) =$$

$$= \overline{m}_\alpha\Bigg(\Bigg(\Bigg(\sum_{i=1}^{k} -\mu_{A_i}\log\mu_{A_i}\Bigg)\Bigg(\sum_{j=1}^{k}\mu_{A_j} \circ T\Bigg) +$$

$$+ \Bigg(\sum_{j=1}^{k} -(\mu_{A_j} \circ T)\log(\mu_{A_j} \circ T)\Bigg)\Bigg(\sum_{i=1}^{k}\mu_{A_i}\Bigg),$$

$$k^2 - \Bigg(\sum_{i=1}^{k} -(1 - \nu_{A_i})\log(1 - \nu_{A_i})\Bigg)\Bigg(\sum_{j=1}^{k}(1 - \nu_{A_j} \circ T)\Bigg) -$$

$$-\Bigg(\sum_{j=1}^{k} -(1 - \nu_{A_j} \circ T)\log(1 - \nu_{A_j} \circ T)\Bigg)\Bigg(\sum_{i=1}^{k} 1 - \nu_{A_i}\Bigg) - k^2 + 1\Bigg)\Bigg) =$$

$$= \overline{m}_\alpha\Bigg(\Big(\sum_{i=1}^{k}\varphi(\mu_{A_i})+\sum_{j=1}^{k}\varphi(\mu_{A_i}\circ T), 1-\sum_{i=1}^{k}\varphi(1-\nu_{A_i})-\sum_{j=1}^{k}\varphi(1-\nu_{A_i}\circ T)\Big)\Bigg) =$$

$$= (1-\alpha)\int_\Omega \sum_{i=1}^{k}\varphi(\mu_{A_i})dP + (1-\alpha)\int_\Omega \varphi(\mu_{A_i}\circ T)dP +$$

$$+\alpha\int_\Omega \sum_{i=1}^{k}\varphi(1-\nu_{A_i})dP + \alpha\int_\Omega \sum_{j=1}^{k}\varphi(1-\nu_{A_i}\circ T)dP =$$

$$= 2\sum_{i=1}^{k}\left((1-\alpha)\int_\Omega \varphi(\mu_{A_i})dP + \int_\Omega \Big(1-(1-\varphi(1-\nu_{A_i}))\Big)dP\right) =$$

$$= 2\sum_{i=1}^{k}\overline{m}_\alpha\Big((\varphi(\mu_{A_i}), 1-\varphi(1-\nu_{A_i}))\Big) =$$

$$= 2\sum_{i=1}^{k}\overline{m}_\alpha\Big(\varphi((\mu_{A_i},\nu_{A_i}))\Big) = 2.\overline{m}_\alpha\left(\bigoplus_{i=1}^{k}\varphi(A_i)\right)$$

Therefore

$$\widehat{H_\alpha}(\mathcal{A}\vee\tau(\mathcal{A})) = H_\alpha(\mathcal{A}\vee\tau(\mathcal{A})) - 2.\overline{m}_\alpha\left(\bigoplus_{i=1}^{k}\varphi(A_i)\right).$$

Similarly

$$\widehat{H_\alpha}\left(\bigvee_{i=0}^{n-1}\tau^i(\mathcal{A})\right) = H_\alpha\left(\bigvee_{i=0}^{n-1}\tau^i(\mathcal{A})\right) - n.\overline{m}_\alpha\left(\bigoplus_{i=1}^{k}\varphi(A_i)\right),$$

hence

$$\lim_{n\to\infty}\frac{1}{n}\widehat{H_\alpha}\left(\bigvee_{i=0}^{n-1}\tau^i(\mathcal{A})\right) = h_\alpha(\mathcal{A},\tau) - \overline{m}_\alpha\left(\bigoplus_{i=1}^{k}\varphi(A_i)\right).$$

Definition 8 *For any IF-partition $\mathcal{A}$ define*

$$\widehat{h_\alpha}(\mathcal{A},\tau) = \lim_{n\to\infty}\frac{1}{n}\widehat{H_\alpha}\left(\bigvee_{i=0}^{n-1}\tau^i(\mathcal{A})\right),$$

and for arbitrary $G \subset \mathcal{M}$ the Hudetz entropy of an IF-dynamical system $(\mathcal{F}, m_\alpha, \tau)$ by the formula

$$_G\widehat{h_\alpha}(\tau) = \{\widehat{h_\alpha}(\mathcal{A},\tau); \mathcal{A} \text{ is an IF-partition, } \mathcal{A} \subset G\}.$$

Theorem 3 *Let $\mathcal{C} = \{C_1, ..., C_t\}$ be a measurable partition of Ω being a generator, $\mathcal{C} \subset G$. Then*

$$_G\widehat{h_\alpha}(\tau) = \widehat{h_\alpha}(\mathcal{C},\tau) = h_\alpha(\mathcal{C},\tau).$$

Proof. We have to prove that $\widehat{h_\alpha}(\mathcal{A},\tau) \le h_\alpha(\mathcal{C},\tau)$ for every IF-partition $\mathcal{A}$. Of course, by Theorem 2

$$h_\alpha(\mathcal{A},\tau) \le h_\alpha(\mathcal{C},\tau) + \int_\Omega \left(\sum_{i=1}^{k}(1-\alpha)\varphi(\mu_{A_i}) + \alpha\varphi(1-\nu_{A_i})\right) dP, \qquad (3)$$

where $\mathcal{A} = \{(\mu_{A_1}, \nu_{A_1}), ..., (\mu_{A_k}, \nu_{A_k})\}$. Since

$$\overline{m}_\alpha\left(\bigoplus_{i=1}^{k}\varphi(A_i)\right) = \int_\Omega \left(\sum_{i=1}^{k}(1-\alpha)\varphi(\mu_{A_i}) + \alpha\varphi(1-\nu_{A_i})\right) dP,$$

then by Proposition 6

$$\widehat{h_\alpha}(\mathcal{A},\tau) = h_\alpha(\mathcal{A},\tau) - \int_\Omega \left(\sum_{i=1}^{k}\varphi((1-\alpha)\mu_{A_i} + \alpha(1-\nu_{A_i}))\right) dP. \qquad (4)$$

Combining (3) and (4) we obtain the assertion.

6 Conclusions

We have presented the definition of Kolmogorov-Sinaj type of the entropy of dynamical systems based on the family of all IF-events. Also, we have defined the Hudetz modification of this notion of the entropy, which eliminate infinity problem with this entropy.

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The papers presented in this Volume 1 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.
It may be viewed as a result of fruitful discussions held during the Eighth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2009) organized in Warsaw on October 16, 2009 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Centre for Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT – Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bistrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, and the University of Westminster, Harrow, UK:

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The Eighth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2009) has been meant to commence a new series of scientific events primarily focused on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Moreover, other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems are discussed.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

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