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Volume I: Foundations

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A new method for computing relative cardinality of intuitionistic fuzzy sets

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Abstract

An efficient and general method for calculating an exact value of relative cardinality of Atanassov's intuitionistic fuzzy sets (IFSs) is still an open problem. In this paper we make a step towards the solution of this problem by proposing an algorithm for computing relative cardinality of IFSs based on algebraic t-norm.

Keywords: Atanassov's intuitionistic fuzzy sets, relative cardinality, subsethood measure, similarity measure .

1 Introduction

Intuitionistic fuzzy set (IFS) theory was proposed by Atanassov [2] as an intuitive and straightforward extension of Zadeh's fuzzy sets theory [16]. An assertion that is assumed in fuzzy set theory claims that from the fact that an element $x \in X$ belongs to a given degree $A(x)$ to a fuzzy set A, follows naturally that x should not belong to A to a degree $1 - A(x)$. On the contrary, an IFS $\mathcal{A} = (A^+, A^-)$ assigns to each element x both a degree of membership $A^+(x)$ and a degree of non-membership $A^-(x)$ such that $A^-(x) \leq 1 - A^+(x)$. Such an approach has the virtue of complementing fuzzy sets, that are able to model vagueness, with an ability to model uncertainty as well. A measure of this uncertainty (non-determinacy, hesitation) is given by $1 - A^+(x) - A^-(x)$.

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Adding the possibility of modeling uncertainty poses new challenges that do not occur in classical fuzzy set theory, concerning proper definition of IFSs operations and relations. In general it is not true that operating on an IFS $\mathcal{A}$ is nothing more than operating separately on each of the fuzzy sets A^+ and A^-. As an example of such operation that is not a straightforward extension of its fuzzy analogue, and desires a deeper insight into the nature of uncertainty of IFS, we discuss a relative cardinality of IFSs. The main objective of this paper is to introduce an efficient and exact algorithm that allows for calculating this value.

To this end, in the second section we indicate motivation showing why it is worth to address the problem of relative cardinality and we consider approaches existing so far. In the main, third section, we present a new algorithm for determining the value of relative cardinality of IFSs based on an algebraic t-norm. We finish with some concluding remarks.

It must be noted that IFS theory is equivalent to some other concepts like interval-valued fuzzy sets or interval type-2 fuzzy sets theories ([1, 7]). All of these approaches have given rise to an extensive literature covering their respective applications, proving their maturity and usefulness in solving real-life problems. Consequently, all the results presented in this paper are valid as well under interval-valued and interval type-2 fuzzy sets theories.

2 Relative cardinality of IFSs

The notion of relative cardinality, denoted as $\sigma(A|B)$, undoubtedly deserves an attention, regardless of A, B being crisp, fuzzy or intuitionistic fuzzy sets. This is because it provides a theoretical basis for many important concepts that are discussed below.

- Inclusion measure (subsethood)

  Primarily, relative cardinality $\sigma(A|B)$ can be interpreted as the degree of truth of a sentence: "B is a subset of A" (cf. [3, 5, 6]):

$$[\![B \text{ is a subset of } A]\!] = \sigma(A|B)$$

- Implication

  Relative cardinality $\sigma(A|B)$ may form a basis for implication operator:

$$[\![\text{ If } B \text{ then } A]\!] = \sigma(A|B)$$

- Similarity

  One of the most common approach to measure similarity between A and B may be formulated on the basis of relative cardinality measure ([4]):

$$[\![A \text{ is similar to } B]\!] = \sigma(A \cap B | A \cup B)$$

- Quantified sentences

  As a means of obtaining the degree of truth of linguistically quantified sentences of a form: "Q B x's are A" (e.g. "Most of my friends trust John", "About half of important experts prefers option i"), relative cardinality is applied in a large number of applications such as data mining, fuzzy expert and recommender systems, decision making processes, fuzzy queries in databases and linguistic summarization of databases (see e.g. [15, 17]):

$$[\![Q\ B\ x \text{ are } A]\!] = Q(\sigma(A|B)).$$

In a fuzzy set theory, a relative cardinality of fuzzy sets A and B is calculating according to the following formula:

$$\sigma_t(A|B) = \frac{\sigma(A \cap_t B)}{\sigma(B)} \tag{1}$$

where cardinality of fuzzy set $\sigma(A)$ is typically defined by so-called sigma-count ([10, 13]):

$$\sigma(A) = \sum_{x \in X} A(x). \tag{2}$$

Moreover, an intersection $A \cap_t B$ is defined as:

$$(A \cap_t B)(x) = A(x)\ t\ B(x) \tag{3}$$

where t is a t-norm, i.e. an increasing, commutative and associative mapping $[0,1]^2 \to [0,1]$ satisfying $t(1,x) = x$ for all $x \in [0,1]$. The most common examples of t-norms are: t-norm minimum $t_\wedge(a,b) = a \wedge b$, algebraic t-norm $t_a(a,b) = a \cdot b$ and Łukasiewicz t-norm $t_L(a,b) = 0 \vee (a+b-1)$. Additionally, by A^c we denote a complement of A defined as $1 - A(x)$ for each x.

Lately, some attempts have been made to define a relative cardinality of IFSs. One approach that draws on interval calculus, was mentioned for example in [6], where a relative cardinality was used to define inclusion measure, and in [12] in the context of linguistic summarization:

$$\sigma_{IF}(\mathcal{A}|\mathcal{B}) = \left[\frac{\sigma(A^+ \cap B^+)}{\sigma((B^-)^c)}, \frac{\sigma((A^-)^c \cap (B^-)^c)}{\sigma(B^+)}\right].$$

The second idea, adapted from the definition of conditional probability of intuitionistic fuzzy events proposed in [8], is a straightforward generalization of the fuzzy case to the IFS case. It assumes that $\sigma_{IF}(\mathcal{A}|\mathcal{B})$ is an interval estimated by lower and upper bounds of IFSs $\mathcal{A}$ and $\mathcal{B}$:

$$\sigma_{IF}(\mathcal{A}|\mathcal{B}) = \left[\sigma(A^+|B^+), \sigma((A^-)^c|(B^-)^c)\right].$$

Another method is taken from [9], where two indicators of inclusion measure for I-fuzzy sets were considered - a necessary and a possible one. Adapting this approach we obtain:

$$\sigma_{IF}(\mathcal{A}|\mathcal{B}) = \left[\sigma(A^+|(B^-)^c), \sigma((A^-)^c|B^+)\right].$$

All of the above-mentioned methods use only A^+ and A^-, B^+ and B^- in calculations, and consequently, they only approximate a value of relative cardinality of IFSs. Such approximation is easy to compute, however, it can be vitiated by an error that is unacceptable in practical applications. Our goal therefore was to find a relatively easy but exact method to determine a value of $\sigma_{IF}(\mathcal{A}|\mathcal{B})$. We claim that, in order to find an exact value of $\sigma_{IF}(\mathcal{A}|\mathcal{B})$, it is necessary to consider some other fuzzy representations of given IFSs $\mathcal{A}$ and $\mathcal{B}$, i.e. $A^* \in Rep(\mathcal{A})$ and $B^* \in Rep(\mathcal{B})$ where:

$$Rep(\mathcal{A}) = \{A^* \in \mathcal{F} \mid \forall_{x \in X} A^+(x) \leq A^*(x) \leq 1 - A^-(x)\}$$

and

$$Rep(\mathcal{B}) = \{B^* \in \mathcal{F} \mid \forall_{x \in X} B^+(x) \leq B^*(x) \leq 1 - B^-(x)\}.$$

This leads to the following definition of relative cardinality $\sigma_{IF}(\mathcal{A}|\mathcal{B})$.

Definition 1 *The relative cardinality of two IF-sets* $\mathcal{A} = (A^+, A^-)$ *and* $\mathcal{B} = (B^+, B^-)$ *is an interval defined as:*

$$\sigma_{IF}(\mathcal{A}|\mathcal{B}) = \left[\min_{\substack{A^* \in Rep(\mathcal{A}) \\ B^* \in Rep(\mathcal{B})}} \sigma(A^*|B^*), 1 - \max_{\substack{A^* \in Rep(\mathcal{A}) \\ B^* \in Rep(\mathcal{B})}} \sigma(A^*|B^*)\right]. \quad (4)$$

The main aim of this paper is to find an efficient method for calculating a value of $\sigma_{IF}(\mathcal{A}|\mathcal{B})$ given by (4). Our inspiration was a work by Mendel and Nguyen. In [11] and then in [14] they proposed an efficient algorithm of calculating relative cardinality of interval type-2 fuzzy sets with a basic t-norm, minimum. In the next section we present how this idea can be adapted to IFS theory and we propose an algorithm of calculating $\sigma_{IF}(\mathcal{A}|\mathcal{B})$ with an algebraic t-norm.

3 A new method for computing relative cardinality of IFSs with algebraic t-norm

Although Definition 1 is simple and intuitive, there is no easy and general - i.e. t-norm independent - method to calculate the final interval $\sigma_{IF}(\mathcal{A}|\mathcal{B})$. Algorithms proposed by Nguyen and Kreinovich [11] and Wu and Mendel [14] solve the problem for minimum t-norm with a complexity: $O(n \log n)$ and $O(n^{1+\alpha})$, respectively. Algebraic t-norm is as well a popular t-norm commonly used in practical applications. In this section we propose and discuss an algorithm for computing relative cardinality of IFSs based on this t-norm.

A formula (4) for algebraic t-norm takes a form:

$$\sigma_{IF}(\mathcal{A}|\mathcal{B}) = \left[\min_{\substack{A^* \in Rep(\mathcal{A}) \\ B^* \in Rep(\mathcal{B})}} \sigma_{t_a}(A^*|B^*), 1 - \max_{\substack{A^* \in Rep(\mathcal{A}) \\ B^* \in Rep(\mathcal{B})}} \sigma_{t_a}(A^*|B^*) \right] =$$

$$= \left[\min_{\substack{A^* \in Rep(\mathcal{A}) \\ B^* \in Rep(\mathcal{B})}} \frac{\sigma(A^* \cap_{t_a} B^*)}{\sigma(B^*)}, 1 - \max_{\substack{A^* \in Rep(\mathcal{A}) \\ B^* \in Rep(\mathcal{B})}} \frac{\sigma(A^* \cap_{t_a} B^*)}{\sigma(B^*)} \right] =$$

$$= \left[\min_{\substack{A^* \in Rep(\mathcal{A}) \\ B^* \in Rep(\mathcal{B})}} \frac{\sum_{x \in X} A^*(x) \cdot B^*(x)}{\sum_{x \in X} B^*(x)}, 1 - \max_{\substack{A^* \in Rep(\mathcal{A}) \\ B^* \in Rep(\mathcal{B})}} \frac{\sum_{x \in X} A^*(x) \cdot B^*(x)}{\sum_{x \in X} B^*(x)} \right]$$

Lemma 1 *For any two IFSs $\mathcal{A}$ and $\mathcal{B}$ following two properties are satisfied:*

- $\min_{\substack{A^* \in Rep(\mathcal{A}) \\ B^* \in Rep(\mathcal{B})}} \sigma_{t_a}(A^*|B^*)$ *is attained when* $A^* = A^+$,
- $\max_{\substack{A^* \in Rep(\mathcal{A}) \\ B^* \in Rep(\mathcal{B})}} \sigma_{t_a}(A^*|B^*)$ *is attained when* $A^* = (A^-)^c$,

Proof Let us notice that $\sigma_{t_a}(A^*|B^*)$ is a non-strictly increasing function of A^* so its minimum (maximum) is attained when A^* takes minimal (maximal) values.

Lemma 2 *For any two IFSs $\mathcal{A}$ and $\mathcal{B}$:*

$$\min_{B^* \in Rep(\mathcal{B})} \sigma_{t_a}(A^+|B^*) \quad \textit{and} \quad \max_{B^* \in Rep(\mathcal{B})} \sigma_{t_a}((A^-)^c|B^*)$$

are attained when $\forall_{x \in X} \quad B^*(x) \in \{B^+(x), (B^-)^c(x)\}$.

Proof We will prove first part. The proof of the second is analogous. Let us state relative cardinality formula in following form

$$\sigma_{t_a}(A^*|B^*) = \frac{A^+(x)B^*(x) + m_x}{B^*(x) + M_x},$$

where $m_x = \sum_{\substack{z\in X \\ z\neq x}} A^+(z)B^*(z)$ and $M_x = \sum_{\substack{z\in X \\ z\neq x}} B^*(z)$. By making further transformations we obtain

$$\sigma_{t_a}(A^*|B^*) = A^+(x) - \frac{A^+(x)M_x - m_x}{B^*(x) + M_x}.$$

From this form it is easy to see that, if $A^+(x) > \frac{m_x}{M_x}$ then relative cardinality attains its smallest value when $B^*(x)$ is the smallest. Otherwise smallest value is attained when $B^*(x)$ is the largest. ∎

From Lemmas 1 and 2 we know that for any $x \in X$ we have only two possible states (both for minimum and maximum case). This yields total complexity of $O(2^{|X|})$ which is not satisfactory for practical applications.

Intuition behind our algorithm for computing the minimum is as follows. Assign high values of $B^*(x)$ whenever $A^+(x)$ is low, thus elements with "low potential" have greater weight. In case of maximum, assign high values of $B^*(x)$ whenever $(A^-)^c$ is also high to assure that "high potential" elements have largest weight. Algorithms 1 and 2 implement this intuition. Elements of domain X are iterated in ascending order of their "potential". Thanks to this it is possible to avoid backtracking. In case of minimum, $B^+(x)$ is changed to $(B^-)^c(x)$ as long as it lowers total ratio. Conversely, in case of maximum $(B^-)^c(x)$ is changed to $B^+(x)$ as long as it increases total ratio.

Computing sums in lines 1 and 2 takes $O(n)$. Each iteration of the loop (lines 4–9) takes constant amount of time thus the whole loop takes $O(n)$. In line 3 set X is iterated in ascending order thus sorting is needed – $O(n \log n)$. This yields total complexity of $O(n \log n)$ for both of algorithms.

4 Conclusions

An efficient and exact algorithm for computing relative cardinality for IFSs based on an algebraic t-norm has been introduced. As we have shown, it can be effectively used in many practical application, among others for determining inclusion measure and similarity measure. Our further work will be focused on extending the proposed method on other t-norms and finally on constructing a universal algorithm for computing relative cardinality for wide range of t-norms.

Algorithm 1 Calculate $\min\limits_{B^* \in Rep(\mathcal{B})} \sigma_{t_a}(A^+|B^*)$

$n \leftarrow \sum_{x \in X} A^+(x) \cdot B^+(x)$
$d \leftarrow \sum_{x \in X} B^+(x)$
for all $x \in X$ in ascending order of $A^+(x)$ **do**
  $r \leftarrow \frac{n}{d}$
  $n \leftarrow n + A^+(x) \cdot ((B^-)^c(x) - B^+(x))$
  $d \leftarrow d + (B^-)^c(x) - B^+(x)$
  if $r \leq \frac{n}{d}$ **then**
    return r
  end if
end for

Algorithm 2 Calculate $\max\limits_{B^* \in Rep(\mathcal{B})} \sigma_{t_a}((A^-)^c|B^*)$

$n \leftarrow \sum_{x \in X} (A^-)^c(x) \cdot B^+(x)$
$d \leftarrow \sum_{x \in X} B^+(x)$
for all $x \in X$ in ascending order of $A^-(x)$ **do**
  $r \leftarrow \frac{n}{d}$
  $n \leftarrow n + (A^-)^c(x) \cdot ((B^-)^c(x) - B^+(x))$
  $d \leftarrow d + (B^-)^c(x) - B^+(x)$
  if $r \geq \frac{n}{d}$ **then**
    return r
  end if
end for

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The papers presented in this Volume 1 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.

It may be viewed as a result of fruitful discussions held during the Twelfth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2013) organized in Warsaw on October 11, 2013 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT - Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bystrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, Prof. Asen Zlatarov University, Burgas, Bulgaria, and the University of Westminster, Harrow, UK:

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The consecutive International Workshops on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGNs) have been meant to provide a forum for the presentation of new results and for scientific discussion on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems have also been included. The Twelfth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2013) is a continuation of this undertaking, and provides many new ideas and results in the areas concerned.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

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