Modern Approaches in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics
Volume I: Foundations

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On modal operators and quasi-orderings for IFSs

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Abstract

Every quasi-ordered set $(Y, \preceq)$ can be naturally split into equivalence classes and its factorization by that equivalence relation turned into partially ordered set as described by Birkhoff [3]. "Necessity" and "possibility" operators (denoted $\Box$ and $\Diamond$ respectively) for intuitionistic fuzzy sets have been introduced by Atanassov. We investigate them in more detail describing the structure of the classes from the corresponding equivalent relation on $IFS(X)$ derived from the modal quasi-orderings. Some new statements about modal operators are introduced and we put light on them from various points of view.

Keywords: Intuitionistic fuzzy sets, Modal operators, Modal quasi-orderings.

1 Introduction to intuitionistic fuzzy sets

A fuzzy set in X (cf. Zadeh [5]) is given by

$$A^{'} = \{\langle x, \mu_{A'}(x)\rangle | x \in X\} \tag{1}$$

where $\mu_{A'}(x) \in [0, 1]$ is the *membership function* of the fuzzy set $A^{'}$. As opposed to the Zadeh's fuzzy set (abbreviated FS), Atanassov (cf. [1], [2]) extended its definition to an intuitionistic fuzzy set (abbreviated IFS) A, given by

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\} \tag{2}$$

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where: $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ such that

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1 \tag{3}$$

and $\mu_A(x)$, $\nu_A(x) \in [0, 1]$ denote a *degree of membership* and a *degree of non-membership* of $x \in A$, respectively. An additional concept for each IFS in X, that is an obvious result of (2) and (3), is called

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \tag{4}$$

a *degree of uncertainty* of $x \in A$. It expresses a lack of knowledge of whether x belongs to A or not (cf. Atanassov [1]). It is obvious that $0 \leq \pi_A(x) \leq 1$, for each $x \in X$. Uncertainty degree turn out to be relevant for both - applications and the development of theory of IFSs. For instance, distances between IFSs are calculated in the literature in two ways, using two parameters only (cf. Atanassov [1]) or all three parameters (cf. Szmidt and Kacprzyk [4]).

Talking about partial ordering in IFSs, we will by default mean $(IFS(X), \leq)$ where $\leq$ stands for the standard partial ordering in IFS(X). That is, for any two A and $B \in IFS(X)$: $A \leq B$ is satisfied if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for any $x \in X$. On Fig. 1 one may see the triangular representation of the two chosen A and B in a particular point $x \in X$, where $f_A(x)$ stands for the point on the plane with coordinates $(\mu_A(x), \nu_A(x))$.

2 Modal operators and quasi-orderings on IFSs

Let us recall the definitions and some properties of the modal operators on intuitionistic fuzzy sets as introduced by Atanassov. For more detailed descriptions and properties the reader may refer to [2], Ch. 4.1., although we introduce now some new statements and put light on them from various points of view. "Necessity" and "possibility" operators (denoted $\Box$ and $\Diamond$ respectively) applied on an intuitionistic fuzzy set $A \in IFS(X)$ have been defined as:

$$\begin{aligned} \Box A &= \{\langle x, \ \mu_A(x), \ 1 - \mu_A(x) \rangle | x \in X\} \\ \Diamond A &= \{\langle x, \ 1 - \nu(x), \ \nu_A(x) \rangle | x \in X\} \end{aligned}$$

From the above definition it is evident that

$$\star\colon IFS(X) \longrightarrow FS(X) \tag{5}$$

where $\star$ is the prefix operator $\star \in \{\Box, \Diamond\}$, operating on the class of intuitionistic fuzzy sets. The reader can now easily check that the functional relation $\star \in \{\Box, \Diamond\}$ defined in (5) is an *non-decreasing* function, that is:

$$(\forall A, B \in IFS(X))(A \leq B \Rightarrow \star A \leq \star B)$$

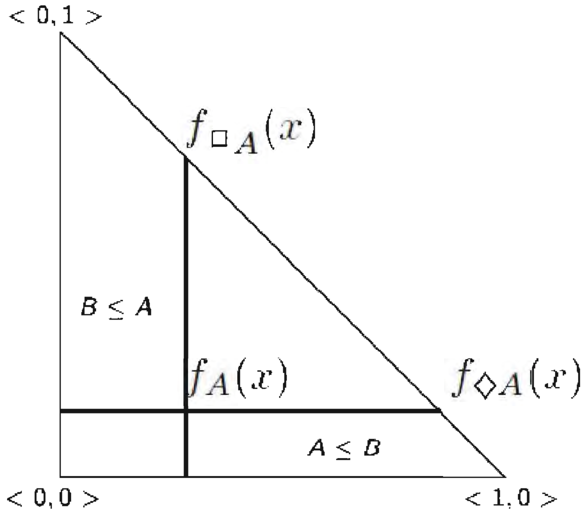

Figure 1: Triangular representation of the the intuitionistic fuzzy sets A and $B \in IFS(X)$ in a particular point $x \in X$, where $f_A(x)$ stands for the point on the plane with coordinates $(\mu_A(x), \nu_A(x))$. $\Box A$ and $\Diamond A$ stand for the two modal operators "necessity" and "possibility" acting on A.

But in general it is not true that $\star$ is increasing, i.e. that for any two $A, B \in IFS(X)$ for which $A \lneq B$ ($A \leq B$ and $A \neq B$) implies $\star A \lneq \star B$.

Example 1 *Taking for instance A and B such that $A \leq B$ with $\mu_A = \mu_B$ ($\nu_A = \nu_B$) on the whole universum X and there exists some $x_0 \in X$ such that $\nu_A(x_0) > \nu_B(x_0)$ ($\mu_A(x_0) < \mu_B(x_0)$). Obviously we have now $A \lneq B$ for which $\Box A = \Box B$ ($\Diamond A = \Diamond B$).*

Fixed point (also known as an invariant point) of a function is an element of the function's domain that is mapped to itself by the function. It is not tough to show that the fixed points of the modal operators $\Box$ and $\Diamond$ with domain IFS(X) are exactly the fuzzy sets FS(X). It is worth summarizing the last observations in the following:

Proposition 1 *The above defined modal operators $\Box$ and $\Diamond$ on IFS(X) are non-decreasing mappings and furthermore their fixed points coincide with the usual fuzzy sets FS(X). That is, $(\forall A \in IFS(X))(\star A = A \Leftrightarrow A \in FS(X))$ where $\star \in \{\Box, \Diamond\}$.*

Remark 1 *Since the two modal operators are idempotent:*

$$(\forall n \in \mathbb{N})(n \geq 1 \Rightarrow \star^n = \star)$$

then for all n ¿ 1 the fixed points of $\star^n$ *coincide with the fixed points of* $\star$*, i.e. FS(X).*

Following Atanassov [2], Ch. 4.1., let remind the *quasi-orderings* (also called *preorderings* by some authors) $\leq_\Box$ and $\leq_\Diamond$ corresponding to the two modal operators on IFSs. Quasi-ordered set is a set Y with a binary relation $\preceq$ satisfying *reflexivity* and *transitivity*, where the *anti-symmetric* property may not be in general satisfied. Thereby a quasi-ordered set $(Y, \preceq)$ is something like a partially ordered set for which it is possible that:

$$(\exists x, y \in Y)(x \preceq y \,\&\, y \preceq x \,\&\, x \neq y) \tag{6}$$

For detailed introduction to quasi-ordered sets the reader can consult Birkhoff [3], Ch. II.1. Let us take any $A, B \in IFS(X)$ and define $A \leq_\Box B$ iff $\mu_A \leq \mu_B$ on X, respectively $A \leq_\Diamond B$ iff $\nu_A \geq \mu_B$ on X. Obviously both $\leq_\Box$ and $\leq_\Diamond$ are reflexive and transitive. That is, they are both quasi-orderings in IFS(X) which will be called *quasi* $\Box$*-ordering* and *quasi* $\Diamond$*-ordering* respectively. Taking any A and B from IFS(X), let us write down some properties of the above defined modal operators and quasi-orderings on IFSs (cf. Atanassov [2], Ch. 4.1.).

1. $\Box A \leq A \leq \Diamond A$
2. $A \leq B$ iff $A \leq_\Box B$ and $A \leq_\Diamond B$
3. $\Box\Box A = \Box A$ and $\Diamond\Diamond A = \Diamond A$ (idempotence)
4. $\Diamond\Box A = \Box A$ and $\Box\Diamond A = \Diamond A$

Every quasi-ordered set $(Y, \preceq)$ can be naturally split into equivalence classes and its factorization by that equivalence relation turned into partially ordered set in the following way. Let us write $y_1 {=_\preceq} y_2$ iff $y_1 \preceq y_2$ and $y_2 \preceq y_1$, for all $y_1, y_2 \in Y$. The reflexivity and transitivity of $\preceq$ imply reflexivity and transitivity of the relation $=_\preceq$. It is also symmetric by definition. And thereby, $=_\preceq$ is an equivalence relation in X. In an obvious way $\preceq$ can be carried over to a quasi-ordering $\precsim$ in $Y_\preceq := Y/ =_\preceq$ (the factorization of Y by $=_\preceq$) which turns out to be now a partial ordering $(\precsim, Y_\preceq)$. In the sequel we will often use the quasi-ordered set $(IFS(X), \leq_\star)$ and its extended partial ordering $\lesssim_\star$ in the factorization of IFS(X): $IFS(X)_\star := IFS(X)/ =_\star$ where $\star \in \{\Box, \Diamond\}$. For simplicity instead of $IFS(X)_{\leq_\star}$ and $=_{\leq_\star}$ we will write $IFS(X)_\star$ and $=_\star$ as we already did. It is worth to be explicitly noted that for any two $A, B \in IFS(X)$:

- $A =_\Box B$ iff $(A \leq_\Box B \,\&\, B \leq_\Box A)$ iff $\Box A = \Box B$ iff $\mu_A = \mu_B$ on X

- $A =_{\Diamond} B$ iff ($A \leq_{\Diamond} B$ & $B \leq_{\Diamond} A$) iff $\Diamond A = \Diamond B$ iff $\nu_A = \nu_B$ on X
- $A \leq B$ iff $A \leq_{\Box} B$ and $A \leq_{\Diamond} B$
- $A = B$ iff $A =_{\Box} B$ and $A =_{\Diamond} B$

Let us show that there is a very natural and intuitive bijective correspondence between the partially ordered sets $(IFS(X)_{\Box}, \lesssim_{\Box})$, $(IFS(X)_{\Diamond}, \lesssim_{\Diamond})$ and the ordinary fuzzy sets in X, $(FS(X), \leq)$, which preserves the corresponding partial orderings. Such bijective and order-preserving maps are called isomorphisms and the partially ordered sets - *isomorphic*. The mentioned correspondence becomes clear from explicitly writing down the factor sets $(IFS(X)_{\star}, \lesssim_{\star})$:

- $IFS(X)_{\Box} := \{A_{\Box} \mid A \in IFS(X)\}$
- $IFS(X)_{\Diamond} := \{A_{\Diamond} \mid A \in IFS(X)\}$

where $A_{\Box} = \{B \mid B \in IFS(X) \,\&\, \Box B = \Box A\}$ and $A_{\Diamond} = \{B \mid B \in IFS(X) \,\&\, \Diamond B = \Diamond A\}$. From the above statements we get the bijective maps, denoted by $\simeq$,

- $IFS(X)_{\Box} \simeq \{\Box A \mid A \in IFS(X)\} = FS(X)$
- $IFS(X)_{\Diamond} \simeq \{\Diamond A \mid A \in IFS(X)\} = FS(X)$

and realize that the elements of $IFS(X)_{\star}, \star \in \{\Box, \Diamond\}$, are exactly the singletons of the ordinary fuzzy sets in X, i.e.

$$IFS(X)_{\Box} = \{\{F\} \mid F \in FS(X)\} = IFS(X)_{\Diamond}$$

The corresponding partial orderings $\lesssim_{\Box}$ and $\lesssim_{\Diamond}$ in $IFS(X)_{\Box}$ and $IFS(X)_{\Diamond}$ respectively also coincide.
Every functional relation f splits its domain of definition, say $Y = Dom(f)$ into equivalence classes in an obvious way: y_1 and y_2 belong to the same class iff $f(y_1) = f(y_2)$. Therefore we can denote by Y/f the factorization of Y by f, i.e. $\{f^{-1}(f(y)) \mid y \in Y\}$. It is now clear that $IFS(X)_{\star}, \star \in \{\Box, \Diamond\}$, could be also expressed in the following way:

- $IFS(X)_{\Box} = \{\Box^{-1}(F) \mid F \in FS(X)\} = IFS(X)/\Box$
- $IFS(X)_{\Diamond} = \{\Diamond^{-1}(F) \mid F \in FS(X)\} = IFS(X)/\Diamond$

The last observations can be summarized in the following:

Proposition 2 *The quasi $\star$-ordering in IFS(X), $\leq_\star$, carries over to a partial ordering $\lesssim_\star$ in $IFS(X)_\star$, where $\star$ stands for any of the two modal operators. The equivalence classes of $IFS(X)_\star$ correspond bijectively to the fixed points of $\star$. $(IFS(X)_\Box, \lesssim_\Box)$, $(IFS(X)_\Diamond, \lesssim_\Diamond)$ and $(FS(X), \leq)$ are isomorphic. And moreover, the factorization of IFS(X) by the function $\star$, i.e. $IFS(X)/\star$, coincides with its factorization by the quasi $\star$-ordering, i.e. $IFS(X)_\star$.*

Let us consider another quasi-metric for IFSs, introduced in Atanassov [2, Ch. 4.1.]. We write $A \sqsubseteq B$ iff for all $x \in X$ we have that $\pi_A(x) \leq \pi_B(x)$. Obviously, this relation satisfies reflexivity and transitivity. The anti-symmetric property does not in general hold. To show this, it is sufficient to take $A, B \in IFS(X)$ such that $\pi_A = \pi_B$ on X with at least one $x_0 \in X$ such that $\mu_A(x_0) \neq \mu_B(x_0)$. Thus we can state the following remark.

Remark 2 *Considering the equivalence relation $=_\sqsubseteq$ associated with the quasi-ordering $\sqsubseteq$ in IFS(X), we have that $A =_\sqsubseteq B$ iff $\pi_A = \pi_B$ on X.*

3 Conclusion

We have investigated "necessity" and "possibility" operators (denoted $\Box$ and $\Diamond$ respectively) for intuitionistic fuzzy sets in more detail. We have described the structure of the classes from the corresponding equivalent relation on $IFS(X)$ derived from the modal quasi-orderings. Many new denotations and statements about modal operators were introduced and proved.

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The papers presented in this Volume 1 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.

It may be viewed as a result of fruitful discussions held during the Twelfth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2013) organized in Warsaw on October 11, 2013 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT - Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bystrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, Prof. Asen Zlatarov University, Burgas, Bulgaria, and the University of Westminster, Harrow, UK:

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The consecutive International Workshops on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGNs) have been meant to provide a forum for the presentation of new results and for scientific discussion on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems have also been included. The Twelfth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2013) is a continuation of this undertaking, and provides many new ideas and results in the areas concerned.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

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