# Modern Approaches in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics
# Volume I: Foundations

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# On intuitionistic fuzzy logics: results and problems

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**Abstract**

In the present paper an overview of the main definitions and results in the area of intuitionistic logic is presented and new open problems are formulated.
**Keywords:** Implications, intuitionistic fuzzy logic, negations

## 1 Introduction

The first researches, related to Intuitionistic Fuzzy Logics (IFLs) started in 1983 together with the researches on Intuitionistic Fuzzy Sets (IFSs), but the first publications in this area are dated to 1988-1990. In them, shortly, ideas for intuitionistic fuzzy propositional calculus [1], intuitionistic fuzzy predicate logic [3], intuitionistic fuzzy modal logic [2] and temporal IFL [4] are introduced. During the next 25 years, these areas were extended essentially. A lot of operations and operators were defined, but up to now there has not been an overall and systematic description of the obtained results. The present paper contains some basic ideas and some unsolved problems in the area of IFLs that will constitute development of this part of fuzzy sets theory.

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# 2 Short remarks on intuitionistic fuzzy propositional calculus

In classical logic, to each proposition (in the classical sense) we can assign its truth value: truth – denoted by $1$, or falsity – $0$. In the case of fuzzy logic, this truth value is a real number in the interval $[0, 1]$ and may be called "truth degree" of a particular proposition. Here, in IFL, we add one more value – "falsity degree" – which is a real number in the interval $[0, 1]$ as well. Thus, two real numbers, $\mu(p)$ and $\nu(p)$, are assigned to the proposition $p$ with the following constraint to hold:

$$0 \leq \mu(p) + \nu(p) \leq 1.$$

Let this assignment be provided by an evaluation function $V$ defined over a set of propositions $S$ in such a way that:

$$V(p) = \langle \mu(p),\ \nu(p) \rangle.$$

Hence, the function $V : S \to [0, 1] \times [0, 1]$ gives the truth and falsity degrees of all propositions in $S$.

We assume that the evaluation function $V$ assigns to the logical truth $T$

$$V(T) = \langle 1, 0 \rangle,$$

and to the logical falsity $F$

$$V(F) = \langle 0, 1 \rangle.$$

Similarly to IFSs theory (see, e.g., [5, 12]), several geometrical interpretations of the results of the function $V$ will be discussed below. It is obvious, that the ordinary fuzzy sets have only one geometrical interpretation, while in the IFSs case, several geometrical interpretations are given.

The first one (which is analogous to the standard fuzzy set interpretation) is shown on Fig. 1.

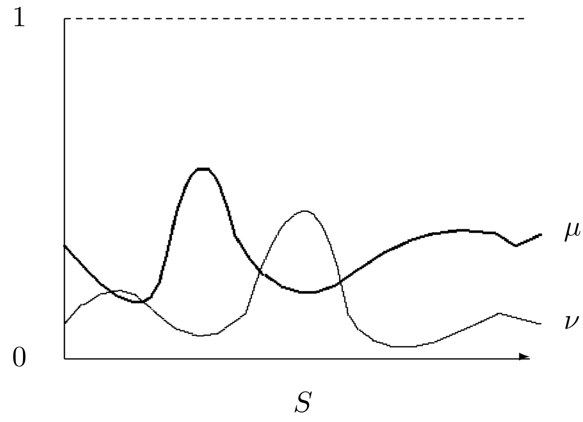



Fig. 1.

Its analogue is given in Fig. 2.

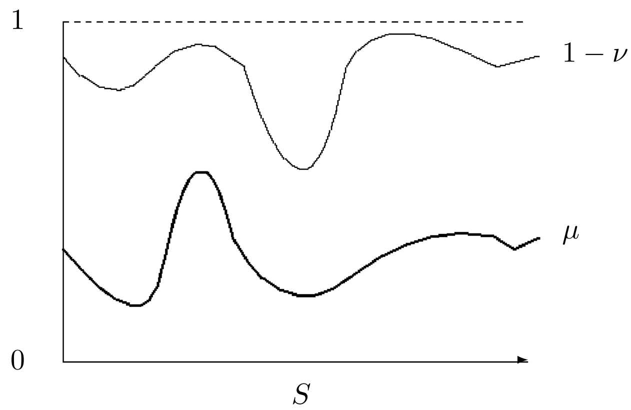
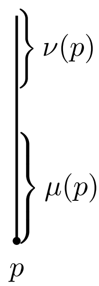



Fig. 2.

Therefore we can map to every proposition $p \in S$ a unit segment of the form:




Let a universe $S$ be given and let us consider the figure $F$ in the Euclidean plane with a Cartesian coordinate system (see Fig. 3).

Then we can construct a function $f$ (an interpretation of function $V$) from $S$ to $F$ such that if $p \in S$, then

$$x = f(p) \in F,$$

the point $x$ has coordinates $\langle a, b \rangle$ for which: $0 \leq a + b \leq 1$ and these coordinates are such that $a = \mu(p), b = \nu(p)$.

We will note that there can exist two different elements $p, q \in S$ for which $\mu(p) = \mu(q)$ and $\nu(p) = \nu(q)$, i.e., for which $f(p) = f(q)$.

About the form and the methods of determining the functions $\mu$ and $\nu$ we must repeat the same as in [5, 12]: *"everywhere below we will assume that these functions are either pre-determined or obtained as a result of the application of some operations or operators over pre-determined membership functions. In the fuzzy set theory there are three basic ways to construct membership functions:*

*i) basing on expert knowledge;*

*ii) explicitly — on the basis of observations collected in advance and processed appropriately (e.g., by statistical methods);*

*iii) analytically — by suitably chosen functions (e.g. probabilistic distribution).*

*Two latter cases are treated in much the same way as with ordinary fuzzy sets; however these methods are now used for the estimation of both the degree of membership and the degree of non-membership of a given element of a fixed universe to a subset of the same universe.* It is clear that a correct method must respect the inequalities

$$0 \leq \mu(p) + \nu(p) \leq 1$$

for every proposition $p$.

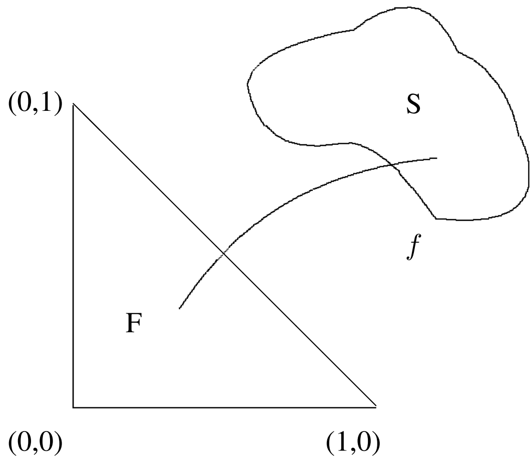



Fig. 3.

When the values $V(p)$ and $V(q)$ of the propositions $p$ and $q$ are known, the evaluation function $V$ can be extended also for the operations "&", "$\vee$" through different (by the moment – two) definitions:

$$V(p\&_1 q) = \langle \min(\mu(p), \mu(q)), \max(\nu(p), \nu(q))\rangle,$$
$$V(p \vee_1 q) = \langle \max(\mu(p), \mu(q)), \min(\nu(p), \nu(q))\rangle$$

and

$$V(p\&_2 q) = \langle \mu(p).\mu(q), \nu(p) + \nu(q) - \nu(p).\nu(q)\rangle,$$
$$V(p \vee_2 q) = \langle \mu(p) + \mu(q) - \mu(p).\mu(q), \nu(p).\nu(q)\rangle.$$

Everywhere below we shall assume that for the two variables $p$ and $q$ there hold the equalities: $V(p) = \langle a, b\rangle, V(q) = \langle c, d\rangle$, $(a, b, c, d, a+b, c+d \in [0, 1])$.

For the needs of the discussion below, we shall define the notion of Intuitionistic Fuzzy Tautology (IFT, see [1, 12]) by:

$p$ is an IFT if and only if $a \geq b$,

while $p$ will be a tautology iff $a = 1$ and $b = 0$.

In some definitions, we use functions sg and $\overline{\text{sg}}$ defined by,

$$\text{sg}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases},$$

$$\overline{\text{sg}}(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x \leq 0 \end{cases}$$

In a series of papers of the author, starting with [1, 2], 138 different intuitionistic fuzzy implications and 34 different intuitionistic fuzzy negations, generated by the intuitionistic fuzzy implications were defined and some of their basic properties were studied. Meantime, in [18, 19, 20], Lilija Atanassova introduced implications $\rightarrow_{139}, ..., \rightarrow_{149}$ and negations $\neg_{35}, ..., \neg_{41}$; and in [21, 22, 23], Piotr Dworniczak introduced implications $\rightarrow_{150}, ..., \rightarrow_{152}$ and negations $\neg_{42}, ..., \neg_{45}$. The author introduced also implication $\rightarrow_{153}$ and negation $\neg_{46}$ in [8].

The list of all existing at the moment intuitionistic fuzzy implications and negations are given in Tables 1 and 2, respectively. The relations between the negations and implications are shown on Table 3.

Table 1: List of known implications

| | |
|---|---|
| $\rightarrow_1$ | $\langle \max(b, \min(a, c)), \min(a, d) \rangle$ |
| $\rightarrow_2$ | $\langle \overline{\text{sg}}(a - c), d.\text{sg}(a - c) \rangle$ |
| $\rightarrow_3$ | $\langle 1 - (1 - c).\text{sg}(a - c)), d.\text{sg}(a - c) \rangle$ |
| $\rightarrow_4$ | $\langle \max(b, c), \min(a, d) \rangle$ |
| $\rightarrow_5$ | $\langle \min(1, b + c), \max(0, a + d - 1) \rangle$ |
| $\rightarrow_6$ | $\langle b + ac, ad \rangle$ |
| $\rightarrow_7$ | $\langle \min(\max(b, c), \max(a, b), \max(c, d)), \max(\min(a, d),$ $\min(a, b), \min(c, d)) \rangle$ |
| $\rightarrow_8$ | $\langle 1 - (1 - \min(b, c)).\text{sg}(a - c), \max(a, d).\text{sg}(a - c), \text{sg}(d - b) \rangle$ |
| $\rightarrow_9$ | $\langle b + a^2 c, ab + a^2 d \rangle$ |
| $\rightarrow_{10}$ | $\langle c.\overline{\text{sg}}(1 - a) + \text{sg}(1 - a).(\overline{\text{sg}}(1 - c) + b.\text{sg}(1 - c)),$ $d.\overline{\text{sg}}(1 - a) + a.\text{sg}(1 - a).\text{sg}(1 - c) \rangle$ |
| $\rightarrow_{11}$ | $\langle 1 - (1 - c).\text{sg}(a - c), d.\text{sg}(a - c).\text{sg}(d - b) \rangle$ |
| $\rightarrow_{12}$ | $\langle \max(b, c), 1 - \max(b, c) \rangle$ |
| $\rightarrow_{13}$ | $\langle b + c - b.c, a.d \rangle$ |
| $\rightarrow_{14}$ | $\langle 1 - (1 - c).\text{sg}(a - c) - d.\overline{\text{sg}}(a - c).\text{sg}(d - b), d.\text{sg}(d - b) \rangle$ |
| $\rightarrow_{15}$ | $\langle 1 - (1 - \min(b, c)).\text{sg}(a - c).\text{sg}(d - b)$ $- \min(b, c).\text{sg}(a - c).\text{sg}(d - b),$ $1 - (1 - \max(a, d)).\text{sg}(\overline{\text{sg}}(a - c) + \overline{\text{sg}}(d - b))$ $- \max(a, d).\overline{\text{sg}}(a - c).\overline{\text{sg}}(d - b) \rangle$ |
| $\rightarrow_{16}$ | $\langle \max(\overline{\text{sg}}(a), c), \min(\text{sg}(a), d) \rangle$ |

*Continued on next page*

Table 1: List of known implications

| | |
|---|---|
| $\rightarrow_{17}$ | $\langle \max(b, c), \min(a.b + a^2, d)\rangle$ |
| $\rightarrow_{18}$ | $\langle \max(b, c), \min(1 - b, d)\rangle$ |
| $\rightarrow_{19}$ | $\langle \max(1 - \text{sg}(\text{sg}(a) + \text{sg}(1 - b)), c), \min(\text{sg}(1 - b), d)\rangle$ |
| $\rightarrow_{20}$ | $\langle \max(\overline{\text{sg}}(a), \text{sg}(c)), \min(\text{sg}(a), \overline{\text{sg}}(c))\rangle$ |
| $\rightarrow_{21}$ | $\langle \max(b, c.(c + d)), \min(a.(a + b), d.(c^2 + d + c.d))\rangle$ |
| $\rightarrow_{22}$ | $\langle \max(b, 1 - d), 1 - \max(b, 1 - d)\rangle$ |
| $\rightarrow_{23}$ | $\langle 1 - \min(\text{sg}(1 - b), \overline{\text{sg}}(1 - d)), \min(\text{sg}(1 - b), \overline{\text{sg}}(1 - d))\rangle$ |
| $\rightarrow_{24}$ | $\langle \overline{\text{sg}}(a - c).\overline{\text{sg}}(d - b), \text{sg}(a - c).\text{sg}(d - b)\rangle$ |
| $\rightarrow_{25}$ | $\langle \max(b, \overline{\text{sg}}(a).\overline{\text{sg}}(1 - b), c.\overline{\text{sg}}(d).\overline{\text{sg}}(1 - c)), \min(a, d)\rangle$ |
| $\rightarrow_{26}$ | $\langle \max(\overline{\text{sg}}(1 - b), c), \min(\text{sg}(a), d)\rangle$ |
| $\rightarrow_{27}$ | $\langle \max(\overline{\text{sg}}(1 - b), \text{sg}(c)), \min(\text{sg}(a), \overline{\text{sg}}(1 - d))\rangle$ |
| $\rightarrow_{28}$ | $\langle \max(\overline{\text{sg}}(1 - b), c), \min(a, d)\rangle$ |
| $\rightarrow_{29}$ | $\langle \max(\overline{\text{sg}}(1 - b), \overline{\text{sg}}(1 - c)), \min(a, \overline{\text{sg}}(1 - d))\rangle$ |
| $\rightarrow_{30}$ | $\langle \max(1 - a, \min(a, 1 - d)), \min(a, d)\rangle$ |
| $\rightarrow_{31}$ | $\langle \overline{\text{sg}}(a + d - 1), d.\text{sg}(a + d - 1)\rangle$ |
| $\rightarrow_{32}$ | $\langle 1 - d.\text{sg}(a + d - 1), d.\text{sg}(a + d - 1)\rangle$ |
| $\rightarrow_{33}$ | $\langle 1 - \min(a, d), \min(a, d)\rangle$ |
| $\rightarrow_{34}$ | $\langle \min(1, 2 - a - d), \max(0, a + d - 1)\rangle$ |
| $\rightarrow_{35}$ | $\langle 1 - a.d, a.d\rangle$ |
| $\rightarrow_{36}$ | $\langle \min(1 - \min(a, d), \max(a, 1 - a), \max(1 - d, d)),$<br>$\max(\min(a, d), \min(a, 1 - a), \min(1 - d, d))\rangle$ |
| $\rightarrow_{37}$ | $\langle 1 - \max(a, d).\text{sg}(a + d - 1), \max(a, d).\text{sg}(a + d - 1)\rangle$ |
| $\rightarrow_{38}$ | $\langle 1 - a + (a^2.(1 - d)), a.(1 - a) + a^2.d\rangle$ |
| $\rightarrow_{39}$ | $\langle (1 - d).\overline{\text{sg}}(1 - a) + \text{sg}(1 - a).(\overline{\text{sg}}(d) + (1 - a).\text{sg}(d)),$<br>$d.\overline{\text{sg}}(1 - a) + a.\text{sg}(1 - a).\text{sg}(d)\rangle$ |
| $\rightarrow_{40}$ | $\langle 1 - \text{sg}(a + d - 1), 1 - \overline{\text{sg}}(a + d - 1)\rangle$ |
| $\rightarrow_{41}$ | $\langle \max(\overline{\text{sg}}(a), 1 - d), \min(\text{sg}(a), d)\rangle$ |
| $\rightarrow_{42}$ | $\langle \max(\overline{\text{sg}}(a), \text{sg}(1 - d)), \min(\text{sg}(a), \overline{\text{sg}}(1 - d))\rangle$ |
| $\rightarrow_{43}$ | $\langle \max(\overline{\text{sg}}(a), 1 - d), \min(\text{sg}(a), d)\rangle$ |
| $\rightarrow_{44}$ | $\langle \max(\overline{\text{sg}}(a), 1 - d), \min(a, d)\rangle$ |
| $\rightarrow_{45}$ | $\langle \max(\overline{\text{sg}}(a), \overline{\text{sg}}(d)), \min(a, \overline{\text{sg}}(1 - d))\rangle$ |
| $\rightarrow_{46}$ | $\langle \max(b, \min(1 - b, c)), 1 - \max(b, c)\rangle$ |
| $\rightarrow_{47}$ | $\langle \overline{\text{sg}}(1 - b - c), (1 - c).\text{sg}(1 - b - c)\rangle$ |
| $\rightarrow_{48}$ | $\langle 1 - (1 - c).\text{sg}(1 - b - c), (1 - c).\text{sg}(1 - b - c)\rangle$ |
| $\rightarrow_{49}$ | $\langle \min(1, b + c), \max(0, 1 - b - c)\rangle$ |

*Continued on next page*

Table 1: List of known implications

| | |
|---|---|
| $\to_{50}$ | $\langle b+c-b.c, 1-b-c+b.c\rangle$ |
| $\to_{51}$ | $\langle \min(\max(b,c), \max(1-b,b), \max(c,1-c)),$ $\max(1-\max(b,c), \min(1-b,b), \min(c,1-c))\rangle$ |
| $\to_{52}$ | $\langle 1-(1-\min(b,c)).\text{sg}(1-b-c), 1-\min(b,c).\text{sg}(1-b-c)\rangle$ |
| $\to_{53}$ | $\langle b+(1-b)^2.c, (1-b).b+(1-b)^2.(1-c)\rangle$ |
| $\to_{54}$ | $\langle c.\overline{\text{sg}}(b)) + \text{sg}(b).(\overline{\text{sg}}(1-c)+b.\text{sg}(1-c)),$ $(1-c).\overline{\text{sg}}(b)+(1-b).\text{sg}(b).\text{sg}(1-c)\rangle$ |
| $\to_{55}$ | $\langle 1-\text{sg}(1-b-c), 1-\overline{\text{sg}}(1-b-c)\rangle$ |
| $\to_{56}$ | $\langle \max(\overline{\text{sg}}(1-b), c), \min(\text{sg}(1-b), (1-c))\rangle$ |
| $\to_{57}$ | $\langle \max(\overline{\text{sg}}(1-b), \text{sg}(c)), \min(\text{sg}(1-b), \overline{\text{sg}}(c))\rangle$ |
| $\to_{58}$ | $\langle \max(\overline{\text{sg}}(1-b), \overline{\text{sg}}(1-c)), 1-\max(b,c)\rangle$ |
| $\to_{59}$ | $\langle \max(\overline{\text{sg}}(1-b), c), (1-\max(b,c))\rangle$ |
| $\to_{60}$ | $\langle \max(\overline{\text{sg}}(1-b), \overline{\text{sg}}(1-c)), \min((1-b), \overline{\text{sg}}(c))\rangle$ |
| $\to_{61}$ | $\langle \max(c, \min(b,d)), \min(a,d)\rangle$ |
| $\to_{62}$ | $\langle \overline{\text{sg}}(d-b), a.\text{sg}(d-b)\rangle$ |
| $\to_{63}$ | $\langle 1-(1-b).\text{sg}(d-b), a.\text{sg}(d-b)\rangle$ |
| $\to_{64}$ | $\langle c+b.d, a.d\rangle$ |
| $\to_{65}$ | $\langle 1-(1-\min(b,c)).\text{sg}(d-b), \max(a,d).\text{sg}(d-b).\text{sg}(a-c)\rangle$ |
| $\to_{66}$ | $\langle c+d^2.b, b.d+d^2.a\rangle$ |
| $\to_{67}$ | $\langle b.\overline{\text{sg}}(1-d)+\text{sg}(1-d).(\overline{\text{sg}}(1-b)+c.\text{sg}(1-b)),$ $a.\overline{\text{sg}}(1-d)+d.\text{sg}(1-d).\text{sg}(1-b)\rangle$ |
| $\to_{68}$ | $\langle 1-(1-b).\text{sg}(d-b), a.\text{sg}(d-b).\text{sg}(a-c)\rangle$ |
| $\to_{69}$ | $\langle 1-(1-b).\text{sg}(d-b)-a.\overline{\text{sg}}(d-b).\text{sg}(a-c), a.\text{sg}(a-c)\rangle$ |
| $\to_{70}$ | $\langle \max(\overline{\text{sg}}(d), b), \min(\text{sg}(d), a)\rangle$ |
| $\to_{71}$ | $\langle \max(b,c), \min(c.d+d^2, a)\rangle$ |
| $\to_{72}$ | $\langle \max(b,c), \min(1-c, a)\rangle$ |
| $\to_{73}$ | $\langle \max(1-\max(\text{sg}(d), \text{sg}(1-c)), b), \min(\text{sg}(1-c), a)\rangle$ |
| $\to_{74}$ | $\langle \max(\text{sg}(b), \overline{\text{sg}}(d)), \min(\overline{\text{sg}}(b), \text{sg}(d))\rangle$ |
| $\to_{75}$ | $\langle \max(c, b.(a+b)), \min(d.(c+d), a.(b^2+a)+a.b)\rangle$ |
| $\to_{76}$ | $\langle \max(c, 1-a), \min(1-c, a)\rangle$ |
| $\to_{77}$ | $\langle (1-\min(\overline{\text{sg}}(1-a), \text{sg}(1-c))), \min(\overline{\text{sg}}(1-a), \text{sg}(1-c))\rangle$ |
| $\to_{78}$ | $\langle \max(\overline{\text{sg}}(1-c), b), \min(\text{sg}(d), a)\rangle$ |
| $\to_{79}$ | $\langle \max(\overline{\text{sg}}(1-c), \text{sg}(b)), \min(\text{sg}(d), \overline{\text{sg}}(1-a))\rangle$ |
| $\to_{80}$ | $\langle \max(\overline{\text{sg}}(1-c), b), \min(d, a)\rangle$ |
| $\to_{81}$ | $\langle \max(\overline{\text{sg}}(1-b), \overline{\text{sg}}(1-c)), \min(d, \overline{\text{sg}}(1-a))\rangle$ |

*Continued on next page*

Table 1: List of known implications

| | |
|---|---|
| $\to_{82}$ | $\langle \max(1-d, \min(d, 1-a)), \min(d, a)\rangle$ |
| $\to_{83}$ | $\langle \overline{\text{sg}}(a+d-1), a.\text{sg}(a+d-1)\rangle$ |
| $\to_{84}$ | $\langle 1-a.\text{sg}(a+d+1), a.\text{sg}(a+d+1)\rangle$ |
| $\to_{85}$ | $\langle 1-d+d^2.(1-a), d.(1-d)+d^2.\rangle$ |
| $\to_{86}$ | $\langle (1-a).\overline{\text{sg}}(1-d)+\text{sg}(1-d).\overline{\text{sg}}(a+\min(1-d, \text{sg}(a))),$ $a.\overline{\text{sg}}(1-d)+d.\text{sg}(1-d).\text{sg}(a)\rangle$ |
| $\to_{87}$ | $\langle \max(\overline{\text{sg}}(d), 1-a), \min(\text{sg}(d), a)\rangle$ |
| $\to_{88}$ | $\langle \max(\overline{\text{sg}}(d), \text{sg}(1-a)), \min(\text{sg}(d), \overline{\text{sg}}(1-a))\rangle$ |
| $\to_{89}$ | $\langle \max(\overline{\text{sg}}(d), 1-a), \min(d, a)\rangle$ |
| $\to_{90}$ | $\langle \max(\overline{\text{sg}}(a), \overline{\text{sg}}(d)), \min(d, \overline{\text{sg}}(1-a))\rangle$ |
| $\to_{91}$ | $\langle \max(c, \min(1-c, b)), 1-\max(b, c)\rangle$ |
| $\to_{92}$ | $\langle \overline{\text{sg}}(1-b-c), \min(1-b, \text{sg}(1-b-c))\rangle$ |
| $\to_{93}$ | $\langle (1-\min(1-b, \text{sg}(1-b-c)), \min(1-b, \text{sg}(1-b-c))\rangle$ |
| $\to_{94}$ | $\langle c+(1-c)^2.b, (1-c).c+(1-c)^2.(1-b)\rangle$ |
| $\to_{95}$ | $\langle \min(b, \overline{\text{sg}}(c))+\text{sg}(c).(\overline{\text{sg}}(1-b)+\min(c, \text{sg}(1-b))),$ $(\min(1-b, \overline{\text{sg}}(c))+\min(1-c, \text{sg}(c), \text{sg}(1-b))\rangle$ |
| $\to_{96}$ | $\langle \max(\overline{\text{sg}}(1-c), b), \min(\text{sg}(1-b), 1-c)\rangle$ |
| $\to_{97}$ | $\langle \max(\overline{\text{sg}}(1-c), \text{sg}(b)), \min(\text{sg}(1-c), \overline{\text{sg}}(b))\rangle$ |
| $\to_{98}$ | $\langle \max(\overline{\text{sg}}(1-c), b), 1-\max(b, c)\rangle$ |
| $\to_{99}$ | $\langle \max(\overline{\text{sg}}(1-c), \overline{\text{sg}}(1-b)), \min(1-c, \overline{\text{sg}}(b))\rangle$ |
| $\to_{100}$ | $\langle \max(\min(b, \text{sg}(a)), c), \min(a, \text{sg}(b), d)\rangle$ |
| $\to_{101}$ | $\langle \max(\min(b, \text{sg}(a)), \min(c, \text{sg}(d))), \min(a, \text{sg}(b), \text{sg}(c), d)\rangle$ |
| $\to_{102}$ | $\langle \max(b, \min(c, \text{sg}(d))), \min(a, \text{sg}(c), d)\rangle$ |
| $\to_{103}$ | $\langle \max(\min(1-a, \text{sg}(a)), 1-d), \min(a, \text{sg}(1-a), d)\rangle$ |
| $\to_{104}$ | $\langle \max(\min(1-a, \text{sg}(a)), \min(1-d, \text{sg}(d))),$ $\min(a, \text{sg}(1-a), d, \text{sg}(1-d))\rangle$ |
| $\to_{105}$ | $\langle \max(1-a, \min(1-d, \text{sg}(d))), \min(a, d, \text{sg}(1-d))\rangle$ |
| $\to_{106}$ | $\langle \max(\min(b, \text{sg}(1-b)), c), \min(1-b, \text{sg}(b), 1-c)\rangle$ |
| $\to_{107}$ | $\langle \max(\min(b, \text{sg}(1-b)), \min(c, \text{sg}(1-c))),$ $\min(1-b, \text{sg}(b), 1-c, \text{sg}(c))\rangle$ |
| $\to_{108}$ | $\langle \max(b, \min(c, \text{sg}(1-c))), \min(1-b, 1-c, \text{sg}(c))\rangle$ |
| $\to_{109}$ | $\langle b+\min(\overline{\text{sg}}(1-a), c), a.b+\min(\overline{\text{sg}}(1-a), d))\rangle$ |
| $\to_{110}$ | $\langle \max(b, c), \min(a.b+\overline{\text{sg}}(1-a), d)\rangle$ |
| $\to_{111}$ | $\langle \max(b, c.d+\overline{\text{sg}}(1-c)), \min(a.b+\overline{\text{sg}}(1-a),$ $d.(c.d+\overline{\text{sg}}(1-c))+\overline{\text{sg}}(1-d))\rangle$ |

*Continued on next page*

Table 1: List of known implications

| | |
|---|---|
| $\rightarrow_{112}$ | $\langle b+c-b.c, a.b+\overline{\text{sg}}(1-a).d\rangle$ |
| $\rightarrow_{113}$ | $\langle b+c.d-b.(c.d+\overline{\text{sg}}(1-c)),$ $(a.b+\overline{\text{sg}}(1-a)).(d.(c.d+\overline{\text{sg}}(1-c))+\overline{\text{sg}}(1-d))\rangle$ |
| $\rightarrow_{114}$ | $\langle 1-a+\min(\overline{\text{sg}}(1-a), 1-d), a.(1-a)+\min(\overline{\text{sg}}(1-a), d)\rangle$ |
| $\rightarrow_{115}$ | $\langle 1-\min(a,d), \min(a.(1-a)+\overline{\text{sg}}(1-a), d)\rangle$ |
| $\rightarrow_{116}$ | $\langle \max(1-a, (1-d).d+\overline{\text{sg}}(d)),$ $\min(a.(1-a)+\overline{\text{sg}}(1-a), d.((1-d).d+\overline{\text{sg}}(d))+\overline{\text{sg}}(1-d))\rangle$ |
| $\rightarrow_{117}$ | $\langle 1-a-d+a.d, (a.(1-a)+\overline{\text{sg}}(1-a)).d\rangle$ |
| $\rightarrow_{118}$ | $\langle 1-a+(1-d).d-(1-a).((1-d).d+\overline{\text{sg}}(d)),$ $(a.(1-a)+\overline{\text{sg}}(1-a)).d.((1-d).d+\overline{\text{sg}}(d))+\overline{\text{sg}}(1-d)\rangle$ |
| $\rightarrow_{119}$ | $\langle b+\min(\overline{\text{sg}}(b), c), (1-b).b+\min(\overline{\text{sg}}(b), 1-c)\rangle$ |
| $\rightarrow_{120}$ | $\langle \max(b,c), \min((1-b).b+\overline{\text{sg}}(b), 1-c)\rangle$ |
| $\rightarrow_{121}$ | $\langle \max(b, c.(1-c)+\overline{\text{sg}}(1-c)),$ $\min((1-b).b+\overline{\text{sg}}(b), (1-c).(c.(1-c)+\overline{\text{sg}}(1-c)))+\overline{\text{sg}}(c)\rangle$ |
| $\rightarrow_{122}$ | $\langle b+c-b.c, ((1-c).b+\overline{\text{sg}}(b)).(1-c)\rangle$ |
| $\rightarrow_{123}$ | $\langle b+c.(1-c)-(b.(c.(1-c)+\overline{\text{sg}}(1-c))),$ $((1-b).b+\overline{\text{sg}}(b)).(((1-c).(c.(1-c)+\overline{\text{sg}}(1-c)))+\overline{\text{sg}}(c))\rangle$ |
| $\rightarrow_{124}$ | $\langle c+\min(\overline{\text{sg}}(1-d), b), c.d+\min(\overline{\text{sg}}(1-d), a)\rangle$ |
| $\rightarrow_{125}$ | $\langle \max(b,c), \min(c.d+\overline{\text{sg}}(1-d), a)\rangle$ |
| $\rightarrow_{126}$ | $\langle \max(c, a.b+\overline{\text{sg}}(1-b)), \min(c.d+\overline{\text{sg}}(1-d),$ $a.(a.b+\overline{\text{sg}}(1-b))+\overline{\text{sg}}(1-a))\rangle$ |
| $\rightarrow_{127}$ | $\langle b+c-b.c, (c.d+\overline{\text{sg}}(1-d)).a\rangle$ |
| $\rightarrow_{128}$ | $\langle c+a.b-c.(a.b+\overline{\text{sg}}(1-b)),$ $(c.d+\overline{\text{sg}}(1-d)).(a.(a.b+\overline{\text{sg}}(1-b))+\overline{\text{sg}}(1-a))\rangle$ |
| $\rightarrow_{129}$ | $\langle 1-d+\min(\overline{\text{sg}}(1-d), 1-a), d.(1-d)+\min(\overline{\text{sg}}(1-d), a)\rangle$ |
| $\rightarrow_{130}$ | $\langle 1-\min(d,a), \min(d.(1-d)+\overline{\text{sg}}(1-d), a)\rangle$ |
| $\rightarrow_{131}$ | $\langle \max(1-d, (1-a).a+\overline{\text{sg}}(a)),$ $\min(d.(1-d)+\overline{\text{sg}}(1-d), a.((1-a).a+\overline{\text{sg}}(a))+\overline{\text{sg}}(1-a))\rangle$ |
| $\rightarrow_{132}$ | $\langle 1-a.d, (d.(1-d)+\overline{\text{sg}}(1-d)).a\rangle$ |
| $\rightarrow_{133}$ | $\langle 1-d+(1-a).a-(1-d).((1-a).a+\overline{\text{sg}}(a)),$ $(d.(1-d)+\overline{\text{sg}}(1-d)).(a.((1-a).a+\overline{\text{sg}}(a))+\overline{\text{sg}}(1-a))\rangle$ |
| $\rightarrow_{134}$ | $\langle c+\min(\overline{\text{sg}}(c), b), (1-c).c+\min(\overline{\text{sg}}(c), (1-b))\rangle$ |
| $\rightarrow_{135}$ | $\langle \max(b,c), \min((1-c).c+\overline{\text{sg}}(c), 1-b)\rangle$ |
| $\rightarrow_{136}$ | $\langle \max(c, (b.(1-b)+\overline{\text{sg}}(1-b))),$ $\min((1-c).c+\overline{\text{sg}}(c), (1-b).(b.(1-b)+\overline{\text{sg}}(1-b))+\overline{\text{sg}}(b))\rangle$ |

*Continued on next page*

Table 1: List of known implications

| | |
|---|---|
| $\rightarrow_{137}$ | $\langle b + c - b.c, ((1 - c).c + \overline{\text{sg}}(c)).(1 - b)\rangle$ |
| $\rightarrow_{138}$ | $\langle c + b.(1 - b) - c.(b.(1 - b) + \overline{\text{sg}}(1 - b)),$ <br> $((1 - c).c + \overline{\text{sg}}(c)).((1 - b).(b.(1 - b) + \overline{\text{sg}}(1 - b)) + \overline{\text{sg}}(b))\rangle$ |
| $\rightarrow_{139}$ | $\langle \frac{b+c}{2}, \frac{a+d}{2}\rangle$ |
| $\rightarrow_{140}$ | $\langle \frac{b+c+\min(b,c)}{3}, \frac{a+d+\max(a,d)}{3}\rangle$ |
| $\rightarrow_{141}$ | $\langle \frac{b+c+\max(b,c)}{3}, \frac{a+d+\min(a,d)}{3}\rangle$ |
| $\rightarrow_{142}$ | $\langle \frac{3-a-d-\max(a,d)}{3}, \frac{a+d+\max(a,d)}{3}\rangle$ |
| $\rightarrow_{143}$ | $\langle \frac{1-a+c+\min(1-a,c)}{3}, \frac{2+a-c-\min(1-a,c)}{3}\rangle$ |
| $\rightarrow_{144}$ | $\langle \frac{1+b-d+\min(b,1-d)}{3}, \frac{2-b+d+\min(b,1-d)}{3}\rangle$ |
| $\rightarrow_{145}$ | $\langle \frac{b+c+\min(b,c)}{3}, \frac{3-b-c-\min(b,c)}{3}\rangle$ |
| $\rightarrow_{146}$ | $\langle \frac{3-a-d-\min(a,d)}{3}, \frac{a+d+\min(a,d)}{3}\rangle$ |
| $\rightarrow_{147}$ | $\langle \frac{1-a+c+\max(1-a,c)}{3}, \frac{2+a-c-\max(1-a,c)}{3}\rangle$ |
| $\rightarrow_{148}$ | $\langle \frac{1+b-d+\max(b,1-d)}{3}, \frac{2-b+d-\max(b,1-d)}{3}\rangle$ |
| $\rightarrow_{149}$ | $\langle \frac{b+c+\max(b,c)}{3}, \frac{3-b-c-\max(b,c)}{3}$ |
| $\rightarrow_{150,\lambda}$ | $\langle \frac{b+c+\lambda-1}{2\lambda}, \frac{a+d+\lambda-1}{2\lambda}$, where $\lambda \geq 1$ |
| $\rightarrow_{151,\gamma}$ | $\langle \frac{b+c+\gamma}{2\gamma+1}, \frac{a+d+\gamma-1}{2\gamma+1}$, where $\gamma \geq 1$ |
| $\rightarrow_{152,\alpha,\beta}$ | $\langle \frac{b+c+\alpha-1}{\alpha+\beta}, \frac{a+d+\beta-1}{\alpha+\beta}$, where $\alpha \geq 1, \beta \in [0, \alpha]$ |
| $\rightarrow_{153,\varepsilon,\eta}$ | $\langle x, \min(1, \max(\mu_B(x), \nu_A(x) + \varepsilon)),$ <br> $\max(0, \min(\nu_B(x), \mu_A(x) - \eta))\rangle$ <br> where $\varepsilon, \eta \in [0, 1]$ and $\varepsilon \leq \eta$ |

Table 2: List of known negations

| | |
|---|---|
| $\neg_1$ | $\langle x, b, a\rangle$ |
| $\neg_2$ | $\langle x, \overline{\text{sg}}(a), \text{sg}(a)\rangle$ |
| $\neg_3$ | $\langle x, b, a.b + a^2\rangle$ |
| $\neg_4$ | $\langle x, b, 1 - b\rangle$ |
| $\neg_5$ | $\langle x, \overline{\text{sg}}(1 - b), \text{sg}(1 - b)\rangle$ |
| $\neg_6$ | $\langle x, \overline{\text{sg}}(1 - b), \text{sg}(a)\rangle$ |
| $\neg_7$ | $\langle x, \overline{\text{sg}}(1 - b), a\rangle$ |
| $\neg_8$ | $\langle x, 1 - a, a\rangle$ |
| $\neg_9$ | $\langle x, \overline{\text{sg}}(a), a\rangle$ |
| $\neg_{10}$ | $\langle x, \overline{\text{sg}}(1 - b), 1 - b\rangle$ |

*Continued on next page*

Table 2: List of known negations

| | |
|---|---|
| $\neg_{11}$ | $\langle x, \text{sg}(b), \overline{\text{sg}}(b)\rangle$ |
| $\neg_{12}$ | $\langle x, b.(b+a), \min(1, a.(b^2+a+b.a))\rangle$ |
| $\neg_{13}$ | $\langle x, \text{sg}(1-a), \overline{\text{sg}}(1-a)\rangle$ |
| $\neg_{14}$ | $\langle x, \text{sg}(b), \overline{\text{sg}}(1-a)\rangle$ |
| $\neg_{15}$ | $\langle x, \overline{\text{sg}}(1-b), \overline{\text{sg}}(1-a)\rangle$ |
| $\neg_{16}$ | $\langle x, \overline{\text{sg}}(a), \overline{\text{sg}}(1-a)\rangle$ |
| $\neg_{17}$ | $\langle x, \overline{\text{sg}}(1-b), \overline{\text{sg}}(b)\rangle$ |
| $\neg_{18}$ | $\langle x, b.\text{sg}(a), a.\text{sg}(b)\rangle$ |
| $\neg_{19}$ | $\langle x, b.\text{sg}(a), 0\rangle$ |
| $\neg_{20}$ | $\langle x, b, 0\rangle$ |
| $\neg_{21}$ | $\langle x, \min(1-a, \text{sg}(a)), \min(a, \text{sg}(1-a))\rangle$ |
| $\neg_{22}$ | $\langle x, \min(1-a, \text{sg}(a)), 0\rangle$ |
| $\neg_{23}$ | $\langle x, 1-a, 0\rangle$ |
| $\neg_{24}$ | $\langle x, \min(b, \text{sg}(1-b)), \min(1-b, \text{sg}(b))\rangle$ |
| $\neg_{25}$ | $\langle x, \min(b, \text{sg}(1-b)), 0\rangle$ |
| $\neg_{26}$ | $\langle x, b, a.b + \overline{\text{sg}}(1-a)\rangle$ |
| $\neg_{27}$ | $\langle x, 1-a, a.(1-a) + \overline{\text{sg}}(1-a)\rangle$ |
| $\neg_{28}$ | $\langle x, b, (1-b).b + \overline{\text{sg}}(b)\rangle$ |
| $\neg_{29}$ | $\langle x, \max(0, b.a + \overline{\text{sg}}(1-b)),$ $\min(1, a.(b.a + \overline{\text{sg}}(1-b)) + \overline{\text{sg}}(1-a))\rangle$ |
| $\neg_{30}$ | $\langle x, a.b, a.(a.b + \overline{\text{sg}}(1-b)) + \overline{\text{sg}}(1-a)\rangle$ |
| $\neg_{31}$ | $\langle x, \max(0, (1-a).a + \overline{\text{sg}}(a)),$ $\min(1, a.((1-a).a + \overline{\text{sg}}(a)) + \overline{\text{sg}}(1-a))\rangle$ |
| $\neg_{32}$ | $\langle x, (1-a).a, a.((1-a).a + \overline{\text{sg}}(a)) + \overline{\text{sg}}(1-a)\rangle$ |
| $\neg_{33}$ | $\langle x, b.(1-b) + \overline{\text{sg}}(1-b), (1-b).(b.(1-b) + \overline{\text{sg}}(1-b)) + \overline{\text{sg}}(b))\rangle$ |
| $\neg_{34}$ | $\langle x, b.(1-b), (1-b).(b.(1-b) + \overline{\text{sg}}(1-b)) + \overline{\text{sg}}(b)\rangle$ |
| $\neg_{35}$ | $\langle \frac{b}{2}, \frac{1+a}{2}\rangle$ |
| $\neg_{36}$ | $\langle \frac{b}{3}, \frac{2+a}{3}\rangle$ |
| $\neg_{37}$ | $\langle \frac{2b}{3}, \frac{2a+1}{3}\rangle$ |
| $\neg_{38}$ | $\langle \frac{1-a}{3}, \frac{2+a}{3}\rangle$ |
| $\neg_{39}$ | $\langle \frac{b}{3}, \frac{3-b}{3}\rangle$ |
| $\neg_{40}$ | $\langle \frac{2-2a}{3}, \frac{1+2a}{3}\rangle$ |
| $\neg_{41}$ | $\langle \frac{2b}{3}, \frac{3-2b}{3}\rangle$ |
| $\neg_{42,\lambda}$ | $\langle \frac{b+\lambda-1}{2\lambda}, \frac{a+\lambda}{2\lambda}$, where $\lambda \geq 1$ |
| $\neg_{43,\gamma}$ | $\langle \frac{b+\gamma}{2\gamma+1}, \frac{a+\gamma}{2\gamma+1}$, where $\gamma \geq 1$ |

*Continued on next page*

Table 2: List of known negations

| | |
|---|---|
| $\neg_{44,\alpha,\beta}$ | $\langle\frac{b+\alpha-1}{\alpha+\beta}, \frac{a+\beta}{\alpha+\beta}$, where $\alpha \geq 1, \beta \in [0, \alpha]$ |
| $\neg_{45,\varepsilon,\eta}$ | $\langle\min(1, \nu_A(x) + \varepsilon), \max(0, \mu_A(x) - \eta)\rangle$<br>where $\varepsilon, \eta \in [0, 1]$ and $\varepsilon \leq \eta$ |

Table 3: Relations between negations and implications

| | |
|---|---|
| $\neg_1$ | $\rightarrow_1, \rightarrow_4, \rightarrow_5, \rightarrow_6, \rightarrow_7, \rightarrow_{10}, \rightarrow_{13}, \rightarrow_{61}, \rightarrow_{63}, \rightarrow_{64}, \rightarrow_{66}, \rightarrow_{67},$<br>$\rightarrow_{68}, \rightarrow_{69}, \rightarrow_{70}, \rightarrow_{71}, \rightarrow_{72}, \rightarrow_{73}, \rightarrow_{78}, \rightarrow_{80}, \rightarrow_{124}, \rightarrow_{125}, \rightarrow_{127}$ |
| $\neg_2$ | $\rightarrow_2, \rightarrow_3, \rightarrow_8, \rightarrow_{11}, \rightarrow_{16}, \rightarrow_{20}, \rightarrow_{31}, \rightarrow_{32}, \rightarrow_{37}, \rightarrow_{40}, \rightarrow_{41}, \rightarrow_{42}$ |
| $\neg_3$ | $\rightarrow_9, \rightarrow_{17}, \rightarrow_{21}$ |
| $\neg_4$ | $\rightarrow_{12}, \rightarrow_{18}, \rightarrow_{22}, \rightarrow_{46}, \rightarrow_{49}, \rightarrow_{50}, \rightarrow_{51}, \rightarrow_{53}, \rightarrow_{54}, \rightarrow_{91}, \rightarrow_{93},$<br>$\rightarrow_{94}, \rightarrow_{95}, \rightarrow_{96}, \rightarrow_{98}, \rightarrow_{134}, \rightarrow_{135}, \rightarrow_{137}$ |
| $\neg_5$ | $\rightarrow_{14}, \rightarrow_{15}, \rightarrow_{19}, \rightarrow_{23}, \rightarrow_{47}, \rightarrow_{48}, \rightarrow_{52}, \rightarrow_{55}, \rightarrow_{56}, \rightarrow_{57}$ |
| $\neg_6$ | $\rightarrow_{24}, \rightarrow_{26}, \rightarrow_{27}, \rightarrow_{65}$ |
| $\neg_7$ | $\rightarrow_{25}, \rightarrow_{28}, \rightarrow_{29}, \rightarrow_{62}$ |
| $\neg_8$ | $\rightarrow_{30}, \rightarrow_{33}, \rightarrow_{34}, \rightarrow_{35}, \rightarrow_{36}, \rightarrow_{38}, \rightarrow_{39}, \rightarrow_{76}, \rightarrow_{82}, \rightarrow_{84}, \rightarrow_{85},$<br>$\rightarrow_{86}, \rightarrow_{87}, \rightarrow_{89}, \rightarrow_{129}, \rightarrow_{130}, \rightarrow_{132}$ |
| $\neg_9$ | $\rightarrow_{43}, \rightarrow_{44}, \rightarrow_{45}, \rightarrow_{83}$ |
| $\neg_{10}$ | $\rightarrow_{58}, \rightarrow_{59}, \rightarrow_{60}, \rightarrow_{92}$ |
| $\neg_{11}$ | $\rightarrow_{74}, \rightarrow_{97}$ |
| $\neg_{12}$ | $\rightarrow_{75}$ |
| $\neg_{13}$ | $\rightarrow_{77}, \rightarrow_{88}$ |
| $\neg_{14}$ | $\rightarrow_{79}$ |
| $\neg_{15}$ | $\rightarrow_{81}$ |
| $\neg_{16}$ | $\rightarrow_{90}$ |
| $\neg_{17}$ | $\rightarrow_{99}$ |
| $\neg_{18}$ | $\rightarrow_{100}$ |
| $\neg_{19}$ | $\rightarrow_{101}$ |
| $\neg_{20}$ | $\rightarrow_{102}, \rightarrow_{108}$ |
| $\neg_{21}$ | $\rightarrow_{103}$ |
| $\neg_{22}$ | $\rightarrow_{104}$ |
| $\neg_{23}$ | $\rightarrow_{105}$ |
| $\neg_{24}$ | $\rightarrow_{106}$ |
| $\neg_{25}$ | $\rightarrow_{107}$ |
| $\neg_{26}$ | $\rightarrow_{109}, \rightarrow_{110}, \rightarrow_{111}, \rightarrow_{112}, \rightarrow_{113}$ |

*Continued on next page*

Table 3: Relations between negations and implications

| | |
|---|---|
| $\urcorner 27$ | $\rightharpoonup 114$, $\rightharpoonup 115$, $\rightharpoonup 116$, $\rightharpoonup 117$, $\rightharpoonup 118$ |
| $\urcorner 28$ | $\rightharpoonup 119$, $\rightharpoonup 120$, $\rightharpoonup 121$, $\rightharpoonup 122$, $\rightharpoonup 123$ |
| $\urcorner 29$ | $\rightharpoonup 126$ |
| $\urcorner 30$ | $\rightharpoonup 128$ |
| $\urcorner 31$ | $\rightharpoonup 131$ |
| $\urcorner 32$ | $\rightharpoonup 133$ |
| $\urcorner 33$ | $\rightharpoonup 136$ |
| $\urcorner 34$ | $\rightharpoonup 138$ |

In [7], the following forms of the De Morgan's Laws

$$\neg(\neg x \vee \neg y) = x \wedge y$$

$$\neg(\neg x \wedge \neg y) = x \vee y$$

and

$$\neg(\neg x \vee \neg y) = \neg\neg x \wedge \neg\neg y$$

$$\neg(\neg x \wedge \neg y) = \neg\neg x \vee \neg\neg y$$

and the following forms of the Law for Excluded Third (Middle)

$$x \vee \neg x$$

$$\neg\neg x \vee \neg x$$

are discussed.

**Open problem 1.** Having in mind the above modified formulas, modify the Law of contraposition and study which implications and the respective negations satisfy the standard and modified forms of this Law.

If some implication $\supset$ and negation $\neg$ satisfy the Law of contraposition in its standard (modified) form, we call that pair $(\supset, \neg)$ is respectively compatible with the standard (modified) Law of contraposition.

**Open problem 2.** Which pairs $(\supset, \neg)$ are respectively compatible with the standard (modified) Law of contraposition?

In [13], it is proved for the first time that the axiom

$$(\neg A \supset \neg B) \supset ((\neg A \supset B) \supset A)$$

is an IFT, when $\supset$ is $\rightarrow_4$, while it is not valid for implication $\rightarrow_3$.

Having in mind the new forms of the De Morgan's Laws and the Law for Excluded Third, the above axiom can be modified to the form

$$(\neg A \supset \neg B) \supset ((\neg A \supset \neg\neg B) \supset \neg\neg A).$$

In [13], we introduced the four theorems given below and proved one of them. Now it is clear that this proof had to be given some 25 years ago.

**Theorem 1.** For every two variables $A$ and $B$, the expression

$$(\neg A \supset \neg B) \supset ((\neg A \supset B) \supset A) \quad (1)$$

is an IFT for implications $\rightarrow_1$, $\rightarrow_4$, $\rightarrow_5$, $\rightarrow_7$, $\rightarrow_9$, $\rightarrow_{13}$, $\rightarrow_{18}$, $\rightarrow_{20}$, ..., $\rightarrow_{23}$, $\rightarrow_{25}$, $\rightarrow_{27}$, ..., $\rightarrow_{29}$, $\rightarrow_{61}$, $\rightarrow_{66}$, $\rightarrow_{71}$, $\rightarrow_{74}$, $\rightarrow_{76}$, $\rightarrow_{77}$, $\rightarrow_{79}$, $\rightarrow_{81}$, $\rightarrow_{100}$, ..., $\rightarrow_{102}$, $\rightarrow_{104}$, $\rightarrow_{105}$, $\rightarrow_{107}$, $\rightarrow_{109}$, ..., $\rightarrow_{113}$, $\rightarrow_{118}$, $\rightarrow_{124}$, ..., $\rightarrow_{128}$, $\rightarrow_{133}$ .

**Theorem 2.** For every two variables $A$ and $B$, the expression

$$(\neg A \supset \neg B) \supset ((\neg A \supset B) \supset A)$$

is a tautology for implications $\rightarrow_{20}$, $\rightarrow_{23}$, $\rightarrow_{74}$, $\rightarrow_{77}$.

**Theorem 3.** For every two variables $A$ and $B$, the expression

$$(\neg A \supset \neg B) \supset ((\neg A \supset \neg\neg B) \supset \neg\neg A)$$

is an IFT for implications $\rightarrow_1$, ..., $\rightarrow_5$, $\rightarrow_7$, ..., $\rightarrow_9$, $\rightarrow_{11}$, ..., $\rightarrow_{38}$, $\rightarrow_{40}$, ..., $\rightarrow_{43}$, $\rightarrow_{45}$, ..., $\rightarrow_{53}$, $\rightarrow_{55}$, ..., $\rightarrow_{57}$, $\rightarrow_{61}$, $\rightarrow_{66}$, $\rightarrow_{71}$, $\rightarrow_{74}$, ..., $\rightarrow_{77}$, $\rightarrow_{79}$, $\rightarrow_{81}$, ..., $\rightarrow_{83}$, $\rightarrow_{85}$, $\rightarrow_{88}$, $\rightarrow_{91}$, $\rightarrow_{94}$, $\rightarrow_{97}$, $\rightarrow_{99}$, ..., $\rightarrow_{119}$, $\rightarrow_{121}$, $\rightarrow_{124}$, ..., $\rightarrow_{137}$ .

**Theorem 4.** For every two variables $A$ and $B$, the expression

$$(\neg A \supset \neg B) \supset ((\neg A \supset \neg\neg B) \supset \neg\neg A)$$

is a tautology for implications $\rightarrow_2$, $\rightarrow_3$, $\rightarrow_8$, $\rightarrow_{11}$, $\rightarrow_{14}$, ..., $\rightarrow_{16}$, $\rightarrow_{19}$, $\rightarrow_{20}$, $\rightarrow_{23}$, $\rightarrow_{31}$, $\rightarrow_{32}$, $\rightarrow_{37}$, $\rightarrow_{40}$, ..., $\rightarrow_{43}$, $\rightarrow_{45}$, $\rightarrow_{47}$, $\rightarrow_{48}$, $\rightarrow_{52}$, $\rightarrow_{55}$, ..., $\rightarrow_{57}$, $\rightarrow_{74}$, $\rightarrow_{77}$, $\rightarrow_{83}$, $\rightarrow_{88}$, $\rightarrow_{97}$, $\rightarrow_{99}$ .

**Open problem 3.** Which pairs $(\supset, \neg)$ are compatible in respect with the standard (modified) Law from (1)?

Let $\mathcal{F}$ be a set of formulas with the property that for all $\langle a, b\rangle \in [0,1] \times [0,1]$ such that $a + b \leq 1$, there exists a formula $f \in \mathcal{F}$ such that $V(F) = \langle a, b\rangle$. In [6, 9], the following assertion is proved for the cases of implications $\rightarrow_3$ and $\rightarrow_4$

**Theorem 5.** Let $F$ and $G$ be different formulas in $\mathcal{F}$ and let $F \rightarrow G$ be a tautology. Then there exists a formula $H \in \mathcal{F}$ different from $F$ and $G$, such that $F \rightarrow H$ and $H \rightarrow G$ are tautologies.

**Open problem 4.** Check for which other implications a similar (as in Theorem 5) result is valid.

The most important problem is related to T- and S-norms, defined for intuitionistic fuzzy case. In the present moment, they satisfy the standard De Morgan Laws and therefore, they are based on the classical negation $\neg_1$.
**Open problem 5.** Develop a new theory of T- and S-norms for intuitionistic fuzzy case, that are based on the modified forms of De Morgan's Laws.

## 3 Short remarks on IF predicate logic

Let $x$ be a variable, obtaining values in set $E$ and let $P(x)$ be a predicate with a variable $x$. Let

$$V(P(x)) = \langle \mu(P(x)), \nu(P(x)) \rangle.$$

The IF-interpretations of the quantifiers *for all* ($\forall$) and *there exists* ($\exists$) are introduced in [14] by the formulas:

$$V(\forall x P(x)) = \langle \sup_{y \in E} \mu(P(y)), \inf_{y \in E} \nu(P(y)) \rangle,$$

$$V(\exists x P(x)) = \langle \inf_{y \in E} \mu(P(y)), \sup_{y \in E} \nu(P(y)) \rangle.$$

Their geometrical interpretations are illustrated in Figs. 4 and 5, respectively, where $x_1, ..., x_5$ are the possible values of variable $x$ and $V(x_1), ..., V(x_5)$, their IF-estimations.

The most important property of the two quantifiers is that each of them juxtaposes to predicate $P$ a point (exactly one for each quantifier) in the IF-interpretational triangle.

In [11], we introduced the following six quantifiers and studied some of their properties.

$$V(\forall_\mu x P(x)) = \{\langle x, \inf_{y \in E} \mu(P(y)), \nu(P(x)) \rangle | x \in E\},$$

$$V(\forall_\nu x P(x)) = \{\langle x, \min(1 - \sup_{y \in E} \nu(P(y)), \mu(P(x)), \sup_{y \in E} \nu(P(y)) \rangle | x \in E\},$$

$$V(\exists_\mu x P(x)) = \{\langle x, \sup_{y \in E} \mu(P(y)), \min(1 - \sup_{y \in E} \mu(P(y)), \nu(P(x)) \rangle | x \in E\},$$

$$V(\exists_\nu x P(x)) = \{\langle x, \mu(P(x)), \inf_{y \in E} \nu(P(y)) \rangle | x \in E\},$$

$$V(\forall_\nu^* x P(x)) = \{\langle x, \min(1 - \sup_{y \in E} \nu(P(y)), \mu(P(x)),$$

$$\min(\sup_{y \in E} \nu(P(y)), 1 - \mu(P(x))\rangle | x \in E\},$$

$$V(\exists^*_\mu x P(x)) = \{\langle x, \min(\sup_{y \in E} \mu(P(y)), 1 - \nu(P(x)),$$

$$\min(1 - \sup_{y \in E} \mu(P(y)), \nu(P((x))\rangle | x \in E\}.$$

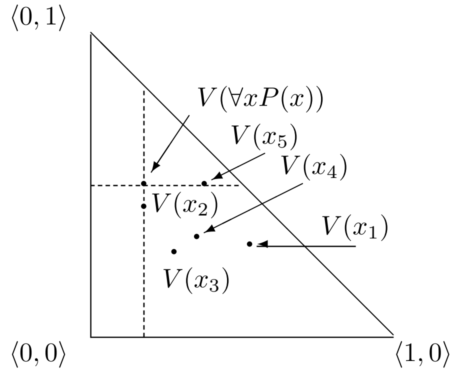



Fig. 4.

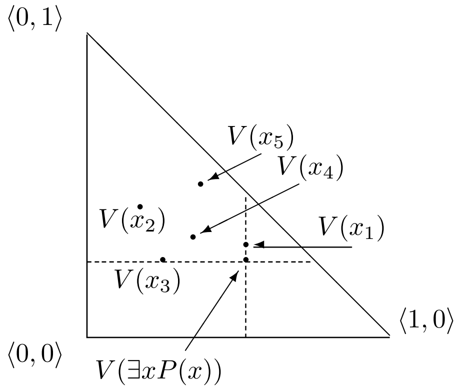



Fig. 5.

Let the possible values of variable $x$ be $a, b, c$ and let their IF-estimations $V(a), V(b), V(c)$ be shown on Fig. 6. The geometrical interpretations of the new quantifiers are shown in Figs. 7-12.

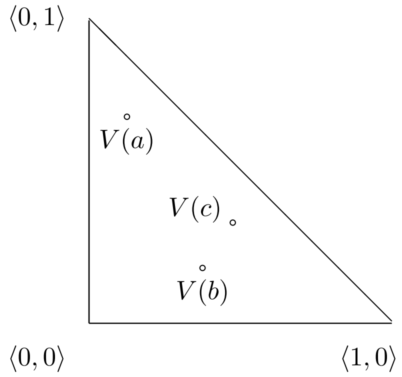



Fig. 6.

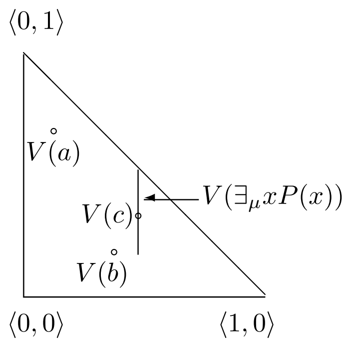



Fig. 7

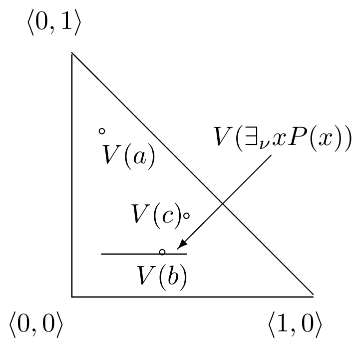



Fig. 8.

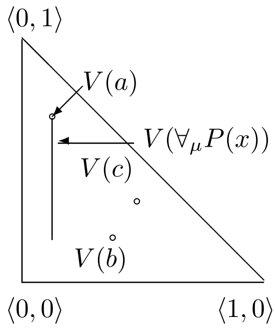



Fig. 9.

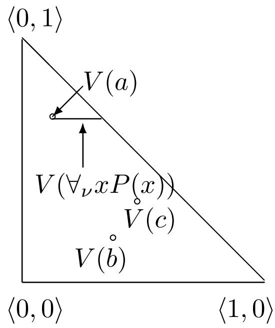



Fig. 10.

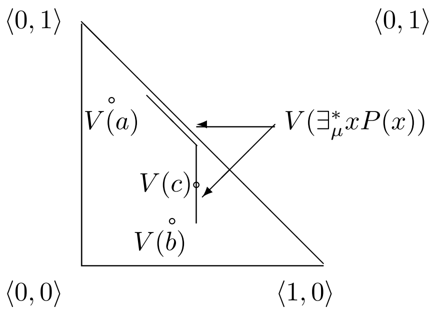



Fig. 11.

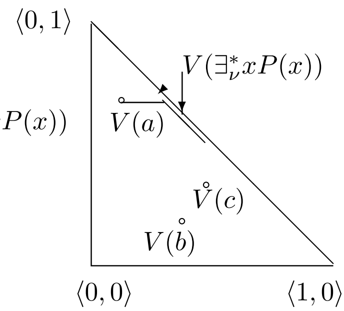



Fig. 12.

Now, we see that we can change the forms of the first two quantifiers to the forms

$$V(\forall xP(x)) = \{\langle x, \inf_{y\in E} \mu(P(y)), \sup_{y\in E} \nu(P(y))\rangle | x \in E\},$$

$$V(\exists xP(x)) = \{\langle x, \sup_{y\in E} \mu(P(y)), \inf_{y\in E} \nu(P(y))\rangle | x \in E\}.$$

Obviously, for every predicate $P$,

$$V(\forall xP(x)) \subseteq V(\forall_\mu xP(x)) \subseteq V(\forall_\nu xP(x)) \subseteq V(\exists_\nu xP(x))$$

$$\subseteq V(\exists_\mu xP(x)) \subseteq V(\exists xP(x))$$

and

$$V(\forall xP(x)) \subseteq V(\forall_\nu xP(x)) \subseteq V(\forall_\nu^* xP(x)) \subseteq V(\exists_\mu^* xP(x))$$

$$\subseteq V(\exists_\mu x P(x)) \subseteq V(\exists x P(x)).$$

The axioms of the logical system **K** (see, e.g., [25]) related to quantifiers, are:
(A4) $(\forall x)(B(x) \supset B(t))$ is a well-formed formula and $t$ is a term that is free for $x$ in $B(x)$
(A5) $(\forall x)(B \supset A) \supset (B \supset (\forall x)A)$ if $B$ contains no free occurrences of $x$.
**Open problem 6.** Which implications satisfy axioms (A4), (A5)?

# 4 Short remarks on IF modal logic

For a proposition $x$ for which:

$$V(x) = \langle a, b \rangle$$

the following analogues of the classical modal operators are defined:

$$V(\Box x) = \langle a, 1-a \rangle,$$

$$V(\Diamond x) = \langle 1-b, b \rangle.$$

Let the truth value function $V$ be defined:

$$\Box V(x) = V(\Box x),$$

$$\Diamond V(x) = V(\Diamond x).$$

For them is proved that following assertions are tautologies (24.0 and 24.1 in [24]):

(a) $\neg \Box A \equiv \Diamond \neg A,$

(b) $\Box A \equiv \neg \Diamond \neg A,$

(c) $\neg \Diamond A \equiv \Box \neg A,$

(d) $\Diamond A \equiv \neg \Box \neg A.$

(e) $\Box A \supset A,$

(f) $A \supset \Diamond A,$

(g) $\Box A \supset \Diamond A.$

Let $A$ be a fixed propositional form and $\alpha, \beta, \gamma, \delta, \varepsilon, \eta \in [0, 1]$. We define operators $D_\alpha$, $F_{\alpha,\beta}$ (for $\alpha + \beta \leq 1$), $G_{\alpha,\beta}$, $H_{\alpha,\beta}$, $H^*_{\alpha,\beta}$, $J_{\alpha,\beta}$, $J^*_{\alpha,\beta}$ and $X_{\alpha,\beta,\gamma,\delta,\varepsilon,\eta}$

by:

$$\begin{aligned}
&V(D_\alpha(A)) = \langle a+\alpha.(1-a-b), b+(1-\alpha).(1-a-b)\rangle,\\
&V(F_{\alpha,\beta}(A)) = \langle a+\alpha.(1-a-b), b+\beta.(1-a-b)\rangle, \text{ for } \alpha+\beta\le 1,\\
&V(G_{\alpha,\beta}(A)) = \langle \alpha.a, \beta.b\rangle,\\
&V(H_{\alpha,\beta}(A)) = \langle \alpha.a, b+\beta.(1-a-b)\rangle,\\
&V(H^*_{\alpha,\beta}(A)) = \langle \alpha.a, b+\beta.(1-\alpha.a-b)\rangle,\\
&V(J_{\alpha,\beta}(A)) = \langle a+\alpha.(1-a-b), \beta.b\rangle,\\
&V(J^*_{\alpha,\beta}(A)) = \langle a+\alpha.(1-a-\beta.b), \beta.b\rangle,\\
&V(X_{\alpha,\beta,\gamma,\delta,\varepsilon,\eta}(A)) = \langle \alpha.a+\beta.(1-a-\gamma.b), \delta.b+\varepsilon.(1-\eta.a-b)\rangle,
\end{aligned}$$

for $\alpha+\varepsilon-\varepsilon.\eta\le 1$, $\beta+\delta-\beta.\gamma\le 1$ and $\beta+\varepsilon\le 1$. We must mention, that the latest inequality is omitted in [5, 12] and it is added to the definition of operator $X_{\alpha,\beta,\gamma,\delta,\varepsilon,\eta}$ in [10].

Obviously,

$$\begin{aligned}
&\Box A = D_0(A),\\
&\Diamond A = D_1(A),\\
&D_\alpha(A) = F_{\alpha,1-\alpha}(A),
\end{aligned}$$

and

$$\begin{aligned}
&\Box A = X_{1,0,r,1,1,1}(A),\\
&\Diamond A = X_{1,1,1,1,0,r}(A),\\
&D_\alpha(A) = X_{1,\alpha,1,1,1-\alpha,1}(A),\\
&F_{\alpha,\beta}(A) = X_{1,\alpha,1,1,\beta,1}(A), \text{ for } \alpha+\beta\le 1,\\
&G_{\alpha,\beta}(A) = X_{\alpha,0,r,\beta,0,r}(A),\\
&H_{\alpha,\beta}(A) = X_{\alpha,0,r,1,\beta,1}(A),\\
&H^*_{\alpha,\beta}(A) = X_{\alpha,0,r,\beta,0,\alpha}(A),\\
&J_{\alpha,\beta}(A) = X_{1,\alpha,1,\beta,0,r}(A),\\
&J^*_{\alpha,\beta}(A) = X_{1,\alpha,\beta,\beta,0,r}(A),
\end{aligned}$$

where $r$ is an arbitrary real number in $[0, 1]$.

**Open problem 7.** Construct an axiomatic system of the IFML.

Such axiomatic system will contain special axioms related to the new modal-type of operators $(D, F, G, H, H^*, J, J^*)$. Some of the first order axioms must be changed with axioms, related to some of the above operators.

For example, the axiom $\neg\neg A \supset A$ (see [25]) can be changed with the the axiom

$H_{\alpha,\beta}(A) \supset A$ or with the axiom $H^*_{\alpha,\beta}(A) \supset A$ for every $0 \leq \alpha \leq 1$ and for every $\beta \geq 0$. We must note that $V(H_{1,0}(A)) = V(H^*_{1,0}(A)) = V(A) = V(\neg\neg A)$ and $V(H^*_{0,1}(A)) = \langle 0, 1\rangle \equiv$ "$FALSE$".

# 5 Temporal Intuitionistic Fuzzy Logic (TIFL)

Let $p$ be a proposition and $V$ be a truth-value function, which maps the ordered pair:

$$V(p,t) = \langle \mu(p,t), \nu(p,t)\rangle$$

to the proposition $p$ and to the time-moment $t \in T$ (where $T$ is a fixed set which we shall call "time-scale" and it is strictly oriented by the relation "$<$").

Let

$$T' = \{t'/t' \in T \& t' < t\}$$

$$T'' = \{t''/t'' \in T \& t'' > t\}$$

In [4], the author defined for given $p$ and $t$ the operators $P, F, H, G$, for which:

$$V(P(p,t)) = \langle \mu(p,t'), \nu(p,t')\rangle,$$

where $t' \in T'$ satisfies the conditions:

(a)

$$\mu(p,t') - \nu(p,t') = \max_{t* \in T'} (\mu(p,t*) - \nu(p,t*)),$$

(b) if there exists more than one such element of $T'$, then $t'$ is the maximal;

$$V(F(p,t)) = \langle \mu(p,t''), \nu(p,t'')\rangle,$$

where $t'' \in T''$ satisfies the conditions:

(a)

$$\mu(p,t'') - \nu(p,t'') = \max_{t* \in T''} (\mu(p,t*) - \nu(p,t*)),$$

(b) if there exist more than one such element of $T''$, then $t''$ is the minimal;

$$V(H(p,t)) = \langle \mu(p,t'), \nu(p,t')\rangle,$$

where $t' \in T'$ satisfies the conditions:

(a)

$$\mu(p,t') - \nu(p,t') = \min_{t* \in T'} (\mu(p,t*) - \nu(p,t*)),$$

(b) if there exis more than one such elements of $T'$, then $t'$ is the maximal;

$$V(G(p,t)) = \langle \mu(p,t''), \nu(p,t'') \rangle,$$

where $t'' \in T''$ satisfies the conditions:

(a)

$$\mu(p,t'') - \nu(p,t'') = \min_{t* \in T''} (\mu(p,t*) - \nu(p,t*)),$$

(b) if there exist more than one such elements of $T''$, then $t''$ is the minimal.

**Theorem 6.** For every proposition $p$ and for every time moment $t$:

$$(a) V(H(p,t)) = V(\neg(P(\neg p),t));$$

$$(b) V(G(p,t)) = V(\neg(F(\neg p),t)).$$

A geometrical interpretation of the temporal IFL is given on Fig. 13.

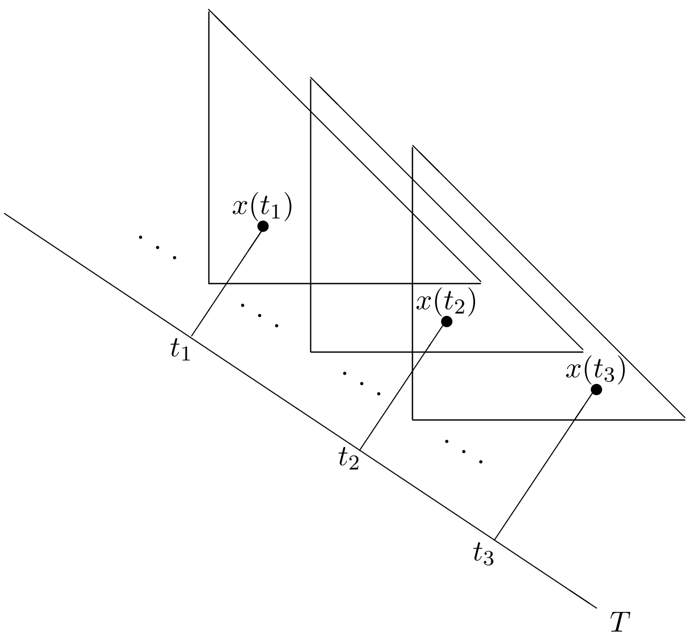



Fig. 13.

Let sets $Z_1, Z_2, ..., Z_n$ be fixed and let for each $i$ $(1 \le i \le n) : z_i \in Z_i$.

By analogy with intuitionistic fuzzy multi-dimensional sets, introduced in [15, 16, 17], here for a first time, we define intuitionistic fuzzy multi-dimensional evaluation function $V$ in the form

$$V(x, z_1, z_2, ..., z_n) = \langle x, \mu_A(x, z_1, z_2, ..., z_n), \nu_A(x, z_1, z_2, ..., z_n) \rangle,$$

where $x$ is a variable, $z_1, z_2, ..., z_n$ are additional arguments,

$$\mu_A(x, z_1, z_2, ..., z_n) + \nu_A(x, z_1, z_2, ..., z_n) \leq 1,$$

and $\mu_A(x, z_1, z_2, ..., z_n)$ and $\nu_A(x, z_1, z_2, ..., z_n)$ are the degrees of membership and non-membership, respectively, of the element $x \in E$ in respect of the additional arguments $z_1, z_2, ..., z_n$.

In the particular case, when $n = 1$, we obtain the case of temporal IIFL.

**Open problem 8.** Which are the forms of the temporal operators $P, F, H, G$, when $n > 1$?

Having in mind the results from [15], we can define for the $(n+1)$-dimensional predicate $P$, the (partial) $i$-quantifiers

$$V(\exists_i(x, z_1, z_2, ..., z_n)P(x, z_1, z_2, ..., z_n)$$
$$= \langle \sup_{t_i \in Z_i} \mu(x, z_1, z_2, ..., z_{i-1}, t_i, z_{i+1}, ..., z_n),$$
$$\inf_{t_i \in Z_i} \nu(x, z_1, z_2, ..., z_{i-1}, t_i, z_{i+1}, ..., z_n)\rangle,$$
$$V(\forall_i(x, z_1, z_2, ..., z_n)P(x, z_1, z_2, ..., z_n)$$
$$= \langle \inf_{t_i \in Z_i} \mu(x, z_1, z_2, ..., z_{i-1}, t_i, z_{i+1}, ..., z_n),$$
$$\sup_{t_i \in Z_i} \nu(x, z_1, z_2, ..., z_{i-1}, t_i, z_{i+1}, ..., z_n)\rangle$$

and the general quantifiers

$$V(\exists(x, z_1, z_2, ..., z_n)P(x, z_1, z_2, ..., z_n)$$
$$= \langle \max_{1 \leq i \leq n} \sup_{t_i \in Z_i} \mu(x, z_1, z_2, ..., z_{i-1}, t_i, z_{i+1}, ..., z_n),$$
$$\min_{1 \leq i \leq n} \inf_{t_i \in Z_i} \nu(x, z_1, z_2, ..., z_{i-1}, t_i, z_{i+1}, ..., z_n)\rangle,$$
$$V(\forall(x, z_1, z_2, ..., z_n)P(x, z_1, z_2, ..., z_n)$$
$$= \langle \min_{1 \leq i \leq n} \inf_{t_i \in Z_i} \mu(x, z_1, z_2, ..., z_{i-1}, t_i, z_{i+1}, ..., z_n),$$
$$\max_{1 \leq i \leq n} \sup_{t_i \in Z_i} \nu(x, z_1, z_2, ..., z_{i-1}, t_i, z_{i+1}, ..., z_n)\rangle$$

It can be directly proved, that both quantifiers have similar properties as the quantifiers of the predicate IFL, but it will be interesting to find their specific properties.

**Open problem 9.** Construct an axiomatic system of the temporal and multidimensional IFLs.

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The papers presented in this Volume 1 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.

It may be viewed as a result of fruitful discussions held during the Twelfth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2013) organized in Warsaw on October 11, 2013 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT - Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bystrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, Prof. Asen Zlatarov University, Burgas, Bulgaria, and the University of Westminster, Harrow, UK:

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The consecutive International Workshops on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGNs) have been meant to provide a forum for the presentation of new results and for scientific discussion on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems have also been included. The Twelfth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2013) is a continuation of this undertaking, and provides many new ideas and results in the areas concerned.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

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