New Trends in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics
Volume I: Foundations

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ISBN 83-894-7546-4

Some connections among interval-valued fuzzy relations

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Abstract

The goal of this paper is to consider properties of the composition of intervalvalued fuzzy relations which were introduced by L.A. Zadeh in 1975. Fuzzy set theory turned out to be a useful tool to describe situations in which the data are imprecise or vague. Interval-valued fuzzy set theory is a generalization of fuzzy set theory which was introduced also by Zadeh in 1965. We examine some properties of interval-valued fuzzy relations in the context of certain Atanassov's operator, lattice operations and connections among considered properties of interval-valued fuzzy relations.

Keywords: Fuzzy relations, interval-valued fuzzy relations, properties of interval-valued fuzzy relations

1 Introduction

The idea of a fuzzy relation was defined in [22]. An extension of fuzzy set theory is interval-valued fuzzy set theory. Any interval-valued fuzzy set is defined by an interval-valued membership function: a mapping from the given universe to the set of all closed subintervals of [0,1] (it means that information is incomplete). In this work we study preservation of properties of interval-valued fuzzy relations by lattice operations and certain Atanassov's operator. We also study

New Trends in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics. Volume I: Foundations *(K.T. Atanassow, M. Baczyński, J. Drewniak, J. Kacprzyk, M. Krawczak, E. Szmidt, M. Wygralak, S. Zadrożny, Eds.), IBS PAN - SRI PAS, Warsaw, 2013.*

properties of the composition of interval-valued fuzzy relations. Consideration of diverse properties of the composition is interesting not only from a theoretical point of view but also for the application reasons since the composition of interval-valued fuzzy relations has proved to be useful in several fields, see for example, [14] (performance evaluation), [20] (genetic algorithm), [13] (approximate reasoning) or in other areas (see [1, 16, 21, 6, 7, 9]). In Section 2, we recall elementary properties of the composition of interval-valued fuzzy relations. Next, we consider some properties of interval-valued fuzzy relations and we study connections between the considered properties, connections between these properties and lattice operations and certain Atanassov's operator. We consider preservation of some properties of interval-valued fuzzy relations by lattice operations and certain Atanassov's operator ([2, 3]). Finally, we consider some property which guarantee the convergence of powers of an interval-valued fuzzy relation.

2 Basic definitions

First we recall the notion of the lattice operations and the order in the family of interval-valued fuzzy relations. Let X, Y, Z be non-empty sets and $L^I = \{[\underline{x}, \overline{x}] : \underline{x}, \overline{x} \in [0,1],\ \underline{x} \leq \overline{x}\}$. We know (see [17]) that $(L^I, \leq)$ is complete, infinitely distributive lattice, where $[\underline{x}, \overline{x}] \leq [\underline{y}, \overline{y}] \Leftrightarrow \underline{x} \leq \underline{y}, \overline{x} \leq \overline{y}$ for all $\underline{x}, \underline{y}, \overline{x}, \overline{y} \in [0,1]$.

Definition 1 (cf. [23])**.** *An interval-valued fuzzy relation R between universes X, Y is a mapping $R : X \times Y \to L^I$ such that*

$$R(x,y) = [\underline{R}(x,y), \overline{R}(x,y)] \in L^I \tag{1}$$

for all pairs $(x,y) \in (X \times Y)$.
The class of all interval-valued fuzzy relations between universes X, Y will be denoted by IVFR(X,Y) or IVFR(X) for $X = Y$.

Interval-valued fuzzy relations reflect the idea that membership grades are often not precise and the intervals represent such uncertainty.

The boundary elements in *IVFR*(X,Y) are $\mathbf{1} = [1,1]$ and $\mathbf{0} = [0,0]$.

Let $S, R \in$ *IVFR*(X,Y). Then for every $(x,y) \in (X,Y)$ we can define

$$S(x,y) \leq R(x,y) \Leftrightarrow \underline{S}(x,y) \leq \underline{R}(x,y), \overline{S}(x,y) \leq \overline{R}(x,y), \tag{2}$$

$$(S \vee R)(x,y) = [\max(\underline{S}(x,y), \underline{R}(x,y)), \max(\overline{S}(x,y), \overline{R}(x,y))], \tag{3}$$

$$(S \wedge R)(x,y) = [\min(\underline{S}(x,y), \underline{R}(x,y)), \min(\overline{S}(x,y), \overline{R}(x,y))], \tag{4}$$

where operations $\vee$ and $\wedge$ are the supremum and the infimum in *IVFR*(X,Y), respectively. Similarly for arbitrary set $T \neq \emptyset$

$$(\bigvee_{t\in T} R_t)(x,y) = [\bigvee_{t\in T} \underline{R_t}(x,y), \bigvee_{t\in T} \overline{R_t}(x,y)], \tag{5}$$

$$(\bigwedge_{t\in T} R_t)(x,y) = [\bigwedge_{t\in T} \underline{R_t}(x,y), \bigwedge_{t\in T} \overline{R_t}(x,y)]. \tag{6}$$

From ([5],[6]) we know that the pair $(IVFR(X,Y), \leq)$ is a partially ordered set. Moreover, the family $(IVFR(X,Y), \vee, \wedge)$ is a lattice. The lattice $IVFR(X,Y)$ is complete. This fact follows from the notion of the supremum $\bigvee$ and the infimum $\bigwedge$ and from the fact that the values of fuzzy relations are from the interval $[0,1]$ which, with the operations maximum and minimum, forms a complete lattice. As a result (*IVFR*$(X,Y), \vee, \wedge)$ is a complete, infinitely distributive lattice. For our further considerations we need the following properties

Definition 2 (cf. [12, 4])**.** *Let* $* : [0,1]^2 \to [0,1]$. *Operation* $*$ *is infinitely distributive (sup-distributive, inf-distributive), if*

$$\bigvee_{t\in T}(x_t * y) = (\bigvee_{t\in T} x_t) * y, \quad \bigvee_{t\in T}(y * x_t) = y * (\bigvee_{t\in T} x_t)$$

and

$$\bigwedge_{t\in T}(x_t * y) = (\bigwedge_{t\in T} x_t) * y, \quad \bigwedge_{t\in T}(y * x_t) = y * (\bigwedge_{t\in T} x_t).$$

3 Properties of Interval-Valued Fuzzy Relations

In [19] we considered preservation of local reflexivity and local irreflexivity of R by some operations and certain operator. Now, we consider another properties, i.e. local connectedness and local asymmetry which have some connections with local reflexivity and local irreflexivity.

Definition 3 (cf. [10])**.** *Let* $S, Q \in IVFR(X)$. *An interval-valued fuzzy relation* $R \in IVFR(X)$, $R(x,y) = [\underline{R}(x,y), \overline{R}(x,y)]$ *is:*

- *locally asymmetric, if for* $S = R \wedge R^{-1}$

$$\forall x,y \in X \left(S(x,y) = \bigwedge_{z\in X} S(x,z) \text{ or } S(x,y) = \bigwedge_{z\in X} S(z,y) \right), \tag{7}$$

- *locally connected, if for* $Q = R \vee R^{-1}$

$$\forall x, y \in X \left(Q(x,y) = \bigvee_{z \in X} Q(x,z) \text{ or } Q(x,y) = \bigvee_{z \in X} Q(z,y) \right), \quad (8)$$

where $R^{-1}(x,y) = R(y,x) = [\underline{R}(y,x), \overline{R}(y,x)]$.

In our future considerations we will use some very interesting relation of equivalence between interval-valued fuzzy relations.

Definition 4 (cf. [10]). *Interval-valued fuzzy relations* $R, S \in IVFR(X)$ *are equivalent* ($R \sim S$)*, if*

$$\forall x, y, u, v \in X \ \underline{R}(x,y) \leq \underline{R}(u,v) \ \Leftrightarrow \ \underline{S}(x,y) \leq \underline{S}(u,v), \quad (9)$$

$$\forall x, y, u, v \in X \ \overline{R}(x,y) \leq \overline{R}(u,v) \ \Leftrightarrow \ \overline{S}(x,y) \leq \overline{S}(u,v). \quad (10)$$

Corollary 1. *If Interval-valued fuzzy relations* $R, S \in IVFR(X)$ *are equivalent* $R \sim S$*, then*

$$R = R^{-1} \Leftrightarrow S = S^{-1}, \ \ R > R^{-1} \Leftrightarrow S > S^{-1}, \ \ R||R^{-1} \Leftrightarrow S||S^{-1}.$$

Lemma 1 (cf. [10]). *Let* $R, S \in IFVR(X)$*. If* $R \sim S$*, then for every non-empty subset* P *of* $X \times X$ *and each* $(x,y),(u,v) \in P$ *the following conditions are fulfilled:*

$$R(x,y) = \bigwedge_{(u,v) \in P} R(u,v) \ \Leftrightarrow \ S(x,y) = \bigwedge_{(u,v) \in P} S(u,v) \quad (11)$$

$$(R(x,y) = \bigvee_{(u,v) \in P} R(u,v) \ \Leftrightarrow \ S(x,y) = \bigvee_{(u,v) \in P} S(u,v)). \quad (12)$$

Lemma 2. *Let* $R, S \in IVFR(X)$*. If* $R \sim S$*, then*

$$(R \vee S) \wedge (R^{-1} \vee S^{-1}) = (R \wedge R^{-1}) \vee (S \wedge S^{-1}). \quad (13)$$

Proof. Let $R \sim S$. We consider following cases:
1. $R(x,y) \leq R(y,x)$.
From assumption we have $S(x,y) \leq S(y,x)$. So
$(R \vee S) \leq (R^{-1} \vee S^{-1})$, and $(R \vee S) \wedge (R^{-1} \vee S^{-1}) = R \vee S$.
Moreover
$(R \wedge R^{-1}) \leq (S \wedge S^{-1})$, so $(R \wedge R^{-1}) \vee (S \wedge S^{-1}) = R \vee S$. Thus

$(R \vee S) \wedge (R^{-1} \vee S^{-1}) = (R \wedge R^{-1}) \vee (S \wedge S^{-1})$.
Similarly we proof the second condition:
2. $R(x, y) > R(y, x)$.
3. $R(x, y) || R(y, x)$.
If $R(x, y) \leq S(x, y)$, then $R^{-1}(x, y) \leq S^{-1}(x, y)$ and
$(R \wedge R^{-1}) \leq (S \wedge S^{-1})$, so
$(R \wedge R^{-1}) \vee (S \wedge S^{-1}) = S \wedge S^{-1}$ and $(R \vee S) \wedge (R^{-1} \vee S^{-1}) = S \wedge S^{-1}$. Thus
$(R \vee S) \wedge (R^{-1} \vee S^{-1}) = (R \wedge R^{-1}) \vee (S \wedge S^{-1})$.
By similar way we obtain this condition for $R(x, y) > S(x, y)$. Moreover, if $R(x, y) || S(x, y)$, then by the Corollary 1 we observe that $R^{-1} || S^{-1}$ and $R \wedge R^{-1} \sim S \wedge S^{-1}$. Thus by the Lemma 1 we have the same values in
$(R \vee S) \wedge (R^{-1} \vee S^{-1})$ and $(R \wedge R^{-1}) \vee (S \wedge S^{-1})$. What finishes the proof. □

Now we examine connection between local connectedness (local asymmetry) with lattice operations.

Theorem 1. *Let $R, S \in IVFR(X)$. If R, S are locally asymmetric and $R \sim S$, then*

$$R \vee S \quad and \quad R \wedge S$$

are also locally asymmetric.

Proof. Let $x, y \in X$. We assume, that $R \sim S$ and R, S are locally asymmetric.
We examine locally asymmetry of $R \vee S$, i.e. we want:
$(R \vee S) \wedge (R \vee S)^{-1}(x, y) = \bigwedge_{z \in X} (R \vee S) \wedge (R^{-1} \vee S^{-1})(x, z)$.
From above lemma we have:

$$(R \vee S) \wedge (R \vee S)^{-1} = (R \wedge R^{-1}) \vee (S \wedge S^{-1}).$$

By the Definition 3, from local asymmetry of R and S, Corollary 1 and (11) for $P \subset X \times X$, $(x, z) \in P$ we obtain:

$$((R \wedge R^{-1}) \vee (S \wedge S^{-1}))(x, y) = \bigwedge_{z \in X} (R \wedge R^{-1})(x, z) \vee \bigwedge_{z \in X} (S \wedge S^{-1})(x, z).$$

Moreover, from infinite distributivity in $(IVFR, \vee, \wedge)$ by $R \sim S$ we have

$$\bigwedge_{z \in X} (R \wedge R^{-1})(x, z) \vee \bigwedge_{z \in X} (S \wedge S^{-1})(x, z) = \bigwedge_{z \in X} ((R \wedge R^{-1}) \vee (S \wedge S^{-1}))(x, z) =$$

$$\bigwedge_{z \in X} (R \vee S) \wedge (R^{-1} \vee S^{-1})(x, z).$$

Similarly, we prove
$((R \wedge R^{-1}) \vee (S \wedge S^{-1}))(x,y) = \bigwedge_{z \in X}(R \vee S) \wedge (R^{-1} \vee S^{-1})(z,y)$.
So $R \vee S$ has local asymmetry property.
Now we examine $R \wedge S$. Thus by (the Definition 3) local asymmetry R, S and by (11) for $P \subset X \times X$, $(x,z) \in P$ we obtain:

$$((R \wedge S) \wedge (R^{-1} \wedge S^{-1}))(x,y) = (R \wedge R^{-1})(x,y) \wedge (S \wedge S^{-1})(x,y) =$$

$$\bigwedge_{z \in X}((R \wedge R^{-1})(x,z) \wedge (S \wedge S^{-1}))(x,z) =$$

$$\bigwedge_{z \in X}((R \wedge S) \wedge (R^{-1} \wedge S^{-1}))(x,z).$$

thus we have

$$((R \wedge S) \wedge (R^{-1} \wedge S^{-1}))(x,y) = \bigwedge_{z \in X}((R \wedge S) \wedge (R^{-1} \wedge S^{-1}))(x,z).$$

Similarly we have

$$((R \wedge S) \wedge (R^{-1} \wedge S^{-1}))(x,y) = \bigwedge_{z \in X}((R \wedge R^{-1})(x,y) \wedge (S \wedge S^{-1}))(z,y).$$

Thus $R \wedge S$ has local asymmetry property. □

Similarly to the Lemma 2 we have

Lemma 3. *Let $R, S \in IVFR(X)$. If $R \sim S$, then*

$$(R \wedge S) \vee (R^{-1} \wedge S^{-1}) = (R \vee R^{-1}) \wedge (S \vee S^{-1}). \tag{14}$$

Moreover by analogy to the last theorem we can prove following

Theorem 2. *Let $R, S \in IVFR(X, Y)$. If R, S are locally connected and $R \sim S$, then*

$$R \vee S \quad \mathit{and} \quad R \wedge S$$

are also locally connected.

We observe that without adequate assumption in the above theorems lattice operations may not preserve local asymmetry and local connectedness.

Example 1. *Let* $cardX = 3$, $R, S \in IVFR(X)$, $R = [\underline{R}, \overline{R}]$ $S = [\underline{S}, \overline{S}]$, *where*

$$R = \begin{bmatrix} [0.2, 0.4] & [0.6, 0.7] & [1, 1] \\ [0.7, 0.8] & [0.6, 0.7] & [0.8, 0.9] \\ [0.8, 0.9] & [0.9, 1] & [0.8, 0.9] \end{bmatrix},$$

$$S = \begin{bmatrix} [0.3, 0.6] & [0.6, 0.8] & [0.7, 0.8] \\ [0, 4, 0.7] & [0, 4, 0.7] & [0.9, 0.9] \\ [1, 1] & [0.7, 0.8] & [0.7, 0.8] \end{bmatrix},$$

$$R \wedge R^{-1} = \begin{bmatrix} [0.2, 0.4] & [0.6, 0.7] & [0.8, 0.9] \\ [0.6, 0.7] & [0.6, 0.7] & [0.8, 0.9] \\ [0.8, 0.9] & [0.8, 0.9] & [0.8, 0.9] \end{bmatrix},$$

$$S \wedge S^{-1} \begin{bmatrix} [0.3, 0.6] & [0, 4, 0.7] & [0.7, 0.8] \\ [0, 4, 0.7] & [0, 4, 0.7] & [0.7, 0.8] \\ [0.7, 0.8] & [0.7, 0.8] & [0.7, 0.8] \end{bmatrix}.$$

By the Definition 3 relations R, S *are locally asymmetric. Moreover, these relations are not equivalent, because* $maxR = r_{1,3}$ *and* $maxS = s_{3,1}$. *If we consider* $R \vee S$ *we observe*

$$R \vee S = \begin{bmatrix} [0.3, 0.6] & [0.6, 0.8] & [1, 1] \\ [0.7, 0.8] & [0.6, 0.7] & [0.9, 0.9] \\ [1, 1] & [0.9, 1] & [0.8, 0.9] \end{bmatrix}$$

and

$$T = (R \vee S) \wedge (R \vee S)^{-1} = \begin{bmatrix} [0.3, 0.6] & [0.6, 0.8] & [1, 1] \\ [0.6, 0.8] & [0.6, 0.7] & [0.9, 0.9] \\ [1, 1] & [0.9, 0.9] & [0.8, 0.9] \end{bmatrix}.$$

So $R \vee S$ *is not locally asymmetric, because* $t_{1,3}$ *is not minimal in row 1 and column 3. But* $R \wedge S$ *is locally asymmetric, i.e.*

$$R \wedge S = \begin{bmatrix} [0.2, 0.4] & [0.6, 0.7] & [0.7, 0.8] \\ [0.4, 0.7] & [0.4, 0.7] & [0.8, 0.9] \\ [0.8, 0.9] & [0.7, 0.8] & [0.7, 0.8] \end{bmatrix},$$

$$(R \wedge S) \wedge (R \wedge S)^{-1} = \begin{bmatrix} [0.2, 0.4] & [0.4, 0.7] & [0.7, 0.8] \\ [0.4, 0.7] & [0.4, 0.7] & [0.7, 0.8] \\ [0.7, 0.8] & [0.7, 0.8] & [0.7, 0.8] \end{bmatrix}.$$

Example 2. *Let* $card X = 3$, $T, U \in IVFR(X)$, $T = [\underline{T}, \overline{T}]$ $U = [\underline{U}, \overline{U}]$, *where*

$$T = \begin{bmatrix} [0.1, 0.2] & [0, 0] & [0.1, 0.2] \\ [0.1, 0.2] & [0.3, 0.4] & [0.3, 0.4] \\ [0.1, 0.2] & [0.2, 0.3] & [0.5, 0.5] \end{bmatrix},$$

$$U = \begin{bmatrix} [0.1, 0.2] & [0.1, 0.2] & [0.1, 0.2] \\ [0, 0] & [0.3, 0.4] & [0.3, 0.4] \\ [0.1, 0.2] & [0.2, 0.3] & [0.5, 0.5] \end{bmatrix},$$

$$T \vee T^{-1} = \begin{bmatrix} [0.1, 0.2] & [0.1, 0.2] & [0.1, 0.2] \\ [0.1, 0.2] & [0.3, 0.4] & [0.3, 0.4] \\ [0.1, 0.2] & [0.3, 0.4] & [0.5, 0.5] \end{bmatrix},$$

$$U \vee U^{-1} \begin{bmatrix} [0.1, 0.2] & [0.1, 0.2] & [0.1, 0.2] \\ [0.1, 0.2] & [0.3, 0.4] & [0.3, 0.4] \\ [0.1, 0.2] & [0.3, 0.4] & [0.5, 0.5] \end{bmatrix},$$

By the Definition 3 relations T, U *are locally connected. Moreover, these relations are not equivalent, because* $minT =_{1,2}$ *and* $minU = u_{2,1}$. *If we consider* $T \wedge U$ *we observe*

$$T \wedge U = \begin{bmatrix} [0.1, 0.2] & [0, 0] & [0.1, 0.2] \\ [0, 0] & [0.3, 0.4] & [0.3, 0.4] \\ [0.1, 0.2] & [0.2, 0.3] & [0.5, 0.5] \end{bmatrix}$$

and

$$V = (T \wedge U) \vee (T \wedge U)^{-1} = \begin{bmatrix} [0.1, 0.2] & [0, 0] & [0.1, 0.2] \\ [0, 0] & [0.3, 0.4] & [0.3, 0.4] \\ [0.1, 0.2] & [0.3, 0.4] & [0.5, 0.5] \end{bmatrix}.$$

So $T \wedge U$ *is not locally connected, because* $v_{1,2}$ *is not maximal in row 1 or column 2.*

Now, we will prove that all equivalent interval-valued fuzzy relations have the same local properties.

Theorem 3. *Let* $R, S \in IVFR(X, Y)$. *If* $R \sim S$, *then*
- R *is locally asymmetric if and only if* S *is locally asymmetric.*
- R *is locally connected if and only if* S *is locally connected.*

Proof. Let $R \sim S$. If R is locally asymmetric and $T = R \wedge R^{-1}$, $Q = S \wedge S^{-1}$, then for $x, y \in X$ one has

$$T(x, y) = \bigwedge_{z \in X} R(x, z) \text{ or } T(x, y) = \bigwedge_{z \in X} R(z, x),$$

This implies by (11):

$$T(x,y) = \bigwedge_{z\in X} R(x,z) \Leftrightarrow Q(x,y) = \bigwedge_{z\in X} S(x,z)$$

or

$$T(x,y) = \bigwedge_{z\in X} R(z,x) \Leftrightarrow Q(x,y) = \bigwedge_{z\in X} S(z,x).$$

Thus

$$Q(x,y) = \bigwedge_{z\in X} S(x,z) \text{ or } Q(x,y) = \bigwedge_{z\in X} S(z,x).$$

So R and S are simultaneously locally asymmetric. Similarly we can prove the case of the local connectedness property. □

So in Theorem 1, 2 we may omit assumption about locally asymmetry and locally connectivity of S.

Definition 5 (cf. [10])**.** *An interval-valued fuzzy relation* $R(x,y) = [\underline{R}(x,y), \overline{R}(x,y)] \in IVFR(X)$ *is:*

- *locally reflexive, if*

$$\forall x \in X \left(R(x,x) = \bigvee_{z\in X} R(x,z) \ \text{ and } \ R(x,x) = \bigvee_{z\in X} R(z,x) \right),$$

- *locally irreflexive, if*

$$\forall x \in X \left(R(x,x) = \bigwedge_{z\in X} R(x,z) \ \text{ and } \ R(x,x) = \bigwedge_{z\in X} R(z,x) \right).$$

We observe the following connection between local asymmetry, local connectedness, local reflexivity and local irreflexivity.

Theorem 4. *Let* $R \in IVFR(X)$*.*
• *If* R *is locally asymmetric, then it is locally irreflexive.*
• *If* R *is locally connected, then it is locally reflexive.*

Proof. If $R \in IVFR(X)$ has locally asymmetric property, then $B = R \wedge R^{-1}$, and we have (11)

$$R(x,x) = B(x,x) = \bigwedge_{y\in X} B(x,y) = \bigwedge_{y\in X} (R(x,y) \wedge R(y,x)) =$$

$$\bigwedge_{y\in X} R(x,y) \wedge \bigwedge_{y\in X} R(y,x),$$

so

$$R(x,x) \leq \bigwedge_{y\in X} R(x,y) \leq R(x,x) \text{ and } R(x,x) \leq \bigwedge_{y\in X} R(y,x) \leq R(x,x).$$

Thus R has local irreflexivity property, which proves the first condition. The second condition can be justified in a similar way. □

We examine connections of the above properties with some Atanassov's operator

Definition 6 (cf. [3], Definition 1.63)**.** *Let* $R \in IVFR(X,Y)$, $\alpha,\ \beta \in [0,1],\ \alpha + \beta \leq 1$.
We define the operator $F_{\alpha,\beta} : IVFR \to IVFR$ *such that*

$$F_{\alpha,\beta}(R) = [\underline{R} + \alpha(\overline{R} - \underline{R}), \overline{R} - \beta(\overline{R} - \underline{R})].$$

In [19] we proved that operator $F_{\alpha,\beta}$ preserve local irreflexivity and reflexivity properties, but for local asymmetry and connectedness we must add adequate condition. Before the next discussion we observe

Lemma 4. *Let* $p, q \in [0,1],\ p + q = 1$. *If* $\underline{R} \sim \overline{R}$, *then for all* $x, y \in X$

$$(p\underline{R} + q\overline{R}) \wedge (p\underline{R}^{-1} + q\overline{R}^{-1}) =$$
$$(p(\underline{R} \wedge \underline{R}^{-1}) + q(\overline{R} \wedge \overline{R}^{-1}).$$

Proof. By distributivity $+, \cdot$ over $\wedge$ ($\vee$) (see [8]) we have

$$(p(\underline{R} \wedge \underline{R}^{-1}) + q(\overline{R} \wedge \overline{R}^{-1}) =$$
$$(p\underline{R} + q\overline{R}) \wedge (p\underline{R} + q\overline{R}^{-1}) \wedge (p\underline{R}^{-1} + q\overline{R}) \wedge (p\underline{R}^{-1} + q\overline{R}^{-1}).$$

• If $\underline{R} \leq \underline{R}^{-1}$ (from $\underline{R} \sim \overline{R}$ also $\overline{R} \leq \overline{R}^{-1}$), then
$(p\underline{R} + q\overline{R}^{-1}) \wedge (p\underline{R}^{-1} + q\overline{R}) \geq (p\underline{R} + q\overline{R})$,
• If $\underline{R} \geq \underline{R}^{-1}$ (from $\underline{R} \sim \overline{R}$ also $\overline{R} \geq \overline{R}^{-1}$), then
$(p\underline{R} + q\overline{R}^{-1}) \wedge (p\underline{R}^{-1} + q\overline{R}) \geq (p\underline{R}^{-1} + q\overline{R}^{-1})$. So

$$(p(\underline{R} \wedge \underline{R}^{-1}) + q(\overline{R} \wedge \overline{R}^{-1}) =$$
$$(p\underline{R} + q\overline{R}) \wedge (p\underline{R} + q\overline{R}^{-1}) \wedge (p\underline{R}^{-1} + q\overline{R}) \wedge (p\underline{R}^{-1} + q\overline{R}^{-1}).$$

□

Theorem 5. *Let $R \in IVFR(X)$. If R is locally connected (locally asymmetric) and $\underline{R} \sim \overline{R}$, then $F_{\alpha,\beta}(R)$ is also locally connected (locally asymmetric).*

Proof. By the Lemma 4 we obtain:

$$[((1-\alpha)\underline{R}+\alpha\overline{R})\wedge((1-\alpha)\underline{R}^{-1}+\alpha\overline{R}^{-1}),((1-\beta)\overline{R}+\beta\underline{R})\wedge((1-\beta)\overline{R}^{-1}+\beta\underline{R}^{-1})]$$

$$= [((1-\alpha)(\underline{R}\wedge\underline{R}^{-1})+\alpha(\overline{R})\wedge\overline{R}^{-1}),((1-\beta)(\overline{R}\wedge\overline{R}^{-1})+\beta(\underline{R}\wedge\underline{R}^{-1})]$$

By the Definition 3 we have:

$$= [((1-\alpha)\bigwedge_{z\in X}(\underline{R}\wedge\underline{R}^{-1})(x,z)+\alpha\bigwedge_{z\in X}(\overline{R})\wedge\overline{R}^{-1})(x,z),$$

$$((1-\beta)\bigwedge_{z\in X}(\overline{R}\wedge\overline{R}^{-1})(x,z)+\beta\bigwedge_{z\in X}(\underline{R})\wedge\underline{R}^{-1})(x,z)]$$

¿From distributivity $+$ with respect infimum and the Definition 6 we have:

$$[((1-\alpha)\bigwedge_{z\in X}(\underline{R}\wedge\underline{R}^{-1})(x,z)+\alpha\bigwedge_{z\in X}(\overline{R})\wedge\overline{R}^{-1})(x,z),$$

$$((1-\beta)\bigwedge_{z\in X}(\overline{R}\wedge\overline{R}^{-1})(x,z)+\beta\bigwedge_{z\in X}(\underline{R})\wedge\underline{R}^{-1})(x,z)] =$$

$$[\bigwedge_{z\in X}(((1-\alpha)(\underline{R}\wedge\underline{R}^{-1})(x,z)+\alpha(\overline{R})\wedge\overline{R}^{-1})(x,z)),$$

$$\bigwedge_{z\in X}(((1-\beta)(\overline{R}\wedge\overline{R}^{-1})(x,z)+\beta(\underline{R})\wedge\underline{R}^{-1})(x,z))] =$$

$$\bigwedge_{z\in X}F_{\alpha,\beta}(R\wedge R^{-1})(x,z).$$

Similarly we prove $(F_{\alpha,\beta}(R)\wedge F_{\alpha,\beta}(R^{-1}))(x,y) = \bigwedge_{z\in X}F_{\alpha,\beta}(R\wedge R^{-1})(x,z)$. So the operator $F_{\alpha,\beta}$ preserves local asymmetry. The case of the local connectedness property may be proved analogously. □

4 Powers of Interval-Valued Fuzzy Relations

Let us recall the notion of the composition in $IVFR$.

Definition 7 (cf. [5],[12]). *Let* $S \in IVFR(X,Y)$, $R \in IVFR(Y,Z)$. *By the* $\sup - \min$ *composition of the relations* S *and* R *we call the relation* $S \circ R \in IVFR(X \times Z)$,

$$(S \circ R)(x,z) = [(\underline{S} \circ \underline{R})(x,z), (\overline{S} \circ \overline{R})(x,z)],$$

where

$$(\underline{S} \circ \underline{R})(x,z) = \bigvee_{y \in Y} (\underline{S}(x,y) \wedge \underline{R}(y,z)), \;\; (\overline{S} \circ \overline{R})(x,z) = \bigvee_{y \in Y} (\overline{S}(x,y) \wedge \overline{R}(y,z))$$

for $x \in X, z \in Z$.

Lemma 5 ([18]). *The* $\sup - \min$ *composition in* $IVFR(X)$ *is isotonic, associative and have neutral element.*

Directly by isotonicity we have

Lemma 6. *Let* $S \in IVFR(X,Y)$, $R \in IVFR(Y,Z)$.

$$(\underline{S} \circ \underline{R})(x,z) \leq (\overline{S} \circ \overline{R})(x,z)$$

for $x \in X, z \in Z$.

Thus by the Lemma 5 we obtain that

Corollary 2. $(IVFR(X), \circ)$ *is a semigroup.*

In a semigroup $(IVFR(X), \circ)$ we can consider the powers of its elements and analogously to [15] we define

Definition 8 ([5]). *By a powers of relation* $R \in IVFR(X)$ *we mean*

$$R^1 = R, \; R^{n+1} = R^n \circ R, \; n \in N.$$

The sequence (R^n) *is called stable, if*

$$\exists_{k \in \mathbb{N}} R^{k+1} = R^k.$$

Example 3. *Let* T *be the relation from Example 2. We calculate powers of this relation and we have*

$$T^2 = \begin{bmatrix} [0.1, 0.2] & [0.1, 0.2] & [0.1, 0.2] \\ [0.1, 0.2] & [0.2, 0.4] & [0.3, 0.4] \\ [0.1, 0.2] & [0.2, 0.3] & [0.5, 0.5] \end{bmatrix} = T^3.$$

So the sequence (T^n) *of the locally connected relation is stable.*

Moreover, by Theorem 4 the locally connected relation is also locally reflexive, so in a finite set we call this property by dominating diagonal. Because $R^n = [\underline{R}^n, \overline{R}^n]$ and by convergence of lower and upper fuzzy relations (matrices) (see [11]) we obtain stability of an interval-valued fuzzy relation R.

Corollary 3. *Let $R \in IVFR(X)$, $\overline{\overline{X}} = m$, $m \in \mathbb{N}$. If R is locally connected, then (R^n) is stable.*

5 Conclusion

In this paper we considered properties of interval-valued fuzzy relations in the context of preservation of these properties by some operations, including lattice operations, the composition and some Atanassov's operator. We observed very interesting connections between these properties and dependence between these properties and the convergence of powers of relations having these properties. In our further considerations we want to examine other properties, more general composition of interval-valued fuzzy relations and which properties guarantee that the sequence (R^n) oscillate, i.e.

$$\exists_{d \in \mathbb{N}} R^{k+d} = R^k.$$

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The papers presented in this Volume 1 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.
It may be viewed as a result of fruitful discussions held during the Eleventh International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2012) organized in Warsaw on October 12, 2012 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT - Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bystrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, Prof. Asen Zlatarov University, Burgas, Bulgaria, and the University of Westminster, Harrow, UK:

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The Workshop has also been in part technically supported by COST Action IC0806 ``Intelligent Monitoring, Control and Security of Critical Infrastructure Systems" (INTELLICIS).

The consecutive International Workshops on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGNs) have been meant to provide a forum for the presentation of new results and for scientific discussion on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems have also been included. The Eleventh International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2012) is a continuation of this undertaking, and provides many new ideas and results in the areas concerned.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

ISBN-13 9788389475466

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