# New Trends in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics
# Volume I: Foundations


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# Aggregation of fuzzy equivalences

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**Abstract**

The article deals with fuzzy $C$-equivalences and the problem of the preservation of their properties by some aggregation functions. This paper gives a contribution to the discussion of tolerance analysis in soft computing, decision making, approximate reasoning, and fuzzy control.

**Keywords:** fuzzy equivalence, fuzzy conjunction, domination, fuzzy $C$-equivalence, aggregation function

## 1 Introduction

The problem of aggregation of diverse mathematical objects is rather well known. We may aggregate for example fuzzy relations and consider the problem of preservation of fuzzy relation properties during aggregation process (e.g. [16, 22, 23]) or examine fuzzy connectives [11] and preservation of their properties by aggregations [10].

If it comes to fuzzy relations, first of all the transitivity property is of the most interest because of the application reasons. The standard transitivity property is too strong for many relations and this is why this property is modified in different ways [17, 5, 19, 26] according to demands of real-life problems. Diverse types of transitivity properties are strictly connected with the preference

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and choice procedures. For example, they are applied to guarantee consistency of fuzzy preferences [5]. The aggregation of information in preference modelling and decision-making mainly concerned fuzzy preference relations with examination of particular classes of binary fuzzy relations. Fundamental properties of aggregation functions were collected by J.C. Fodor and M. Roubens [16]. They asked if the aggregation result has properties of the aggregated relations. Similar question concerns the correctness of algorithms in computational mathematics. An algorithm for determination of the values of a function $F$ is called correct if its result is equal to the value of a function $G$ from the same class of functions and from a convex neighbourhood of a function $F$. Thus, convex classes of operations are useful in computational mathematics.

In this paper we consider aggregation of fuzzy equivalences as one of the fuzzy connectives. We take into account only one type of the possible definition of a fuzzy equivalence, namely the one which is based on the notion of fuzzy equivalence relation which is reflexive, symmetric and transitive. We discuss three kinds of transitivity properties and in this way we obtain three kinds of fuzzy equivalences. We take into account transitivity property and its weaker versions - weak and semi transitivity. The idea of a weak and semi properties appeared in [6] and for transitivity it was developed later in [8, 9, 12, 14].

In section 2, basic notions useful in the paper are presented. In section 3, diverse types of fuzzy equivalences are described, and in Section 4, aggregation of fuzzy equivalences are discussed.

# 2 Preliminaries

Here we recall basic notions and their properties which will appear in the sequel. We consider aggregation functions, relation of domination between operations, fuzzy conjunctions and fuzzy equivalences.

## 2.1 Aggregation functions

Now we present useful information about aggregation functions.

**Definition 1** (cf. [3], pp. 6-22, [18], pp. 216-218)**.** Let $n \in \mathbb{N}$. A function $A : [0,1]^n \to [0,1]$ which is increasing, i.e.

$$A(x_1, \ldots, x_n) \leqslant A(y_1, \ldots, y_n) \quad \text{for} \quad x_i, y_i \in [0,1], x_i \leqslant y_i, i = 1, \ldots, n$$

is called an aggregation function if

$$A(0, \ldots, 0) = 0, \ A(1, \ldots, 1) = 1. \tag{1}$$

Moreover, an aggregation $A$ we call a mean if it is idempotent, i.e.

$$A(x, \ldots, x) = x, \quad x \in [0, 1]. \tag{2}$$

**Definition 2.** Let $n \in \mathbb{N}$. We say that function $A : [0, 1]^n \to [0, 1]$:
• has a zero element $z \in [0, 1]$ (cf. [3], Definition 10) if

$$\underset{1 \leqslant k \leqslant n}{\forall} \quad \underset{x_1,...,x_{k-1},x_{k+1},...,x_n \in [0,1]}{\forall} \quad A(x_1, ..., x_{k-1}, z, x_{k+1}, ..., x_n) = z, \tag{3}$$

• is without zero divisors if

$$\underset{x_1,...,x_n \in [0,1]}{\forall} \quad (A(x_1, ..., x_n) = z \;\Rightarrow\; (\underset{1 \leqslant k \leqslant n}{\exists} x_k = z)), \tag{4}$$

• fulfils strong 1-boundary condition if

$$\underset{x_1,...,x_n \in [0,1]}{\forall} \quad (A(x_1, ..., x_n) = 1 \;\Leftrightarrow\; (\underset{1 \leqslant k \leqslant n}{\forall} x_k = 1)). \tag{5}$$

Let us notice that by putting $n = 2$ in the above definition we get the respective well-known conditions for binary operations. It is easy to check that the following statements are true.

**Lemma 1** ([12])**.** *Let $n \in \mathbb{N}$. An aggregation $A : [0, 1]^n \to [0, 1]$ has a zero element $z = 0$ if and only if it fulfils condition*

$$\underset{x_1,...,x_n \in [0,1]}{\forall} ((\underset{1 \leqslant k \leqslant n}{\exists} x_k = 0) \Rightarrow A(x_1, ..., x_n) = 0). \tag{6}$$

**Lemma 2** ([12])**.** *Let $n \in \mathbb{N}$. An aggregation $A : [0, 1]^n \to [0, 1]$ has a zero element $z = 0$ and is without zero divisors if and only if*

$$\underset{x_1,...,x_n \in [0,1]}{\forall} (A(x_1, ..., x_n) > 0 \Leftrightarrow (\underset{1 \leqslant k \leqslant n}{\forall} x_k > 0)). \tag{7}$$

A description of other families of aggregation functions can be found in [3].

**Example 1** (cf. [3], pp. 44-56, [13])**.** $A_0, A_1$ are the least and the greatest aggregation functions, where

$$A_0(x_1, \ldots, x_n) = \begin{cases} 1, & (x_1, \ldots, x_n) = (1, \ldots, 1) \\ 0, & (x_1, \ldots, x_n) \neq (1, \ldots, 1) \end{cases},$$

$$A_1(x_1, \ldots, x_n) = \begin{cases} 0, & (x_1, \ldots, x_n) = (0, \ldots, 0) \\ 1, & (x_1, \ldots, x_n) \neq (0, \ldots, 0) \end{cases},$$

$x_1, \ldots, x_n \in [0, 1]$. Simple examples of aggregation function are given by standard means such as lattice operations $\min, \max$ and

- projections

$$P_k(x_1, \ldots, x_n) = x_k, \quad for \quad k = 1, 2, \ldots, \tag{8}$$

- geometric mean

$$G(x_1, \ldots, x_n) = \sqrt[n]{x_1 \cdot \cdots \cdot x_n}, \tag{9}$$

- weighted means

$$A_w(x_1, \ldots, x_n) = \sum_{k=1}^{n} w_k x_k, \quad for \quad w_k > 0, \sum_{k=1}^{n} w_k = 1, \tag{10}$$

- quasi-arithmetic means

$$M_\varphi(x_1, \ldots, x_n) = \varphi^{-1}(\frac{1}{n}\sum_{k=1}^{n} \varphi(x_k)), \tag{11}$$

- quasi-linear means

$$F(x_1, ..., x_n) = \varphi^{-1}(\sum_{k=1}^{n} w_k \varphi(x_k)), \tag{12}$$

where $w_k > 0$, $k = 1, ..., n$, $\sum_{k=1}^{n} w_k = 1$, $x_1, \ldots, x_n \in [0, 1]$ and $\varphi : [0, 1] \to \mathbb{R}$ is a continuous, strictly increasing function.

We may also notice that the following property is true

**Theorem 1** ([12]). *Let $A : [0, 1]^n \to [0, 1]$ be an aggregation function, $\varphi : [0, 1] \to [0, 1]$ be an increasing bijection. If A has a zero element $z = 0$ and it is an operation without zero divisors then operation $A_\varphi$, isomorphic to A, has the same properties, where*

$$A_\varphi(x_1, ..., x_n) = \varphi^{-1}(A(\varphi(x_1), ..., \varphi(x_n))), \ \ x_1, ..., x_n \in [0, 1]. \tag{13}$$

## 2.2 Fuzzy conjunctions

Now, the definition of a fuzzy conjunction is presented.

**Definition 3** ([10]). An operation $C : [0,1]^2 \to [0,1]$ is called a fuzzy conjunction if it is increasing with respect to each variable and

$$C(1,1) = 1, \quad C(0,0) = C(0,1) = C(1,0) = 0.$$

Let us observe that fuzzy conjunctions are aggregation functions for $n = 2$. Directly from the definition we obtain a useful property of a fuzzy conjunction.

**Corollary 1.** *A fuzzy conjunction has a zero element* $z = 0$.

**Example 2.** Consider the following family of fuzzy conjunctions for $\alpha \in [0,1]$

$$C^{\alpha}(x,y) = \begin{cases} 1, & \text{if } x = y = 1 \\ 0, & \text{if } x = 0 \text{ or } y = 0 \\ \alpha & \text{otherwise} \end{cases}.$$

Operations $C^0$ and $C^1$ are the least and the greatest fuzzy conjunction, respectively.

We may also distinguish subfamilies of fuzzy conjunctions.

**Definition 4** ([25], [15]). A triangular seminorm (t-seminorm, semicopula) is a fuzzy conjunction with a neutral element $e = 1$.

Semicopulas generalize t-norms. Namely

**Definition 5** ([18], p. 4). A triangular norm (t-norm) $T : [0,1]^2 \to [0,1]$ is an arbitrary associative, commutative, increasing in each variable operation with a neutral element $e = 1$.

**Definition 6** ([18], p. 28). A strict t-norm $T : [0,1]^2 \to [0,1]$ is a t-norm which is continuous and strictly increasing in $(0,1]^2$.

**Example 3.** In the table below there are some examples of conjunctions. Among them we recall the well-known triangular norms: minimum, product, Łukasiewicz, drastic, which are denoted in the traditional way $T_M$, $T_P$, $T_L$, $T_D$, respectively, where

| | |
|---|---|
| $C_2(x,y) = \begin{cases} y, & \textit{if } x = 1 \\ 0, & \textit{if } x < 1 \end{cases}$ | $T_M(x,y) = \min(x,y)$ |
| $C_3(x,y) = \begin{cases} x, & \textit{if } y = 1 \\ 0, & \textit{if } y < 1 \end{cases}$ | $T_P(x,y) = xy$ |
| $C_4(x,y) = \begin{cases} 0, & \textit{if } x+y \leqslant 1 \\ y, & \textit{if } x+y > 1 \end{cases}$ | $T_L(x,y) = \max(x+y-1, 0)$ |
| $C_5(x,y) = \begin{cases} 0, & \textit{if } x+y \leqslant 1 \\ x, & \textit{if } x+y > 1 \end{cases}$ | $T_D(x,y) = \begin{cases} x, & \textit{if } y = 1 \\ y, & \textit{if } x = 1 \\ 0 & \textit{otherwise} \end{cases}$ |

Functions $T_P$, $T_M$, $T_L$ and $T_D$ have the zero element $z = 0$. Functions $T_P$ and $T_M$ are operations without zero divisors, $T_L$ and $T_D$ have zero divisors.

**Lemma 3** ([16], p. 11)**.** *Any strict t-norm is an operation isomorphic (in the meaning of formula* (13)*) to product t-norm* $T_P$*.*

## 2.3 Domination

Domination is one of the interesting dependence between operations.

**Definition 7** (cf. Saminger et al. [23], Definition 2.5; cf. [24], p. 209)**.** Let $m, n \in \mathbb{N}$. An operation $F : P^m \to P$ dominates an operation $G : P^n \to P$ (shortly $F \gg G$), if for any matrix $[a_{i,k}] = A \in P^{m\times n}$ they fulfil the inequality

$$F(G(a_{1,1},\ldots,a_{1,n}),\ldots,G(a_{m,1},\ldots,a_{m,n})) \geqslant \\ \geqslant G(F(a_{1,1},\ldots,a_{m,1}),\ldots,F(a_{1,n},\ldots,a_{m,n})). \tag{14}$$

Let us consider an example of dominating operations.

**Example 4.** Let $n \in \mathbb{N}$. Each two projections from

$$P_k(x_1,...,x_n) = x_k, \quad k \in \{1,...,n\}, \quad x_1,...,x_n \in [0,1] \tag{15}$$

dominate each other.

In this paper we consider domination between $n$-ary aggregation $A$ and binary fuzzy conjunction $C$. In this case the domination $A \gg C$ means fulfilling the condition

$$A(C(a_{1,1},a_{1,2}),\ldots,C(a_{n,1},a_{n,2})) \geqslant C(A(a_{1,1},\ldots,a_{n,1}),A(a_{1,2},\ldots,a_{n,2})). \tag{16}$$

We may also characterize aggregations $A$ which dominate $C$ for some given conjunction $C$. We recall here characterization theorem of all aggregation functions which dominate $T_D$.

**Theorem 2** (cf. [23], Proposition 5.2). *Let $A : [0,1]^n \to [0,1]$ be aggregation function. Then $A \gg T_D$ if and only if there exists a non-empty subset $I = \{k_1, ..., k_m\} \subset \{1, ..., n\}$, $k_1 < ... < k_m$, and an increasing mapping $B : [0,1]^m \to [0,1]$ satisfying the following conditions: $B(0, ..., 0) = 0$ and $B(u_1, ..., u_m) = 1 \Leftrightarrow u_1 = ... = u_m = 1$, such that $A(x_1, ..., x_n) = B(x_{k_1}, ..., x_{k_m})$.*

Observe that function $B$ in Theorem 2 is an aggregation function and concerning triangular norms $T$ we have $T(x_1, x_2) = 1$ if and only if $x_1 = x_2 = 1$ and thus $I = \{1, 2\}$, so $B = T$ and $T \gg T_D$ ([23], p. 32). Moreover, we deduce that the following holds true.

**Example 5.** Quasi-arithmetic means $A$ dominate $T_D$ and t-seminorms $A$ dominate $T_D$. This is due to the fact that for both types of the aggregation functions fulfil strong 1-boundary condition (5).

We put also characterization of aggregations $A$ which dominate $C = \min$.

**Theorem 3** (cf. [23], Proposition 5.1). *An aggregation function $A$ dominates minimum if and only if*

$$A(x_1, ..., x_n) = \min(f_1(x_1), ..., f_n(x_n)), \quad x_1, ..., x_n \in [0,1], \tag{17}$$

*where $f_k : [0,1] \to [0,1]$ are increasing, $k = 1, ..., n$.*

**Example 6.** Here are examples of functions which fulfil (17):
if $f_k(x) = x$, $k = 1, ..., n$, then $A = \min$,
if for some $k \in \{1, ..., n\}$, $f_k(x) = x$, $f_i(x) = 1$ for $i \neq k$, then $A = P_k$,
if $f_k(x) = \max(1 - v_k, x)$, $v_k \in [0,1]$, $k = 1, ..., n$, $\max_{1 \leqslant k \leqslant n} v_k = 1$, then $A$ is weighted minimum

$$A(x_1, ..., x_n) = \min_{1 \leqslant k \leqslant n} \max(1 - v_k, x_k), \; v = (v_1, ..., v_n) \in [0,1]^n, \; \max_{1 \leqslant k \leqslant n} v_k = 1. \tag{18}$$

We know that minimum dominates any function which is increasing with respect to each variable ([23], p. 16). As a result we get

**Theorem 4.** *Minimum dominates any fuzzy conjunction.*

## 3 Fuzzy equivalences

In the literature one can meet various definitions of a fuzzy equivalence. A trivial case used in many contributions, for example those concerning generalized logical laws, is an equality, that is the function $E : [0,1]^2 \to [0,1]$ given by the formula (relation of identity)

$$E(x,y) = \begin{cases} 1, & \text{gdy } x = y \\ 0, & \text{gdy } x \neq y \end{cases}. \tag{19}$$

Usually it is expected that such notion of a fuzzy equivalence is a generalization of the equivalence of classical propositional calculus, that is the function $E : [0,1]^2 \to [0,1]$ that fulfils conditions $E(0,1) = E(1,0) = 0$, $E(0,0) = E(1,1) = 1$. We will apply the approach in which definition of a fuzzy equivalence follows from the notion of a fuzzy equivalence relation, namely relation which is reflexive, symmetric and transitive.

**Definition 8** (cf. [20], p. 33)**.** Let $C$ be a fuzzy conjunction. A fuzzy $C$-equivalence is a function $E : [0,1]^2 \to [0,1]$ fulfilling the following conditions

$$E(0,1) = 0 \qquad \text{(boundary property)}, \tag{20}$$

$$E(x,x) = 1, \quad x \in [0,1] \qquad \text{(reflexivity)}, \tag{21}$$

$$E(x,y) = E(y,x), \quad x,y \in [0,1] \qquad \text{(symmetry)}, \tag{22}$$

$$C(E(x,y), E(y,z)) \leqslant E(x,z) \quad x,y,z \in [0,1] \qquad \text{(transitivity)}. \tag{23}$$

**Example 7.** The function (19) is a fuzzy $C$-equivalence for any fuzzy conjunction $C$.

In the cited monograph [20] property (20) is omitted. However, in this case the constant function $E(x,y) = 1$, $x,y \in [0,1]$ fulfils the definition of a fuzzy equivalence although it is not a generalization of crisp equivalence. This is why we add this assumption to the definition.

We may weaken conditions given in the previous definition by replacing transitivity property with the appropriate weaker transitivity conditions introduced for fuzzy relations [6, 8, 9, 12]. Moreover, for example in [2], [19] weak transitivity is considered in the context of fuzzy preference relations. For a fuzzy equivalence as a connective we will apply conjunction $C$ instead of an arbitrary binary operation in the unit interval $[0,1]$. In the literature, triangular norms are often considered instead of this operation. Here applying conjunctions we use a generalization of triangular norms.

**Definition 9.** Let $C$ be a fuzzy conjunction. A fuzzy weak $C$-equivalence is a function $E : [0,1]^2 \to [0,1]$ fulfilling conditions (20)-(22) and

$$\forall_{x,y,z \in X} C(E(x,y), E(y,z)) > 0 \Rightarrow E(x,z) > 0. \tag{24}$$

A fuzzy semi $C$-equivalence is a function $E : [0,1]^2 \to [0,1]$ fulfilling conditions (20)-(22) and

$$\forall_{x,y,z \in X} C(E(x,y), E(y,z)) = 1 \Rightarrow E(x,z) = 1. \tag{25}$$

Directly from definitions of a fuzzy $C$-equivalence, weak and semi $C$-equivalence we see that

**Lemma 4.** *Let $C$ be a fuzzy conjunction. If $E : [0,1]^2 \to [0,1]$ is a fuzzy $C$-equivalence, then it is both weak $C$-equivalence, semi $C$-equivalence.*

**Lemma 5.** *Let $C_1, C_2$ be fuzzy conjunctions, $C_1 \leqslant C_2$. If $E$ is a fuzzy $C_2$-equivalence (weak $C_2$–equivalence, semi $C_2$-equivalence), then it is also $C_1$–equivalence (weak $C_1$-equivalence, semi $C_1$-equivalence).*

*Proof.* It is enough to consider the result from [8] (Lemma 3) for fuzzy relations and arbitrary binary operation in the interval $[0,1]$. □

The following example shows that a weak and semi $C$-equivalence are not equivalent to $C$-equivalence.

**Example 8.** Let us consider a function

$$E(x,y) = \begin{cases} 1, & \text{gdy } x = y \\ \frac{\min(x,y)}{\max(x,y)}, & \text{gdy } x \neq y \end{cases} \tag{26}$$

and conjunction $C = \min$. Obviously $E$ fulfils conditions (20) - (22). We shall show that it does not fulfil (23).

For $x = 0.2$, $y = 0.4$, $z = 0.8$ one has

$$C(E(x,y), E(y,z)) = \min\left(\frac{\min(x,y)}{\max(x,y)}, \frac{\min(y,z)}{\max(y,z)}\right) = \\ = \min\left(\frac{0.2}{0.4}, \frac{0.4}{0.8}\right) = 0.5.$$

Moreover, $E(x,z) = \frac{\min(x,z)}{\max(x,z)} = \frac{0.2}{0.8} = 0.25$. Thus, $C(E(x,y), E(y,z)) > E(x,z)$ and the function (26) is not min-equivalence.

Now, let us observe that from the assumption $\min(E(x,y),E(y,z)) > 0$ it follows that both $E(x,y) > 0$ and $E(y,z) > 0$. This holds if $x = y = z$ or $x,y,z > 0$. In both cases we have $E(x,y) > 0$. This is why the function (26) is a weak min-equivalence.

Let us consider a condition $\min(E(x,y),E(y,z)) = 1$. From this it follows that both $E(x,y) = 1$ and $E(y,z) = 1$ and it means that $x = y$ and $y = z$. Thus, $x = z$ and $E(x,z) = 1$. This means that the function (26) is a semi min-equivalence.

There is no correspondence between properties (24) and (25) what is shown in the next examples.

**Example 9.** Let us consider an arbitrary fuzzy conjunction $C$ and the function

$$E(x,y) = \begin{cases} 1, & \text{if } x = y \\ 0.5, & \text{if } x \neq y, (x,y) \in (0,1)^2 \\ 0, & \text{otherwise} \end{cases}. \tag{27}$$

Directly by the formula of $E$ it follows that it fulfils conditions (20) - (22). Let us consider weak $C$-transitivity condition in its contrapositive form:

$$\underset{x,y,z \in X}{\forall} E(x,z) = 0 \Rightarrow C(E(x,y),E(y,z)) = 0.$$

By the definition of the function $E$ it follows that $E(x,z) = 0$ if:
1) $x = 0$ and $z > 0$ or
2) $x > 0$ and $z = 0$ or
3) $x = 1$ and $z < 1$ or
4) $x < 1$ and $z = 1$
For an arbitrary $y \in [0,1]$ let us consider the value $V := C(E(x,y),E(y,z))$.
1) If $y > 0$, then $E(x,y) = E(0,y) = 0$ and because of the zero element of $C$ (cf. 1) $V = 0$. If $y = 0$, then for $z > 0$ we have $E(y,z) = E(0,z) = 0$ and $V = 0$.
2) If $y > 0$, then $E(y,z) = E(y,0) = 0$, so $V = 0$. If $y = 0$, then for $x > 0$ we have $E(x,y) = E(x,0) = 0$ and $V = 0$.
3) If $y < 1$, then $E(x,y) = E(1,y) = 0$, so $V = 0$. If $y = 1$, then for $z < 1$ we have $E(y,z) = E(1,z) = 0$ and $V = 0$.
2) If $y < 1$, then $E(y,z) = E(y,1) = 0$, so $V = 0$. If $y = 1$, then for $x < 1$ we have $E(x,y) = E(x,1) = 0$ and $V = 0$.
This is why the function (27) is weakly $C$-transitive for any fuzzy conjunction $C$.

However, this function may not be semi $C$-transitive for some fuzzy conjunction $C$. Let us consider a fuzzy conjunction

$$C(x,y)=\begin{cases}\min(x,y), & \text{if } y\leqslant 1-x\\ 1, & \text{otherwise}\end{cases}.$$

For the values $x=y=0.6$, $z=0.8$ we have $C(E(x,y),\, E(y,z)) = C(E(0.6,$ $0.6),\, E(0.6,0.8)) = (1,0.5)=1$, but $E(x,z)=E(0.6,0.8)=0.5$. This is why the function (27) is not semi $C$-transitive for the given fuzzy conjunction $C$.

**Example 10.** Now, let us assume that $C=\min$ and

$$E(x,y)=\begin{cases}1, & \text{if } x=y\\ 0, & \text{if } \{x,y\}=\{0,1\}\\ 0.5, & \text{otherwise}\end{cases}. \tag{28}$$

Directly by the formula of $E$ it follows that it fulfils conditions (20) - (22). Let us consider weak $C$-transitivity. For $x=0$, $y=0.5$, $z=1$ we have $C(E(x,y),$ $E(y,z)) = C(E(0,0.5),E(0.5,1)) = C(0.5,0.5) = 0.5 > 0$, but $C(x,z) =$ $C(0,1)=0$. Thus, the function (28) is not weak $C$-transitive. Now, let us observe that for the given operations $C$ and $E$ we have

$$\begin{aligned}C(E(x,y),E(y,z))=1 \Leftrightarrow E(x,y)=1\wedge E(y,z)=1 \Leftrightarrow\\ \Leftrightarrow x=y\wedge y=z \Rightarrow x=z \Rightarrow E(x,z)=1.\end{aligned}$$

Thus, the function (28) is semi $C$-transitive.

The following theorems indicate examples of fuzzy $C$-equivalences.

**Theorem 5.** *The function*

$$E(x,y)=\begin{cases}1, & \textit{if } x=y\\ \min(x,y), & \textit{otherwise}\end{cases}$$

*is a fuzzy $C$-equivalence if and only if $C\leqslant\min$.*

*Proof.* Obviously, conditions (20) - (22) are fulfilled. Let us examine the property of $C$-transitivity (23). ($\Rightarrow$) Let $x,z\in[0,1]$. If $x\neq z$ then for $y=1$ one has

$$C(x,z)=C(E(x,1),E(1,z))\leqslant E(x,z)=\min(x,z).$$

On the other hand if $x = z$ then for $x = 1$ one has $C(x, x) = C(1, 1) = 1 \leqslant 1 = x$, and for $x \neq 1$ there exists $t \geqslant x$ and then $C(x, x) \leqslant C(x, t) \leqslant \min(x, t) = x$.
($\Leftarrow$) Let $C \leqslant \min$, $x, y, z \in [0, 1]$. If $x \neq y$ and $y \neq z$ then

$$C(E(x, y), E(y, z)) = C(\min(x, y), \min(y, z)) \leqslant$$
$$\leqslant \min(\min(x, y), z) \leqslant \min(x, z) \leqslant E(x, z).$$

For $x = y$ one obtains

$$C(E(x, y), E(y, z)) = C(1, \min(y, z)) \leqslant \min(y, z) = \min(x, z) \leqslant E(x, z).$$

Similarly, for $y = z$ one has

$$C(E(x, y), E(y, z)) = C(\min(x, y), 1) \leqslant \min(x, y) = \min(x, z) \leqslant E(x, z).$$

□

# 4 Aggregation of fuzzy $C$-equivalences

Fundamental properties of aggregation for fuzzy relations were gathered by J.C. Fodor and M. Roubens [16]. We may aggregate diverse objects: fuzzy relations, fuzzy connectives etc. Here we consider aggregation of fuzzy equivalences defined in the previous section.

**Definition 10** (cf. [16], p. 14)**.** Let $n \in \mathbb{N}$ and $A$ be an arbitrary aggregation function. For given fuzzy equivalences $E_1, \ldots, E_n$, we consider an aggregation connective

$$E(x, y) = A(E_1(x, y), \ldots, E_n(x, y)), \quad x, y \in [0, 1]. \tag{29}$$

We say that an aggregation function $A$ preserves a property of the given fuzzy equivalences if the operation $E$ from (29) has such a property for arbitrary $E_1, \ldots, E_n$ fulfilling this property. A class of fuzzy equivqlences is closed under an aggregation $A$ if the result of the aggregation belongs to this class for arbitrary fuzzy equivalences from the class.

For example, any projection (8) preserves fuzzy $C$-equivalence (weak fuzzy $C$-equivalence, semi fuzzy $C$-equivalence) because in the formula (29) with $A = P_k$ we get $E = E_k$ for $k \in \{1, \ldots, n\}$. From condition (1) we get

**Lemma 6** (cf. [10])**.** *Any aggregation function preserves binary truth tables of aggregated equivalences.*

**Lemma 7** (cf. [10]). *Any aggregation function preserves symmetry of aggregated equivalences.*

**Lemma 8.** *Any aggregation function preserves the property* (21) *of the aggregated operations.*

*Proof.* Let $E_1$, $E_2$ be operations fulfilling the conditions $E_1(x,x) = 1$,$E_2(x,x) = 1$ for $x \in [0,1]$. According to (29), for any aggregation $A$ one has

$$E(x,x) = A(E_1(x,x), E_2(x,x)) = A(1,1) = 1, \quad \text{for} \quad x \in [0,1].$$

□

From Lemmas 6-8 it follows that any aggregation function preserves boundary property, symmetry and reflexivity of a fuzzy equivalence. Thus, for aggregation of fuzzy equivalences it is enough to consider only transitivity conditions. We will do it in the sequel.

## 4.1 Fuzzy $C$-equivalences

In this part a result from the paper [9] (Theorem 10) written in other terminology is used.

**Theorem 6.** *Let C be a fuzzy conjunction. An aggregation function $A : [0,1]^n \to [0,1]$ preserves condition* (23) *of the aggregated fuzzy equivalences $E_1, ..., E_n$ if and only if A dominates C ($A \gg C$).*

By Theorem 6 and Lemma 5 we get

**Corollary 2.** *Let $C_1, C_2$ be fuzzy conjunctions, $C_1 \leqslant C_2$. If aggregation function $A$ dominates $C_2$ and we have fuzzy $C_2$-equivalences $E_1, ..., E_n$, then $A(E_1, ..., E_n)$ is a fuzzy $C_1$-equivalence.*

From Theorem 6 and Lemmas 6, 7, 8 one obtains

**Theorem 7.** *The family of all fuzzy $C$-equivalences is closed under aggregation operations $A$ that dominate $C$ ($A \gg C$).*

By Theorem 4 we get

**Theorem 8.** *Minimum preserves fuzzy $C$-equivalence for any fuzzy conjunction $C$.*

In virtue of the results for preservation of transitivity for fuzzy relations ([21], Proposition 1, [23], Example 5.2 and [7], Corollary 6) we obtain respectively the following conclusions.

**Corollary 3.** *Weighted means preserve fuzzy $T_L$-equivalence.*

**Corollary 4.** *Geometric mean preserves fuzzy $T_P$-equivalence.*

**Corollary 5.** *Aggregation $A$ described by the formula*

$$A(x_1, ..., x_n) = \frac{p}{n}\sum_{k=1}^{n} t_k + (1-p)\min_{1\leqslant k\leqslant n} t_k, \quad p \in [0,1], \tag{30}$$

*preserves fuzzy $T_L$-equivalence.*

Now we want to pay attention to preservation of weak $C$-equivalence and semi $C$-equivalence by aggregations. The results which will be presented for weak $C$-equivalences and semi $C$-equivalences were firstly obtained for fuzzy relations [12]. Here we give suitable versions for fuzzy equivalences (connectives).

## 4.2 Fuzzy weak $C$-equivalences

Let us notice that there are aggregations which preserve weak $C$-equivalence for any fuzzy conjunction $C$.

**Theorem 9.** *Let $C$ be a fuzzy conjunction. Minimum preserves weak $C$-equivalence.*

*Proof.* We my apply here a result from [8] (Theorem 12) where this preservation was shown for fuzzy relations where instead of conjunctions, increasing operations were considered. □

**Theorem 10.** *If aggregation function $A : [0,1]^n \to [0,1]$ has a zero element $z = 0$ and both $A$ and a fuzzy conjunction $C$ are operations without zero divisors then $A$ preserves weak $C$-equivalence.*

*Proof.* Let $E_1, ..., E_n$ be weak $C$-equivalences. If $C(E(x,y), E(y,z)) > 0$, then because $C$ as a conjunction has a zero element $z = 0$ and by Lemma 2 we get $E(x,y) > 0$ and $E(y,z) > 0$. As a result from (29) we get

$$A(E_1(x,y), ..., E_n(x,y)) > 0, \;\; A(E_1(y,z), ..., E_n(y,z)) > 0.$$

By Lemma 2 and assumptions about function $A$ we have

$$E_1(x,y) > 0, ..., E_n(x,y) > 0, E_1(y,z) > 0, ..., E_n(y,z) > 0.$$

Again, in virtue of Lemma 2 and by assumptions about $A$ and $C$ we see that $C(E_k(x,y), E_k(y,z)) > 0$, $k = 1, ..., n$. Since $E_k$ for $k = 1, ..., n$ are weak $C$-equivalences we get $E_k(x,z) > 0$, $k = 1, ..., n$ and $A(E_1(x,z), ..., E_n(x,z)) > 0$, so $E(x,z) > 0$ which proves that $E$ is a weak $C$-equivalence. □

**Example 11.** Theorem 10 gives only a sufficient condition for preservation of weak $C$-equivalence. The weighted mean preserves weak $T_L$-equivalence (see Theorem 13 from [8] for fuzzy relations) but the weighted mean does not have a zero element $z = 0$.

By Theorems 1 and 10 we get

**Corollary 6.** *Let $A : [0,1]^n \to [0,1]$ be an aggregation function and $\varphi : [0,1] \to [0,1]$ be an increasing bijection. If $A$ preserves weak $C$-equivalence, then $A_\varphi$ also preserves this property (cf.* (13)*).*

By Example 3, Theorem 10, Theorem 1 and Lemma 3 we get

**Theorem 11.** *Functions $T_P$, $T_M$ and any strict t-norm preserve weak $C$-equivalence for $C = \min$.*

### 4.3 Fuzzy semi $C$-equivalences

Now we will turn to the problem of preservation of semi $C$-equivalence. Firstly, let us notice that there exists an aggregation which preserves semi $C$-equivalence for any conjunction $C$.

**Theorem 12.** *Let $C$ be a fuzzy conjunction. Minimum preserves semi $C$-equivalence.*

*Proof.* Let $E_1, \ldots, E_n$ be semi $C$-equivalences, $E = \min(E_1, \ldots, E_n)$, and $x, y, z \in [0,1]$. If $C(E(x,y), E(y,z)) = 1$, then by monotonicity of $C$

$$1 = C(\min_{1 \leqslant k \leqslant n} E_k(x,y), \min_{1 \leqslant k \leqslant n} E_k(y,z)) \leqslant C(E_i(x,y), E_i(y,z)), \quad i = 1, ..., n,$$

so $C(E_i(x,y), E_i(y,z)) = 1$ and since $E_1, \ldots, E_n$ are semi $C$-equivalences we get $E_i(x,z) = 1$, for $i = 1, ..., n$. Therefore

$$E(x,z) = \min(E_1(x,z), \ldots, E_n(x,z)) = 1$$

and $E$ is a semi $C$-equivalence. □

**Theorem 13.** *If an aggregation* $A : [0,1]^n \to [0,1]$ *and a fuzzy conjunction* $C : [0,1]^2 \to [0,1]$ *fulfil strong* 1*-boundary condition* (5)*, then* $A$ *preserves semi* $C$*-equivalence.*

*Proof.* Let $x, y, z \in [0,1]$, $E_1, ..., E_n$ be semi $C$-equivalences and both $A$ and $C$ have strong 1-boundary property. If $C(E(x,y), E(y,z)) = 1$, then we get $E(x,y) = 1$, $E(y,z) = 1$, so $A(E_1(x,y), ..., E_n(x,y)) = 1$ and $A(E_1(y,z), ..., E_n(y,z)) = 1$. In virtue of (5), we obtain $E_1(x,y) = 1$, ..., $E_n(x,y) = 1$, $E_1(y,z) = 1, ..., E_n(y,z) = 1$. By condition (5) for conjunction $C$ we have $C(E_k(x,y), E_k(y,z)) = 1$, $k = 1, ..., n$ and since $E_k$, $k = 1, ..., n$ are semi $C$-equivalences we obtain $E_k(x,z) = 1$, $k = 1, ..., n$. Using (5) we have $A(E_1(x,z), ..., E_n(x,z)) = E(x,z) = 1$, so $E$ is a semi $C$-equivalence. □

**Example 12.** Theorem 13 gives only a sufficient condition for a preservation of semi $C$-equivalence. We see that the minimum preserves semi $C$-equivalence for $C = C^1$ (the greatest conjunction), where

$$C^1(x,y) = \begin{cases} 0, & \text{if } x = 0 \text{ or } y = 0 \\ 1 & \text{otherwise} \end{cases}.$$

Let $x, y, z \in [0,1]$, $E_1, ..., E_n$ be fuzzy semi $C$-equivalences and $C(E(x,y), E(y,z)) = 1$. Thus we have $E(x,y) > 0$ and $E(y,z) > 0$. So $A(E_1(x,y), ..., E_n(x,y)) > 0$ and $A(E_1(y,z), ..., E_n(y,z)) > 0$, where $A(x_1, \ldots, x_n) = \min_{1 \leqslant k \leqslant n} x_k$. As a result $E_k(x,y) > 0$ for $k = 1, ..., n$ and $E_k(y,z) > 0$ for $k = 1, ..., n$, thus $C(E_k(x,y), E_k(y,z)) = 1$ for $k = 1, ..., n$. By assumption $E_1, ..., E_n$ are fuzzy semi $C$-equivalences, so $E_k(x,z) = 1$ for $k = 1, ..., n$. Finally, $E(x,z) = A(E_1(x,z), ..., E_n(x,z)) = \min(1, ..., 1) = 1$. This proves that $E$ is semi $C$-equivalence for the given aggregation $A$ and conjunction $C$. However, $C = C^1$ does not fulfil condition (5).

**Example 13.** Triangular norms, t-seminorms and the weighted arithmetic means have strong 1-boundary property.

**Theorem 14.** *The weighted arithmetic mean preserves semi* $C$*-equivalence for any t-seminorm (t-norm)* $C$*.*

Analogously to the way presented in Theorem 1 we may prove

**Theorem 15.** *Let* $\varphi : [0,1] \to [0,1]$ *be an increasing bijection. If an aggregation function* $A : [0,1]^n \to [0,1]$ *fulfils* (5)*, then* $A_\varphi$ *also fulfils* (5).

By Example 13, Lemma 3 and Theorem 15 we get

**Example 14.** The quasi-linear means have the strong 1-boundary property.

**Theorem 16.** *Let $C$ be a t-seminorm (t-norm). The quasi-linear means preserve semi $C$-equivalence.*

# 5 Conclusions

Admissible aggregations preserving properties of aggregated connectives such as fuzzy equivalences were presented. There were considered three types of such connectives. In our further work we would like to discuss other equivalence notions (e.g. [16], p. 33) and preservation of their properties by aggregations.

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The papers presented in this Volume 1 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.
It may be viewed as a result of fruitful discussions held during the Eleventh International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2012) organized in Warsaw on October 12, 2012 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT - Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bystrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, Prof. Asen Zlatarov University, Burgas, Bulgaria, and the University of Westminster, Harrow, UK:

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The consecutive International Workshops on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGNs) have been meant to provide a forum for the presentation of new results and for scientific discussion on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems have also been included. The Eleventh International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2012) is a continuation of this undertaking, and provides many new ideas and results in the areas concerned.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

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