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On a class of operations on interval-valued fuzzy sets

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Abstract

In this paper we consider two aspects of binary operations on interval valued fuzzy sets. The first is connected with some methods of construction of such operations. Here we describe decomposable operations (e.g. t-representable t-norms) and their generalizations. Moreover we describe which properties of components are transferred to the decomposable operations. The same problem is considered for the generalization of decomposable operations. The second aspect considered in this paper is connected with algebraic properties of binary operations, i.e. for a given properties (associativity, monotonicity, commutativity and existence of neutral or zero element) we tray describe the structure of operations. In particular, we describe the structure of uninorms and nullnorms on L^I.

Keywords: Interval-valued fuzzy set, Atanassov's intuitionistic fuzzy set, lattices L^* and L^I, uninorm, nullnorm, decomposable operations.

1 Introduction

Binary operations such as triangular norms and triangular conorms are applied in multivalued logic and fuzzy set theory. In this paper we consider two aspects of binary operations on interval valued fuzzy sets. The first is connected with some

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methods of construction of such operations. The second aspect considered in this paper is connected with algebraic properties of binary operations, i.e. for a given properties (associativity, monotonicity, commutativity and existence of neutral or zero element) we tray describe the structure of operations (cf. [4]). In particular, we describe the structure of uninorms and nullnorms on L^I.

Interval valued fuzzy sets have been introduced by Zadeh [19] and form an extension of fuzzy sets. While fuzzy sets give only a degree of membership for each element of the universe, interval valued fuzzy sets maps each element of the universe on an interval of possible membership-degrees. Hence interval valued fuzzy sets are not only capable of modelling vagueness, but also uncertainty.

The another extension of fuzzy sets are intuitionistic fuzzy sets introduced by Atanassov [1]. Atanassov intuitionistic fuzzy sets give not only a degree of membership μ for each element of the universe, but also give a degree of non-membership ν, which only need to satisfy the constraint $\mu + \nu \leq 1$. The number $1 - \mu - \nu$ is called the hesitation degree and hence is also capable of modelling uncertainty.

In the next section we define interval-valued fuzzy sets, intuitionistic fuzzy sets and the lattices L^I and L^*, and we show that both interval-valued fuzzy set theory and intuitionistic fuzzy set theory are equivalent to L^I fuzzy set theory. Further we recall the definition of t-norm and t-conorm on L^I. In Section 3 we describe decomposable operations (e.g. t-representable t-norms) and their generalizations (e.g. pseudo-t-representable operations). Moreover we describe which property of components operations are transferred to the decomposable operations. The same problem is considered for the generalization of decomposable operations. In Section 4 we recall the properties of uninorms in $[0, 1]$ and next we describe the uninorms on L^I. First we show the relationship with t-norms and t-conorms on L^I and next we describe some properties of representable uninorms on L^I, e.g. we discuss the possible values of the neutral element and zero element for these uninorms. In Section 5 we present nullnorms on L^I and consider similar properties as for uninorms.

2 Interval valued and Atanassov's intuitionistic fuzzy sets

First we recall the notion of some extensions of fuzzy set theory. The fuzzy set theory turned out to be a useful tool to describe situations in which the data are imprecise or vague.

Definition 1 ([18]). *A fuzzy set A in a universe X is a mapping*

$$A : X \to [0, 1].$$

Fuzzy set describe the degree to which a certain point belongs to a set. A is also called a membership function and $A(x)$ is called membership degree of $x \in X$.

The natural way of extension the operations on sets to fuzzy sets is by membership functions

$$(A \cup B)(x) = \max\left(A(x), B(x)\right),$$
$$(A \cap B)(x) = \min\left(A(x), B(x)\right),$$

for $x \in X$.

There are many other generalizations of these operations. Some of them are based on triangular norms and triangular conorms (cf.[16]) which we may use instead of operations $\min$, $\max$.

Definition 2 ([16]). *A triangular norm T is an increasing, commutative, associative operation $T : [0, 1]^2 \to [0, 1]$ with neutral element 1.*
A triangular conorm S is an increasing, commutative, associative operation $S : [0, 1]^2 \to [0, 1]$ with neutral element 0.

Example 1 ([16]). *Well-known t-norms and t-conorms are:*

$T_M(x, y) = \min(x, y)$,	$S_M(x, y) = \max(x, y)$,
$T_P(x, y) = x \cdot y$,	$S_P(x, y) = x + y - xy$,
$T_L(x, y) = \max(x + y - 1, 0)$,	$S_L(x, y) = \min(x + y, 1)$,

So, we have the generalization of the sum and the intersection in the following form

$$(A \cup B)(x) = S\left(A(x), B(x)\right),$$
$$(A \cap B)(x) = T\left(A(x), B(x)\right),$$

for $x \in X$.

Intuitionistic fuzzy sets were introduced by Atanassov as an extension of the fuzzy sets in the following way.

Definition 3 (cf. [1], [2]). *An Atanassov intuitionistic fuzzy set A on a universe X is a triple*

$$A = \{(x, \mu(x), \nu(x)) : x \in X\}, \tag{1}$$

where $\mu, \nu : X \to [0, 1]$ and $\mu(x) + \nu(x) \leq 1$, $x \in X$.
$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is called the hesitation degree of x.

An Atanassov intuitionistic fuzzy set assigns to each element of the universe not only a membership degree $\mu(x)$ but also a nonmembership degree $\nu(x)$, $x \in X$.

Now we consider the operations defined on Atanassov intuitionistic fuzzy sets. Namely

$$A \cup B = \{x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\},$$
$$A \cap B = \{x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\}.$$

An Atanassov intuitionistic fuzzy set A on X can be represented by an L^*-fuzzy set in the sense of Goguen. Namely

Definition 4 (cf. [14])**.** *An L-fuzzy set A on a universe X is a function $A : X \to L$ where L is a lattice.*

In this paper by $(L^*, \leq_{L^*})$ we mean the following complete lattice

$$L^* = \{(x_1, x_2) \in [0,1]^2 : x_1 + x_2 \leq 1\}, \tag{2}$$

$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1$ and $x_2 \geq y_2$.

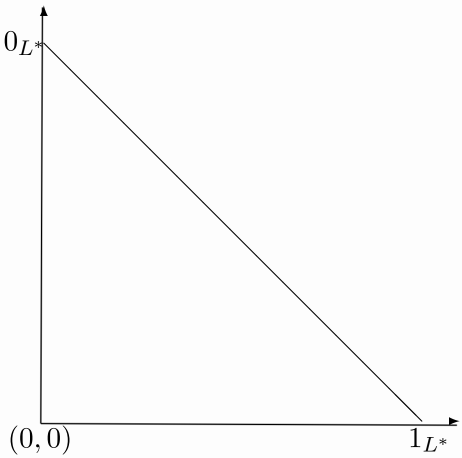

Figure 1: Lattice L^*

Another extension of fuzzy sets are interval-valued fuzzy sets introduced by Zadeh [19]. In interval-valued fuzzy sets to each element of the universe a closed subinterval of the unit interval is assigned and this is the way of describing the unknown membership degree.

Definition 5 ((cf. [15])). *An interval valued fuzzy set A in a universe X is a mapping $A : X \to \mathrm{Int}([0,1])$, where $\mathrm{Int}([0,1])$ denotes the set of all closed subintervals of $[0,1]$, i.e. a mapping which assigns to each element $x \in X$ the interval $[\underline{A}(x), \overline{A}(x)]$, where $\underline{A}(x), \overline{A}(x) \in [0,1]$ and $\underline{A}(x) \leq \overline{A}(x)$.*

An interval valued fuzzy set A on X can be represented by the L^I-fuzzy set A in the sense of Goguen, where

$$L^I = \{[x_1, x_2] : x_1, x_2 \in [0,1] : x_1 \leq x_2\}, \tag{3}$$

with following order

$$[x_1, x_2] \leq_{L^I} [y_1, y_2] \Leftrightarrow x_1 \leq y_1 \wedge x_2 \leq y_2.$$

$(L^I, \leq_L)$ is a complete lattice with operations

$$[x_1, x_2] \wedge [y_1, y_2] = [\min(x_1, y_1), \min(x_2, y_2)],$$

$$[x_1, x_2] \vee [y_1, y_2] = [\max(x_1, y_1), \max(x_2, y_2)].$$

and the boundary elements $1_{L^I} = [1,1]$ and $0_{L^I} = [0,0]$.

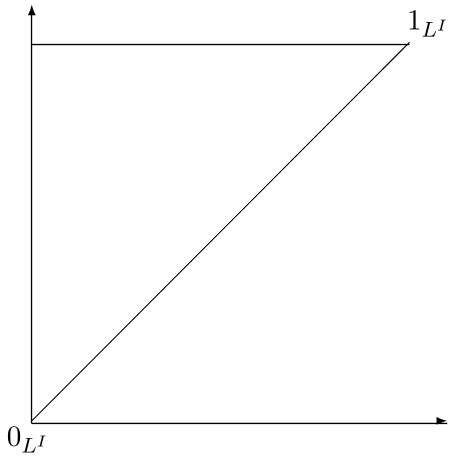

Figure 2: Lattice L^I

Deschrijver and Kerre [5] showed that Atanassov intuitionistic fuzzy sets are equivalent to interval-valued fuzzy sets. The isomorphism assign the Atanassov intuitionistic fuzzy set the interval value fuzzy set as follows: $(x, \mu_A(x), \nu_A(x)) \mapsto [\mu_A(x), 1 - \nu_A(x)]$.

In this article we will develop our investigations for $(L^I, \leq_L)$, since in this case we have the product order and it will be easier to prove the main result.

3 Some methods of construction of binary operations

In this section we put some methods of construction of binary operations. We look for assumptions needed to the construction of the operation from a given class.

First we put some properties of binary operations which will be useful in the further considerations.

Definition 6 ([13]). *A binary operation $\mathcal{F}$ is called idempotent in L^I if*

$$\forall_{x\in L^I} \mathcal{F}(x,x)=x. \tag{4}$$

It is called associative if

$$\forall_{x,y,z\in L^I} \mathcal{F}(x,\mathcal{F}(y,z))=\mathcal{F}(\mathcal{F}(x,y),z). \tag{5}$$

It is called commutative if

$$\forall_{x,y\in L^I} \mathcal{F}(x,y)=\mathcal{F}(y,x). \tag{6}$$

It has a neutral element $e\in L^I$ if

$$\forall_{x\in L^I} \mathcal{F}(x,e)=\mathcal{F}(e,x)=x. \tag{7}$$

It has a zero element $z\in L^I$ if

$$\forall_{x\in L^I} \mathcal{F}(x,z)=\mathcal{F}(z,x)=z. \tag{8}$$

The operation $\mathcal{F}$ is called increasing in $(L^I,\leq)$ if

$$\forall_{x,y,z\in L^I} (x\leq y)\Rightarrow \quad (\mathcal{F}(x,z)\leq\mathcal{F}(y,z),\ \mathcal{F}(z,x)\leq\mathcal{F}(z,y)). \tag{9}$$

Definition 7 ([6], [8]). *A triangular norm $\mathcal{T}$ on L^I is an increasing, commutative, associative operation $\mathcal{T}:(L^I)^2\to L^I$ with a neutral element 1_{L^I}.*
A triangular conorm $\mathcal{S}$ on L^I is an increasing, commutative, associative operation $\mathcal{S}:(L^I)^2\to L^I$ with a neutral element 0_{L^I}.

Example 2. *The following are examples of t-norms on L^I*

$$\inf(x,y)=[\min(x_1,y_1),\min(x_2,y_2)],$$
$$\mathcal{T}(x,y)=[\max(0,x_1+y_1-1),\min(x_2,y_2)],$$

and t-conorm on L^I

$$\sup(x,y)=[\max(x_1,y_1),\max(x_2,y_2)].$$

Now, we recall one of the crucial definition for investigations in this paper.

Definition 8 ([11]). *An operation $\mathcal{F} : (L^I)^2 \to L^I$ is called decomposable if there exist operations $F_1, F_2 : [0,1]^2 \to [0,1]$ such that for all $x, y \in L^I$*

$$\mathcal{F}(x, y) = [F_1(x_1, y_1), F_2(x_2, y_2)], \tag{10}$$

where $x = [x_1, x_2]$, $y = [y_1, y_2]$.

The following lemma characterize certain family of decomposable operations

Lemma 1 (cf. [11]). *Increasing operations $F_1, F_2 : [0,1]^2 \to [0,1]$ in* (10) *gives a decomposable operation $\mathcal{F}$ if and only if $F_1 \leq F_2$.*

Remark 1. *If we use the triangular norms in the construction of decomposable operation, then we obtain decomposable triangular norm. The same situation we have if we use triangular conorms, uninorms or nullnorms. Moreover decomposable triangular norms, triangular conorms, uninorms, nullnorms are also called t-representable triangular norms, triangular conorms, uninorms and nullnorms.*

Example 3. *The operation*

$$\mathcal{T}(x, y) = [\max(x_1 + y_1 - 1, 0), \max(x_2 + y_2 - 1, 0)]$$

is a t-representable t-norm, with the Łukasiewicz t-norm.

Below we give the relationship between properties of decomposable operation and the properties of its component operations.

Theorem 1 (cf. [11]). *Let $\mathcal{F} : (L^I)^2 \to L^I$ be decomposable binary operation such that $\mathcal{F} = [F_1, F_2]$. Decomposable operation $\mathcal{F}$ has neutral element $[e, e]$ if and only if operations F_1 and F_2 have the neutral element e.*

Theorem 2 (cf. [11]). *Operations $F_1,\ F_2 : [0,1]^2 \to [0,1]$ are increasing if and only if, decomposable operation $\mathcal{F}$ is increasing.*

Theorem 3 (cf. [11]). *Operations $F_1,\ F_2 : [0,1]^2 \to [0,1]$ are commutative if and only if, decomposable operation $\mathcal{F}$ is commutative i.e.,*

$$\mathcal{F}(x, y) = \mathcal{F}(y, x) for x, y \in L^I.$$

Theorem 4 (cf. [11]). *Operations $F_1,\ F_2 : [0,1]^2 \to [0,1]$ are associative if and only if, decomposable operation $\mathcal{F}$ is associative i.e.,*

$$\mathcal{F}(x, \mathcal{F}(y, z)) = \mathcal{F}(\mathcal{F}(x, y), z) for x, y, z \in L^I.$$

Corollary 1. *Let $\mathcal{F} : (L^I)^2 \to L^I$ be t-representable binary operation such that $\mathcal{F} = [F_1, F_2]$. t-representable operation $\mathcal{F}$ is t-norm (t-conorm) if and only if F_1 and F_2 are t-norms (t-conorms) and $F_1 \leq F_2$.*

Theorem 5 (cf. [11])**.** *Operations $F_1,\ F_2 : [0,1]^2 \to [0,1]$ are idempotent if and only if, decomposable operation $\mathcal{F}$ is idempotent i.e.,*

$$\mathcal{F}(x,x) = x for x \in L^I.$$

Directly from above we obtain

Corollary 2. *Let $\mathcal{F} : (L^I)^2 \to L^I$ be an t-representale t-norm (t-conorm). If the operation $\mathcal{F}$ is idempotent, then $\mathcal{F} = \wedge$ ($\mathcal{F} = \vee$).*

There are many properties of binary operation preserved by decomposable operations, e.g.

Theorem 6 ([9])**.** *Let $\mathcal{F}, \mathcal{G} : (L^I)^2 \to L^I$ be two decomposable binary operations such that $\mathcal{F} = (F_1, F_2)$, $\mathcal{G} = (G_1, G_2)$. Operation $\mathcal{F}$ is left (right) distributive over the operation $\mathcal{G}$ if and only if operation F_1 is left (right) distributive over the operation G_1 and operation F_2 is left (right) distributive over the operation G_2.*

The other properties can be found in [11].
The another method of construction of binary operation is given as follows

Definition 9 (cf. [6])**.** *The t norm $\mathcal{T}$ (t-conorm $\mathcal{S}$) is called pseudo-t-representable if*

$$\mathcal{T}(x,y) = [T(x_1,y_1), \max(T(x_1,y_2), T(x_2,y_1))]$$
$$(\mathcal{S}(x,y) = [\min(S(x_1,y_2), S(x_2,y_1)), S(x_2,y_2)]) .$$

Theorem 7. *If T and S are arbitrary binary operation on $[0,1]$ then operation given above preserve commutativity, associativity and isotonicity.*

Remark 2. *Pseudo-t-representable t-norms and t-conorms not preserve idempotency.*

The another generalization of decomposable operations are

$$\mathcal{T}(x,y) = [\min(T(x_1,y_2), T(x_2,y_1)), T(x_2,y_2)]$$
$$\mathcal{S}(x,y) = [S(x_1,y_1), \max(S(x_1,y_2), S(x_2,y_1))]$$

Definition 10 (cf. [6]). *The t norm $\mathcal{T}$ is called generalized pseudo-t-representable if*

$$\mathcal{T}_{T_1,T_2,t}(x,y) = [T_1(x_1,y_1), \max(T_2(t,T_2(x_2,y_2)), T_2(x_1,y_2), T_2(x_2,y_1))],$$

where T_1 and T_2 additionally satisfy, for all $x_1, y_1 \in [0,1]$,

$$T_2(x_1,y_1) > T_2(t,T_2(x_1,y_1)) \Rightarrow T_1(x_1,y_1) = T_2(x_1,y_1).$$

We will consider the following generalization of pseudo t-representable t-norms

$$\mathcal{T}(x,y) = [T(x_1,y_1), S(T(x_1,y_2), T(x_2,y_1))] \tag{11}$$

Theorem 8. *Function $\mathcal{T}$ in* (11) *is a t-norm if and only if $S = \max$.*

Proof. If $S = \max$, then $\mathcal{T}$ is a pseudo t-representable t-norm.
If $\mathcal{T}$ is a t-norm, then

$$\mathcal{T}(x, 1_{L^I}) = [T(x_1,1), S(T(x_1,1), T(x_2,1))] = [x_1, S(x_1,x_2)] = [x_1,x_2].$$

So, for all $x_1 \leq x_2$ we have $S(x_1,x_2) = x_2$, therefore $S = \max$. □

Remark 3. *Operation given by* (11) *preserve isotonicity and commutativity.*

4 Uninorms

In this section we recall the definition and properties of a uninorms on $[0,1]$ and next we describe the uninorms on L^I. First we show the relationship with t-norms and t-conorms on L^I and next we describe some properties of representable uninorms on L^I, e.g. we discuss the possible values of the neutral element and zero element for these uninorms some properties of these operations.

Definition 11 ([17]). *Operation $U : [0,1]^2 \to [0,1]$ is called a uninorm if it is commutative, associative, increasing and has the neutral element $e \in [0,1]$.*

Theorem 9 ([12]). *If a uninorm U has the neutral element $e \in (0,1)$, then there exist a triangular norm T and a triangular conorm S such that*

$$U(x,y) = \begin{cases} T^*(x,y) & \text{if} \quad x,y \leq e, \\ S^*(x,y) & \text{if} \quad x,y \geq e, \end{cases} \tag{12}$$

where

$$\begin{cases} T^*(x,y) = \varphi^{-1}\left(T\left(\varphi(x),\varphi(y)\right)\right), \ \varphi(x) = x/e, & x,y \in [0,e] \\ S^*(x,y) = \psi^{-1}\left(S\left(\psi(x),\psi(y)\right)\right), \ \psi(x) = (x-e)/(1-e), & x,y \in [e,1] \end{cases}. \tag{13}$$

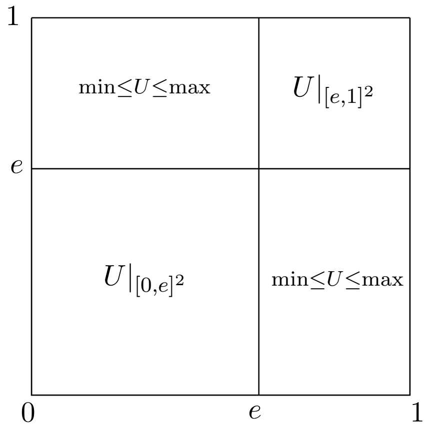

Figure 3: The structure of uninorms on $[0, 1]$.

Lemma 2 (cf. [12]). *If U is a uninorm with the neutral element $e \in (0, 1)$ then*

$$\min(x, y) \leq U(x, y) \leq \max(x, y) \text{ for } x \leq e \leq y \text{ or } y \leq e \leq x. \quad (14)$$

Lemma 3 (cf. [12]). *If U is an uninorm with the neutral element $e \in (0, 1)$ then $U(0, 1) \in \{0, 1\}$ and $U(0, 1)$ is the zero element of operation U.*

Definition 12 (cf. [7]). *Operation $\mathcal{U} : (L^I)^2 \to L^I$ is called a uninorm if it is commutative, associative, increasing and has the neutral element $e \in L^I$.*

In Theorem 9 there is given the structure of uninorms on $[0, 1]$ which show the relationship with t-norms and t-conorms. To provide a similar description we define the following sets on L^I:

$$E_e = \{x \in L^I : x \leq_{L^I} e\},$$
$$E'_e = \{x \in L^I : x \geq_{L^I} e\},$$
$$D = \{[x, x] : x \in [0, 1]\}.$$

Theorem 10 (cf. [7]). *Let $e \in L^I \setminus \{0_{L^I}, 1_{L^I}\}$. If $e \notin D$, then there does not exist an increasing bijection $\Phi_e : L^I \to E_e$ such that Φ_e^{-1} is increasing and there does not exist an increasing bijection $\Psi_e : L^I \to E'_e$ such that Ψ_e^{-1} is increasing.*

Because of the above theorem there is no a description of uninorms with t-norms and t-conorms on L^I when neutral element is outside the the set D. However, if the neutral element is from the set D we obtain the following description

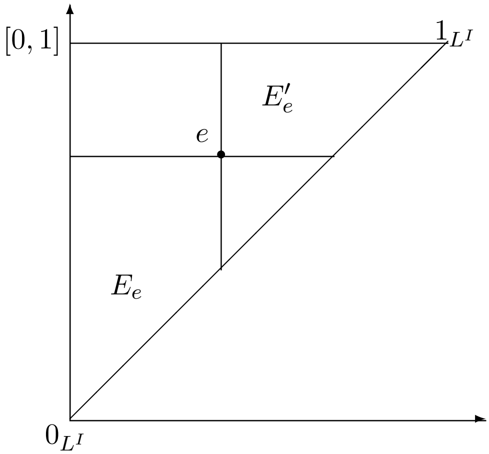

Figure 4: The sets E_e and E'_e

Theorem 11 (cf. [7]). *If a uninorm $\mathcal{U}$ has the neutral element $e = [e_1, e_1] \in D \setminus \{0_{L^I}, 1_{L^I}\}$, then there exist a t-norm $\mathcal{T}$ and a t-conorm $\mathcal{S}$ such that*

$$\mathcal{U}(x,y) = \begin{cases} \mathcal{T}^*(x,y) \text{ if } x,y \leq e \\ \mathcal{S}^*(x,y) \text{ if } x,y \geq e \end{cases},$$

where

$$\begin{cases} \mathcal{T}^*(x,y) = \Phi_e^{-1}\left(\mathcal{T}\left(\Phi_e(x), \Phi_e(y)\right)\right), \\ \Phi_e(x) = (e_1 x_1, e_1(x_2)),\ x,y \in E_e; \\ \mathcal{S}^*(x,y) = \Psi_e^{-1}\left(\mathcal{S}\left(\Psi_e(x), \Psi_e(y)\right)\right), \\ \Psi_e(x) = (e_1 + x_1 - e_1 x_1, e_1 + (1-e_1)x_2)\ x,y \in E'_e. \end{cases} \tag{15}$$

Lemma 4 (cf.[10]). *If $\mathcal{U}$ is a uninorm with the neutral element $e \in L^I$ then for all $x, y \in L^I$ such that $x \leq e \leq y$ we have*

$$x \leq \mathcal{U}(x,y) \leq y.$$

Lemma 5 (cf.[10]). *If $\mathcal{U}$ is a uninorm with the neutral element $e \in L^I$ then for all $x, y \in L^I$ such that $x \leq e \leq y$ or $y \leq e \leq x$ we have*

$$\min(x,y) \leq \mathcal{U}(x,y) \leq \max(x,y).$$

Lemma 6 (cf. [7]). *If $\mathcal{U}$ is a uninorm with the neutral element $e \in L^I \setminus \{0_{L^I}, 1_{L^I}\}$ then for all $x \in L^I$ we have*

$$\mathcal{U}(0_{L^I}, 1_{L^I}) = \mathcal{U}(\mathcal{U}(0_{L^I}, 1_{L^I}), x),$$

i.e. $\mathcal{U}(0_{L^I}, 1_{L^I})$ is a zero element of uninorm $\mathcal{U}$.

Lemma 7 (cf. [7]). *If $\mathcal{U}$ is a uninorm with the neutral element $e \in L^I \setminus \{0_{L^I}, 1_{L^I}\}$ then $\mathcal{U}(0_{L^I}, 1_{L^I}) = 0_{L^I}$ or $\mathcal{U}(0_{L^I}, 1_{L^I}) = 1_{L^I}$ or $\mathcal{U}(0_{L^I}, 1_{L^I}) \| e$.*

Example 4. *Let U_1, U_2 be uninorm given by*

$$U_1(x,y) = \begin{cases} \max(x,y), & \textit{if } x,y \in [0.1,1], \\ \min(x,y) & \textit{else}, \end{cases}$$

$$U_2(x,y) = \begin{cases} \min(x,y), & \textit{if } x,y \in [0,0.1], \\ \max(x,y) & \textit{else}, \end{cases}$$

then for uninorm

$$\mathcal{U}(x,y) = [U_1(x_1,y_1), U_2(x_2,y_2)]$$

we have $\mathcal{U}(0_{L^I}, 1_{L^I}) = [U_1(0,1), U_2(0,1)] = [0,1]$ and $\mathcal{U}$ is neither conjunctive nor disjunctive.

We can also consider decomposable uninorms.

Example 5. *Let U be a uninorm given by*

$$U(x,y) = \begin{cases} \max(x,y), & \textit{if } x,y \in [0.5,1] \\ \min(x,y), & \textit{otherwise} \end{cases} .$$

Operation

$$\mathcal{U}(x,y) = [U(x_1,y_1), U(x_2,y_2)]$$

is decomposable uninorm.

Example 6. *Let U be an arbitrary uninorm. Operation*

$$\mathcal{U}(x,y) = [\min(U(x_1,y_2), U(y_1,x_2)), U(x_2,y_2)]$$

is not decomposable.

For arbitrary uninorm the zero element is equal 0_{L^I}, 1_{L^I} or it is incomparable with neutral element. If we consider decomposable uninorm then we have the following results

Lemma 8. *If $\mathcal{U}$ is a decomposable uninorm with the neutral element $e \in L^I$ then $\mathcal{U}(0_{L^I}, 1_{L^I}) = 0_{L^I}$ or $\mathcal{U}(0_{L^I}, 1_{L^I}) = 1_{L^I}$ or $\mathcal{U}(0_{L^I}, 1_{L^I}) = [0,1]$.*

Proof. Since $\mathcal{U}$ is decomposable, then there exist U_1 and U_2, such that

$$\mathcal{U}(x,y) = [U_1(x_1,y_1), U_2(x_2,y_2)].$$

We consider the four possible cases:

- $U_1(0,1) = 0, U_2(0,1) = 0$, then $\mathcal{U}(0_{L^I}, 1_{L^I}) = [U_1(0,1), U_2(0,1)] = [0,0] = 0_{L^I}$
- $U_1(0,1) = 0, U_2(0,1) = 1$, then $\mathcal{U}(0_{L^I}, 1_{L^I}) = [U_1(0,1), U_2(0,1)] = [0,1]$
- $U_1(0,1) = 1, U_2(0,1) = 1$, then $\mathcal{U}(0_{L^I}, 1_{L^I}) = [U_1(0,1), U_2(0,1)] = [1,1] = 1_{L^I}$
- $U_1(0,1) = 1, U_2(0,1) = 0$ not occur, according to the Lemma 1.

□

Remark 4. *We cannot use the pair of disjunctive and conjunctive uninorms for construction of a decomposable uninorm, because this leads to the fourth case in the above lemma.*

Supposition 1. *If is a uninorm $\mathcal{U}$ has the neutral element $e \in D$ then $\mathcal{U}(0_{L^I}, 1_{L^I}) = 0_{L^I}$ or $\mathcal{U}(0_{L^I}, 1_{L^I}) = 1_{L^I}$ or $\mathcal{U}(0_{L^I}, 1_{L^I}) = [0,1]$.*

Supposition 2. *If is a uninorm $\mathcal{U}$ has the neutral element $e \in L^I$ then $\mathcal{U}(0_{L^I}, 1_{L^I}) = 0_{L^I}$ or $\mathcal{U}(0_{L^I}, 1_{L^I}) = 1_{L^I}$ or $\mathcal{U}(0_{L^I}, 1_{L^I}) = [0,1]$.*

If we consider decomposable uninorms, then we obtain some dependencies between neutral elements of its components uninorms.

Theorem 12. *If a uninorm $\mathcal{U}$ is decomposable then $e_1 \geq e_2$, where e_1 and e_2 are the neutral element of uninorms U_1 and U_2.*

Proof. Let e_1 and e_2 be the neutral element of uninorms U_1 and U_2. Then

$$\mathcal{U}([e_1, e_1], [e_2, e_2]) = [U_1(e_1, e_2), U_2(e_1, e_2)] = [e_2, e_1] \in L^I.$$

So, $e_2 \leq e_1$. □

Since $[e_1, e_2] \in L^I$, then directly from above we obtain

Theorem 13. *If $\mathcal{U}$ is a decomposable uninorm with a neutral element $e = [e_1, e_2]$, then $e \in D$.*

Supposition 3. *If $\mathcal{U}$ is a uninorm with a neutral element $e \in L^I$, then $e \in D$.*

5 Nullnorms on L^I

Another generalization of triangular norms and conorms are nullnorms.

Definition 13 ([3]). *An operation $V : [0,1]^2 \to [0,1]$ is called a nullnorm if it is increasing, commutative, associative, has a zero element $z \in [0,1]$, and satisfies*

$$V(0,x) = x \quad \text{for all } x \le z, \tag{16}$$

$$V(1,x) = x \quad \text{for all } x \ge z. \tag{17}$$

Remark 5. *In previous definition we may omit the assumption that the element z is the zero element of operation V, because it follows from the conditions* (16) *and* (17).

Theorem 14 ([3]). *Let $z \in (0,1)$. A binary operation V is a nullnorm with a zero element z if and only if there exist a triangular norm T and a triangular conorm S such that*

$$V(x,y) = \begin{cases} S^*(x,y) \text{ if } x,y \in [0,z] \\ T^*(x,y) \text{ if } x,y \in [z,1] \\ z \quad otherwise \end{cases}, \tag{18}$$

where

$$\begin{cases} S^*(x,y) = \varphi^{-1}\left(S\left(\varphi(x),\varphi(y)\right)\right), & \\ \varphi(x) = x/z, & x,y \in [0,z], \\ T^*(x,y) = \psi^{-1}\left(T\left(\psi(x),\psi(y)\right)\right), & \\ \psi(x) = (x-z)/(1-z), & x,y \in [z,1]. \end{cases} \tag{19}$$

We may straightforward transform the definition of nullnorm from $[0,1]$ into the lattice L^I.

Definition 14. *Operation $\mathcal{V} : (L^I)^2 \to L^I$ is called a nullnorm if it is commutative, associative, increasing and has a zero element $z \in L^I$ and satisfies*

$$\mathcal{V}(0_{L^I},x) = x \quad \text{for all } x \le z, \tag{20}$$

$$\mathcal{V}(1_{L^I},x) = x \quad \text{for all } x \ge z. \tag{21}$$

Theorem 15. *If a nullnorm $\mathcal{V}$ has a zero element $z \in D \setminus \{0_{L^I}, 1_{L^I}\}$, then there exist a t-norm $\mathcal{T}$ and a t-conorm $\mathcal{S}$ such that*

$$\mathcal{V}(x,y) = \begin{cases} \mathcal{T}^*(x,y) \text{ if } x,y \in E_z' \\ \mathcal{S}^*(x,y) \text{ if } x,y \in E_z \end{cases} \tag{22}$$

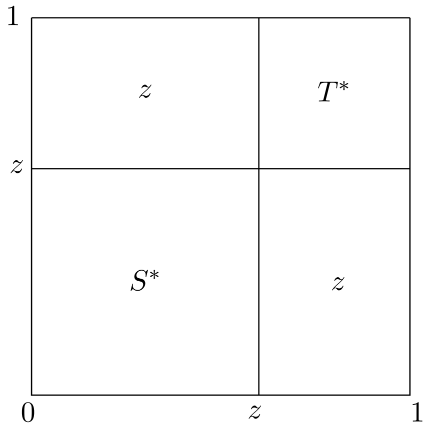

Figure 5: The structure of nullnorm

where

$$\begin{cases} \mathcal{S}^*(x,y) = \Phi_z\left(\mathcal{S}\left(\Phi_z^{-1}(x), \Phi_z^{-1}(y)\right)\right), & x,y \in E_z \\ \mathcal{T}^*(x,y) = \Psi_z\left(\mathcal{T}\left(\Psi_z^{-1}(x), \Psi_z^{-1}(y)\right)\right), & x,y \in E_z' \end{cases}. \tag{23}$$

and Ψ_z*,* Φ_z *are given as in* (15)*.*

Remark 6. *Similarly as for ordinary nullnorm we may omit in the above definition the assumption that* z *is a zero element of operation* $\mathcal{V}$*.*

Lemma 9. *Let* $\mathcal{V} : (L^I)^2 \to L^I$ *be an increasing, associative operation. An element* $z \in L^I$ *is the zero of operation* $\mathcal{V}$ *if and only if*

$$z = \mathcal{V}(0,1) = \mathcal{V}(1,0). \tag{24}$$

We can also consider the decomposable nullnorms.

Example 7. *Let* V *be nullnorm given by*

$$V(x,y) = \begin{cases} \max(x,y), & \text{if } x,y \in [0,0.5], \\ \min(x,y) & \text{if } x,y \in [0.5,1], \\ 0.5 & \text{otherwise}, \end{cases}$$

then the operation

$$\mathcal{V}(x,y) = [V(x_1,y_1), V(x_2,y_2)]$$

is a decomposable nullnorm.

If we consider decomposable nullnorms, then we obtain some dependencies between zero elements of its component, which are similar to those obtained for the neutral element of decomposable uninorm.

Theorem 16. *If a nullnorm $\mathcal{V}$ is decomposable then $z_1 \geq z_2$, where z_1 and z_2 are the zero element of nullnorms V_1 and V_2.*

Proof. Let z_1 and z_2 be the zero element of nullnorms V_1 and V_2. Then

$$\mathcal{V}([z_1, z_1], [z_2, z_2]) = [V_1(z_1, z_2), V_2(z_1, z_2)] = [z_2, z_1] \in L^I.$$

So, $z_2 \leq z_1$. □

Since $[z_1, z_2] \in L^I$, then directly from above we obtain

Theorem 17. *If $\mathcal{V}$ is a decomposable nullnorm with a zero element $z = [z_1, z_2]$, then $z \in D$.*

Supposition 4. *If $\mathcal{V}$ is a nullnorm with a zero element $z \in L^I$, then $z \in D$.*

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The papers presented in this Volume 1 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.
It may be viewed as a result of fruitful discussions held during the Eleventh International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2012) organized in Warsaw on October 12, 2012 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT - Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bystrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, Prof. Asen Zlatarov University, Burgas, Bulgaria, and the University of Westminster, Harrow, UK:

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The consecutive International Workshops on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGNs) have been meant to provide a forum for the presentation of new results and for scientific discussion on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems have also been included. The Eleventh International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2012) is a continuation of this undertaking, and provides many new ideas and results in the areas concerned.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

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