Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics. Volume I: Foundations

Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics Volume II: Applications

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ISBN 9788389475305

Interval type-2 fuzzy logic: theory and applications

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Abstract

Type-2 fuzzy sets are used for modeling uncertainty and imprecision in a better way. These type-2 fuzzy sets were originally presented by Zadeh in 1975 and are essentially "fuzzy fuzzy" sets where the fuzzy degree of membership is a type-1 fuzzy set. The new concepts were introduced by Mendel and Liang allowing the characterization of a type-2 fuzzy set with a superior membership function and an inferior membership function; these two functions can be represented each one by a type-1 fuzzy set membership function. The interval between these two functions represents the footprint of uncertainty (FOU), which is used to characterize a type-2 fuzzy set.

Keywords: interval type-2 fuzzy logic, type-2 fuzzy models, high order fuzzy sets.

1 Introduction

On the past decade, fuzzy systems have displaced conventional technology in different scientific and system engineering applications, especially in pattern recognition and control systems. The same fuzzy technology, in approximation reasoning form, is resurging also in the information technology, where it is now giving support to decision making and expert systems with powerful reasoning capacity and a limited quantity of rules.

The fuzzy sets were presented by L.A. Zadeh in 1965 [10,11,12] to process / manipulate data and information affected by unprobabilistic uncertainty / imprecision. These were designed to mathematically represent the vagueness and uncertainty of linguistic problems; thereby obtaining formal tools to work with intrinsic imprecision in different type of problems; it is considered a generalization of the classic set theory.

Intelligent Systems based on fuzzy logic are fundamental tools for nonlinear complex system modeling. The fuzzy sets and fuzzy logic are the base for fuzzy systems, where their objective has been to model how the brain manipulates inexact information [4].

Type-2 fuzzy sets are used for modeling uncertainty and imprecision in a better way. These type-2 fuzzy sets were originally presented by Zadeh in 1975 and are essentially "fuzzy fuzzy" sets where the fuzzy degree of membership is a type-1 fuzzy set [5, 13]. The new concepts were introduced by Mendel and Liang [6, 7] allowing the characterization of a type-2 fuzzy set with a superior membership function and an inferior membership function; these two functions can be represented each one by a type-1 fuzzy set membership function. The interval between these two functions represents the footprint of uncertainty (FOU), which is used to characterize a type-2 fuzzy set.

The uncertainty is the imperfection of knowledge about the natural process or natural state. The statistical uncertainty is the randomness or error that comes from different sources as we use it in a statistical methodology. There are different sources of uncertainty in the evaluation and calculus process. The five types of uncertainty that emerge from the imprecise knowledge natural state are:

- Measurement uncertainty. It is the error on observed quantities.
- Process uncertainty. It is the dynamic randomness.
- Model uncertainty. It is the wrong specification of the model structure.
- Estimate uncertainty. It is the one that can appear from any of the previous uncertainties or a combination of them, and it is called inexactness and imprecision.
- Implementation uncertainty. It is the consequence of the variability that results from sorting politics, i.e. incapacity to reach the exact strategic objective.

2 Interval type-2 fuzzy set theory

A type-2 fuzzy set [5,15] expresses the non-deterministic truth degree with imprecision and uncertainty for an element that belongs to a set. A type-2 fuzzy set denoted by $\tilde{\tilde{A}}$, is characterized by a type-2 membership function $\mu_{\tilde{\tilde{A}}}(x,u)$, where $x \in X$, $u \in J_x^u \subseteq [0,1]$ and $0 \le \mu_{\tilde{\tilde{A}}}(x,u) \le 1$ defined in equation (1).

$$\begin{aligned}\tilde{\tilde{A}} &= \left\{(x, \mu_{\tilde{\tilde{A}}}(x)) \mid x \in X\right\} \\ &= \left\{\left(x, u, \mu_{\tilde{\tilde{A}}}(x,u)\right) \mid \forall x \in X, \forall u \in J_x^u \subseteq [0,1]\right\}\end{aligned} \tag{1}$$

An example of a type-2 membership function constructed in the IT2FLS toolbox was composed by a Pi primary and a Gbell secondary type-1 membership functions, these are depicted in Figure 1.

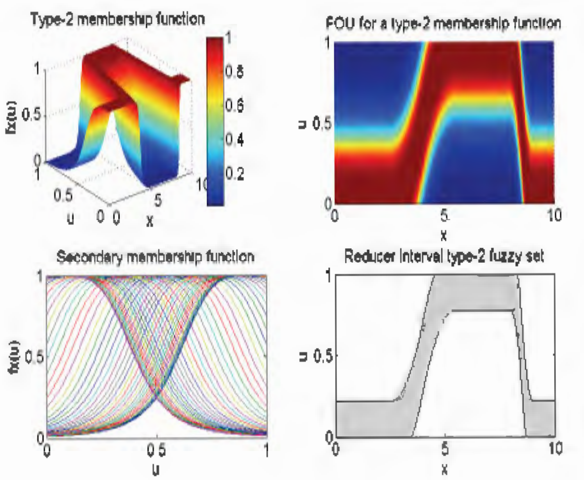

Figure 1: FOU for type-2 membership functions

If $\tilde{\tilde{A}}$ is continuous it is denoted in equation (2).

$$\tilde{\tilde{A}} = \left\{ \int_{x\in X} \left[\int_{u \in J_x^u \subseteq [0,1]} f_x(u)/u \right] / x \right\} \tag{2}$$

where $\int\int$ denotes the union of x and u. If $\tilde{\tilde{A}}$ is discrete then it is denoted by equation (3).

$$\tilde{\tilde{A}} = \left\{ \sum_{x\in X} \mu_{\tilde{A}}(x)/x \right\} = \left\{ \sum_{i=1}^{N} \left[\sum_{k=1}^{M_i} f_{x_i}(u_{ik})/u_{ik} \right] / x_i \right\} \tag{3}$$

where $\sum\sum$ denotes the union of x and u.

If $f_x(u) = 1, \forall u \in [\underline{J}_x^u, \bar{J}_x^u] \subseteq [0,1]$, the type-2 membership function $\mu_{\tilde{A}}(x,u)$ is expressed by one type-1 inferior membership function,

$\underline{J}_x^u \equiv \underline{\mu}_A(x)$ and one type-1 superior, $\overline{J}_x^u \equiv \overline{\mu}_A(x)$ (Fig. 2), then it is called an interval type-2 fuzzy set [6] denoted by equations (4) and (5).

$$\tilde{\tilde{A}} = \begin{cases} (x,u,1) \mid \forall x \in X, \\ \qquad \forall u \in [\underline{\mu}_{\tilde{A}}(x), \overline{\mu}_{\tilde{A}}(x)] \subseteq [0,1] \end{cases} \tag{4}$$

or

$$\begin{aligned} \tilde{\tilde{A}} &= \left\{ \int_{x\in X} \left[\int_{u\in[\underline{J}_x^u, \overline{J}_x^u]\subseteq[0,1]} 1/u \right] / x \right\} \\ &= \left\{ \int_{x\in X} \left[\int_{u\in[\underline{\mu}_{\tilde{A}}(x), \overline{\mu}_{\tilde{A}}(x)]\subseteq[0,1]} 1/u \right] / x \right\} \end{aligned} \tag{5}$$

If $\tilde{\tilde{A}}$ is a type-2 fuzzy Singleton, the membership function is defined by equation (6).

$$\mu_{\tilde{\tilde{A}}}(x) = \begin{cases} 1/1 & si\ x = x' \\ 1/0 & si\ x \neq x' \end{cases} \tag{6}$$

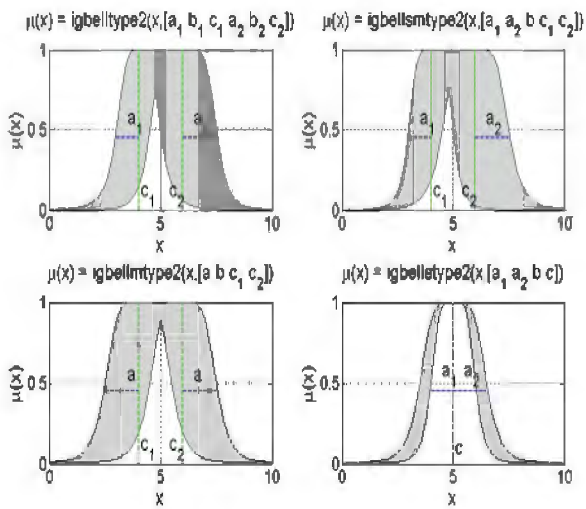

Figure 2: FOU for Gbell primary interval type-2 membership functions

We can apply some operators to the fuzzy sets, or we can make some operations between them [7,8,13]. When we apply an operator to one fuzzy set we obtain another fuzzy set; by the same manner when we combine an operation with two or more sets we obtain another fuzzy set. If we have two type-2 fuzzy subsets identified by the letters $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$, associated to a linguistic variable, we can define three basic operations: complement, union and intersection (Table 1).

Table 1: Interval type-2 fuzzy set operations

NAME	OPERATOR	OPERATION
Union	$\sqcup$ = join	$\tilde{\tilde{A}} \sqcup \tilde{\tilde{B}} = \{\int_{x\in X} \mu_{\tilde{\tilde{A}}}(x) \sqcup \mu_{\tilde{\tilde{B}}}(x)/x\}$ $= \left\{ \int_{x\in X} \left[\int_{\alpha\in[\underline{\mu}_{\tilde{A}(x)} \vee \underline{\mu}_{\tilde{B}(x)}, \overline{\mu}_{\tilde{A}(x)} \vee \overline{\mu}_{\tilde{B}(x)}]} 1/\alpha \right] / x \right\}$
Intersection	$\sqcap$ = meet	$\tilde{\tilde{A}} \sqcap \tilde{\tilde{B}} = \{\int_{x\in X} \mu_{\tilde{\tilde{A}}}(x) \sqcap \mu_{\tilde{\tilde{B}}}(x)/x\}$ $= \left\{ \int_{x\in X} \left[\int_{\alpha\in[\underline{\mu}_{\tilde{A}(x)} \wedge \underline{\mu}_{\tilde{B}(x)}, \overline{\mu}_{\tilde{A}(x)} \wedge \overline{\mu}_{\tilde{B}(x)}]} 1/\alpha \right] / x \right\}$
Negation	$\neg$	$\neg\tilde{\tilde{A}} = \left\{ \int_{x\in X} \mu_{\neg\tilde{\tilde{A}}}(x)/x \right\}$ $= \left\{ \int_{x\in X} \left[\int_{\alpha\in[1-\overline{\mu}_{\tilde{A}}(x), 1-\underline{\mu}_{\tilde{A}}(x)]} 1/\alpha \right] / x \right\}$

The human knowledge is expressed in fuzzy rule terms with the next syntax:

IF <*fuzzy proposition*> **THEN** <*fuzzy proposition*>

The fuzzy propositions are divided in two types, the first one is named **atomic: x is A**, where x is a linguistic variable and A is a linguistic value; the second one is called **compounded: x is A AND y is B OR z is NOT C**, this is a compounded atomic fuzzy proposition with the "AND", "OR" and "NOT" connectors, representing fuzzy intersection, union and complement respectively. The compounded fuzzy propositions are fuzzy relationships. The membership function of the rule IF-THEN is a fuzzy relation determined by a fuzzy implication operator. The fuzzy rules combine one or more fuzzy sets of entry, called antecedent, and are associated with one output fuzzy set, called consequents. The Fuzzy Sets of the antecedent are associated by fuzzy operators AND, OR, NOT and linguistic modifiers. The fuzzy rules permit expressing the available knowledge about the relationship between antecedent and consequents. To express this knowledge completely we normally have several rules, grouped to form what it is known a rule base, that is, a set of rules that express the known relationships between antecedent and consequents. The fuzzy rules are basically IF <Antecedent> THEN <Consequent> and expresses a fuzzy relationship or proposition.

In fuzzy logic the reasoning is imprecise, it is approximated, that means that we can infer from one rule a conclusion even if the antecedent doesn't comply completely. We can count on two basic inference methods between rules and inference laws, Generalized Modus Ponens (GMP) [5,6,13,14] and Generalized Modus Tollens (GMT), that represent the extensions or generalizations of classic reasoning. The GMP inference method is known as direct reasoning and is resumed as:

Rule	*IF x is A THEN y is B*
Fact	*x is A'*
Conclusion	*y es B'*

Where A, A', B and B' are fuzzy sets of any kind. This relationship is expressed as $B' = A' \circ (A \rightarrow B)$. Figure 3 shows an example of Interval Type-2 direct reasoning with Interval Type-2 Fuzzy Inputs.

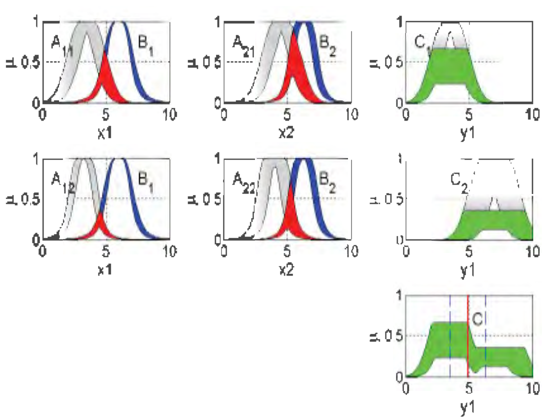

Figure 3: Interval type-2 fuzzy reasoning

An Inference Fuzzy System is a rule base system that uses fuzzy logic, instead of Boolean logic utilized in data analysis [3, 7, 13]. Its basic structure includes four components (Fig. 4):

- ***Fuzzifier.*** Translates inputs (real values) to fuzzy values.
- ***Inference System.*** Applies a fuzzy reasoning mechanism to obtain a fuzzy output.
- ***Type Defuzzifier/Reducer.*** The defuzzifier traduces one output to precise values; the type reducer transforms a Type-2 Set into a Type-1 Fuzzy Set.
- ***Knowledge Base.*** Contains a set of fuzzy rules, and a membership functions set known as the database.

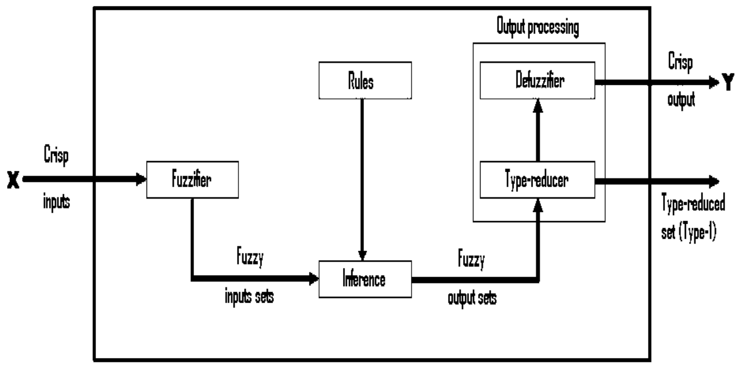

Figure 4: Type-2 inference fuzzy system structure

The decision process is a task that identifies parameters by the inference system using the rules of the rule base data. These fuzzy rules define the connection between the input and output fuzzy variables. A fuzzy rule has the form: IF <Antecedent> THEN <Consequent>, where antecedent is a compound fuzzy logic expression of one or more simple fuzzy expressions connected with fuzzy operators; and the consequent is an expression that assigns fuzzy values to output variables. The inference system evaluates all the rules of the rule base and combines the weights of the consequents of all relevant rules in one fuzzy set using the aggregate operation. This operation is analog in fuzzy logic to the S-norm operator.

Fuzzy modeling is a task for parameter identification in a fuzzy inference system to obtain an adequate behavior. A fuzzy model with the direct view is constructed with the knowledge of an expert. This task becomes more difficult when the available knowledge is incomplete or when space is a problem, then the use of automatic views are recommended for the fuzzy model. It can be considered different point of views for fuzzy modeling, based on neural networks, genetic algorithms and hybrid methods. The selection of relevant variables and adequate rules is critical for generating a good system. One of the biggest problems occurring in fuzzy modeling is dimensionality, that is, when the computational requirements grow exponentially in relation of the quantity of variables.

3 Interval type-2 fuzzy system design

The Mamdani and Takagi-Sugeno-Kang (TSK) Interval Type-2 Fuzzy Inference Models [1] and the design of Interval Type-2 membership functions and operators are implemented in the IT2FLS Toolbox (Interval Type-2 Fuzzy Logic Systems) reused from the Matlab® commercial Fuzzy Logic Toolbox [3].

The Interval Type-2 Fuzzy Inference Systems (IT2FIS) structure is the MATLAB object that contains all the interval type-2 fuzzy inference system information. This structure is stored inside each GUI tool. Access functions such as getifistype2 and setifistype2 make it easy to examine this structure.

All the information for a given fuzzy inference system is contained in the IT2FIS structure, including variable names, membership function definitions, and so on. This structure can itself be thought of as a hierarchy of structures, as shown in the following diagram (Figure 5).

The implementation of the IT2FLS GUI is analogous to the GUI used for Type-1 FLS in the Matlab® Fuzzy Logic Toolbox, thus permitting the experienced user to adapt easily to the use of IT2FLS GUI. Figures 6 and 7 show the main screen of the Interval Type-2 Fuzzy Inference Systems Structure Editor called IT2FIS (Interval Type-2 Fuzzy Inference Systems).

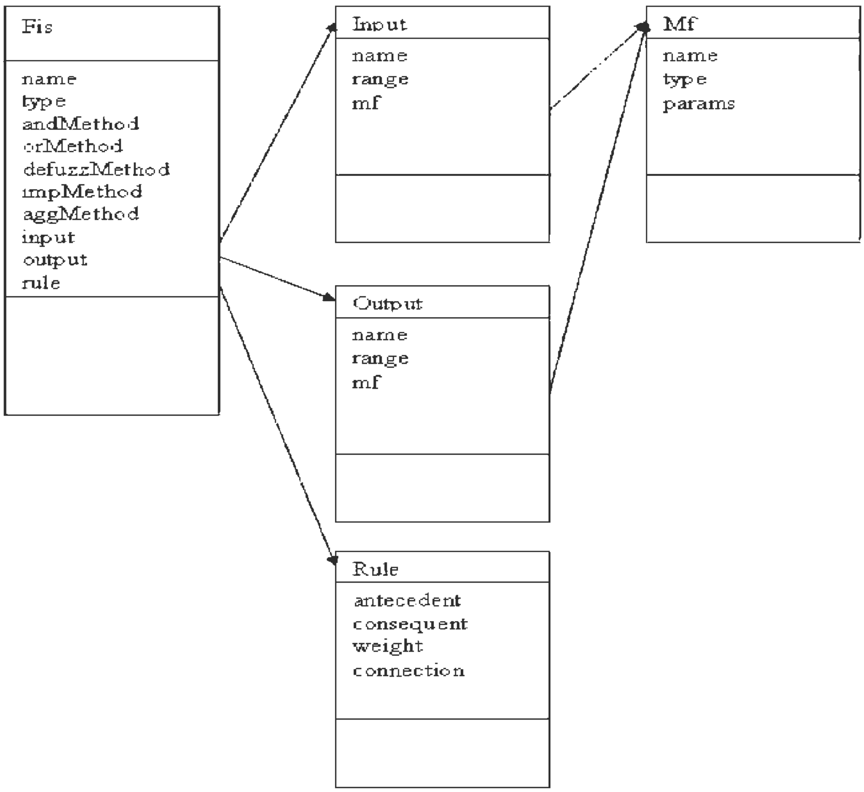

Figure 5: Hierarchy of IT2FIS structures diagram

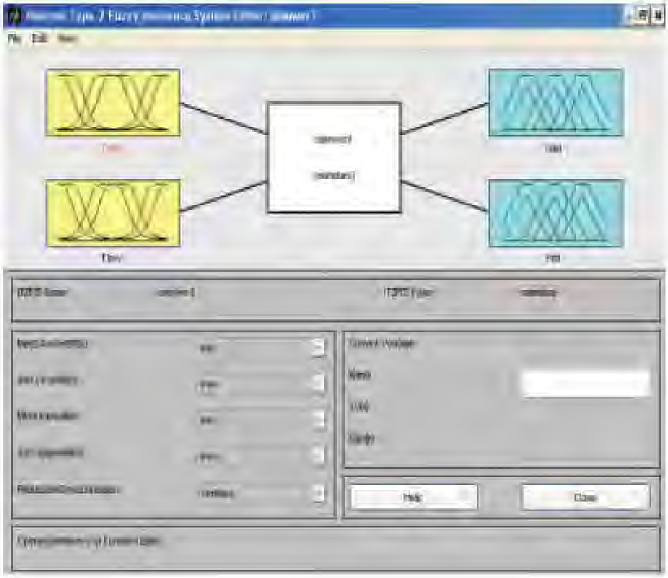

Figure 6: IT2FIS Editor

Figure 7: Interval type-2 MF's editor

The Mamdani IT2FIS, is designed with **n** inputs, **m** outputs and **r** rules. The *k*th rule with interval type-2 fuzzy antecedents $\tilde{\tilde{A}}_{k,i} \in \{\mu_{i,l_{k,i}}\}$, interval type-2 fuzzy consequent $\tilde{\tilde{C}}_{k,j} \in \{\sigma_{j,l_{k,j}}\}$ and interval type-2 fuzzy facts $\tilde{\tilde{A}}_i^{'}$ are inferred as a direct reasoning [1, 7]. The evaluation of this type of reasoning is as follows:

$$\tilde{\tilde{R}}_{k,j} = \tilde{\tilde{A}}_{k,1} \times \ldots \times \tilde{\tilde{A}}_{k,n} \to \tilde{\tilde{C}}_{k,j} = (\tilde{\tilde{A}}_{k,1} \to \tilde{\tilde{C}}_{k,j}) \times \ldots \times (\tilde{\tilde{A}}_{k,n} \to \tilde{\tilde{C}}_{k,j}),\ k\text{th rule.}$$

$$\tilde{\tilde{H}} = \tilde{\tilde{A}}_1^{'} \times \ldots \times \tilde{\tilde{A}}_n^{'},\ \text{facts.}$$

$$\tilde{\tilde{C}}_{k,j}^{''} = \tilde{\tilde{H}} \circ \tilde{\tilde{R}}_{k,j} = \bigcap_{i=1}^{n} [\tilde{\tilde{A}}_i^{'} \circ (\tilde{\tilde{A}}_{k,i} \to \tilde{\tilde{C}}_{k,j})]$$

$$= \left\{ \int_Y \left[\int_{\alpha \in [\underline{\mu}_{\tilde{\tilde{C}}_{k,j}^{''}}(y_j), \overline{\mu}_{\tilde{\tilde{C}}_{k,j}^{''}}(y_j)] \subseteq [0,1]} 1/\alpha \right] / y_j \right\}$$

$$\underline{\mu}_{\tilde{\tilde{C}}_{k,j}^{''}}(y_j) = \left[\mathop{\tilde{*}}_{i=1}^{n} \left(\underline{\mu}_{\tilde{\tilde{A}}_i^{'}}(x_i) \tilde{*} \underline{\mu}_{\tilde{\tilde{A}}_{k,i}}(x_i) \right) \right] \tilde{*} \underline{\mu}_{\tilde{\tilde{C}}_{k,j}}(y_j)$$

$$\overline{\mu}_{\tilde{\tilde{C}}_{k,j}^{''}}(y_j) = \left[\mathop{\tilde{*}}_{i=1}^{n} \left(\overline{\mu}_{\tilde{\tilde{A}}_i^{'}}(x_i) \tilde{*} \overline{\mu}_{\tilde{\tilde{A}}_{k,i}}(x_i) \right) \right] \tilde{*} \overline{\mu}_{\tilde{\tilde{C}}_{k,j}}(y_j)$$

$$\tilde{\tilde{C}}_j^{'} = \bigcup_{k=1}^{r} \tilde{\tilde{C}}_{k,j}^{''} = \bigcup_{k=1}^{r} [\bigcap_{i=1}^{n} [\tilde{\tilde{A}}_i^{'} \circ (\tilde{\tilde{A}}_{k,i} \to \tilde{\tilde{C}}_{k,j})]]$$

$$= \left\{ \int_Y \left[\int_{\alpha \in [\underline{\mu}_{\tilde{\tilde{C}}_j^{'}}(y_j), \overline{\mu}_{\tilde{\tilde{C}}_j^{'}}(y_j)] \subseteq [0,1]} 1/\alpha \right] / y_j \right\}$$

$$\underline{\mu}_{\tilde{\tilde{C}}_j^{'}}(y_j) = \bigvee_{k=1}^{r} \left(\underline{\mu}_{\tilde{\tilde{C}}_{k,j}^{''}}(y_j) \right)$$

$$= \bigvee_{k=1}^{r} \left(\left[\mathop{\tilde{*}}_{i=1}^{n} \left(\underline{\mu}_{\tilde{\tilde{A}}_i^{'}}(x_i) \tilde{*} \underline{\mu}_{\tilde{\tilde{A}}_{k,i}}(x_i) \right) \right] \tilde{*} \underline{\mu}_{\tilde{\tilde{C}}_{k,j}}(y_j) \right)$$

$$\overline{\mu}_{\tilde{\tilde{C}}_j^{'}}(y_j) = \bigvee_{k=1}^{r} \left(\overline{\mu}_{\tilde{\tilde{C}}_{k,j}^{''}}(y_j) \right)$$

$$= \bigvee_{k=1}^{r} \left(\left[\mathop{\tilde{*}}_{i=1}^{n} \left(\overline{\mu}_{\tilde{\tilde{A}}_i^{'}}(x_i) \tilde{*} \overline{\mu}_{\tilde{\tilde{A}}_{k,i}}(x_i) \right) \right] \tilde{*} \overline{\mu}_{\tilde{\tilde{C}}_{k,j}}(y_j) \right)$$

The defuzzification of the interval type-2 fuzzy aggregated output set $\tilde{\tilde{C}}_j^{'}$ is: $\hat{y}_j = idefuzztype2(\mu_{\tilde{\tilde{C}}_j^{'}}(y_j),'type')$ where *type* is the name of the defuzzification technique. If $\tilde{\tilde{A}}_i^{'}$ are interval type-2 fuzzy singletons then:

$$\mu_{\tilde{\tilde{C}}_{k,j}^{''}}(y_j) = \left[\quad \sqcap_{i=1}^{n} [\ \mu_{\tilde{\tilde{A}}_{k,i}}(x_i)] \quad \right] \sqcap \mu_{\tilde{\tilde{C}}_{k,j}}(y_j)$$

$$= \left\{ \int_Y \left[\int_{\alpha \in [\underline{\mu}_{\tilde{\tilde{C}}_{k,j}^{''}}(y_j), \overline{\mu}_{\tilde{\tilde{C}}_{k,j}^{''}}(y_j)] \subseteq [0,1]} 1/\alpha \right] / y_j \right\}$$

$$\underline{\mu}_{\tilde{C}_{k,j}^{''}}(y_j) = [\tilde{\ast}_{i=1}^{n} \underline{\mu}_{\tilde{A}_{k,i}}(\hat{x}_i)] \tilde{\ast} \underline{\mu}_{\tilde{C}_{k,j}}(y_j)$$

$$\overline{\mu}_{\tilde{C}_{k,j}^{''}}(y_j) = [\tilde{\ast}_{i=1}^{n} \overline{\mu}_{\tilde{A}_{k,i}}(\hat{x}_i)] \tilde{\ast} \overline{\mu}_{\tilde{C}_{k,j}}(y_j)$$

$$\mu_{\tilde{\tilde{C}}_j^{'}}(y_j) = \sqcup_{k=1}^{r} \mu_{\tilde{\tilde{C}}_{k,j}^{''}}(y_j)$$

$$= \sqcup_{k=1}^{r} [\ \sqcap_{i=1}^{n} [\ \mu_{\tilde{\tilde{A}}_{k,i}}(x_i)] \ \sqcap \mu_{\tilde{\tilde{C}}_{k,j}}(y_j)]]$$

$$= \left\{ \int_Y \left[\int_{\alpha \in [\underline{\mu}_{\tilde{\tilde{C}}_j^{'}}(y_j), \overline{\mu}_{\tilde{\tilde{C}}_j^{'}}(y_j)] \subseteq [0,1]} 1/\alpha \right] / y_j \right\}$$

$$\underline{\mu}_{\tilde{\tilde{C}}_j^{'}}(y_j) = \bigvee_{k=1}^{r} [\underline{\mu}_{\tilde{\tilde{C}}_{k,j}^{''}}(y_j)] = \bigvee_{k=1}^{r} \left([\tilde{\ast}_{i=1}^{n} \underline{\mu}_{\tilde{\tilde{A}}_{k,i}}(\hat{x}_i)] \tilde{\ast} \underline{\mu}_{\tilde{\tilde{C}}_{k,j}}(y_j) \right)$$

$$\overline{\mu}_{\tilde{\tilde{C}}_j^{'}}(y_j) = \bigvee_{k=1}^{r} [\overline{\mu}_{\tilde{\tilde{C}}_{k,j}^{''}}(y_j)] = \bigvee_{k=1}^{r} \left([\tilde{\ast}_{i=1}^{n} \overline{\mu}_{\tilde{\tilde{A}}_{k,i}}(\hat{x}_i)] \tilde{\ast} \overline{\mu}_{\tilde{\tilde{C}}_{k,j}}(y_j) \right)$$

The IT2FIS de Takagi-Sugeno-Kang system is designed with **n** inputs, **m** outputs and **r** rules. The *k*th rule with interval type-2 fuzzy antecedents $\tilde{\tilde{A}}_{k,i} \in \{\mu_{i,l_{k,i}}\}$, interval type-1 fuzzy set are used for the consequents sets, $f_{j,k} = \theta_{0,j}^{k} + \sum_{i=1}^{n} \theta_{i,j}^{k} \cdot x_i$ and real facts are inferred as a direct reasoning [9, 10].

The evaluation of this reasoning is:

$$\alpha_k = [\underline{\alpha}_k, \overline{\alpha}_k] = \int_{i=1}^{n} [\ \mu_{\tilde{A}_{k,i}}(\hat{x}_i)]$$
$$= \left[\tilde{*}_{i=1}^{n} \left(\underline{\mu}_{\tilde{A}_{k,i}}(\hat{x}_i) \right), \tilde{*}_{i=1}^{n} \left(\overline{\mu}_{\tilde{A}_{k,i}}(\hat{x}_i) \right) \right]$$

where $\alpha_k = [\underline{\alpha}_k, \overline{\alpha}_k]$ is the firing set of the interval type-1 fuzzy antecedent of the *k*th rule.

$$f_{j,k} = \theta_{0,j}^{k} + \sum_{i=1}^{n} \theta_{i,j}^{k} \cdot \hat{x}_i$$

where $f_{j,k} = [{}^{l}f_{j,k}, {}^{r}f_{j,k}]$ is a real function of the interval consequents of the *k*th rule. If $\theta_{i,j}^{k} = [c_{i,j}^{k} - s_{i,j}^{k}, c_{i,j}^{k} + s_{i,j}^{k}]$ $\forall\, i = 0, \ldots, n$, where $c_{i,j}^{k}$ is the center and $s_{i,j}^{k}$ denotes the spread, then ${}^{l}f_{j,k}, {}^{r}f_{j,k}$ is expressed as:

$${}^{l}f_{j,k} = \sum_{i=1}^{n} c_{i,j}^{k} \cdot x_i + c_{0,j}^{k} - \sum_{i=1}^{n} s_{i,j}^{k} \cdot |x_i| - s_{0,j}^{k}$$
$${}^{r}f_{j,k} = \sum_{i=1}^{n} c_{i,j}^{k} \cdot x_i + c_{0,j}^{k} + \sum_{i=1}^{n} s_{i,j}^{k} \cdot |x_i| + s_{0,j}^{k}$$

With the Karnik and Mendel algorithm [10] the ${}^{l}\alpha_k$ and ${}^{r}\alpha_k$ are evaluated to obtain the FIS output variables, these are expressed as

$$\hat{y}_j^{l} = \frac{\sum_{k=1}^{r} {}^{l}\alpha_k \cdot {}^{l}f_{j,k}}{\sum_{k=1}^{r} {}^{l}\alpha_k} = \frac{\sum_{k=1}^{L} \overline{\alpha}_k \cdot {}^{l}f_{j,k} + \sum_{k=L+1}^{r} \underline{\alpha}_k \cdot {}^{l}f_{j,k}}{\sum_{k=1}^{L} \overline{\alpha}_k + \sum_{k=L+1}^{r} \underline{\alpha}_k}$$

$$\hat{y}_j^{r} = \frac{\sum_{k=1}^{r} {}^{r}\alpha_k \cdot {}^{r}f_{j,k}}{\sum_{k=1}^{r} {}^{r}\alpha_k} = \frac{\sum_{k=1}^{R} \underline{\alpha}_k \cdot {}^{r}f_{j,k} + \sum_{k=R+1}^{r} \overline{\alpha}_k \cdot {}^{r}f_{j,k}}{\sum_{k=1}^{R} \underline{\alpha}_k + \sum_{k=R+1}^{r} \overline{\alpha}_k}$$

$$\hat{y}_j = \frac{\hat{y}_j^{l} + \hat{y}_j^{r}}{2}$$

There are also other methods to obtain the previous upper and lower values of the output variables, which include approximate methods based on genetic algorithms or neural networks [1, 2].

4 Simulation Results

The results show a comparative analysis of a Mackey-Glass chaotic time-series. The forecasting study considers using intelligent hybrid methods, with neural networks, (Mamdani, Takagi-Sugeno-Kang) type-1 fuzzy inference systems and genetic algorithms (neuro-genetic, fuzzy-genetic and neuro-fuzzy) and an interval type-2 fuzzy logic model, for the implicit knowledge acquisition in a time series behavioral data history. To identify the model we use 5 delays, $L^5x(t)$, 6 periods and 500 training data values to forecast 500 output values. The IT2FLS system works with 4 inputs, 2 interval type-2 Gaussian membership functions for each input, 16 rules and one output with 16 interval linear functions. The root mean square error (RMSE) forecasted is 0.0335 (Table 2).

Table 2: Forecasting of time series

Methods	**Mackey-Glass**			
	RMSE	***trn/chk***	***epoch***	***cpu(s)***
NNFF****†**	0.0595	500/500	200	13.36
CANFIS	0.0016	500/500	50	7.34
NNFF-GA†	0.0236	500/500	150	98.23
FLS(TKS)-GA†	0.0647	500/500	200	112.01
FLS(MAM)-GA†	0.0693	500/500	200	123.21
IT2FLS	0.0335	500/500	6	20.23

5 Conclusions

We have presented in this paper the basic concepts of interval type-2 fuzzy logic. Also, the use of a toolbox for type-2 fuzzy logic, developed by our group, to apply the theory in solving real-world problems is illustrated. Simulation results in time series prediction show the feasibility of the approach. Future researh work includes applying interval type-2 fuzzy logic to other applications areas, and also considering generalized or non-singleton type-2 fuzzy sets.

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The papers presented in this Volume 2 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.
It may be viewed as a result of fruitful discussions held during the Eighth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2009) organized in Warsaw on October 16, 2009 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Centre for Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT – Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bistrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, and the University of Westminster, Harrow, UK:

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The Eighth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2009) has been meant to commence a new series of scientific events primarily focused on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Moreover, other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems are discussed.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

ISBN-13 9788389475305
ISBN 838947530-8
9 788389 475305