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PROCEEDINGS OF THE 3rd ITALIAN-POLISH CONFERENCE ON APPLICATIONS OF SYSTEMS THEORY TO ECONOMY, MANAGEMENT AND TECHNOLOGY

WARSZAWA 1977

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PREFACE

The present volume comprises the papers from the 3rd Italian-Polish Conference which was held in Warsaw in [illegible]. The conference organized by the Centre of Control Systems and Computation of the Italian National Research Council in Rome and the Institute for Organization, Management and Control Sciences of Polish Academy of Sciences and the Ministry of Science, Higher Education and Technology in Warsaw was the third conference of this type.

The first one took place in Kraków (Poland) in 1972, the second in Pugnochiuso (Italy) in 1974.

This conference, like both the previous ones, gathered the leading representatives of many scientific centres from Italy as well as from Poland, representing also the different aspects of the systems analysis applied to the economic, management, data processing and technological systems.

From the past, such meetings enrich the exchange of views and experiences between the specialists from both countries what is the condition advantageous to both sides. Shown in the several parts, the summarization of the seminars and conferences especially enrich the communications concerning the essential development of the quantitative analysis and the problems of modelling of large complex systems.

The meaning of such meetings, however, goes far beyond that, what is happening during the conference because the established contacts are also fruitful in form of common scientific research works and — last but not least — begin the friendly links between the scientists of both countries which last for years.

The proceedings of the conference were divided into three parts:

1. Optimization and Control Theory;
2. Systems Theory in Economics;
3. Technological, Managerial and Information Systems.

While the first two parts are of rather homogeneous character, the third part contains the papers concerning the different types of models — for the economic, technological, management and data processing systems.

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MODELS OF COMMUNICATION NETWORKS IN LARGE-SCALE SYSTEMS

1. INTRODUCTION

As the structure of an engineering process or a socio-economical organization grows beyond a certain complexity, the so-called phenomena of the "dispersion of authority and information" occur more and more frequently.

This can be due to technological, behavioural or topological reasons. Industrial plants with remote observation posts and controllers, large organizations characterized by many decision centers, air, railway and road traffic control systems are only some of the many examples which can be given to illustrate such a dispersion.

There is no doubt that ideal communication networks, with no costs, delays and noises, interconnecting the points where information is handy and decisions are taken, should greatly decrease the level and the importance of decentralization in large scale systems. However, since real communication links are characterized by costs, noises and interruptions, communication problems play a central role in the synthesis of decentralized control structures.

Problems of this type are the following: to select the instants at which messages must be sent from observation to decision posts, to decide what data are worth transmitting and, in general, to define the communication procedure for the information interchange. These problems lead directly to a new class of control actions, that is, to the control of data flow within a communication network.

It follows that an overall optimization problem is in general to be solved, in which an optimal compromise must be sought between the cost of a communication structure and the expected pay-off that the set of data made available by such a communication structure can provide to the decision makers who exert control actions on the process.

The concept of "data flow control" is strictly related to the introduction of two subteams of decision makers. The decision makers of the first subteam are given the task of gathering, processing and transmitting data, whereas the decision makers of the second subteam are given the usual task of generating control actions. To be more specific, a passive measurement device becomes a decision maker of the first subteam whenever it is given the task of evaluating whether, and in what form, the gathered information is worth transmitting to the controllers. In a distributed information system, for example, this decision maker might be a smart terminal transmitting data to a central computer.

The benefit (i.e., the decrease in the expected loss function in controlling the process) that might be obtained by assigning the above tasks to the measurements posts will be called "expected value of task decentralization" (EVTD) throughout the paper.

The decision makers of the two classes will be considered as the cooperating members of a team [1]. The communication network interconnecting the agents of the team constitutes, in general, a graph which specifies the topology of the control organization. Two special cases will be considered in the paper. In Section 2, a two-person team will be dealt with, in which an observing agent sends data to a controlling agent through a point-to-point communication link. In section 3, we shall consider a star-shaped network connecting n peripheral observing agents to a central controller.

Extension of these two cases to more general situations is currently under investigation. It is worth noting that in recent years, the coordination of many controllers acting on the same dynamic systems has been extensively investigated in the literature (see, for instance, the papers by Yoshikawa [2] and Kurtaran [3] also for references). The case of a unique decision maker sharing information among several agents has been considered in [4].

However, to the best of the authors' knowledge, the on-line control of the communication network interconnecting the team agents had not been explicitly dealt with before [5] and [6]. The dynamic control problem of a measurement channel with observation costs and noises has been discussed in more than ten papers (see [5] also for references). Such a problem is somehow related to the online optimal adjustment of a communication channel. However, no intelligence is assumed for the measurement device, and then no task decentralization problem is posed.

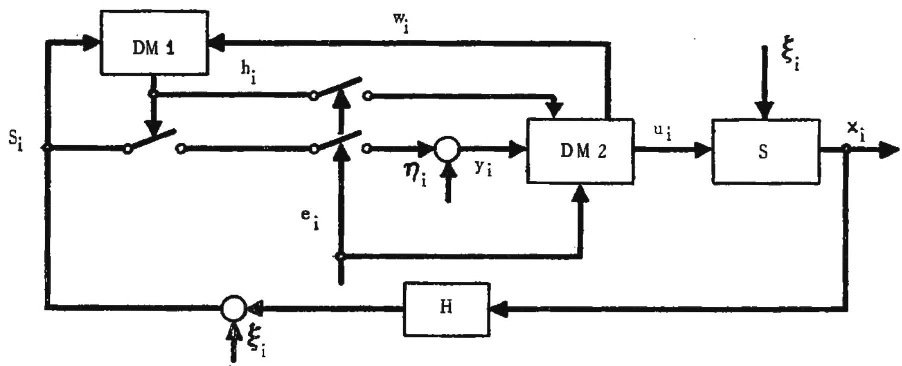

Fig. 1

2.1. PROBLEM STATEMENT FOR THE POINT-TO-POINT COMMUNICATION LINK

Consider the team structure shown in Fig. 1. A linear dynamic system is given, which we assume time-invariant for the sake of simplicity

$$x_{i+1} = Ax_i + Bu_i + \xi_i, \quad i = 0, 1, \dots, N-1 \tag{1}$$

where x_i is the n-dimensional state vector at stage i, u_i is the r-dimensional control vector exerted on the system by $DM2$ (the receiver and controller), and ξ_i is a noise vector.

$DM1$ (the observer and transmitter) is assumed to take noisy observations s_i on the system state given by

$$s_i = Hx_i + \zeta_i, \quad i = 0, 1, \dots, N-1 \tag{2}$$

where y_i is an m-dimensional vector ($m < n$) and ξ_i is a random vector. A, B, H are matrices of suitable dimensions. $DM1$ controls the information flow from $DM1$ to $DM2$ by means of a communication channel of the form

$$y_i = h_i e_i \, s_i + \eta_i, \quad i = 0, 1, \dots, N-1 \tag{3}$$

$h_i \in \{0, 1\}$ is a binary variables selected by $DM1 : h_i = 1$ means that a message is sent, $h_i = 0$ means that no message is sent. More complex communication models might be conceived by assuming that matrix H depends on a control variable h_i selected by $DM1$, $e_i \in \{0, 1\}$ is a binary random variable that takes into account the possibility of stochastic interruptions. The random event $e_i = 1$ (no interruption takes place) occurs with a known probability p.

η_i is a transmission noise vector.

$\xi_i, \varsigma_i, \eta_i (i = 0, 1, \dots, N\text{-}1)$ are zero-mean, mutually independent, Gaussian random vectors with covariance matrices $\operatorname{cov}(\xi_i) = Q$, $\operatorname{cov}(\varsigma_i) = W$, $\operatorname{cov}(\eta_i) = R$ x_0 is a Gaussian random vector, independent of the random vectors previously defined, with $E(x_0) = \alpha$, $\operatorname{cov}(x_0) = \sum$.

We assume that $DM1$, before choosing h_i, may receive some message vector w_i, which will be specified later on. We also assume that at each stage $DM2$ can derive the exact values of the variables e_i and h_i (if $e_i = 1$). This fact can be considered realistic enough since simple transmission codes can allow $DM1$ to transmit noisefree messages containing the binary variable h_i, and enable $DM2$ to recognize interruptions in the communication link.

On the basis of the above hypotheses, at stage i observations, decisions and messages are then performed in the following sequence:
1) $DM1$ observes s_i and receives w_i; 2) $DM1$ selects h_i; 3) $DM2$ recognizes the state of the communication channel and receives h_i and y_i, if $e_i = 1$; 4) $DM2$ selects the control action u_i. Therefore, the information set I_i^j of DMj at stage i can be defined as follows

$$I_i^1 \triangleq \{s^i, w^i, h^{i-1}\}, \quad I_i^2 \triangleq \{(eh)^i_, \, y^i, u^{i-1}\} \tag{4}$$

where $s^i \triangleq \{s_0, ..., s_i\}$, $(eh)^i \triangleq \{e_0 h_0, ..., e_i h_i\}$, and so on. We want then to determine optimal decision laws $\gamma^0_{1i}, \gamma^0_{2i}$ of the form

$$h_i = \gamma_{1i}(I^1_i), \quad u_i = \gamma_{2i}(I^2_i) \tag{5}$$

that minimize the expected value of the process cost

$$J(\gamma_1^{N-1}, \gamma_2^{N-1}) = \sum_{i=0}^{N-1} [\|u_i\|^2_P + ch_i] + \|x_N\|^2_V \tag{6}$$

where $c \geqslant 0$ is a cost associated to the transmission of a message, $\|u_i\|^2_P = u_i^T P u_i$, $P = P^T > 0$, $V = V^T > 0$. All the parameters defining the optimization problem are assumed to be known to both *DMs*. It follows that each *DM* can derive the other *DM's* decision law. For the sake of simplicity, we assume that *DM*2 does not change his probability density on x_i, whenever he receives the message $h_i = 0$.

We want to remark explicitly that the communication channel model (2) is comprehensive of the following two practical cases, which will be considered separately in some detail.

a) The analog communication model (*ACM*), in which a Gaussian noise is superimposed on the message, but no stochastic interruption takes place, h_i is received in any case. The channel is then described by the equation

$$y_i = h_i s_i + \eta_i, \quad i = 0, 1, ..., N-1 \tag{7}$$

b) The digital communication model (*DCM*). The observed vector s_i and h_i are encoded in a redundant digital message which enables error detection. Whenever an error is detected by *DM*2, we say that a stochastic interruption takes place. *DM*2 sends a positive or a negative acknowledgement signal (message w_{i+1}) back to *DM*1 depending whether he has received a correct message s_i or not. The communication model is then given by

$$y_i = s_i \quad \text{if} \quad h_i e_i = 1, \qquad y_i = \varnothing \quad \text{if} \quad h_i e_i = 0 \tag{8}$$

Since *DM*1 and *DM*2 act as the cooperating members of a team, in the next Section we shall derive their person-by-person satisfactory (p.b.p.s.) strategies [1] that will also prove to be optimal.

2.2. PERSON-BY-PERSON SATISFACTORINESS AND OPTIMALITY

There is a groving literature concerning the application of dynamic programming in the so-called "nonclassical" optimal control problems [2, 7, 8]. Let us consider the applicability of this algorithm to the general communication model (3).

Stage *N*-1. A necessary condition for the optimality of $\gamma_{2,N-1}$ is the following

$$EJ(\gamma_1^{*N-1}, \gamma_2^{*N-1}) \leqslant EJ(\gamma_1^{*N-1}, \gamma_2^{*N-2}, \gamma_{2,N-1}) \tag{9}$$

(9) is one of the 2*N* necessary conditions for optimality that define p.b.p.s. strategies $\gamma^*_{1i}, \gamma^*_{2i}$ in team-theory. We show now that (9) is by itself sufficient

to make $\gamma^*_{2,N-1}$ optimal. By assuming γ_1^{*N-1} to be fixed, dynamic programming can be applied to derive $DM2's$ p.b.p.s. strategies. Then we have

$$u^*_{N-1} = \gamma^*_{2,N-1}(I^2_{N-1}) = -L_{N-1}\mu_{N-1} \tag{10}$$

where $L_{N-1} = (P+B^TVB)^{-1}B^TVA$ and $x_{N-1} = E(x_{N-1}I^2_{N-1})$ can be derived by $DM2$ via a Kalman filter, since he knows the communication channel state. Observe that the unique strategy $\gamma^*_{2,N-1}$ depends on the sequence h_1^{*N-1}, but not on the particular form of strategies γ_1^{*N-1}. The same holds true as regards the dependence on γ_2^{*N-2}. Then $\gamma^*_{2,N-1}$ is optimal. Let $\gamma^*_{2,N-1} = \gamma^0_{2,N-1}$.

A necessary condition for the optimality of $\gamma_{1,N-1}$ is the following

$$E[J(\gamma_1^{*N-2}, \gamma^*_{1,N-1}, \gamma_2^{*N-2}, \gamma^\circ_{2,N-1})] \leqslant E[J(\gamma_1^{*N-2}, \gamma_{1,N-1}, \gamma_2^{*N-2}, \gamma^\circ_{2,N-1})] \tag{11}$$

Since $DM2's$ strategies are fixed, we can apply dynamic programming to derive $\gamma_{1,N-2}$, that is

$$J^*_{1,N-1}(I^1_{N-1}) = \min_{h_{N-1}}[ch_{N-1} + E(\|u^0_{N-1}\|^2_P + \|x_N\|^2_V | I^1_{N-1}) =$$

$$= \min_{h_{N-1}}\{ch_{N-1} + \hat\mu^T_{N-1}L^T_{N-1}PL_{N-1}\hat\mu_{N-1} + tr[L^T_{N-1}PL_{N-1}\operatorname{cov}(\mu_{N-1}|I^1_{N-1})] +$$

$$+\|A\hat x_{N-1} - BL_{N-1}\hat\mu_{N-1}\|^2_V + tr[\operatorname{cov}(Ax_{N-1} - BL_{N-1}\mu_{N-1}|I^1_{N-1})V]\} +$$

$$+tr(VQ) \tag{12}$$

where $\hat x_{N-1} \triangleq E(x_{N-1}|I^1_{N-1})$, $\hat\mu_{N-1} \triangleq E(\mu_{N-1}|I^1_{N-1})$.

After some algebraic manipulations, (12) becomes

$$J^*_{1,N-1}(I^1_{N-1}) = \hat x^T_{N-1}T_{N-1}\hat x_{N-1} + tr(VQ) + tr(VA\sum_{1,N-1}A^T) +$$

$$+\min_{h_{N-1}}\{ch_{N-1} + \hat\mu^T_{N-1}L^T_{N-1}(P+B^TVB)L_{N-1}\hat\mu_{N-1} + \hat x^T_{N-1}(A^TVA - T_{N-1})x_{N-1} +$$

$$-2\hat x^T_{N-1}A^TVBL_{N-1}\hat\mu_{N-1} + tr\,L^T_{N-1}(P+B^TVB)L_{N-1}\operatorname{cov}(\mu_{N-1}|I^1_{N-1}) +$$

$$-2\,tr\,VAE[(x_{N-1} - \hat x_{N-1})(\mu_{N-1} - \hat\mu_{N-1})^T|I^1_{N-1}]L^T_{N-1}B^T]\} =$$

$$= \hat x^T_{N-1}T_{N-1}\hat x_{N-1} + tr(VQ) + tr(VA\sum_{1,N-1}A^T) +$$

$$+\min_{h_{N-1}}\{\|\hat x_{N-1} - \hat\mu_{N-1}\|^2_{S_{N-1}}, + tr\,S_{N-1}[\operatorname{cov}(\mu_{N-1}|I^1_{N-1})] +$$

$$-2\,tr[S_{N-1}E[(x_{N-1} - \hat x_{N-1})(\mu_{N-1} - \hat\mu_{N-1})^T|I^1_{N-1}] + ch_{N-1}\} \tag{13}$$

where $\sum_{1,N-1} \triangleq \operatorname{cov}(x_{N-1}I^1_{N-1})$, $T_{N-1} \triangleq A^TV - VB(P+B^TVB)^{-1}B^TVA$ and $S_{N-1} \triangleq A^TVA - T_{N-1}$.

To compute the estimates $\hat{x}_{N-1}$, $\hat{\mu}_{N-1}$ and the other conditional expectations in (13), we need now to specify the type of messages w^{N-1} received by $DM1$. In this computation a central role is played by the following

Assertion 1: If the set of messages w^{N-1} are such that the information set I^2_{N-1} is nested in I^1_{N-1} [9], the control law $h^*_{N-1}=\gamma^*_{1,N-1}(I^1_{N-1})$ does not depend on u^{N-2}.

From Assertion 1 it immediately follows that $\gamma^*_{1,N-1}$ is not influenced by γ_2^{*N-2}. Since $DM1$ needs to retain h^{*N-2}, but not γ_1^{*N-1}, in order to compute h_2^{*N-2}, $\gamma^*_{1,N-1}$ turns out to be optimal. Assertion 1 and the following results can be considered an extension of the theorem presented in [5], p. 117.

To prove Assertion 1, observe first that the nested information structure we have assumed enables $DM1$ to derive u^{N-2} exactly, and then to compute x_{N-1} via the Kalman filter.

$$\hat{x}_{N-1} = A\hat{x}_{N-2}+Bu_{N-2}+K_{1,N-1}\nu_{1,N-1} \tag{14}$$

On the other hand, $DM2$ knows the state of the communication channel, and then he can apply the Kalman filter

$$\mu_{N-1} = A\mu_{N-2}+Bu_{N-2}+h_{N-1}e_{N-1}K_{2,N-1}\nu_{2,N-1} \tag{15}$$

where $K_{1,N-1}$, $K_{2,N-1}$ are the filters gains ($K_{2,N-1}$ is the gain for a communication link without interruptions at stage $N-1$) and the innovations are given by

$$\begin{aligned} \nu_{1,N-1} &= s_{N-1}-H(A\hat{x}_{N-2}+Bu_{N-2}), \\ \nu_{2,N-1} &= y_{N-1}-e_{N-1}h_{N-1}H(Au_{N-1}+Bu_{N-2}) \end{aligned} \tag{16}$$

Observe now that in (13) $x_{N-1}-\hat{x}_{N-1}$ does not depend on u^{N-2} because of a well known property of innovations [10]. Also observe that the assumption of Assertion 1 allows $DM1$ to derive μ_{N-2}. Then $DM1$ can compute μ_{N-1} as follows

$$\begin{aligned} \hat{\mu}_{N-1} &= A\mu_{N-2}+Bu_{N-2}+h_{N-1}pK_{2,N-1}[E(y_{N-1})I^1_{N-1},\ e_{N-1}=1)- \\ &\quad -H(A\mu_{N-2}+Bu_{N-2})] = A\mu_{N-2}+Bu_{N-2}+h_{N-1}pK_{2,N-1}[s_{N-1}- \\ &\quad -H(A\mu_{N-2}+Bu_{N-2})] \end{aligned} \tag{17}$$

From properties which are similar to those of innovations, it is easy to see that $\mu_{N-1}-\hat{\mu}_{N-1}$ does not depend on u^{N-2}. The same is true for $\text{cov}(\mu_{N-1}| |I^1_{N-1})$ and for $\hat{x}_{N-1}-\hat{\mu}_{N-1}$, as can be shown by subtracting (17) from (14).

Let us briefly discuss some cases where the hypothesis of Assertion 1 is fulfilled. Three cases are worth noting.

1) Communications are costly, but neither noisy ($\eta_i=0$) nor stochastically interrupted ($p=1$). In this case, it is obviously unnecessary to send back messages from $DM2$ to $DM1$.

2) ACM: the assumption of Assertion 1 is satisfied if $w_i=y_{i-1}$.

3) DCM: the above assumption is satisfied if $w_i=e_{i-1}$.

Clearly, case 1 is a particular form of the *ACM*. Such a model has been partially discussed in [5]. Then, to avoid too tedious algebra, we shall abandon the general communication model (3) and go on applying the p.b.p. satisfactoriness criterion and dynamic programming by focusing attention on the *DCM*, which, on the other hand, seems to be more interesting in practical cases.

2.3. THE DIGITAL COMMUNICATION MODEL

a) Stage N-1

DM2. We have already found *DM2's* optimal strategy at stage $N-1$ in the general case. This strategy holds true also in the present case without any modification. Thus we start from relation (13), obtained in the general case for $DM1$.

DM1. With reference to the *DCM* case, let us first specify the various quantities which appear in the from to be minimized in (13).

For $\hat{x}_{N-1}-\hat{\mu}_{N-1}$: form the Kalman filter equation (15) for μ_{N-1}, since μ_{N-2}, u_{N-2} and s_{N-1} are known to $DM1$, it immediately follows:

$$\hat{\mu}_{N-1} = A\mu_{N-2}+Bu_{N-2}+h_{N-1}\,pK_{2,N-1}[s_{N-1}-H(A_{N-2}+Bu_{N-2})] =$$

$$= A\mu_{N-2}+Bu_{N-2}+h_{N-1}\,pK_{2,N-1}\,\bar{v}_{2,N-1} \tag{18}$$

where $\bar{v}_{2,\,N-1}\triangleq s_{N-1}-H(A\mu_{N-2}+Bu_{N-2})$ is the innovation for a communication link without interruptions at stage $N-1$. Consequently:

$$\hat{x}_{N-1}-\hat{\mu}_{N-1} = \hat{x}_{N-1}-A\mu_{N-2}-Bu_{N-2}-h_{N-1}\,pK_{2,N-1}\,\bar{v}_{2,N-1}$$

and defining

$$\lambda_{N-1}\triangleq\hat{x}_{N-1}-A\mu_{N-2}-Bu_{N-2} \tag{19}$$

$$z_{N-1}\triangleq K_{2,N-1}\,\bar{v}_{2,N-1} \tag{20}$$

we can write

$$\hat{x}_{N-1}-\hat{\mu}_{N-1} = \lambda_{N-1}-h_{N-1}\,pz_{N-1} \tag{21}$$

Of the two terms, λ_{N-1} and z_{N-1}, the former represents the difference between *DM1's* filtered estimate and *DM2's* predicted estimate of the state vector x_{N-1}; the latter represents the correction of *DM2's* predicted estimate, which is introduced in $DM2$ receives the message s_{N-1}. Moreover, it is easily seen that both terms do not depend on u^{N-2}.

For $\mathrm{cov}(\mu^1_{N-1}/I_{\hat{N}-1})$: substracting (18) from (15) and using definition (20), we obtain

$$\mu_{N-1}-\hat{\mu}_{N-1} = h_{N-1}\,z_{N-1}(e_{N-1}-p) \tag{22}$$

and then:

$$\operatorname{cov}(\mu_{N-1}/I^1_{N-1}) = h_{N-1}\, z_{N-1}\, z^T_{N-1}\, E[(e_{N-1}-p)^2] = z_{N-1}\, z^T_{N-1}\, p(1-p)\, h_{N-1} \tag{23}$$

For $E[(x_{N-1}-\hat{x}_{N-1})(\mu_{N-1}-\hat{\mu}_{N-1})^T/I^1_{N-1}]$: using (22), we get

$$E[(x_{N-1}-\hat{x}_{N-1})(\mu_{N-1}-\hat{\mu}_{N-1})^T/I^1_{N-1}] =$$

$$= E[(x_{N-1}-\hat{x}_{N-1})(e_{N-1}-p)/I^1_{N-1}]\, z^T_{N-1}\, h_{N-1} =$$

$$= E[(x_{N-1}-\hat{x}_{N-1})/I^1_{N-1}]\, E[e_{N-1}-p]\, z^T_{N-1}\, h_{N-1} = 0 \tag{24}$$

To obtain (24), we have exploited the fact that e_{N-1} is independent of $x_{N-1}-\hat{x}_{N-1}$.

Substituting (21), (23) and (24) in (13), the expression of the cost for *DM*1 becomes:

$$J^*_{1,N-1}(I^1_{N-1}) = \|\hat{x}_{N-1}\|^2_{T_{N-1}} + tr(VQ) + tr(A^T VA \textstyle\sum_{1,N-1}) +$$

$$+ \min_{h_N \eta_1} \{\|\lambda_{N-1} - z_{N-1}\, p h_{N-1}\|^2_{S_{N-1}} + tr(S_{N-1} z_{N-1} z^T_{N-1})\, p(1-p)\, h_{N-1} + c h_{N-1}\} \tag{25}$$

We define:

$$F_{N-1} \triangleq tr(VQ) + tr(A^T VA \textstyle\sum_{1,N-1}) \tag{26}$$

It is worth noting that F_{N-1} is independent of $h^{*\,N-2}$ (and also of u^{N-2}). This fact will be useful to further developments in the following stages. Adding and subtracting $\|\lambda_{N-1}\|^2_{S_{N-1}} p$ in (25), after some algebraic manipulations, we get

$$J^*_{1,N-1}(I^1_{N-1}) = F_{N-1} + \|\hat{x}_{N-1}\|^0_{T_N} + \|\lambda_{N-1}\|^2_{S_{N-1}}(1-p) +$$

$$+ p \min_{h_{N-1}} \left\{ \|\lambda_{N-1} - z_{N-1} h_{N-1}\|^2_{S_{N-1}} + \frac{c h_{N-1}}{p} \right\} \tag{27}$$

In the preceding expression, we define:

$$G_{N-1}(\lambda_{N-1}, z_{N-1}) \triangleq \min_{h_{N-1}} \left\{ \|\lambda_{N-1} - z_{N-1} h_{N-1}\|^2_{S_{N-1}} + \frac{c h_{N-1}}{p} \right\} \tag{28}$$

$$z_{N-1}(\lambda_{N-1}, z_{N-1}) \triangleq \|\lambda_{N-1}\|^2_{S_{N-1}}(1-p) + p G_{N-1}(\lambda_{N-1}, z_{N-1}) \tag{29}$$

From (28) it follows that the decision law for *DM1* takes on the form:

$$h^0_{N-1} = \gamma^0_{1,N-1}(I^1_{N-1}) = f_{N-1}(\lambda_{N-1}, z_{N-1}) = f_{N-1}(-\lambda_{N-1}, -z_{N-1}) \tag{30}$$

Thus, h^0_{N-1} is symmetrical with respect to the vector $t_{N-1} = [\lambda^T_{N-1}, z^T_{N-1}]^T$. As has been shown in Section 2.2, *DM*1*'s* p.b.p.s. decision law (30) is also optimal.

b) Stage N-2

DM2. A necessary condition for the optimality of $\gamma_{2,N-2}$ is the following:

$$E[J(\gamma_1^{*N-2}, \gamma_{1,N-1}^0, \gamma_2^{*N-3}, \gamma_{2,N-2}^*, \gamma_{2,N-1}^0)] \leqslant$$
$$\leqslant E[J(\gamma_1^{*N-2}, \gamma_{1,N-1}^0, \gamma_2^{*N-3}, \gamma_{2,N-2}, \gamma_{2,N-1}^0)] \tag{31}$$

Since $DM1$'s strategies are fixed, we can go on applying dynamic programming to derive $\gamma_2{}^*{}_{N-2}$, i.e.,

$$J_{2,N-2}^*(I_{N-2}^2) = \min_{u_{N-2}} \{\|u_{N-2}\|_P^2 + E[ch_{N-1}^0 + J_{2,N-1}^0(I_{N-1}^2)/I_{N-2}^2]\} =$$
$$= \min_{u_{N-2}} \{\|u_{N-2}\|_P^2 + E[ch_{N-1}^0 + \|\mu_{N-1}\|_{T_{N-1}}^2 + C_{N-1}(\textstyle\sum_{2,N-1})/I_{N-2}^2]\} \tag{32}$$

where $\sum_{2,N-1} = \text{cov}(x_{N-1}/I_{N-1}^2)$ and, as in the classical case [11],

$$J_{2,N-1}^0(I_{N-1}^2) \triangleq \|\mu_{N-1}\|_{T_{N-1}}^2 + C_{N-1}(\textstyle\sum_{2,N-1}) = J_{2,N-1}^*(I_{N-1}^2).$$

The two terms ch_{N-1} and $C_{N-1}(\sum_{2,N-1})$ can be taken out of the minimization, since h_{N-1}^0 and $\sum_{2,N-1}$ do not depend on u^{N-2}. Then we can write:

$$J_{2,N-2}^*(I_{N-2}^2) = E[ch_{N-1}^0 + C_{N-1}(\textstyle\sum_{2,N-1})/I_{N-2}^2] +$$
$$+ \min_{u_{N-2}} \{\|u_{N-2}\|_P^2 + \|E[\mu_{N-1}/I_{N-2}^2]\|_{T_{N-1}}^2 + tr[T_{N-1}\,\text{cov}(\mu_{N-1}/I_{N-2}^2)]\} \tag{33}$$

Let us consider the various quantities which appear in the term to be minimized.

For $E[\mu_{N-1}/I_{N-2}^2]$: from the Kalman filter equation (15), we have:

$$E[\mu_{N-1}/I_{N-2}^2] = A\mu_{N-2} + Bu_{N-2} + E[e_{N-1}\,h_{N-1}^0\,K_{2,N-1}\,\bar{v}_{2,N-1}/I_{N-2}^2] \tag{34}$$

As $e_{N-1}h_{N-1}^0$ is a binary random variable which can attain only the values 1 or 0, we can write

$$E[e_{N-1}\,h_{N-1}^0\,K_{2,N-1}\,\bar{v}_{2,N-1}/I_{N-2}^2] =$$
$$= K_{2,N-1}\,E[\bar{v}_{2,N-1}/I_{N-2}^2, e_{N-1}\,h_{N-1}^0 = 1]\,Pr\{e_{N-1}\,h_{N-1}^0 = 1\} = 0 \tag{35}$$

$v_{2,N-1}$ being a zero-mean random vector according to the well-known property of innovations [10]. Thus

$$E[\mu_{N-1}/I_{N-2}^2] = A\mu_{N-2} + Bu_{N-2} \tag{36}$$

For $\text{cov}(\mu_{N-1}/I_{N-2}^2)$: from (34) and (36) we get

$$\text{cov}(\mu_{N-1}/I_{N-2}^2) = E[e_{N-1}\,h_{N-1}^0\,K_{2,N-1}\,\bar{v}_{2,N-1}\,\bar{v}_{2,N-1}^T\,K_{2,N-1}^T/I_{N-2}^2] \tag{37}$$

and since $e_{N-1}, h_{N-1}^0, K_{2,N-1}, \bar{v}_{2,N-1}$ do not depend on u^{N-2}, also (37) can be taken out of the minimization, Then, defining

$$D_{N-2} \triangleq E[ch_{N-1}^0 + C_{N-1}(\textstyle\sum_{2,N-1})/I_{N-2}^2] + tr[T_{N-1}\,\text{cov}(\mu_{N-1}/I_{N-2}^2)] \tag{38}$$

and substituting (36) in the expression of the cost (33), it results

$$J^*_{2,N-2}(I^2_{N-2}) = D_{N-2} + \min_{u_{N-2}} \{\|u_{N-2}\|^2_P + \|A\mu_{N-2} + Bu_{N-2}\|^2_{T_{N-1}}\} \tag{39}$$

Minization of (39) yields

$$u^*_{N-2} = \gamma^*_{2,N-2}(I^2_{N-2}) = -L_{N-2}\,\mu_{N-2} \tag{40}$$

$$J^*_{2,N-2}(I^2_{N-2}) = D_{N-2} + C_{N-2}(\textstyle\sum_{2,N-2}) + \|\mu_{N-2}\|^2_{T_{N-2}} \tag{41}$$

where the quantities L_{N-2}, T_{N-2}, $C_{N-2}(\sum_{2,\,N-2})$ are the same as in the classical case; namely, for the first two of them, $L_{N-2} = (P+B^T T_{N-1} B)^{-1} B^T T_{N-1} A$, $T_{N-2} = A^T[T_{N-1} - T_{N-1} B(P+B^T T_{N-1} B)^{-1} B^T T_{N-1}]A$.

Again, we can observe that the unique strategy $\gamma^*_{2,\,N-2}$ depends on the sequences $h^{*\,N-2}$ and $u^{*\,N-3}$, but it does not depend on the particular form of the strategies γ_1^{*N-2} and γ_2^{*N-3}. Thus, $\gamma^*_{2,\,N-2}$ is optimal. Let $\gamma^*_{2,\,N-2} = \gamma^*_{2,\,N-2}$ Besides, it is worth noting that the quantity $D_{N-2} + C_{N-2}(\sum_{2,\,N-2})$ is independent of u^{N-2}. This fact will be useful in the following stages.

*DM*1. A necessary condition for the optimality of $\gamma_{1,\,N-2}$ is the following:

$$E[J(\gamma_1^{*N-2}, \gamma^0_{1,N-1}, \gamma^*_{2,N-2}, \gamma^0_{2,N-1})] \leqslant$$
$$\leqslant E[J(\gamma_1^{*N-3}, \gamma_{1,N-2}, \gamma^0_{1,N-1}, \gamma_2^{*N-3}, \gamma^0_{2,N-2}, \gamma^0_{2,N-1})] \tag{42}$$

Since *DM2's* strategies are fixed, we can apply dynamic programming and obtain

$$J^*_{1,N-2}(I^1_{N-2}) = \min_{h_{N-2}} \{ch_{N-2} + E[\|u^0_{N-2}\|^2_P + J^0_{1,N-1}(I^1_{N-1})/I^1_{N-2}]\} =$$
$$= \min_{h_{N-2}} \{ch_{N-2} + E[\|u^0_{N-2}\|^2_P + F_{N-1} + \|\hat{x}_{N-1}\|^2_{T_{N-1}} +$$
$$+ z_{N-1}(\lambda_{N-1}, z_{N-1})/I^1_{N-2}]\} = F_{N-1} + \min_{h_{N-2}} \{ch_{N-2} +$$
$$+ \|\hat{\mu}_{N-2}\|^2_{L^T_{N-2} PL_{N-2}} + tr\,[L^T_{N-2}\,PL_{N-2}\,\mathrm{cov}\,(\mu_{N-2}/I^1_{N-2})] +$$
$$+ \|E[\hat{x}_{N-1}/I^1_{N-2}]\|^2_{T_{N-1}} + tr\,[T_{N-1}\,\mathrm{cov}\,(\hat{x}_{N-1}/I^1_{N-2})] +$$
$$+ E[z_{N-1}(\lambda_{N-1}, z_{N-1})/I^1_{N-2}]\} \tag{43}$$

We have now to specify some of the quantities which appear in the preceding expression of the cost.

For $E[\hat{x}_{N-1}/I^1_{N-2}]$: as the innovation $\nu_{1,\,N-1}$ is zero-mean, we have

$$E[\hat{x}_{N-1}\; I^1_{N-2}] = E[A\hat{x}_{N-2} + Bu^0_{N-2} + K_{1,N-1}\,\nu_{1,N-1}/I^1_{N-1}] =$$
$$= A\hat{x}_{N-2} - BL_{N-2}\,\hat{\mu}_{N-2} \tag{44}$$

For $\text{cov}(\hat{x}_{N-1}/I^1_{N-2})$:

$$\text{cov}(\hat{x}_{N-1}/I^1_{N-2}) = \text{cov}\,[(A\hat{x}_{N-2} - BL_{N-2}\,\mu_{N-2} + K_{1,N-1}\,\nu_{1,N-1})/I^1_{N-2}] =$$
$$= BL_{N-2}\,\text{cov}(\mu_{N-2}/I^1_{N-2})\,L^T_{N-2}\,B^T + K_{1,N-1}\,\text{cov}(\nu_{1,N-1}/I^1_{N-2})\,K^T_{1,N-1} +$$
$$- BL_{N-2}\,\text{cov}(\mu_{N-2}, \nu_{1,N-1}/I^1_{N-2})\,K^T_{1,N-1} -$$
$$- K_{1,N-1}\,\text{cov}(\nu_{1,N-1}, \mu_{N-2}/I^1_{N-2})\,L^T_{N-2}\,B^T \tag{45}$$

where $\text{cov}(a, b) = E\{[a - E(a)]\,[b - E(b)]^T\}$. Now we shall show that the two cross covariances in (45) are zero, namely:

$$\text{cov}(\mu_{N-2}, \nu_{1,N-1}/I^1_{N-2}) = \text{cov}\,[\mu_{N-2}, (HAx_{N-2} + H\xi_{N-2} + \zeta_{N-1} -$$
$$- HA\hat{x}_{N-2})/I^1_{N-2}] \tag{46}$$

Since $\hat{x}_{N-2}$ is known to $DM1$, and ξ_{N-2}, ζ_{N-1} are uncorrelated with μ_{N-2}, we have:

$$\text{cov}(\mu_{N-2}, \nu_{1,N-1}/I^1_{N-2}) = \text{cov}(\mu_{N-2}, x_{N-2}/I^1_{N-2})\,A^T =$$
$$= \text{cov}\,[(A\mu_{N-3} + Bu_{N-3} + e_{N-2}h_{N-2}K_{2,N-2}\bar{\nu}_{2,N-2}), x_{N-2}/I^1_{N-2}]\,A^T H^T \tag{47}$$

Again, $\mu_{N-3}, u_{N-3}, K_{N-2}, \bar{\nu}_{2,N-2}$ being quantities known to $DM1$ at the present stage, and being e_{N-2} independent of x_{N-2}, it turns out that $\text{cov}(\mu_{N-2}, \nu_{1,N-1}/I^1_{N-2}) = 0$.

As regards the second term in (45), we have:

$$\text{cov}(\nu_{1,N-1}/I^1_{N-2}) = \text{cov}\,[HA(x_{N-2} - \hat{x}_{N-2}) + H\xi_{N-2} + \zeta_{N-1}(I^1_{N-2}] =$$
$$= H(A\textstyle\sum_{1,N-2} A^T + Q)H^T + R = H\sum^p_{1,N-1} H^T + W \triangleq N_{1,N-1} \tag{48}$$

where $\sum^p_{1,N-1}$ is the covariance of the one-step predicted estimate of x_{N-1} made by $DM1$ at stage N—2. Substitution of the preceding results in (45) yields

$$\text{cov}(\hat{x}_{N-1}/I^1_{N-2}) = BL_{N-2}\,\text{cov}(\mu_{N-2}/I^1_{N-2})\,L^T_{N-2}\,B^T +$$
$$+ K_{1,N-1}\,N_{1,N-1}\,K^T_{1,N-1} \tag{49}$$

where the last term does not depend on $h^{*\,N-2}$.

Substituting (44) and (49) in (43), and adding and subtracting the quantity $\|\hat{x}_{N-2}\|^2_{T_{N-2}}$, after some algebraic manipulations, we have

$$J^*_{1,N-2}(I^1_{N-2}) = F_{N-1} + tr\,[T_{N-1}\,K_{1,N-1}\,N_{1,N-1}\,K^T_{1,V-1}] +$$
$$+ \min_{h_{N-2}}\{ch_{N-2} + \|\hat{x}_{N-2} - \hat{\mu}_{N-2}\|^2_{S_{N-2}} + tr\,[S_{N-2}\,\text{cov}(\mu_{N-2}/I^1_{N-2})] +$$
$$+ E\,[Z_{N-1}(\lambda_{N-1}, z_{N-1})/I^1_{N-2}]\} \tag{50}$$

where $S_{N-2} \triangleq A^T T_{N-1}\,A - T_{N-2}$.

Define

$$F_{N-2} \triangleq F_{N-1} + tr\left[T_{N-1} K_{1,N-1} N_{1,N-1} K^T_{1,N-1}\right] \tag{51}$$

and observe that in (50) the part to be minimized is analogous to the one appearing in (13). Then, following the same procedure used to obtain (27), we can finally write

$$J^*_{1,N-2}(I^1_{N-2}) = F_{N-2} + \|\hat{x}_{N-2}\|^2_{T_{N-2}} + \|\lambda_{N-2}\|^2_{S_{N-2}}(1-p) +$$

$$+ p \min_{h_{N-2}} \left\{ \|\lambda_{N-2} - z_{N-2} h_{N-2}\|^2_{S_{N-2}} + \frac{c h_{N-2}}{p} + \frac{1}{p} E\left[Z_{N-1}(\lambda_{N-1}, z_{N-1})/I^1_{N-2}\right] \right\} \tag{52}$$

The form of (54) differs from that of (27) because of the presence of the expectation in the part which has to be minimized. To compute this expectation, we must first find the probability density $p(\lambda_{N-1}, z_{N-1}/I^1_{N-2}) = p(t_{N-1}/ /I_{N-2})$. This will be the subject of the following discussion.

For λ_{N-1}, using the Kalman filter equation of $\hat{x}_{N-1}$ in (19) we can write

$$\lambda_{N-1} = A(\hat{x}_{N-2} - \mu_{N-2}) + K_{1,N-1} v_{1,N-1} =$$

$$= A(\hat{x}_{N-2} - A\mu_{N-3} - Bu_{N-3} - e_{N-2} h_{N-2} K_{2,N-2} \bar{v}_{2,N-2}) + K_{1,N-1} v_{1,N-1} =$$

$$= A(\lambda_{N-2} - e_{N-2} h_{N-2} z_{N-2}) + K_{1,N-1} v_{1,N-1} \tag{53}$$

The random variables in (53) are the binary random variable e_{N-2} and the innovation term $v_{1,N-1}$, which is Gaussian with zero mean and covariance $N_{1,N-1}$ given by (48).

Putting definitions (20) and (53) together, we can write:

$$t_{N-1} = \begin{bmatrix} \lambda_{N-1} \\ z_{N-1} \end{bmatrix} = \begin{bmatrix} A & -e_{N-2} h_{N-2} A \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_{N-2} \\ z_{N-2} \end{bmatrix} + \begin{bmatrix} K_{1,N-1} & 0 \\ 0 & K_{2,N-1} \end{bmatrix} \begin{bmatrix} v_{1,N-1} \\ \bar{v}_{2,N-1} \end{bmatrix} \tag{54}$$

In (54), the gain $K_{2,N-1}$ depends on the value assumed by the product $e_{N-2} h_{N-2}$, that is:

$$K_{2,N-1} = \sum^{P}_{2,N-1} H^T \Big(H \sum^{P}_{2,N-1} H^T + W\Big)^{-1} \tag{55}$$

where

$$\sum^{P}_{2,N-1} = A\Big[\sum^{P}_{2,N-2} - h_{N-2} e_{N-2} K_{2,N-2} H \sum^{P}_{2,N-2}\Big] A^T + Q \tag{56}$$

It follows that $K_{2,N-1}$ can assume two different values, which are easily derived from (55) and (56). Define

$$K_{2,N-1} \triangleq \begin{cases} K^a_{2,N-1}, & e_{N-2} h_{N-2} = 1 \\ K^b_{2,N-1}, & e_{N-2} h_{N-2} = 0 \end{cases}$$

according to this definition, we can rewrite (54) in the following form:

$$\begin{bmatrix}\lambda_{N-1}\\ z_{N-1}\end{bmatrix}=\left\{\begin{bmatrix}A & -h_{N-2}A\\ 0 & 0\end{bmatrix}\begin{bmatrix}\lambda_{N-2}\\ z_{N-2}\end{bmatrix}+\right.$$

$$\left.+\begin{bmatrix}K_{1,N-1} & 0\\ 0 & K^a_{2,N-1}h_{N-2}+K^b_{2,N-1}(1-h_{N-2})\end{bmatrix}\begin{bmatrix}v_{1,N-1}\\ \bar{v}_{2,N-1}\end{bmatrix}\right\}e_{N-2}+$$

$$+\left\{\begin{bmatrix}A & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\lambda_{N-2}\\ z_{N-2}\end{bmatrix}+\begin{bmatrix}K_{1,N-1} & 0\\ 0 & K^b_{2,N-1}\end{bmatrix}\begin{bmatrix}v_{1,N-1}\\ \bar{v}_{2,N-1}\end{bmatrix}\right\}(1-e_{N-2}) \tag{57}$$

Now, we can write:

$$p(t_{N-1}/I^1_{N-2})=p(t_{N-1}/I^1_{N-2},\, e_{N-2}=1)\,p+$$
$$+p(t_{N-1}/I^1_{N-2},\, e_{N-2}=0)\,(1-p) \tag{58}$$

We see immediately from (57) that the two probability densities appearing in (58) are Gaussian with mean

$$\begin{bmatrix}A(\lambda_{N-2}-h_{N-2}z_{N-2})\\ 0\end{bmatrix};\ \begin{bmatrix}A\lambda_{N-2}\\ 0\end{bmatrix}$$

respectively (since $v_{1,\,N-1}$ and $v_{2,\,N-1}$ for a given e_{N-2} are Gaussian and zero-mean in any case). Their covariances depend respectively on

$$\text{cov}\,[(v^T_{1,N-1},\, \bar{v}^T_{2,N-1})^T/I^1_{N-2},\, e_{N-2}=0];$$

$$\text{cov}\,[(v^T_{1,N-1},\, \bar{v}^T_{2,N-1})^T/I^1_{N-2},\, e_{N-2}=1]$$

To compute the two preceding covariances, let us consider separately the various submatrices into which they can be decomposed. We first note that the covariance of $\bar{v}_{2,\,N-1}$ depends on the value assumed by the product $e_{N-2}h_{N-2}$, that is:

$$\text{cov}\,(\bar{v}_{2,N-1}/I^1_{N-2},\, e_{N-2}\,h_{N-2})=H\sum^{P}_{2,N-1}H^T+W \tag{59}$$

where $\sum^P_{2,\,N-1}$ is given by (56). Hence the covariance can assume two distinct values, which are easily derived from (56) and (59). Define

$$\text{cov}\,(\bar{v}_{2,N-1}/I^1_{N-2},\, e_{N-2}\,h_{N-2})\triangleq\begin{cases}N^a_{2,N-1}, & e_{N-2}\,h_{N-2}=1\\ N^b_{2,N-1}, & e_{N-2}\,h_{N-2}=0\end{cases}$$

Therefore, since h_{N-2} is not a random variable, but a parameter to be chosen, we can write:

$$\text{cov}\,(\bar{v}_{2,N-1}/I^1_{N-2},\, e_{N-2})=\begin{cases}N^a_{2,N-1}\,h_{N-2}+N^b_{2,N-1}(1-h_{N-2}), & e_{N-2}=1\\ N^b_{2,N-1} & ,\, e_{N-2}=0\end{cases} \tag{60}$$

For the covariance of $\nu_{1,N-1}$ we have:

$$\operatorname{cov}(\nu_{1,N-1}/I^1_{N-2}, e_{N-2}) = H \sum^{p}_{1,N-1} H^T + W = \operatorname{cov}(\nu_{1,N-1}/I^1_{N-2}) = N_{1,N-1} \quad (61)$$

$\sum^p_{1,N-1}$ being independent of the product $e_{N-2}h_{N-2}$.

The last term to be computed is the cross covariance between $\nu_{1,N-1}$ and $\bar{\nu}_{2,N-1}$. We have

$$\operatorname{cov}(\nu_{1,N-1}, \bar{\nu}_{2,N-1}/I^1_{N-2}, e_{N-2}) = \operatorname{cov}[HA(x_{N-2}-\hat{x}_{N-2})+$$

$$+H\xi_{N-2}+\zeta_{N-1}, HA(x_{N-2}-\mu_{N-2})+H\xi_{N-2}+\zeta_{N-1}/I^1_{N-2}, e_{N-2}] =$$

$$= HA\operatorname{cov}[(x_{N-2}-\hat{x}_{N-2}), (x_{N-2}-\mu_{N-2})/I^1_{N-2}, e_{N-2}]A^TH^T+HQH^T+W \quad (62)$$

Taking the expectations, we can write:

$$\operatorname{cov}(\nu_{1,N-1}, \bar{\nu}_{2,N-1}/I^1_{N-2}, e_{N-2}) = HAE\{(x_{N-2}-\hat{x}_{N-2})[(x_{N-2}-\hat{x}_{N-2})^T-$$

$$-(\mu_{N-2}-\hat{\mu}_{N-2})^T]/I^1_{N-2}, e_{N-2}\}A^TH^T+HQH^T+W =$$

$$= H[A\textstyle\sum_{1,N2} A^T+Q]H^T+W+HAE[(x_{N-2}-\hat{x}_{N-2})\mu^T_{N-2}/I^1_{N-2}, e_{N-2}] \quad (63)$$

The last term in (63) is zero for the following reasons: μ_{N-2} is a linear combination of observations s_i, $x_{N-2}-\hat{x}_{N-2}$ (estimation error) is always orthogonal to all the observations s_i (orthogonality principle). Moreover, we note that the sum of the first two terms in (63) is the covariance $N_{1,N-1}$ of $\nu_{1,N-1}$ previously defined. Hence we have:

$$\operatorname{cov}(\nu_{1,N-1}, \bar{\nu}_{2,N-1}\, I^1_{N-2}, e_{N-2}) = N_{1,N-1} \quad (64)$$

independent of the value assumed by e_{N-2}.

Finally, using (60), (61) and (64), we can write explicitly the expression of the probability density (58) as follows:

$$p(t_{N-1}/I^1_{N-1}) = n\left(\begin{bmatrix} A(\lambda_{N-2}-h_{N-2}z_{N-2}) \\ \hline 0 \end{bmatrix}; \begin{bmatrix} K_{1,N-1} & 0 \\ 0 & K^a_{2,N-1}h_{N-2}+ \\ & K^b_{2,N-1}(1-h_{N-2}) \end{bmatrix} \times\right.$$

$$\left.\times\begin{bmatrix} N_{1,N-1} & 0 \\ N_{1,N-1} & N^a_{2,N-1}h_{N-2}+ \\ & N^b_{2,N-1}(1-h_{N-2}) \end{bmatrix}\begin{bmatrix} K_{1,N-1} & 0 \\ 0 & K^a_{2,N-1}h_{N-2}+ \\ & N^b_{2,N-1}(1-h_{N-2}) \end{bmatrix}^T\right)p+$$

$$+n\left(\begin{bmatrix} A\lambda_{N-2} \\ 0 \end{bmatrix}; \begin{bmatrix} K_{1,N-1} & 0 \\ 0 & K^b_{2,N-1} \end{bmatrix}\begin{bmatrix} N_{1,N-1} & N_{1,N-1} \\ N_{1,N-1} & N^b_{2,N-1} \end{bmatrix}\begin{bmatrix} K_{1,N-1} & 0 \\ 0 & K^b_{2,N-1} \end{bmatrix}^T\right)(1-p) \quad (65)$$

By means of (65) it is possible to compute the expectation which appears in the expression of the cost (52), that is:

$$E[Z_{N-1}(\lambda_{N-1} Z_{N-1})/I^1_{N-2}] = p$$
$$Z_{N-1}(t_{N-1})p(t_{N-1}/I^1_{N-2})e_{N-2}=1)dt_{N-1}+$$
$$+(1-p)\int Z_{N-1}(t_{N-1})\,p(t_{N-1}\ I^1_{N-2},e_{N-2}=0)\,dt_{N-1} \tag{66}$$

We note that (66) is a function of $\lambda_{N-2}, z_{N-2}, \sum^P_{2,\,N-2}$ (see (65)); moreover, (65) is an even function of λ_{N-2} and z_{N-2}.

The results obtained allow us to go back to the cost (52), and define:

$$G_{N-2}(\lambda_{N-2}, z_{N-2}, \sum^P_{2,N-2}) \triangleq \min_{h_{N-2}}\Big\{\|\lambda_{N-2}-z_{N-2}\,h_{N-2}\|^2_{S_{N-2}}+$$
$$+\frac{1}{p}\,E[Z_{N-1}(\lambda_{N-1}, z_{N-1})\ I^1_{N-2}]+\frac{ch_{N-2}}{p}\Big\} \tag{67}$$

$$Z_{N-2}(\lambda_{N-2}, z_{N-2}, \sum^P_{2,N-2}) = \|\lambda_{N-2}\|^2_{S_{N-2}}(1-p)+pG_{N-2}(\lambda_{N-2}, z_{N-2}, \sum^P_{2,N-2})$$

so that the expression of the cost becomes

$$J^*_{1,N-2}(I^1_{N-2}) = F_{N-2}+\|\hat{x}_{N-2}\|^2_{T_{N-2}}+Z_{N-2}(\lambda_{N-2}, z_{N-2}, \sum^P_{2,N-2}) \tag{69}$$

From (67) it follows that the decision law for $DM1$ at stage N—2 takes on the form

$$h^*_{N-2} = \gamma^*_{1,N-2}(I^1_{N-2}) = f_{N-2}(\lambda_{N-2}, z_{N-2}, \sum^P_{2,N-2}) =$$
$$= f_{N-2}(-\lambda_{N-2}, -z_{N-2}, \sum^P_{2,N-2}) \tag{70}$$

We note that, at stage N—2, the information set I^1_{N-2} reduces to the quantities $\lambda_{N-2}, z_{N-2}, \sum^P_{2,\,N-2}$. At stage N—1, I^1_{N-1} consisted only of the two quantities λ_{N-1} and z_{N-1}. Also observe that the unique strategy $\gamma^*_{1,\,N-2}$ turns out to be independent of the form of the strategies γ_1^{*N-3} and γ_2^{*N-3}. Hence $\gamma^*_{1,\,N-2}$ is optimal, and we write $\gamma^*_{1,\,N-2}=\gamma^0_{1,\,N-2}$.

It is clear now that the calculations performed at stage N—2 can be extended to the preceding stages, so that we can formally state the results obtained in the following

Theorem: If, in the *DCM* problem, $w_i=e_{i-1}$, the optimal strategies of the two *DMs* may be determined as follows:

— *DM2's* optimal strategy is given by

$$u^0_i(I^2_i) = -L_i\,\mu_i, \quad i=0,1,\ldots,N-1 \tag{71}$$

where

$$L_i = (P+B^T T_{i+1} B)^{-1} B^T T_{i+1} A \tag{72}$$

$$T_i = A^T[T_{i+1} - T_{i+1} B (P + B^T T_{i+1} B)^{-1} B T_{i+1}] A\,, \quad T_N = V \tag{73}$$

$$\mu_i \triangleq E(x_i/I_2^2) = A\mu_{i-1} + Bu_{i-1} + e_i h_i K_{2i} \bar{v}_{2i} \tag{74}$$

$$\bar{v}_{2i} \triangleq s_i - H(A\mu_{i-1} + Bu_{i-1}) \tag{75}$$

$$\sum_{2,i+1}^{P} \triangleq \text{cov}(x_{i+1}\; I_i^2) = A[\sum_{2i}^{P} - e_i h_i K_{2i} H \sum_{2i}^{P}] A^T + Q \tag{76}$$

$$K_{2i} = \sum_{2i}^{P} H^T (H \sum_{2i}^{P} H^T + W)^{-1} \tag{77}$$

under the initial conditions

$$\mu_0 = \alpha + e_0 h_0 K_{20}(y_0 - H\alpha)\,, \quad \sum_{20}^{P} = \sum$$

— *DM*1*'s* optimal strategy is given by

$$h_i^0 = f_i(\lambda_i, z_i, \sum_{2i}^{P})\,, \qquad i = 0, 1, \dots, N-2 \tag{78}$$

$$h_{N-1}^0 = f_{N-1}(\lambda_{N-1}, z_{N-1}) \tag{79}$$

where

$$\lambda_i = \hat{x}_i - A\mu_{i-1} - Bu_{i-1} \tag{80}$$

$$z_i = K_{2i}\bar{v}_{2i} = K_{2i}[s_i - H(A\mu_{i-1} + Bu_{i-1})] \tag{81}$$

with $\hat{x}_i \triangleq E(x_i/I_i^1)$ obtained by *DM*1 via a Kalman filter. The optimal decision laws (78), (79) are obtained by solving the following recursive equations:

$$G_i(\lambda_i, z_i, \sum_{2i}^{P}) = \min_{h_2} \Big\{ \|\lambda_i - z_i h_i\|_{S_i}^2 + \frac{1}{p} E[Z_{i+1}(\lambda_{i+1}, z_{i+1}, \sum_{2,i+1}^{P})/I_i^1] + \frac{ch_i}{p} \Big\} \tag{82}$$

$$Z_i(\lambda_i, z_i, \sum_{2i}^{P}) = \|\lambda_i\|_{S_i}^2 (1-p) + pG_i(\lambda_i, z_i, \sum_{2i}^{P})\,, \qquad i = 0, 1, \dots, N-2 \tag{83}$$

where $z_N(.) = 0$, so that $G_{N-1}(.) = G_{N-1}(\lambda_{N-1}, z_{N-1})$, $Z_{N-1}(.) = Z_{N-1}(\lambda_{N-1}, z_{N-1})$, and

$$S_i = A^T T_{i+1} A - T_i \qquad i = 0, 1, \dots, N-1 \tag{84}$$

In (82), expectations are computed by means of the probability densities $p(\lambda_{i+1}, z_{i+1}/I_i^1)$ given by (65) after the appropriate subsitution of indexes.

The main results of the above Theorem are the following. *DM*2*'s* optimal decision strategy is linear and can be implemented on the basis of a well known separation property. This attractive result is a direct consequence of the fact that for the *DCM* the information set I^2_{i-1} is nested in I^1_i (see also Assertion 1).

On the contrary, *DM*1's optimal decision strategy cannot be derived in an analytical form but must be computed by solving numerically the nonlinear stochastic optimization problem outlined by (82), (83).

2.4. A SUBOPTIMAL COMMUNICATION SCHEME FOR THE ACM

While in the *DCM* it is realistic to assume that a message $w_i = e_{i-1}$ is received by *DM*1 in any case, since this message reduces to a positive or a negative acknowledgement signal, in the *ACM* it seems rather artificial to suppose that a non-noisy message $w_i = y_{i-1}$ is always received by *DM*1 (this might be reasonable whenever *DM*1 could use a more powerful receiving device than *DM*2's).

On the other hand, noisy or interrupted or incomplete messages from *DM*2 to *DM*1 lead in general to a non-nested information structure, and it has been shown in [5] that, in such a case, *DM*2 cannot generate control actions on the system by means of a linear control law based on the usual separation property. Since it can be demonstrated that the computation of *DM*2's optimal control law entails heavy complications, it seems interesting to examine whether the EVTD is still positive for an *ACM* in which *DM*2 is constrained to use the classical estimation and control scheme of *LQC* stochastic optimal control.

To be more specific, we assume that 1) a communication channel from *DM*1 to *DM*2 (as described in Section 2.1, point a)) is given; 2) no message w_i is sent back to *DM*1; 3) *DM*2 is constrained to generate u_i via the suboptimal control law $u_i = -L_i \mu_i$, where μ_i is estimated by means of the usual Kalman filter adapted to the received sequence h^i. This problem has some aspects in common with the case considered in [12], in which, however, a simpler controlling structure is assumed.

For the sake of brevity, we shall only consider a static scalar problem characterized by the one-stage decision process

$$x_1 = ax_0 + bu_0 + \xi_0, \quad p(x_0) = N(0, \textstyle\sum) \tag{85}$$

The measurement and communication links are given by

$$y_0 = h_0 s_0 + \eta_0 = h_0(x_0 + \zeta_0) + \eta_0, \; h_0 \in \{0, 1\} \tag{86}$$

No cost c is involved in transmitting the message s_0. Later on, we shall sketch the extension of this problem to the dynamic vectorial case.

We are then interested in computing the quantity

$$EVTD = E(\tilde{J}^0 - J^0) \tag{87}$$

where $J^0 = \min_{h_0(s_0)} E(Pu_0^2 + Vx_1^2)$ (88)

whith the constraint $u_0 = -L_0\mu_0$, μ_0 derived via a Kalman filter, and

$$\tilde{J}^0 = \min_{u_0(y_0)} E(Pu_0^2 + Vx_1^2), \quad \text{with } h_0 = 1. \tag{89}$$

In other words, $\tilde{J}^0$ is the classical minimum expected cost of LQG stochastic optimal control [11], in which $DM1$ is reduced to a passive observing — transmitting device, while J^0 is the optimal cost for a control structure, in which $DM2$ obeys the classical decision law. But $DM1$ is now "active", in the sense that he is given the responsibility of deciding the convenience of transmitting the observed data (task decentralization).

From the stochastic control theory we have

$$\tilde{J}^0 = T_0\mu_0^2 + VQ + a^2V \ \text{var}(x_0\ y_0) \tag{90}$$

where $\mu_0 = E(x_0/y_0) = K_0 y_0$, $K_0 = \sum(\sum + R + W)]$ Then

$$E(\tilde{J}^0) = T_0 K_0^2 E(y_0^2) + VQ + a^2V[1/\sum + 1/(R+W)]^{-1} =$$

$$= (a^2V - S_0)\sum^2 (\sum + R + W)^{-1} + VQ + a^2V\sum(R+W)(\sum + R + W)^{-1} =$$

$$= a^2V\sum + VQ - S_0\sum^2(\sum + R + W)^{-1} \tag{91}$$

To compute J^0, consider (13) in a static case and obtain

$$J^0 = T_0\hat{x}_0^2 + VQ + a^2V\,\text{var}(x_0\ s_0) + S_0 \min_{h_0(s_0)} \{(\hat{x}_0 - \hat{\mu}_0)^2 + \text{var}(\mu_0/s_0) -$$

$$-2E[(x_0 - \hat{x}_0)(\mu_0 - \hat{\mu}_0)/s_0]\} \tag{92}$$

It is easy to see that in (92) the cross product vanishes, and that $\text{var}(\mu_0/s_0) = h_0 K_{20}^2 R$, where K_{20} is $DM2$'s Kalman filter gain when $h_0 = 1$. Then we have

$$E(J^0) = T_0 K_{10}^2(\sum + W) + VQ + a^2V\sum_{10} +$$

$$+ S_0 E \min \begin{cases} K_{10}^2 s_0^2, & \text{if} \quad h_0 = 0 \\ (K_{10} - K_{20})^2 s_0^2 + K_{20}^2 R, & \text{if} \quad h_0 = 1 \end{cases} =$$

$$= T_0\sum^2(\sum + W)^{-1} + VQ + a^2V\sum W(\sum + W)^{-1} +$$

$$+ S_0 E\left[K_{10}^2 s_0^2 + \min \begin{cases} 0 \\ (K_{20}^2 - 2K_{10}K_{20})s_0^2 + K_{20}^2 R \end{cases}\right] =$$

$$= a\ V\sum[\sum(\sum + W)^{-1} + W(\sum + W)^{-1}] + VQ +$$

$$+S_0 E \min \begin{cases} 0 \\ \{[\sum/(\sum+R+W)]^2 - 2\sum^2(\sum+R+W)^{-1}(\sum+W)^{-1}\}s_0^2 + K_{20}^2 R \end{cases} =$$

$$= a^2 V \sum + VQ + S_0 E \min \begin{cases} 0 \\ -\psi_0 s_0^2 + K_{20}^2 R \end{cases} \tag{93}$$

where $\psi_0 = \sum^2 (2R + \sum + W)[(\sum + R + W)^2(\sum + W)]^{-1} > 0$.

It can easily be shown that $E(-\psi_0 s_0^2) + K_{20}^2 R = -\sum^2 (\sum + R + W)^{-1}$ (see (91)). This corresponds to the quite obvious fact that $E(J^0)|_{h_0(s_0)=1} = E(\tilde{J}^0)$. Therefore we have

$$EVTD = E(\tilde{J}^0) - E(J^0) =$$

$$= S_0 E\left[-\psi_0 s_0^2 + K_{20}^2 R - \min \begin{cases} 0 \\ -\psi_0 s_0^2 + K_{20}^2 R \end{cases}\right] =$$

$$= S_0 E[\max(0, -\psi_0 s_0^2 + K_{20}^2 R)] > 0 \tag{94}$$

The positive quantity (94) yields then the maximum cost is convenient to spend in order to give the observing device decision responsibilities concerning the transmission of data.

Let us briefly discuss how the scalar problem can be generalized to the dynamic vectorial case. This general case can be solved by observing that a unique decision maker (i.e., the observing — transmitting device $DM1$) is acting, and that two dynamic subsystems characterize the process, namely, the plant subsystem (1) and the Kalman filter implemented by the controller.

The Kalman filter equation can be rewritten as follows

$$\mu_i = (I - h_i K_{2i} H)(A\mu_{i-1} + Bu_{i-1}) + h_i K_{2i} y_i =$$

$$= \varphi_{2i}(h_i)(A - BL_{i-1})\mu_{i-1} + h_i K_{2i} H x_i + h_i K_{2i} \zeta_i + h_i K_{2i} \eta_i \tag{95}$$

where $\varphi_{2i}(h_i) = I - h_i K_{2i} H$ and K_{2i} is again the controller's filter gain when $h_i = 1$. By introducing the controller's decision law, the state equation (1) becomes

$$x_{i+1} = Ax_i - BL_i \mu_i + \xi_i = -BL_i \varphi_{2i}(h_i)(A - BL_{i-1})\mu_{i-1} +$$

$$+ (A - h_i BL_i K_{2i} H)x_i + \xi_i - h_i BL_i K_{2i} \zeta_i - h_i BL_i K_{2i} \eta_i \tag{96}$$

Define the $2n$-dimensional augmented state vector $X_i \triangleq (\mu_{i-1}^T, x_i^T)^T$. Then, using more compact notations, we can write

$$X_{i+1} = \hat{A}_i(h_i) X_i + \hat{c}_i(h_i) \psi_i \tag{97}$$

where $\psi_i = (\xi_i^T, \zeta_i^T, \eta_i^T)^T$ and matrices $\hat{A}_i(h_i)$, $\hat{c}_i(h_i)$ are immediately derived from (95), (96). The observing equation is given by

$$s_i = [0|H]\, X_i + \zeta_i \tag{98}$$

Finally, cost (6) can be rewritten as

$$J(\gamma_1^{N-1}) = \sum_{i=0}^{N-1} (\|\mu_i\|_{\hat{P}_i}^2 + c_i h_i) + \|x_N\|_V^2 \tag{99}$$

where $\hat{P}_i \triangleq L_i^T P L_i$.

Relationships (97), (98) and (99) lead to a single-person stochastic optimization problem, in which h_i is the control variable. This problem is not *LQG*, but can be solved via dynamic programming in a conventional way.

The computational aspects of the above stated problem will be analyzed in a forthcoming paper. Some preliminary results can be summarized in the following

Assertion 2: Although *DM2's* strategy $u_i = -L_i \mu_i$ leads to a suboptimal solution to the *ACM* problem with no communication link from *DM*2 to *DM*1, it turns out that $EVTD \geqslant 0$.

Moreover, it can be shown that $EVTD \geqslant 0$ even if messages are not penalized by transmission costs c. This is an interesting result, as it means that improvements can be obtained in the process cost (with respect to the classical *LQC* stochastic optimization) by "giving intelligence" to the observing-transmitting device.

3.1. PROBLEM FORMULATION FOR THE STAR-SHAPED COMMUNICATION NETWORK

In this Section, the team structure differs from that discussed in Section 2 for two reasons: 1) a peripheral subteam with more than one observing transmitting agent is dealt with; 2) the problem is static, i.e., a single-stage decision process is considered. Despite of the simplification induced by the second assumption, the coordination mechanism among the peripheral agents leads to severe computational problems.

An *ACM* will be discussed in which, without loss of generality, only two peripheral agents, *PA*1 and *PA*2, are active. The team structure is shown in Fig. 2. Let us state the corresponding problem.

Let an n-dimensional random vector $r_j (i = 1, 2)$ describe the influence exerted by a j-th stochastic environment sector on a decision process, r_1, r_2 are mutually independent, Gaussian, zero-mean with $\mathrm{cov}(r_1) = \sum_1$, $\mathrm{cov}(r_2) = \sum_2$. We assume that r_j is observed by $PA\,j$ through a noisy measurement channel

$$s_j = H_j r_j + \zeta_j, \qquad j = 1, 2 \tag{100}$$

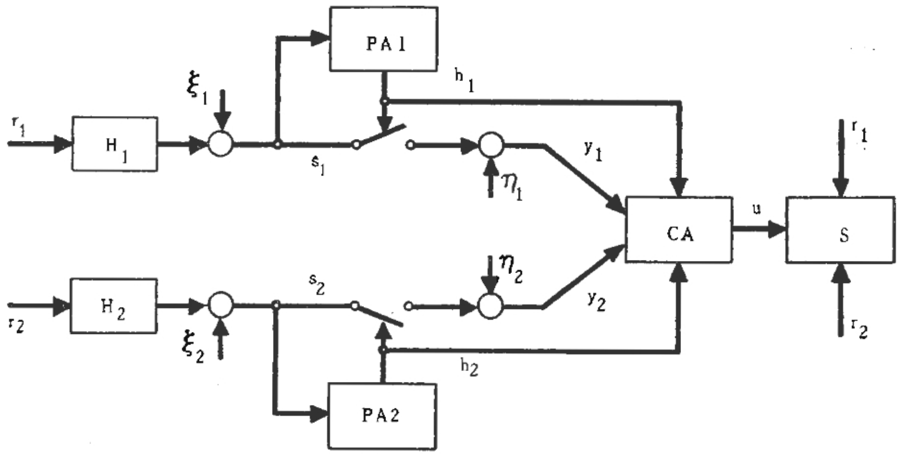

Fig. 2. Single equation regression analysis

PAj is given the task of deciding whether the observed vector s_j is worth transmitting to a central agent CA by means of a noisy communication link characterized by the equation

$$y_j = h_j s_j + \eta_j, \quad h_j \in \{0, 1\}, \quad j = 1, 2 \tag{101}$$

Then a message s_j is sent from PAj to CA or not depending on whether $h_j = 1$ or $h_j = 0$, respectively. Transmission of s_j is penalized by a cost c_j. Noises $\zeta_1, \xi_2, \eta_1, \eta_2$ are mutually independent and independent of r_1, r_2, Gaussian, zero-mean with $E(\xi_1) = E(\xi_2) = E(\eta_1) = E(\eta_2) = 0$ with $\text{cov}(\zeta_1) = W_1$, $\text{cov}(\zeta_2) = W_2$, $\text{cov}(\eta_1) = R_1$, $\text{cov}(\eta_2) = R_2$. All covariances are > 0. We also assume that CA can tell the exact value of h_j. The process cost is given by

$$J = c_1 h_1 + c_2 h_2 + u^T Q u + 2(r_1 + r_2)^T D u \tag{102}$$

where $Q = Q^T > 0$ and D are matrices of suitable dimensions.

Let $I \triangleq \{h_1, h_2, y_1, y_2\}$ be CA's information set. We want then to determine optimal decision laws $\gamma_1^0, \gamma_2^0, \varphi^0$ of the form

$$h_1 = \gamma_1(s_1), \quad h_2 = \gamma_2(s_2), \quad u = \varphi(I) \tag{103}$$

which minimize the expected value of cost (102).

It is worth noting that an N-stage decision process that can be reduced to the above stated static case has been presented in [13]. To be more specific, it has been shown that such a reduction is possible whenever CA can observe the dynamic system state vector x_i exactly at each stage.

3.2. DERIVATION OF THE P.B.P.S. STRATEGIES FOR THE PERIPHERAL AGENTS

A necessary condition for the optimality of strategies (103) is given by

$$E[J(\gamma_1^*, \gamma_2^*, \varphi^*)] \leqslant E[J(\gamma_1^*, \gamma_2^*, \varphi)] \tag{104}$$

or, equivalently,

$$E\{E[J(\gamma_1^*, \gamma_2^*, \varphi^*)/I]\} \leqslant E\{E[J(\gamma_1^*, \gamma_2^*, \varphi)/I]\} \tag{105}$$

From (105) the following problem is derived

$$\min_u E[J(\gamma_1^*, \gamma_2^*, u)/I] = \min_u \{u^T Q u + 2[E(r_1/I) + E(r_2/I)]^T\} Du +$$

+ terms independent of u.

Thus, for fixed γ_1^*, γ_2^*, we obtain

$$u^* = \varphi^*(I) = -Q^{-1}D^T[E(r_1/I) + E(r_2/I)] = -Q^{-1}D^T(h_1 K_1 y_1 + h_2 K_2 y_2) \tag{107}$$

where

$$K_j = [\textstyle\sum_j^{-1} + H_j^T(W_j + R_j)^{-1} H_j]^{-1} H_j(W_j + R_j)^{-1}, \quad j = 1, 2 \tag{108}$$

The unique strategy (107) does not depend on the particular form of γ_1^*, γ_2^* and then it is optimal. Let $\varphi^* = \varphi^0$. Substitution of φ^0 in the cost (102) yields

$$J = c_1 h_1 + c_2 h_2 + \|r_1 - h_1 K_1 y_1 + r_2 - h_2 K_2 y_2\|_S^2 - \|r_1 + r_2\|_S^2 \tag{109}$$

where $S \triangleq DQ^{-1}D^T$. To derive the optimal strategies γ_1^0, γ_2^0, we must then solve the following problem

$$\min_{\gamma_1, \gamma_2} E(c_1 h_1 + c_2 h_2 + \|r_1 - h_1 K_1 y_1 + r_2 - h_2 K_2 y_2\|_S^2) \tag{110}$$

In order to show what difficulties may be encountered in solving problem (110), and not to be involved in too lengthy algebra, let us simplify the network model by assuming that all measurement and communication channels are noise-free and that $H_1 = H_2 = I$. Then, problem (110) becomes

$$\min_{\gamma_1, \gamma_1} E[c_1 h_1 + c_2 h_2 + \|(1 - h_1) r_1 + (1 - h_2) r_2\|_S^2] \tag{111}$$

This minimization has been discussed in [13]. We summarize here some results. The p.b.p.s. strategies $\gamma_1^* = h_1^*(r_1)$, $\gamma_2^* = h_2^*(r_2)$ which are candidates for optimality must satisfy the following conditions

$$E[J_T(\gamma_1^*, \gamma_2^*)] \leqslant E[J_T(\gamma_1, \gamma_2^*)], \; E[J_T(\gamma_1^*, \gamma_2^*)] \leqslant E[J_T(\gamma_1^*, \gamma_2)] \tag{112}$$

where $J_T(\gamma_1, \gamma_2) = c_1 h_1 + c_2 h_2 + \|(1 - h_1) r_1 + (1 - h_2) r_2\|_S^2$.

In [13], it has been shown that to derive the strategy pairs γ_1^*, γ_2^* (these pairs may not be unique) we can use the following

Assertion 3: the p.b.p.s. strategy pairs γ_1^*, γ_2^* (if they exist) are given by

$$h_j^*(r_j) = 1\left(\|r_j - k_j\|_S^2 - c_j - \|k_j\|_S^2\right), \quad j = 1, 2 \tag{113}$$

where k_1, k_2 are n-dimensional vectors that must satisfy the following system of $2n$ (nonlinear) equations

$$\begin{aligned} k_1 &= \underset{r_2}{E}\left[h_2^*(r_2)\, r_2\right] = f_1(k_2) \\ k_2 &= \underset{r_2}{E}\left[h_1^*(r_1)\, r_1\right] = f_2(k_1) \end{aligned} \tag{114}$$

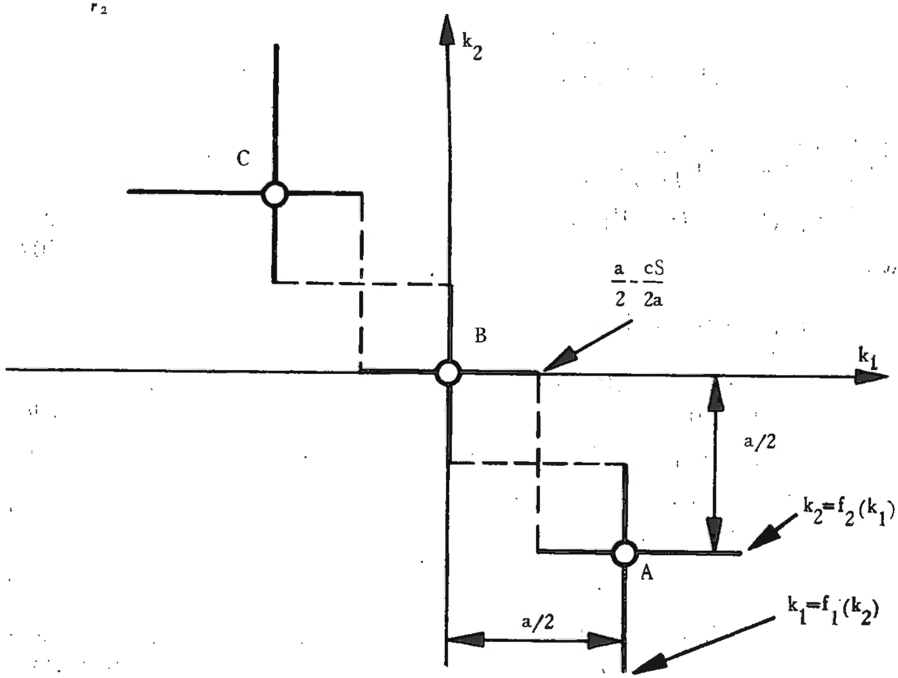

Fig. 3. Simultaneous estimation results

In order to solve system (114), the knowledge of $p(r_1), p(r_2)$ is required. These probability densities, however, need not be Gaussian. A possible numerical method to find the optimal strategy pair is then the following: a) system (114) is solved and the set of p.b.p.s. strategy pairs is determined; b) if more then one solution is found, the cost $E[J_T(\gamma_1, \gamma_2)]$ is evaluated for all pairs and the globally optimal solution is derived. Let us illustrate this procedure by means of an example given in [13].

Suppose that r_j is a scalar binary variable taking on values a and $-a$ with probability 0.5. Let $c_1 = c_2 = c$ and $c/S < a^2$. (113) yields the form of strategy $h_j^*(r_j)$: if r_j belongs to a certain interval centered on k_j, PA_j must not send any message, otherwise a message must be sent. The length of the interval depends on k_j.

By using (114), it is immediate to determine functions $k_1 = f_1(k_2)$ and $k_2 = f_2(k_1)$, which are shown in fig. 3. Three intersections A, B, C are found, each giving a p.b.p.s. strategy pair. Point A yields $h_1^*(a) = 0, h_1^*(-a) = 1$ and $h_2^*(a) = 1, h_2^*(-a) = 0$. At point C, $PA1$ and $PA2$ exchange these strategies. At point B, it is always convenient to send a message. Evaluation of cost $E[J_T(\gamma_1, \gamma_2)]$ for the three pairs gives: $E(J_T) = c + Sa^2/2$ for points A, C, $E(J_T) = 2c$ for point B. Therefore, the pairs corresponding to points A, C are optimal if $a^2 < 2c/S$, otherwise the pair corresponding to point B is optimal.

Clearly the numerical procedure illustrated in the example may turn out to be too cumbersome for a large dimension of the random vectors and for more than two PAs. Assignment of local costs to $PA1$ and $PA2$ (see [14]) may allow a direct computation of the optimal values of k_1, k_2 provided that particular conditions are met for $p(r_1), p(r_2)$. These conditions have been discussed in [13], but only qualitative (and rather conservative) results have been obtained.

In any case, Assertion 3 is not devoid of interest, since a functional problem has been reduced to a much simpler one and this regardless of the form of $p(r_j)$. Moreover, the particular structure of system (114) can be exploited in step a) of the numerical procedure, whereas a direct comparison of the average costs for the various p.b.p.s. strategy pairs (if more than one solution has been found) does not seem a formidable problem.

4. CONCLUSIONS

It is deemed that the communication problems presented in the paper and their generalization to more complex and realistic models constitute a central point in the theory of large scale systems for the following reasons.

a) Costly, noisy and stochastically interrupted communications are among the main reasons for "dispersion of authority and information" in large scale decision processes.

b) Significant savings in controlling an informationally decentralized process can be obtained by introducing (and optimizing) a new class of control actions, that is, the on-line control of data flow within the decentralized structure, and more specifically from the posts where information is handy to the posts where decisions are taken.

c) The growing employment of low-cost numerical devices (like microprocessors, microcomputers, etc.) will certainly accelerate the tendency to "distribute intelligence" in the information structure. Practical examples may be suggested by the problems presented in this paper, that is, by the convenience of giving the measurement devices responsibilities on the selection of optimal trnasmission instants and of the form of communication procedure.

It follows that simplified models of communication networks can be quite useful to make aggregate analyses of the costs and benefits of process control oriented information structures (costs and benefits mean respectively the costs that must be paid to distribute intelligence and the decrease in the expected cost function obtained by introducing the corresponding intelligent devices). Actually, the concept of *EVTD* has been introduced with the aim of framing such analyses in quantitative terms. Evaluation of the costs and benefits of an information structure is certainly facilitated in the area of industrial processes, where any loss due to non-optimal data handling can be directly compared with the intrinsic cost of the process itself.

As regards the theoretical aspects of the paper, a few comments may be useful to a more synthetical understanding of the results. In the point-to-point communication link, the central problem is concerned with the possibility of planning communication channels, for which the assumption of nested information structure is satisfied. If this assumption is met, the controller's optimal decision law is linear according to a classical separation property, whereas the transmitter's optimal strategy can be derived by solving numerically a certain nonlinear stochastic optimization problem. In the case of a non-nested information structure, a linear, although suboptimal, decision law for the controller is still convenient with respect to the conventional control scheme, in which no intelligence is assumed for the observing-transmitting device.

In the star-shaped communication network, it has been shown that non-trivial computational aspects also arise in the case of static decision processes. The extension of this communication network to the general dynamic case seems to pose formidable computational problems. No results are available either for a star-shaped network, in which a unique observer shares information among several controllers, of for more general communication structures described by a bipartite graph connecting an observing subteam to a controlling subteam.

Finally, it is worth noting that an interesting area for both theoretical and applicative research should concern the asymptotical behaviour of decision makers' strategies in an infinite time-control horizon.

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SUMMARY

Costly and/or noisy communications constitute one of the main reasons for dispersion of information among different decision makers in large scale systems. Because of the technical constraints of communication links, other responsibilities may be given to decision makers besides the usual task of generating control actions on a process; for instance, to select the instants at which messages must be sent to other agents of the organization, to decide data are worth transmitting, and, in general, to define the communication, procedure for the information interchange.

In the paper, a rather frequent communication network is considered, in which peripheral agents gather and communicate data to a central agent,

who controls a single process operating in a stochastic environment. In a distributed information system, for example, the peripheral agents might be smart terminals transmitting data to a central computer which controls an industrial or an administrative process. The peripheral and the central agents are considered as the cooperating decision makers of a team.

Two special cases are examined, which provide introductory elements for a more integrated view of those decentralized systems in which both control of data flow and control actions on the process must be taken into account. Actually, it is deemed that simplified models of this type can be useful to make aggregate analyses of costs and benefits in general information structures.

A crucial point throughout the paper is the optimal assignment of tasks among the team agents. In this regard, the concept of "expected value of task decentralization" (EVTD) is defined and evaluated for all problems.

In the first case, a static decision problem is dealt with. A central agent controls the process, on which $n > 1$ sectors of a stochastic environment exert their influence by means of n random vectors. The i-th random vector is only known to the i-th peripheral agent. Under the usual L-Q-G assumptions, necessary conditions for the optimal control of data flow from the peripheral agents to the central controller are established by means of the so-called person-by-person satisfactoriness (p.b.p.s.) principle.

In the second case, a dynamic decision problem is considered but, in this case, a single peripheral agent is supposed to be given the task of controlling the communication channel. This agent takes observations on the state vector of the dynamic system. Here again the problem is approached via the p.b.p.s. principle, and a dynamic programming algorithm is derived for the two agents' decision laws.

In both case, the obtained algorithms do not yield the transmission strategies in a closed analytical form except for trivial cases. However, they are constructive for numerical computations and provide useful simulation guidelines. Extensions of these two basic problems to the n-agent peripheral subteam are finally discussed in the dynamic case.

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