POLSKA AKADEMIA NAUK
INSTYTUT BADAŃ SYSTEMOWYCH

PROCEEDINGS OF THE 3rd ITALIAN-POLISH CONFERENCE ON APPLICATIONS OF SYSTEMS THEORY TO ECONOMY, MANAGEMENT AND TECHNOLOGY

WARSZAWA 1977

Redaktor techniczny
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PREFACE

The present volume comprises the papers from the 3rd Italian-Polish Conference which was held in [illegible] 10 11. The conference organized by the Centre of Control Systems and Computation of the Italian National Research Council in [illegible] and the Institute for Organization, Management and Control Sciences of Polish Academy of Sciences and the Ministry of Science, Higher Education and Technology in Warsaw was the third conference of this type.

The first one took place in Kraków (Poland) in 1972, the second in Pugnochiuso (Italy) in 1974.

This conference, like both the previous ones, gathered the leading representatives of many scientific centres from Italy as well as from Poland, representing also the different aspects of the systems analysis applied to the economic, management, data processing and technological systems.

From the past such meetings enrich the exchange of views and experiences between the specialists from both countries what is the [illegible] to both sides, shows the [illegible] of the systems and control theory [illegible] the essential development of the quantitative methods and [illegible] of large complex systems.

The meaning of such meetings, however, goes far beyond this; what is happening during the conference between the established contacts are also fruitful in form of common scientific research works and — last but not least — form the friendly links between the scientists of both countries which last for years.

The proceedings of the conference were divided into three parts:

1. Optimization and Control Theory;
2. Systems Theory in Economics;
3. Technological, Managerial and Information Systems.

While the first two parts are of rather homogeneous character, the third part contains the papers concerning the different types of models — for the economic, technological, management and data processing systems.

G. P. Szegö
University of Bergamo

GENERALIZED SEPARATION IN PORTFOLIO THEORY*)

1. INTRODUCTION

We shall be concerned with some problems arising in the mathematical theory of portfolio selection i.e. in the problem, given n risk investments and a capital w, of finding an allocation of the capital among the n investments which maximizes the expected utility of the investor.

One of the most peculiar properties which have been found is the so-called separation property, i.e. the separation of the original set of n investments into a certain number of "portfolios" which are linear combinations of certain nonintersecting subsets of the original sets of investments and are such that the original set of investments can be replaced by these portfolios.
The classical result is the following.

Consider the boundary of the region of admissible portfolios in the (σ, π), plane, where

$$\sigma = (x^T V x)^{1/2} = v^2 \tag{1.1}$$

$$\pi = x^T r \tag{1.2}$$

in which we have denoted by σ—the standard deviation of the portfolios, by π its expected return, by r the expected return vector of the set of investments, and by V—the variance-covariance matrix*).

The region of admissible portfolios includes, in the plane (v, π), or (σ, π) the set of all points which can be reached by dividing the given unit capital in all possible ways among the given n-investments. Notice that in this formulation we allow a negative value of some competent x_i of the allocation vector, in this case the economic interpretation that a certain quantity x_i of the investment i has been borrowed.

In this case the boundary of the region of admissible portfolios is defined by the following minimization problem:

*) This research has been partially supported by CNR, Comitato per le scienze economiche, sociologiche e statistiche.

*) $V_{ij} = \sigma_i \sigma_j \varrho_{ij}$ where ϱ_{ij} is the correlation coefficient between the i-th and the j-th investment.

$$\min v = x^T V x \tag{1.3}$$

subject to

$$\pi = x^T r \tag{1.4}$$

and

$$1 = x^T e \tag{1.5}$$

where e is the unit vector.

The last equality constraint 1.5. shows that x is indeed an allocation vector i.e. obtained by dividing the given unit capital 1.

It is easy to solve the minimization problem analitically and to obtain the following [2].

Theorem (1.6)

If rank $V = n$, and there does not exist any real number δ with $x = \delta e$, in the plane (v, π) the boundary of the region of admissible portfolios is given by the following parabole

$$v = (\gamma\pi^2 - 2\beta\pi + \alpha)/(\alpha\gamma - \beta^2) \tag{1.7}$$

where

$$\alpha = r^T V^{-1} r, \quad \beta = r^T V^{-1} e, \quad \gamma = e^T V^{-1} e \tag{1.8}$$

The parabola 1.7. is the locus of all and only points of the plane (v, π) to which there corresponds a unique allocation vector x, vector which is given by the equation

$$x = [(\pi\gamma - \beta) V^{-1} r + (\alpha - \pi\beta) V^{-1} e]/(\alpha\gamma - \beta^2) \tag{1.9}$$

which is linear in π.

The vertex of the parabola is given by the values

$$v = 1/\gamma, \quad \pi = \beta/\gamma, \quad x_v = \frac{1}{\gamma} V^{-1} e = \frac{V^{-1} r}{e^T V^{-1} e} \tag{1.10}$$

The vertex of the parabola has the important property.

Theorem (1.11)

All portfolios on the boundary 1.9. are positively linearly correlated with the vertex portfolio x_v (1.10).

In the case in which instead of the plane (v, π) one is interested in computing the boundary in the plane (δ, π), where $\sigma = (v)^{1/2}$ this boundary is given by the following branch of hyperbola

$$\sigma = [(\gamma\pi^2 - 2\beta\pi + \alpha)/(\alpha\gamma - \beta^2)]^{1/2} \tag{1.12}$$

Assume now that one of the investments, say the n-th is riskless, i.e. $\varrho_N = 0$. In this case the variance-covariance matrix V is singular since its n-th row and column are identically zero.

If the matrix V has rank n—1, and again there does not exists any real number δ with $\boldsymbol{x} = \delta e$, in the plane (σ, π) the boundary of the region of admissible portfolios, degenerates into the following two straight lines:

$$\pi = r_n \pm \sigma (r_n^2 \tilde{\gamma} - 2 r_n \tilde{\beta} + \tilde{\alpha}) \tag{2.1}$$

through the point r_n, the return of the riskless investment.

In 2.1. we have used the following notations:

$$\tilde{\alpha} = \tilde{r}^T \tilde{V}^{-1} \tilde{r}, \quad \tilde{\beta} = \tilde{r}^T \tilde{V}^{-1} \tilde{e}, \quad \tilde{\gamma} = e^T \tilde{V}^{-1} \tilde{e} \tag{2.2}$$

in which $\tilde{r}$ denotes the expected return vector of the first $(n$—1) risky investments, with $\tilde{V}$ the non-singular n-1 — minor of V, with $\tilde{e}$ the n-1 -dimensional unit vector.

Under these assumptions we have the following theorem:

Theorem (2.3)

If $r_n \neq \hat{\varrho}/\hat{\gamma}$ one of the two straight lines 2.1. is tangent to the branch of hyperbola

$$\gamma = [(\tilde{\gamma}\pi^2 - 2\tilde{\beta}\pi + \tilde{\alpha})/(\tilde{\alpha}\tilde{\gamma} - \tilde{\beta}^2)]^{1/2} \tag{2.4}$$

which is the boundary of the region of attraction corresponding to the first n-1 investment.

The tangency point is given by:

$$\pi^* = \frac{(r_n \tilde{e} - \tilde{r})^T \tilde{V}^{-1} \tilde{r}}{(r_n \tilde{e} - \tilde{r})^T \tilde{V}^{-1} \tilde{e}} \tag{2.5}$$

$$v^* = \frac{(r_n \tilde{e} - \tilde{r})^T \tilde{V}^{-1} (r_n \tilde{e} - \tilde{r})}{[(r_n \tilde{e} - \tilde{r})^T \tilde{V}^{-1} \tilde{e}]^2} \tag{2.6}$$

The corresponding allocation vector has the form:

$$\boldsymbol{x}^* = \frac{\tilde{V}^{-1}(r_n \tilde{e} - \tilde{r})}{(\tilde{r}_n \tilde{e} - \tilde{r})^T \tilde{V}^{-1} \tilde{e}} \tag{2.7}$$

Now, due to the linearity of the vector $\boldsymbol{x}$ as a function of π, which still holds on the boundary 2.1. it is easy to see that the following theorem holds:

Theorem (separation) (2.8)

The original set of n investments is equivalent to the combination of the riskless investment $\boldsymbol{x}_n$ and of the "optimal portfolio of risky investments" x^* (2.7).

Notice that in this case "separation" i.e. the equivalence between the set of the given n investments and a certain number of portfolios each of which containing different assets, coincide with the property of "return to scale" i.e. linearity of the boundary, which as shown by Stone [1] depends entirely on the homogenity of σ as a function of x.

3. SEPARATION WITH ONLY RISKY ASSETS: THE SINGULAR CASE

In the case in which the matrix V is singular, without any null rows (no riskless investments) and there does not exist any investment which is a mutual fund, i.e. which is a linear combination of other investment then ν in the plane (σ, π), boundary is still defined by two straight lines

$$\pi = \pm\sigma(\bar{r}_n^2\tilde{\gamma} - 2\bar{r}_n\tilde{\beta} + \tilde{\alpha} + \bar{r}_n)^{1/2} \tag{3.1}$$

Trough the point $\bar{r}_n$ which is the return of a particular riskless portfolio (equivalent riskless asset). Indeed the problem can be formally reduced to the same problem investigated in § 2.

In 3.1. we have denoted by $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ the same constants as in 2.2. where now $\tilde{V}$ denotes a non singular n-1 — minor of the matrix V.

The value of the equivalent riskless portfolio x_r can be easily computed.

Notice however that even the boundary is a straight line (return to scale), it does not necessarily follow that there exists separation and in particular that the investments contained in the equivalent riskless portfolio are completely different from those contained in the portfolio x^* (2.7) corresponding to our problem i.e. to the portfolio

$$x^* = \frac{V^{-1}(\bar{r}_n e - r)}{(\bar{r}_n e - r)^T V^{-1} e} \tag{3.2}$$

4. SEPARATION WITH ONLY RISKY ASSETS, SOME GENERAL RESULTS

Since separation is completely independent from the linearity of the boundary of the region of admissible portfolios it is worthwhile to investigate some conditions which ensure separation.

We can make first the following preliminary remark:

Remark (4.1)

If at the boundary of the region of admissible portfolios the allocation vector x is a linear function of π, then if it exist separation it must be among exactly two portfolios. Then, since the two portfolios must define the whole boundary, in the plane (σ, π) the boundary must be a branch of hyperbola (possibly degenerated its asymptotes) and one of the two portfolios must lay in the vertex.

From the expressions 1.10 it immediately follows:

Theorem (4.2)

A necessary condition for separation is that

$$(V^{-1}e)_i = 0 \tag{4.3}$$

for some $i = 1, \ldots, n$.

Theorem (4.4)

A sufficient condition for separation is that

$$(V^{-1}e)_i = 0 \tag{4.5}$$

for n-1 values of the index i.

For the possibility constructing cases of separation in which the sufficient condition 4.4. is not met we are still forced to investigate conditions under which more then one components of the vector x simultaneously vanish at the same point of the boundary of the region of admissible portfolios.

For that we shall rewrite the expression 1.9 in the form:

$$x = [\pi(\gamma V^{-1}r - \beta V^{-1}e) + (\alpha V^{-1}e - \beta V^{-1}r)]/(\alpha\gamma - \beta^2) \tag{4.6}$$

The conditions under which the components i and j of the vector x vanish at the same point of the boundary are then the following:

$$\frac{\alpha(V^{-1}e)_i - \beta(V^{-1}r)_i}{\gamma(V^{-1}r)_i - \beta(V^{-1}e)_i} = \frac{\alpha(V^{-1}r)_j - \beta(V^{-1}r)_j}{\gamma(V^{-1}r)_j - \beta(V^{-1}e)_j} \tag{4.7}$$

which shows that the sufficient condition 4.4. is not necessary.

5. CONCLUSIONS

The possibility of separation between assets is one of the main problems in portfolio selection.

Beside the problem which we have treated in this paper it is of interest in the case in which the risk is not measured by an homogeneous function of the allocation vector x. In this case the whole structure of the problem changes, and it is in general impossible to find the analytical expression for the boundary of admissible portfolios.

In certain cases [3] however even in these more general models separation can still be proved.

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SUMMARY

The work concerns some problems arising in the mathematical theory of portfolio selection i.e. in the problem, given n risk investments and a capital w, of finding an allocation of the capital among the n investments which maximizes the expected utility of the investor.

One of the most peculiar properties which have been found is the so-called separation property, i.e. the separation of the original set of n investments into a certain number of "portfolios" which are linear combinations of certain nonintersecting subsets of the original sets of investments and are such that the original set of investments can be replaced by these portfolios.

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