POLSKA AKADEMIA NAUK
INSTYTUT BADAŃ SYSTEMOWYCH

PROCEEDINGS OF THE 3rd ITALIAN-POLISH CONFERENCE ON APPLICATIONS OF SYSTEMS THEORY TO ECONOMY, MANAGEMENT AND TECHNOLOGY

WARSZAWA 1977

Redaktor techniczny
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PREFACE

The present volume comprehends the papers from the 3rd Italian-Polish Conference which was held in Białowieża in May 30 to 31. The conference organized by the Centre of Control Systems and Computation of the Italian National Research Council in Rome and the Institute for Organization, Management and Control Sciences of Polish Academy of Sciences and the Ministry of Science, Higher Education and Technology in Warsaw was the third conference of this type.

The first one took place in Kraków (Poland) in 1972, the second in Pugnochiuso (Italy) in 1974.

This conference, like both the previous ones, gathered the leading representatives of many scientific centres from Italy as well as from Poland, representing also the different aspects of the systems analysis applied to the economic, management, data processing and technological systems.

From the past, such meetings enrich the exchange of views and experiences between the specialists from both countries what is the condition of advantage to both sides, above the several years the summarization of the systems and control theory results with the accompanying concentration on the essential development of the quantitative methods and the particularly rapidly development of large computer systems.

The meaning of such meetings, however, goes far beyond that what is happening during the conference because the established contacts are also fruitful in form of common scientific research works and — last but not least — form the friendly ties between the scientists of both countries which last for years.

The proceedings of the conference were divided into three parts:

1. Optimization and Control Theory.
2. Systems Theory in Economics.
3. Technological, Managerial and Information Systems.

While the first two parts are of rather homogeneous character, the third part contains the papers concerning the different types of models — for the economic, technological, management and data processing systems.

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MATHEMATICAL MODEL OF ECONOMIC EXCHANGE

INTRODUCTION

The goal of this paper is an attempt to present a model of open national economy using mathematical methods and investigate an influence of economic exchange on its development. There is considered a group of N-states leading production activity and performing an economic exchange. It is assumed that each state can lead an exchange with all others or only some of them. The possibility of creation a group of states acting against other group or states is excluded. Each state has his utility function which can be for exampie the State revenue.

The exchange has to pass in a way how to maximize utility function. We will have N-functionals maximized with respect to certain decision variables. From mathematical point of view we will seek equilibrium points in such stated problem. After the introducing information we can pass to the construction and describing the model.

DESCRIPTION OF THE MODEL

We consider a group of N-states taking part in the exchange of goods between one another. A model of the state economy is similar to others. They differ only in parameters of individual parts of economy. For this reason, in further considerations we restrict ourselves to the case of presenting the model of one state economy and to showing relations with others.

Let us suppose, we have taken the state with number j. Further assumption is that all parts of the model are linear. The economic process is considered on interval $[0, T]$. During this period it is assumed that a structure of economy is changing according to the given function. There are assumed following condition of economic exchange. Each state establishes its export possibility U_j^0, but it does not get insight into the fashion it is established. A criterion can be of economic type as for example: we can export only excess crossing a prescribed limit of a given good. So if the state number j has established maxifunctio quantity of export, then each other state can make a decision about purchases in th framework of these possibilities. Let $u_j=(u_{ji}, \ldots u_{JN})$ be a vector whose i-th component is vector of purchases u_{ji} of goods which are sold by the state j to the state i.

In a similar way behave remaining states. If the export limit of the state j from state i we denote by V^0_{ji}, then v_{ji} means a purchase vector of state j in state i and $v^k_j=(v_{ji}, \ldots, v_{JN})$. We divide vector v_j into two parts v^k_j and v_j relating to consumer and investment purchases. It ought to be emphasized that the model is created with regard to decision variables related to export and import.

Now we can pass to the mathematical deseribing of the model. For this aim, we introduce a few notations. Let R_0 be a set of all real nonnegative the numbers and R^n_0 their n-time Cartesian product. In the following V_{ji} be a set of the form

$$V_{ji} \overset{df}{=} \{v_{ji}, v_{ji} \quad V^0_{ji} > v_{ji} \in R^n_0\} \tag{1}$$

where

$$x \leqslant y \rightarrow x = (x_1 \ldots x_n) \quad y = (y_1 \ldots y_n) \quad \text{and} \quad x_i \leqslant y_i \quad \text{for} \quad i \leqslant n$$

and

$$V_j = \sum_{i=1}^{N} V_{ji} \tag{2}$$

means the Cartesian product of sets defined above.

Let

$$u_j = \{u; u \leqslant U^0_j \; u \in R^n_0\} \tag{3}$$

From above it follows that V_{ji}, $U_j \subset R^n_0$ are convex, closed and bounded. The equation describing economic process has linear form, namely

$$z^j(t) = P_{0j}(t)\, z(t) + P_j(t)(v^I_j(t))\, z^j(t) + M_j(t)\,(v^I_j(t)\, z^j(t) + \\ + \sum_{i=1}^{N} (v^k_{ji} - u_{ji}) \quad j = 1, \ldots, N \tag{4}$$

where

$P_{oj}(t)$ — is a measurable $n \times n$ function matrix of variable t.

Matrix $P_{oj}(t) = P_j(t)\,(I - K_j(t))$ where $K_j(t)$ is a consumption matrix, that means $K_j(t)\; z$ is part of production which is appropriate for consumption.

Under fixed t, $P_j(t)(v)$ and $M_j(t)(v)$ are linear operators mapping $R^{n \cdot N}$ into a space of $n \times n$ matrices. In the end for fixed we have measurable bounded function of t the values of which are $n \times n$ matrices.

Let us define $v = [v^I, v^k]$. Matrices $P_j(t)\,(v)$ and $M_j(t)\,(v)$ have similar structure to $P_{oj}(t)$, but they are related to that part of economy which can be controlled.

Moreover matrix $M_j(t)\,(v)$ meets the case, when some part of production is introduced to the circulation with some time lag h.

$z^j(t)$ — is a n-dimensional vector, representing goods made by the j-th economy and aggregated to same level.

Let $v_{ji} \in V_{ji}$ and $\sum_{i=1}^{N} u_{ji} \in U_j$

A function $v_{ji}(t)$ and $u_{ji}(t)$ are measurable with respect to t. Initial condition for equation (4) has the form

$$z^j(t) = \varkappa_j(t) \quad \text{fer} \quad t \in [-h, 0] \quad j = 1, \dots N \tag{5}$$

Equation (4) with initial condition (5) is a linear differential equation with time lag. From a practical point of view the stability of this equation is important. An equation is called stable, if his spectrum lay on the left hand side of imagine axis.

In the paper we dont deal with stability because it is difficult to say anything interesting about this equation under so general assumptions. It should be noted that the set of all controls v for which it remains stable, creates certain convex cone S.

Thus for $V \in V_j \cap S$ solutions stil stay stable. From this it follows that changes of V_j can cause instability of equation (4). Last fact can be interpreted as that instability of the world market can cause instability of national economy. An other important problem is the export-import balance. If we denote by $p_i(t)$ $i = 1 \dots N$ prices of goods in the i-th state at the moment t then the import-export balance can be stated in the form

$$\int_0^T \langle p_i(t), v_{ji}(t) \rangle dt = \int_0^T c_{ji} \langle p_j(t), u_{ji}(t) \rangle dt \tag{6}$$

for $i \quad 1, \dots N$

where c_{ji} — is a monetary conversion.

The system of equations (6) says that there ought to be economic balance each state with each other. But it is possible to assume less restrictive assumption as

$$\sum_{i=1}^{N} \int_0^T \langle p_i(t), v_{ji}(t) \rangle dt = \sum_{i=0}^{N} \int_0^T c_{ji} \langle p_j(t), u_{ji}(t) \rangle dt \quad j = 1 \dots N \tag{6'}$$

The condition (6′) says, that the value of all of this what we sell to other states must be equal to our entire purchases. Now we pass to the problem of valuation of economic activity of each state.

As utility functional we can take value of production during the period $[0, T]$,
so get

$$K_j = \max_{\vartheta \in V_j} \int_0^T \langle p_j(t), z^j(t) \rangle dt \quad j = 1 \dots N \tag{7}$$

So we get a problem of polioptimization.

If V_{ji} is given by (1) and V_j by (3) then we have

$$U_j = \sum_{i=1}^{N} V_{ij} \tag{8}$$

where a sum is understood in the algebraic sense. Let V_j^s denote the set of all measurable selectors of V_j.

We define a function of the form

$$f(V_j) \stackrel{df}{=} \max_{\vartheta_j \in V_j} \sum_{i=1}^{N} \int_0^T \langle p_i(t), v_{ji}(t) \rangle \, dt \tag{9}$$

where $v_j = (v_{ji} \dots V_{JN})$

f is a linear continuous functional on the family of all compact convex sets with Hausdorf distance. Since we can come with max under the sign of the sum thus we get

$$f(V_{j1}, \dots, V_{jN}) = \sum_{i=1}^{N} f_i(V_{ji}) \tag{10}$$

where $\quad f_i(V_{ji}) = \max_{v_{ji} \in V_{ji}} \int_0^T \langle p_i(t), v_{ji}(t) \rangle \, dt$

$$\varphi(V_{1j}, \dots, V_{Nj}) = \max_{u \in (V_{1j}, \dots, V_{Nj})} \sum_{i=1}^{N} c_{ji} \int_0^T \langle p_j(t), u_{ji}(t) \rangle \, d! \tag{11}$$

Then the balance equation gets a form

$$f(V_j) - \varphi(V_{1j}, \dots, V_{Nj}) = 0 \quad j = 1, \dots, N \tag{12}$$

or

$$\sum_{i=1}^{N} \left(f_i(V_{ji}) - \varphi_i(V_{ij}) \right) = 0 \quad j = 1, \dots, N \tag{13}$$

So we have a system of N — equations with $2N$ — variables.

Solutions of this system, if they exist, create certain cone E.

Let us define

$$C = \{ (V_{ij})_{j,i=1 \dots N} \sum_{i=1}^{N} V_{ij} = U_j \quad j = 1, \dots, N \} \tag{14}$$

The set C physically means a set of divisions of all offers between states takin part in the economic exchange.

Let us take

$$C_0 = C \cap E \tag{15}$$

and then for $(V_{ji})_{i,j=1 \dots N} \in C_0$ we define

$$V_j' = \{ v_j(t) = (v_{j1}(t), v_{jN}(t)); v_{ji}(t) \in V_{ji} \quad f(V_j) = f(v_j) \} \quad j = 1, \dots, N \tag{16}$$

V_j' — are convex closed sets and such that for each element belonging to them we have economic balance.

In a simple case they can reduce to a single point. Now we can pass to the problem of polioptimization but on (V_j')

$$j=1 \ldots N$$

$$K_j = \sup_{\vartheta \in V_j} \int_0^T \langle p_j(t), z^j(t) \rangle \, dt \qquad j = 1, \ldots, N \tag{17}$$

$$j=1, \ldots, N$$

We will look for an equilibrium in the sense of Nash. For this aim, we write the Hamilton — Jacobi equation

$$\frac{\partial W_j}{\partial t} + \sup_{\vartheta \in V_j} \langle \operatorname{grad}_z W_j, [P_{0j}(t) + P_j(t)(v^I)] z^j(t) + M_j(t)(v^I) z^j(t-h) +$$

$$+ \sum^N (v_{ji}(t) - u_{ji}(t)) \rangle + \langle p_j(t), z^j(t) \rangle = 0 \quad \text{for} \quad j = 1, \ldots, N$$

where W_j is a value of a differential game described by equation (4) with functionals (17). It is possible to show the existence of value W_j of this game, and the existence of optimal strategies. We can show also Lipschitz-continuity of $W_j(t, z^j)$ with respect to (t, z_j), what implies differentiability of W_j and equation (18) is satisfied almost everywhere.

Since functionals $K_j, j = i \ldots N$ and what implies W_j depend also on $(V_j')_j = 1 \ldots N$ we optimize them with respect to these sets in the sense of Pareto on the family C_0 transformed according to (16).

Generally we can take more complex equations and functionals but at same level it occurs impossible to consider them even so generally.

SUMMARY

The problem of n-state exchange economy is considered. The economy of each state is described by linear differential equation of a retarded type. In the model of the economy there are considered two sectors, the sector of means of production and the sector of means of consumption.

Each state designates maximal quantity of desired export. In the framework of this possibility, each state chooses suitable for it quantity of import. This quantity appeares in the equation that describes the model in the form of control.

A linear functional is assumed as utility functional. The functional is maximized with respect to import and moreover there is assumed import-export equilibrium to be fulfilled. From mathematical point of view this problem is a n-person differential game.

Further generalization are discussed as well.

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