POLSKA AKADEMIA NAUK
INSTYTUT BADAŃ SYSTEMOWYCH

PROCEEDINGS OF THE 3rd ITALIAN-POLISH CONFERENCE ON APPLICATIONS OF SYSTEMS THEORY TO ECONOMY, MANAGEMENT AND TECHNOLOGY

WARSZAWA 1977

Redaktor techniczny
Iwona Dobrzyńska

Korekta
Halina Wołyniec

POLSKA AKADEMIA NAUK
INSTYTUT BADAŃ SYSTEMOWYCH

PROCEEDINGS OF THE 3rd ITALIAN-POLISH CONFERENCE ON APPLICATIONS OF SYSTEMS THEORY TO ECONOMY, MANAGEMENT AND TECHNOLOGY

BIAŁOWIEŻA, POLAND
MAY 26—31, 1976

EDITED BY J. GUTENBAUM

WARSZAWA 1977

Redaktor techniczny
Iwona Dobrzyńska

Korekta
Halina Wołyniec

PREFACE

The present volume comprises the papers from the 3rd Italian-Polish Conference which was held in Warszawa in May 1976, 10 to 11. The conference organized by the Centre of Control Systems and Computation of the Italian National Research Council in Rome and the Institute for Organization, Management and Control Sciences of Polish Academy of Sciences and the Ministry of Science, Higher Education and Technology in Warsaw was the third conference of this type.

The first one took place in Kraków (Poland) in 1972, the second in Pugnochiuso (Italy) in 1974.

This conference, like both the previous ones, gathered the leading representatives of many scientific centres from Italy as well as from Poland, representing also the different aspects of the systems analysis enabled by the economic, management, data processing and technological systems.

From one part, such meetings enrich the exchange of views and experiences between the specialists from both countries what is the link, advantageous to both sides, shows the actual gains, the systematization of the systems and control theory useful, first of all, the economical contributions, the essential development of the quantitative methods and the possibilities of large complex systems.

The meaning of such meetings, however, goes far beyond that: what is happening during the conference between the established contacts are also fruitful in form of common scientific research works and — "last but not least" — make the friendly links between the scientists of both countries which last for years.

The proceedings of the conference were divided into three parts:

1. Optimization and Control Theory;
2. Systems Theory in Economics;
3. Technological, Managerial and Information Systems.

While the first two parts are of rather homogeneous character, the third part contains the papers concerning the different types of models — for the economic, technological, management and data processing systems.

O. Gedymin
Institute of Management Organization, Technical University of Warsaw

THE DYNAMIC PARETO'S OPTIMUM AND THE PONTRIAGIN-KRIVIENKOV'S THEORY IN PLANNING OF ECONOMIC GROWTH

1. STATEMENT OF THE PROBLEM

Consider the following problem of economic growth. In finite planning period $[O, T]$, $n+1$ different measurable economic goals are accomplished. These goals are stocks of $n+1$ commodities $x_1^T, \ldots, x_{n+1}^T$ in final moment T. They form a vector called "the final economic activity goal". In every moment of planning period $[O, T]$ $n+1$ commodities are produced at rates $u_1(t), \ldots, u_{n+1}(t)$ with the aid of $n+1$ homogenous factors of production $x_1(t), \ldots, x_{n+1}(t)$. Thus we have $n+1$ industries, each industry producing only one commodity and using $n+1$ stocks of the production factors. Then, the production relationship (production function) can be written in the form:

$$\mathscr{C}_i[u_1(t), \ldots, u_{n+1}(t)] \leqslant x_i(t) \qquad t \in [0, T] \quad i = 1, \ldots, n+1 \tag{1}$$

$$u_i \geqslant 0 \tag{2}$$

We shall require that $\mathscr{C}_i (i=1, \ldots, n+1)$ be twice differentiable and defined over open set in E^{2n+2} containing the set of solutions to (1).

The process of commodity accumulation is described by:

$$\frac{dx_i(t)}{dt} = u_i(t) - \mu_i x_i(t) \qquad i = 1, \ldots, n+1 \tag{3}$$

where $\mu_i (i=1, \ldots, n+1)$ are certain constants and

$$x_i(0) = x_i^0 \qquad i = 1, \ldots, n+1 \tag{4}$$

We say that the control strategy $(u_1(t), \ldots, u_{n+1}(t))$ is ***admissible*** when, for each $\boldsymbol{u}(t) \in U$, where the set U depends on the state variables $\boldsymbol{x} = (x_1, \ldots, x_{n+1})$ and $u(t)$ is a piecewise continuous function of time. It is the set of points u which satisfy (1) for a specified x. Now we will try to formulate the criterion of the selection of such admissible control variables and corresponding state variables, which will be optimal from the point of view of the whole society. This point of view must be explicitly expressed by Central Planning Authority. It seems reasonable that such criterion should be based on Pareto optimality concept. One must assume that the attainment level of "final economic activity goal" at the moment T increases, when the level of at least one goal increases, and the attainment of the other goals does not decrease. Then the dynamic Pareto optimum can be defined as a state of the economy in which it is not possible to increase the level of one goal at moment T, without decreasing the level of at least one of the other goals at moment T. Thus we have the following problem of attaining dynamic Pareto optimum in production process described by (1)—(4): Which conditions must be satisfied by admissible control variables $u_1(t), \ldots, u_{n+1}(t)$ and the corresponding trajectory for the state variables $x_1(t), \ldots, x_{n+1}(t)$ that the stock of one commodity on period $[O, T]$ be maximal without decreasing stocks of other n goods in that period below the prescribed level. The problem is that of maximizing the objective functional

$$J(x, u) = \int_0^T u_{n+1}\, dt \tag{5}$$

subject to the constraints (1)—(3), the initial conditions (4) and

$$x_i(T) \geqslant x_i^T \qquad i = 1, \ldots, n \tag{6}$$

2. NECESSARY CONDITIONS

The constrained maximization problem (1)—(6) is equivalent to the following problem of maximizing the Lagrange functional:

$$\Lambda = \int_0^T F\, dt = \int_0^T \left\{ u_{n+1} + \sum_{i=1}^{n} p_i(t) \left[u_i(t) - \mu_i x_i(t) - \frac{dx_i(t)}{dt} \right] + \right.$$

$$\left. + \sum_{i=1}^{n} q_i(t) \left[x_i(t) - \mathscr{C}_i - \varepsilon_i(t) \right] \right\} dt + \sum_{i=1}^{n} p_i(T) \left[x_i(T) - x_i^T - \alpha_i \right]$$

where ε_i and α_i ($i = 1, \ldots, n$) are nonnegative slack variables and $p_i(t)$, $q_i(t)$, $p_i(T)$, ($i = 1, \ldots, n+1$) are Lagrangian multipliers. One group of necessary conditions of optimal control and state variables are the Euler-Lagrange equations:

$$\frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right) = 0, \tag{7}$$

where y stands for variables $u_j(t)$, $x_i(t)$, $p_j(t)$, $q_i(t)$, $\varepsilon_i(t)$, $\alpha_i(t)$, ($i = 1, \ldots, n$; $j = 1, \ldots, n+1$).

For all variables other than $x_i (i=1, \ldots, n)$ conditions (7) boil down to equations

$$\frac{\partial F}{\partial y} = 0 \tag{8}$$

since no other time derivative appears.

Particular attention must be paid to equations:

$$\frac{\partial F}{\partial x_i} = \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}_i}\right) \tag{9}$$

$$\frac{\partial F}{\partial p_i} = 0 \tag{10}$$

They can be written as:

$$\frac{dp_i^*(t)}{dt} = \mu_i \, p_i^*(t) - q_i^*(t) \qquad i = 1, \ldots, n \tag{11}$$

$$\frac{dx_i^*(t)}{dt} = u_i^*(t) - \mu_i \, x_i^*(t) \qquad i = 1, \ldots, n \tag{12}$$

If we introduce the Hamiltonian function

$$H = p_{n+1} \, u_{n+1} + \sum_{i=1}^{n} p_i(t) \, [u_i(t) - \mu_i \, x_i(t)] \tag{13}$$

we have:

$$A = \int_0^T F \, dt = \int_0^T \left\{ u_{n+1} + H - \sum_{i=1}^{n} p_i(t) \frac{dx_i(t)}{dt} + \sum_{i=1}^{n+1} q_i(t) \times \right.$$

$$\left. \times [x_i(t) - \mathscr{C}_i - \varepsilon_i(t)] \right\} dt + \sum_{i=1}^{n} p_i(T) \, [x_i(T) - x^T - \alpha_i] \tag{14}$$

we can rewrite equations (11), (12) in the following way

$$\frac{dp_i^*(t)}{dt} = -\frac{\partial H[\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t)]}{\partial x_i} - q_i(t) \qquad i = 1, \ldots, n \tag{15}$$

$$\frac{dx_i^*(t)}{dt} = \frac{\partial H[\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t)]}{\partial p_i} \qquad i = 1, \ldots, n \tag{16}$$

Moreover, we have the other necessary conditions:

$$\frac{\partial H[\boldsymbol{x}^*(t), \boldsymbol{u}^*(t), \boldsymbol{p}^*(t)]}{\partial u_j} = \sum_{i=1}^{n+1} \frac{\partial \mathscr{C}_i[\boldsymbol{u}^*(t)]}{\partial u_j} q_i^*(t) \quad j = 1, \ldots, n+1 \tag{17}$$

that is

$$p_i^*(t) = \sum_{i=1}^{n} \frac{\partial \mathscr{C}_i[\boldsymbol{u}^*(t)]}{\partial u_j} q_i^*(t) \quad j = 1, \ldots, n+1 \tag{18}$$

We also have:

$$\sum_{i=1}^{n} q_i^*(t) ([x_i^*(t) - \mathscr{C}_i[\boldsymbol{u}^*(t)]) = 0 \tag{19}$$

$$\sum_{i=1}^{n} p_i^*(T) [x_i^*(T) - x_i^T] = 0 \tag{20}$$

Now we can formulate the necessary conditions for our control problem. They are based on the Pontriagin-Hadley-Kemp theory [7, 4].

Theorem 1: Suppose that $\boldsymbol{u}^*(t)$ is an admissible control for control problem (1)—(6), and let $\boldsymbol{x}^*(t)$, $\boldsymbol{x}^*(0) = \boldsymbol{x}^0$, be corresponding trajectory for the state variables which ends on the smooth manifold defined by $\boldsymbol{x}(T) - \boldsymbol{x}^T \geqslant 0$. Then if program $[\boldsymbol{x}^*(t), \boldsymbol{u}^*(t)]$ maximizes $J(\boldsymbol{x}, \boldsymbol{u})$, it is necessary that there exist a constant $p_{n+1} \geqslant 0$ (which without loss of generality can be taken to be 0 or 1) and a continuous vector-valued function $\boldsymbol{p}(t)$ such that

$$[p_{n+1}, \boldsymbol{p}(t)] \neq 0 \tag{21}$$

for any t, as well as a vector-valued function $\boldsymbol{q}(t) \geqslant 0$ which is continuous except possibly at corners of $\boldsymbol{x}^*(t)$, with such properties that if (13) then the conditions (15)—(20) are satisfied. Furthemore, if $U(\boldsymbol{x})$ is the set of $\boldsymbol{u}$ satisfying (1) and if

$$M[\boldsymbol{x}^*(t), \boldsymbol{p}^*(t)] = \sup_{u \in U} H[\boldsymbol{x}^*(t), \boldsymbol{u}(t), \boldsymbol{p}^*(t)] \tag{22}$$

then for each $t \in [O, T]$

$$H[\boldsymbol{x}^*(t), \boldsymbol{u}(t), \boldsymbol{p}^*(t)] = M[\boldsymbol{x}^*(t), \boldsymbol{p}^*(t)] \tag{23}$$

These conditions were derived on the basis of the results obtained by Hadley and Kemp [4].

Thus we showed that necessary conditions for Pareto optimum for the problem (1)—(6) are a combination of conditions given by Pontriagin and Hadley and Kemp.

Variables $\boldsymbol{p}(t)$ and $\boldsymbol{q}(t)$ are Lagrangian multipliers connected with the flow constraint (3) and the stock constraint (1). $q_i(t)$ $(i = 1, \ldots, n)$ are the interest rates for stocks of commodities $x_i(t)$. Then differential equations (11) or (15) may be interpreted as relations between shadow prices in optimum state of economy when the foresight is perfect. Next we have the optimal balance equations between gross flows of goods $u_i(t)$ and net flows $\frac{dx_i(t)}{dt}$ in every moment $t \in [O, T]$, μ_i being treated as the rates of depreciations of commodity stocks $x(t)$.

The Hamiltonian function (13) may be interpreted as follows: we can put $p_{n+1} = 1$ so flow of commodity $n+1$ that is u_{n+1} may be treated as a numeraire. Thus the Hamiltonian function is a weighted sum of net commodity flows $\frac{dx_i(t)}{dt}$ $(i = 1, \ldots, n+1)$ where weights are the prices $p_i(t)$ expressed in terms of the numeraire, and H may be interpreted as an inputed net product. We could say that these net commodity flows are the **current economic activity goals.** So the condition (23) requires to maximize the weighted sum of current goals in every moment of the planning period $t \in [O, T]$. In such a way a dynamic problem with multiple final goals (final economic activity goal at final moment T) can be decomposed into the series of static problems with multiple current goals, each problem for each moment of the planning period $[O, T]$. The equivalence between the dynamic and static problems is guaranteed if prices $p_i(t)$ $(i = 1, \ldots, n)$ are derived from the equations (11) or (15) and (17)—(19) and the transversality conditions (19), according to the theorem presented before. The prices $p_i(t)$ $(i = 1, \ldots, n+1)$ are the coeffitients of the mutual current goals transformation in every moment $t \in [O, T]$.

Now we give simple interpretation of the prices p_i $(i = 1, \ldots, n)$.

Theorem 2: Suppose that the state of an economic system (1)—(3), in moment $t \in [O, T]$ is defined by the vector-valued function $\boldsymbol{x}(t)$. If in the period $[t, T]$ the system is controlled in an optimal manner, then the value of the objective functional in this period $[t, T]$ depends only on state of the system in the moment t. Let this value be $J^*[x(t)]$. In the problem of maximizing (5) subject to the constraints (1)—(3) and the boundary conditions (4), (6) let the function $J^*[\boldsymbol{x}(t)]$ be continuous and continuously differentiable in the region R; then for all $t \in [O, T]$ for which $\boldsymbol{x}(t) \in R$ the optimal control $\boldsymbol{u}^*(t)$ satisfies. Pontriagin's maximum condition (see Theorem. 1) with respect to to $\boldsymbol{p}(t) = [p_1(t), \ldots, p_n(t)]$, where

$$p_i^*(t) = \frac{\partial J^*[\boldsymbol{x}^*(t)]}{\partial x_i} \qquad i = 1, \ldots, n \tag{24}$$

$(\boldsymbol{x}^*(t)$, is the trajectory corresponding to the optimal control $\boldsymbol{u}^*(t)$, $t \in [O, T])$, and

$$\frac{\partial J^*[\boldsymbol{x}^*(t)]}{\partial t} = H[\boldsymbol{x}^*(t), \boldsymbol{p}^*(t), \boldsymbol{u}^*(t)] \tag{25}$$

This follows from the modified results obtained by L. J. Rozonoer [8] and M. Albouy [1].

Following M. Albouy [1] we may interprete (24) as follows. The dual price $p_i(t)$ is a measure of cumulated utility (gain) J^* increase in the period $[t, T]$

due to a marginal increment of the input stock x_i at the moment t, provided this stock is used in optimal manner.

The maximum value of Hamiltonian function is a weighted sum of the net current goals

$$\left[\frac{dx_i(t)}{dt}, \ldots, \frac{dx_n}{dt}\right] \tag{26}$$

where weights are marginal gains in the period $[t, T]$, due to little changes in input stocks x_i

$$H[x^*(t), p^*(t), u^*(t)] = \sum_{i=1}^{n+1} p_i^*(t)\frac{dx_i^*(t)}{dt} = \sum_{i=1}^{n+1} \frac{\partial J^*[x^*(t)]}{\partial x_i}\frac{dx_i^*(t)}{dt} \tag{27}$$

3. GENERALIZED CONTROL PROBLEM

The question arises how to find the coefficients of transformation between final goals in moment T of the planning period. In order to solve this problem we formulate our previous problem (1)—(6) as a generalized control problem. We can write down the general goals as weighted sum of final goals:

$$\pi_{n+1}\int_0^T u_{n+1}\,dt + \sum_{i=1}^{n} \pi_i x_i(T) \tag{28}$$

Now we assume that $\mathscr{C}_i$ $(i=1,\ldots,n)$ are linear functions of $(u_1, \ldots, u_{n+1})$

We will try to define the coefficients π_i $(i=1, \ldots, n)$ .The primal dynamic problem is to maximize (13) subject to the constraints (1)—(4).

According to the mathematical theory of dynamic control of J. Krivienkov [5] we can formulate a dual dynamic problem as follows [3]: Minimize the weighted sum of the initial stocks of inputs:

$$\sum_{i=1}^{n+1} p_i(0)\,x_i^0, \qquad p_{n+1} = 1 \tag{29}$$

subject to the constraints:

$$\frac{\partial p_i(t)}{dt} = \mu_i\,p_i(t) - q_i(t), \qquad q_i \geqslant 0, \qquad i = 1, \ldots, n+1 \tag{30}$$

$$\frac{\partial \mathscr{C}[u(t)]}{\partial u_i} \geqslant p_i(t) \qquad i = 1, \ldots, n+1 \tag{31}$$

and:

$$p_i(T) = \pi_i \equiv p_i^T \quad i = 1, \dots, n+1 \tag{32}$$

Now we write the **dynamic duality theorem.**

Theorem 3: To every **primal** dynamic problem (28), (1)—(4), (6) there corresponds the dual dynamic problem (29)—(32) and the following dual relation exists:

$$Max\,[p_{n+1}^T \int_0^T u_{n+1}\,dt + \sum_{i=1}^{n} p_i^T x_i(T)] = Min \sum_{i=1}^{n+1} p_i(0)\,x_i^0 \tag{33}$$

We can prove this theorem on the basis of Krivienkov theory.

We may interpret this theorem as follows: If the control variables and the state variables are optimal in dynamic Pareto sense then there exist such trajectories of prices — weights $p_i(t)$, $t \in [O, T]$, $(i = 1, \dots, n+1)$ that weighted sum of final goals

$$\sum_{i=1}^{n+1} p_i^T x_i(T) \quad (\text{where } x_{n+1} = \int_0^T u_{n+1}\,dt)$$

equals to the weighted sum of initial stocks of inputs.

This theorem has very interesting implications, especially in the field of economic growth and theory of investment planning [3]. We note, that the dual constraints (30), (31) resemble very closely the Pontriagin-Hadley-Kemp conditions (14), (16a).

Then next dynamic duality theorem is as follows.

Theorem 4: Suppose that the program $\{x(t)^*, u^*(t), t \in [O, T]\}$ is a feasible solution the problem (28), (1) ,(4), (6) and program to $\{p^*(t), q^*(t), t \in [O, T]\}$ is a feasible solution to the problem (29)—(32). Then satisfying one of the conditions:

$$1)\quad \sum_{i=1}^{n+1} p_i^T x_i^*(T) = \sum_{i=1}^{n+1} p_i^*(0)\,x_i^0 \tag{34}$$

$$2)\quad \int_0^T [\sum_{i=1}^{n+1} p_i^*(t)\,u_i^*(t)]\,dt = \int_0^T [\sum_{i=1}^{n+1} x_i^*(t)\,q_i^*(t)]\,dt \tag{35}$$

is sufficient for optimality of both programs.

This theorem also can be proved on the basis of the Krivienkov's theory.

One may show, that optimal solutions to the static primal problem, in every moment $t \in [O, T]$

$$Max \sum_{i=1}^{n+1} p_i(t)\,u_i(t) \tag{36}$$

subject to the constraints

$$\mathscr{C}_i[u(t)] \leqslant x_i(t) \quad i = 1, \dots, n \tag{37}$$

$$u_i(t) \geqslant 0 \quad i = 1, \dots, n+1 \tag{38}$$

are equivalent to the optimal solution of the primal dynamic problem (28), (1), (4), (6). The problem (36)—(37) could be called the **associated** primal static problem with the multiple goals.

Moreover, optimal solutions to the static dual problem, in every moment $t \in [O, T]$

$$\operatorname{Min} \sum_{i=1}^{n+1} x_i(t)\, q_i(t) \tag{39}$$

subject to the constraints

$$\sum_{k=1}^{n} q_k(t) \frac{\partial \mathscr{C}_k}{\partial u_j} \geqslant p_j(t) \qquad q_j \geqslant 0; \quad j = 1, \dots, n+1 \tag{40}$$

are equivalent to the optimal solutions of the dual dynamic problem (29)—(32).

The problem (39), (40) conld be called the **associated** dual static problem.

One may prove that for both static problems in every moment $t \in [O, T]$ the following condition is satisfied:

$$\operatorname{Max} \sum_{i=1}^{n+1} p_i(t)\, u_i(t) = \operatorname{Min} \sum_{i=1}^{n+1} x_i(t)\, q_i(t) \tag{41}$$

Moreover, fulfillment of the condition

$$\sum_{i=1}^{n+1} p_i(t)\, u\,(t) = \sum_{i=1}^{n+1} x_i(t)\, q_i(t) \qquad \text{for every} \qquad t \in [0, T] \tag{42}$$

is sufficient for optimality of feasible solutions to both static problems.

Thus the dynamic problem (28), (1)—(4), (6) for the planning period $[O, T]$ can be decomposed into the series of the static problems (36)—(37) for each moment $t \in [O, T]$. Similarly this relation is valid for the dual dynamic (29)—(32) and static problems (39), (40).

What relation does exist between the Hamiltonian function (13) (net product) and the objective function (36) (gross product) of the static problem?

It follows from the theorems 1 and 4, that these functions attain their maxima in the same point; then the condition of maximization of the former may be replaced by maximization of the latter. The reverse is also true.

If the conditions of Pontriagin and Krivienkov (see the theorems) are satisfied then we have the equivalence between dynamic Pareto optimum in the period $[O, T]$ with the series of the associated static Pareto optima in every moment $t \in [O, T]$. Decomposing the dynamic problem with multiple final goals into the series of the static problems with multiple current (intermediary, momentary) goals makes possible combination of the short term economic planning with long term planning.

Taking into account (24) we may write:

$$p_i^{*T} = \frac{\partial J^*[x^*(T)]}{\partial x_i} \qquad i = 1, \dots, n+1 \tag{43}$$

Thus the maximum value of the generalized objective functional is

$$J^*(x^* u^*) = \int_0^T u_{n+1}^* \, dt + \sum_{i=1}^{n} \frac{\partial J^*[x^*(T)]}{\partial x_i} x_i^*(T) \tag{44}$$

We may interprete the optimal prices (43) as an efficient price system that reflects the relative desirability of various commodities stocks at the terminal date.

The problem of finding the Pareto optimum can be formulated as the problem of maximization of an arbitrary $x_i (i=1,\dots,n+1)$ in $[OT]$ when other final goaln do not change.

Example

Let us examine a simple model of multisectoral economy, in which the technology is of the discrete type [2]. We can formulate the problem of optimal growth of such economy as the control problem with multiple goals.

Consider an economy producing $n+1$ goods, a consumption good C and n depreciable capital goods $Z_1, \dots, Z_n$ with depreciation rates $\mu_1, \dots, \mu_n$. Each sector uses, as fixed proportion inputs, both capital goods and a labour L, which grows at an exogenously fixed rate n.

We assume that the production technology is given by a coefficient matrix

$$\begin{bmatrix} b_{01} & b_{02} & \dots & b_{0,n+1} \\ b_{11} & b_{12} & \dots & b_{1,n+1} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{n,n+1} \end{bmatrix} \tag{45}$$

Therefore we have the production function as a system of inequalities. The labour constraint:

$$b_{0,n+1} C + \sum_{j=1}^{n} b_{0j} Z_j \leqslant L \tag{46}$$

The capital constraints:

$$b_{i,n+1} C + \sum_{j=1}^{n} b_{ij} Z_j \leqslant K_i \qquad i = 1, \dots, n \tag{47}$$

$$C \quad 0; \qquad Z_j \geqslant 0; \qquad j = 1, \dots, n \tag{48}$$

We now have the following optimal growth problem:

$$\text{Maximize} \int_0^T C e^{-\delta t}\, dt \tag{49}$$

where δ is time rate of discount.

Subject to the inequalities (46), (47) and the flow constraints:

$$\frac{dL(t)}{dt} = vL(t) \tag{50}$$

where v is the rate of labour growth and

$$\frac{dK_i(t)}{dt} = Z_i - \mu_i K_i(t) \qquad i = 1, \dots, n \tag{51}$$

we have also (48) and

$$K_i(0) = K_i^0 \tag{52}$$

$$K_i(T) \geqslant K_i^T \qquad i = 1, \dots, n \tag{53}$$

where K_i^0 and K_i^T $(i = 1, \dots, n)$ are fixed parameters, the former are historically given, the latter given by Central Planning Authority.

Now we apply the Pontriagin-Hadley-Kemp theory and we get the dynamic conditions (15), (16):

$$\frac{dp_i^*(t)}{dt} = -\frac{\partial H}{\partial x_i} = (\mu_i + \delta)\, p_i^*(t) - q_i^*(t) \qquad i = 1, \dots, n \tag{54}$$

$$\frac{dx_i^*(t)}{dt} = \frac{\partial H}{\partial p_i} = Z_i^*(t) - \mu_i K_i^*(t) \qquad i = 1, \dots, n \tag{55}$$

and the static conditions (17)

$$b_{0,n+1} w^* + \sum_{j=0}^{n} b_{j,n+1}\, q_i^* \geqslant p_0^* \tag{56}$$

$$b_{j,n+1} w^* + \sum_{i=1}^{n} b_{ji}\, q_i^* \geqslant p_j^* \qquad j = 1, \dots, n \tag{57}$$

Where w is the dual price (the wage rate) for labour constraint (46).

Moreover if

$$M(\boldsymbol{K}^*(t), \boldsymbol{p}^*(t)] = \sup_{Z \in \mathrm{U}} H[\boldsymbol{K}^*(t), \boldsymbol{p}^*(t), \boldsymbol{Z}(t)] \tag{58}$$

then it is necessary that

$$H[\boldsymbol{K}^*(t), \boldsymbol{Z}^*(t), \boldsymbol{p}^*(t)] = M[\boldsymbol{K}^*(t), \boldsymbol{p}^*(t)] \tag{59}$$

This is the Hamiltonian function, interpreted as net imputed product, being the function of the $Z=(Z_1, \ldots, Z_n)$, then it has its absolute maximum in the point $Z=Z^*(t)$, in every moment of planning period $t \in [O, T]$.

Next, we have [see (19)]:

$$w^*(L-b_{0,n+1}C-\sum_{j=1}^{n} b_{0j}Z_j)=0 \tag{60}$$

$$\sum_{i=1}^{n} q_i^*(K_i-b_{i,n+1}C-\sum_{j=1}^{n} b_{ij}Z_j)=0 \tag{61}$$

and transversality conditions (20)

$$\sum_{i=1}^{n}[K_i(T)-K_i^T]\,p_i(T)\,\mathrm{e}^{-\delta T}=0 \tag{62}$$

Now we can formulate a generalized primal dynamic planning problem as follows [3]:

$$\mathrm{Max}\sum_{i=0}^{n+1} p_i^T K_i(T) \tag{63}$$

subject to the flow constraints

$$\frac{dK_i(t)}{dt}=Z_i(t)-\mu_i K_i(t) \qquad i=0,1,\ldots,n+1 \tag{64}$$

where

$$K_0(t)=L(t), \quad K_{n+1}(t)=J; \quad Z_0(t)=0; \quad Z_{n+1}(t)=C(t)$$

$$\mu_0=-v; \quad \mu_{n+1}=0$$

the stock constraints (46), (47) and the boundary conditions

$$K_i(0)=K_i^0 \qquad i=0,1,\ldots,n \tag{65}$$

$$K_i(T)\geqslant K_i^T \qquad i=1,\ldots,n \tag{66}$$

Generalized dual dynamic problem:

$$\mathrm{Min}\sum_{i=0}^{n+1} K_i^0 p_i(0) \tag{67}$$

subject to the constraints

$$\frac{dp_i}{dt}=(\mu_i+\delta)\,p_i(t)-q_i(t) \tag{68}$$

$$\sum_{i=0}^{n} b_{ji}q_i\geqslant p_j \qquad j=1,\ldots,n+1 \tag{69}$$

$$q_i\geqslant 0 \qquad i=0,\ldots,n \tag{70}$$

where $q_0 = w$

$$p_i(T) = p_i^T \qquad i = 0, 1, \dots, n+1 \tag{71}$$

where $p_{n+1} = 1$.

From the Theorems 3 and 4 we have

$$\text{Max} \sum_{i=0}^{n+1} p_i^T K_i(T) = \text{Min} \sum_{i=0}^{n+1} K_i^0 p_i(0) \tag{72}$$

and

$$1) \quad \sum_{i=0}^{n+1} p_i^T K_i^*(T) = \sum_{i=0}^{n+1} K_i^0 p_i^*(0) \tag{73}$$

$$2) \quad \int_0^T \Big[\sum_{i=0}^{n+1} p_i(t) Z_i(t)\Big] dt = \int_0^T \Big[\sum_{i=0}^{n+1} K_i(t) q_i(t)\Big] dt \tag{74}$$

Then associated primal problem is to maximize:

$$\sum_{i=0}^{n+1} p_i Z_i \tag{75}$$

subject to the constraints (46)—(48) and the associated dual problem is to minimize

$$\sum_{i=0}^{n+1} K_i q_i \tag{76}$$

subject to the constraints (69), (70).

REFERENCES

[1] ALBOUY M.: La regulation économique dans l'entreprise. Deuxième tome. Dunod, Paris 1972.

[2] BRUNO M.: Discrete Capital Models. In: K. Shell (ed.) Essays on the Theory of Optimal Economic Growth. The MTJ Press 1967.

[3] GEDYMIN O.: Sterowanie optymalne stopą akumulacji i wyborem technik produkcji (Optimal control of saving ratio and of choice of production techniques). Instytut Organizacji Przemysłu Maszynowego, Warszawa 1974.

[4] HADLEY G. KEMP M. C.: Variational Methods in Economic. North-Holland, Amsterdam, London 1971.

[5] KRIVIENKOV J. P.: Matiematiczeskije i wyczislitielnyje woprosy liniejnogo dinamiczeskogo programmirovanija. Wyč. isl. Centr AN SSSR. Moskwa 1969.

[6] LANGE O.: Optymalne decyzje (Optimal decisions). PWN, Warszawa 1964.
[7] PONTRIAGIN L. S., BOLTJANSKIJ W. G., GAMKRELIDZE R. W, MISCENKO J. F.: Matiematiceskaja tieorija optimalnych processov. „Nauka" Second edition, Moskva 1969. Eng. Transl.: The Mathematical Theory of Optimal Processes. Interscience, N. York and London 1962.
[8] ROZONOER L. J.: Pontriagin's maximum principle in the theory of optimum systems. I, II and III Automation and Remote Control, Vol. 20, 10, 11, 12.

SUMMARY

The following problem of planning economic growth with multiplicity of goals is the starting point of our considerations. In interval of the social reproduction process, considered in finite planning period (O, T), n different measurable economic goals are realized. They may be expressed as a vector, called the "general economical activity goal".

The dynamic Pareto optimum is defined as the situation in which it is not possible to increase the degree of attainment of one goal without decreasing the degree of attainment of at least one of the other goals in the final moment of planning period. If the stocks of n goods in moment T are the economical activity goals, the choice problem of optimal strategy of the economic growth can be formulated in followin gway: what conditions must be fulfilled to maximize the production of one good during the planning period (O, T) without decreasing stocks of the other goods below the given level in moment T. Our purpose may be formulated as the maximization problem of a Lagrange functional, where the boundary conditions are determined. The strategy which maximizes the functional is the optimal strategy in the sense of dynamic Pareto optimum. The necessary conditions for the maximum of this functional may be obtained from modified Pontriagin conditions. Beside the other conditions, they consist of differential equations describing the dynamics of the conjugate variables which may be treated as specific dual prices. Then the maximization of mentioned Lagrange functional with given boundary conditions may be treated as the maximization of weighted sum in interval (O, T), where the weights are the dual prices in moment T, when the other conditions are fulfilled. It may be proved that this problem is equivalent to the maximization of the hamiltonian which may be treated as the product (income) flow in every moment of planning interval; it is the weighted sum of current goals (particular good flows). We prove that the strategy maximizing the hamiltonian in every moment of planning period (O, T) is the strategy which corresponds to the dynamic Pareto optimum if only the weights in the hamiltonian are the conjugate variables obtained from the Pontriagin conditions. The maximum of hamiltonian in given moment may be treated as a static Pareto optimum in this moment.

Instytut Badań Systemowych PAN
Nakład 300 egz. Ark. wyd. 25,0. Ark. druk. 23,75. Papier druk. sat. kl. III 80 g 61×86. Oddano do składania 8 X 1976 Podpisano do druku w sierpniu 1978 r. Druk ukończono w sierpniu 1978 roku

CDW — Zakład nr 5 w Bielsku-Białej zam. 62/K/77 J-124