POLSKA AKADEMIA NAUK
INSTYTUT BADAŃ SYSTEMOWYCH

PROCEEDINGS OF THE 3rd ITALIAN-POLISH CONFERENCE ON APPLICATIONS OF SYSTEMS THEORY TO ECONOMY, MANAGEMENT AND TECHNOLOGY

WARSZAWA 1977

Redaktor techniczny
Iwona Dobrzyńska

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PREFACE

The present volume comprises the papers from the first Italian-Polish Conference which was held in Warszawa in May 1983 (30 to 3). The conference organized by the Centre of Control Systems and Computation of the Italian National Research Council in Rome and the Institute for Organization, Management and Control Sciences of Polish Academy of Sciences and the Ministry of Science, Higher Education and Technology in Warsaw was the third conference of this type.

The first one took place in Kraków (Poland) in 1972, the second in Pugnochiuso (Italy) in 1974.

This conference, like both the previous ones, gathered the leading representatives of many scientific centres from Italy as well as from Poland, representing also the different aspects of the systems analysis applied to the economic, management, data processing and technological systems.

From the past, such meetings enrich the exchange of views and experiences between the specialists from both countries what is the condition necessary to both sides. Among the several goals the systematization of the systems and control theory results, it is the common interest connected with the essential development of the quantitative methods and the particular applications of large computer systems.

The meaning of such meetings, however, goes far beyond this; what is happening during the conference between the established contacts are also fruitful in form of common scientific research works and — last but not least — begin the friendly links between the scientists of both countries which last for years.

The proceedings of the conference were divided into three parts:

1. Optimization and Control Theory;
2. Systems Theory in Economics;
3. Technological, Managerial and Information Systems.

While the first two parts are of rather homogeneous character, the third part contains the papers concerning the different types of models — for the economic, technological, management and data processing systems.

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SOME PROBLEMS OF OPTIMIZATION OF HIERARCHICAL STRUCTURES

1. INTRODUCTION

This paper takes up the subjects dealt with in an article by Mańczak [1] and in two doctoral dissertations which the author supervised in the years 1972 to 1975 [2], [4].

I shall illustrate the problem I am considering on an example communicated to me by prof. K. Mańczak in 1972.

Suppose we have one hundred healthy men for digging quickly a ditch. We assume for a moment that all these men are equally both capable for digging as for management. We take into account the fact that supervised people work better than not supervised.

We can organize the work in various ways.

First of all we can determine one boss supervising 99 workers.

We may determine a boss who supervises 9 foremen, each of them supervising 10 workers. The global efect will be the result of 90 workers.

We remark that global effect will decrease both when the number of supervisors is too small and too large, because in the latter case there are too few people actually digging the ditch.

We are interested in obtaining a maximal global effect by choosing the proper structure depending on the efficiency of the particular members of the team.

2. THE PROBLEM OF STATIC OPTIMIZATION OF HIERARCHICAL STRUCTURES

The present paper deals only with structures consisting of elements with characteristics of the type shown in Fig. 1.

If we assume that the controlled elements is a human, we can treat

y_i^o as the initiative of the i^{th} element,

y_i^m as the maximal potential of the i^{th} element,

y_i^∞ as the minimum value of the i^{th} element resulting from "over-control" e.g. in the form of frustration,

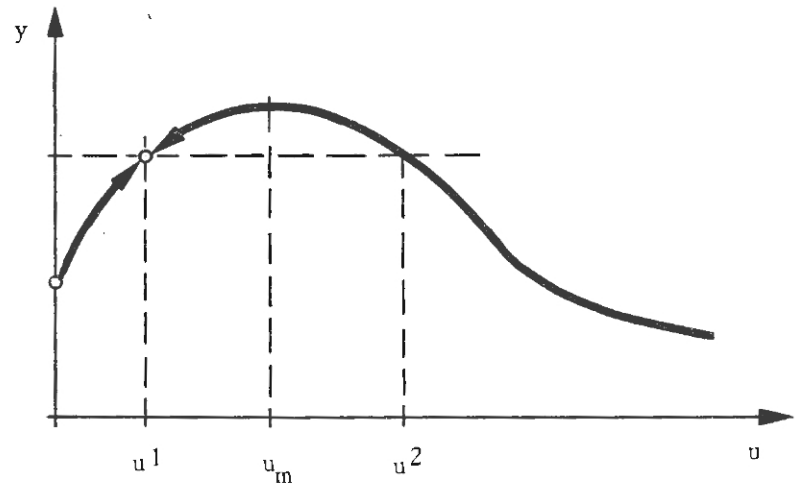


Fig. 1a

$c_i = \mathrm{tg}\ \alpha_i$ as controllability in the general sense i.e. derviative of efect with respect to cause.

The characteristics of elements we shall assume in the form:

$$y_i = y_i^{\infty} + (y_i^0 - y_i)\left[(a_i+1)\,\mathrm{e}^{-\alpha_i u_i} - a_i \mathrm{e}^{-2\alpha_i u_i}\right] \tag{1}$$

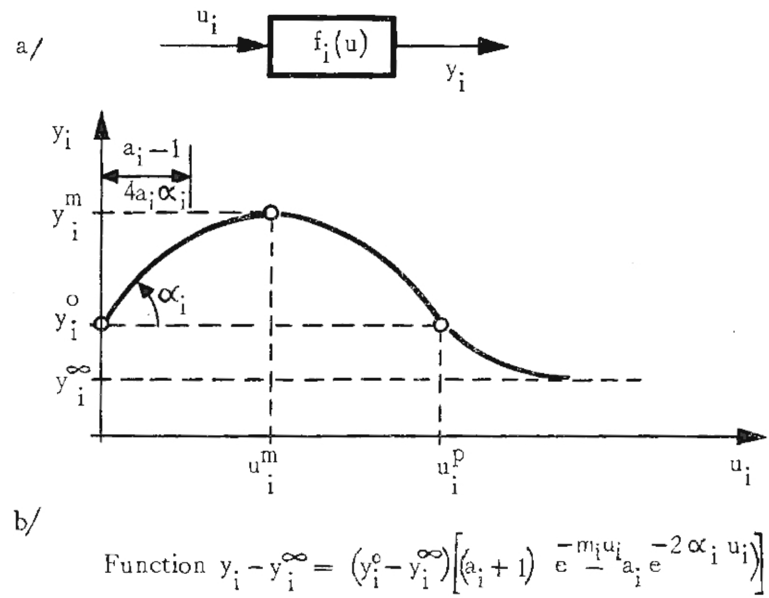


Fig. :

Table 1.

u	0	max point $u_m = \frac{1}{\alpha} \ln \frac{2a}{a+1}$	infl. point. $u_p = \frac{2}{\alpha} \ln \frac{2a}{a+1}$	∞
$\frac{y-y^\infty}{y^0-y^\infty}$	1	$\frac{(a+1)^2}{4a}$	$\frac{3(a+1)^2}{16a}$	0
$\frac{y'}{\alpha(y^0-y^\infty)}$	$a-1$	0	$-\frac{(a+1)^2}{8a}$	0
$\frac{y'}{a^2(y^0-y^\infty)}$	$1-3a$	$-\frac{(a+1)^2}{2a}$	0	0

Elements of this type approximate a fairly large class of real objects. Characteristic points of the curve defined by (1) are shown in Table 1. In the course of the argument we shall assume that

$$y_i^\infty \geqslant 0$$

$$y_i^0 > y_i^\infty$$

$$u_i \geqslant 0$$

$$\alpha_i > 0$$

$$a_i > 0$$

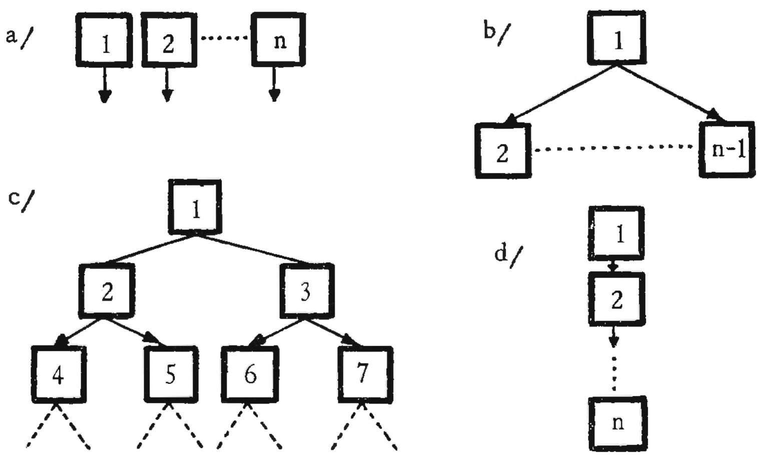


Fig. 2

Having n elements of the type described above we can couple them into different structures (Fig 2a, 2b, 2c, 2d). Each of the structures is characterized by different constraints. The problem consists in selecting the structure which is optimal with respect to a chosen performance index.

Let the performance index be the sum of outputs from the elements of the lowest level

$$F = \sum_{i=1}^{k} y_i \tag{2}$$

subject to constraints resulting from the type of structure. Our goal is to maximize F, which we can regard as productivity of the system. Note that in the case of structure "a" although all the elements contribute to the total productivity F, the elements are not controlled and therefore their individual productivities are minimal, resulting only from their own initiative.

On the other hand, in the case of structure "d" the productivity of the system equals the productivity of the only one element of the lowest level. Although the productivity of it can be maximal because of the control, the total productivity of the system is small, as only one element is engaged in the actual production. It follows from the above that the optimal structure lies somewhere between the two extremes, and the problem consists in finding the optimal number of levels of control and the optimal number of elements of each level subordinated to each element of a higher level. This number of elements in the course of the argument will be referred to as subordination number. We can also maximize the mean productivity "per capita"

$$E = \frac{F}{n} \tag{3}$$

First, we shall solve a simpler problem which is to find the optimal distribution of the control signal controlling k elements which belong to the same type or even are identical. It can be noted that the number of elements that can be controlled depends on the resources, i.e. on the value of control.

2.1. SOLUTION OF THE SUBPROBLEM

Let us consider the problem of distribution of the resources of a higher level element between the subordinated elements in the case of structure shown in Fig. 3. The elements are characterized by

$$\begin{aligned} & y_i = y_i^{\infty} + (y_i^0 - y_i^{\infty})\,[(a+1)\,\mathrm{e}^{-\alpha_i u_i} - a\,\mathrm{e}^{-2\alpha_i u_i}] \\ & y_i^0 > y_i^{\infty} \geqslant 0 \qquad i = 1, 2, \ldots k \\ & \alpha_i > 0 \\ & u_i \geqslant 0 \end{aligned} \tag{4}$$

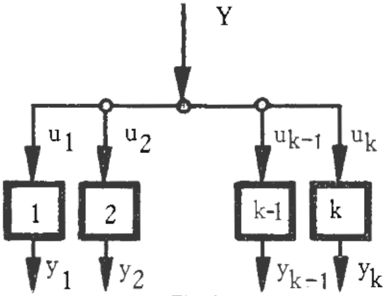


Fig. 3

For the sake of simplicity it is assumed that

$a_1 = a_2 = \dots = a_k = a > 1$

The goal is to maximize a scalar function of k variables

$$\max_{u_1, \dots u_k} \{F(u_1, \dots u_k)\} = \sum_{i=1}^{k} y_i^{\infty} + (y_i^0 - y_i)\,[(a+1)\,\mathrm{e}^{-\alpha_i u_i} - a^{2\alpha_i u_i}] \tag{5}$$

subject to the equality constraint

$$G(u_1, \dots u_k) = w_1 u_1 + w_2 u_2 + \dots + w_k u_k - Y = 0 \tag{6}$$

where $w_1, \dots, w_k > 0$ are weight functions regarding different costs of different controls,
Y is the total cost of control
and inequality constraints

$$\begin{matrix} u_1 \geqslant 0 \\ \vdots \\ u_k > 0 \end{matrix} \tag{7}$$

We shall arrange the elements as follows

$$\alpha_1(y_i^0 - y_i^{\infty}) \geqslant \alpha_2(y_2^0 - y_2^{\infty}) \geqslant \dots \geqslant \alpha_k(y_k^0 - y_k^{\infty}) \tag{8}$$

Let us define a Lagrangian function

$$L(u_1, \dots, u_k, \lambda) = F(u_1, \dots, u_k) + \lambda G(u_1, \dots, u_k) \qquad \lambda > 0 \tag{9}$$

It follows from the necessary conditions of optimality that

$$\frac{\partial L}{\partial u_i} = (y_i^0 - y_i^{\infty})\,[-\alpha_i(a+1)\,\mathrm{e}^{-\alpha_i u_i} + 2\alpha_i\, a\, \mathrm{e}^{-2\alpha_i u_i}] + \lambda w_i = 0 \tag{10}$$

$$\frac{\partial L}{\partial \lambda} = \sum_{i=1}^{k} w_i u_i - Y = 0 \qquad i = 1, 2, \dots k$$

The solutins of (10) are

$$u_i^m = \frac{1}{\alpha_i} \ln \frac{a+1+\sqrt{(a+1)^2 - 8a \dfrac{\lambda w_i}{\alpha_i(y_i^0 - y_i^\infty)}}}{\dfrac{2\lambda w_i}{\alpha_i(y_i^0 - y_i^\infty)}} \qquad i = 1, \dots k \tag{11}$$

or

$$u_i^m = \frac{1}{\alpha_i} \ln \frac{a+1-\sqrt{(a+1)^2 - 8a \dfrac{\lambda w_i}{\alpha_i(y_i^0 - y_i^\infty)}}}{\dfrac{2\lambda w_i}{\alpha_i(y_i^0 - y_i^\infty)}} \qquad i = 1, \dots k \tag{12}$$

For the solution to have a practical significance we shall assume that

$$(a+1)^2 \geqslant 8a \frac{\lambda w_i}{\alpha_i(y_i^0 - y_i^\infty)} \tag{13}$$

Then from the two possible solutions, the one that maximizes the function F is to be chosen.

For the sake of simplicity we shall assume further that the weight functions w_i are such that the inequalities (13) are satisfied as equalities. Then the solutions for u_i are unique

$$u_i^m = \frac{1}{\alpha_i} \ln \frac{\alpha_i(a+1)(y_i^0 - y_i^\infty)}{2\lambda w_i} \qquad i = 1, \dots k \tag{14}$$

Taking into account that resulting from (13)

$$w_i = \frac{\alpha_i(a+1)^2 (y_i^0 - y_i^\infty)}{8a\lambda} \qquad i = 1, \dots k \tag{15}$$

we arrive at

$$u_i^m = \frac{2}{\alpha_i} \ln \frac{2a}{a+1} \qquad i = 1, \dots k \tag{16}$$

(16) substituted in the constrain equation (6) gives

$$\lambda = \frac{(a+1)^2 \ln \dfrac{2a}{a+1} \displaystyle\sum_{i=1}^{k} (y_i^0 - y_i^\infty)}{4aY} \tag{17}$$

From (17) and (15) it follows that

$$w_i = \frac{\alpha_i(y_i^0 - y_i^\infty)}{2\ln\frac{2a}{a+1}\sum_{i=1}^{k}(y_i^0 - y_i^\infty)} Y \tag{18}$$

Then, finally, a controlled weighted resource is

$$u_i^m \cdot w_i = \frac{y_i^0 - y_i^\infty}{\sum_{i=1}^{k}(y_i^0 - y_i^\infty)} Y \tag{19}$$

Thus introducing the appropriate weight function we can find controls for elements with nonconvex characteristics identical with the controls for elements with convex characteristics.

Moreover, if the characteristics of all the elements are identical, then the obtained formulas are fairly simple

$$u^m = \frac{2}{\alpha}\ln\frac{2a}{a+1} \tag{20}$$

$$\lambda = \frac{(a+1)^2 \ln\frac{2a}{a+1}\, k(y^0 - y^\infty)}{4aY} \tag{21}$$

$$w = \frac{\alpha}{2k\ln\frac{2a}{a+1}} Y \tag{22}$$

$$u^m \cdot w = \frac{Y}{k} \tag{23}$$

2.2.2 THE OPTIMIZATION OF STRUCTURES CONSISTING OF IDENTICAL ELEMENTS WITH A CONSTANT SUBORDINATION NUMBER

Let us assume that all the elements of the system are identical and defined by

$$y_i = y^\infty + (y^0 - y^\infty)\left[(a+1)\,e^{-\alpha u_i} - a\,e^{-2\alpha u_i}\right] \tag{24}$$

The productivity of the system per element is

$$E = \frac{F}{n}$$

If we denote the subordination number by k, then for p levels of hierarchy

$$n = 1+k+k^2+ \dots +k^{p-1} = \frac{k^{p-1}}{k-1} \tag{26}$$

and the performance index of the system is

$$F = k^{p-1} y_{(p)} \tag{27}$$

If the number of levels diverges to infinity i.e. $p \to \infty$, we have

$$\lim_{p\to\infty} y_{(p)} = y = y^{\infty}+(y^0-y^{\infty})\left[(a+1)\,\mathrm{e}^{-\frac{\alpha y}{k}} - a\,\mathrm{e}^{-2\frac{\alpha y}{k}}\right] \tag{28}$$

and the productivity of the system per element

$$E_{(\infty)} = \lim_{p\to\infty} \frac{k^{p-1}(k-1)}{k^p-1}\, y = \frac{k-1}{k}\, y \tag{29}$$

Now we can find the value of k which maximizes the productivity of the system per element.

Differentiating (29) and (28) with respect to k we obtain respectively

$$\frac{dE_{(\infty)}}{dk} = \frac{1}{k^2}\, y + \frac{dy}{dk}\,\frac{k-1}{k} = 0 \tag{30}$$

$$\frac{dy}{dk} = \frac{(y^0-y^{\infty})\,\frac{\alpha}{k^2}\,\mathrm{e}^{-\alpha\frac{y}{k}}\, y\left[(a+1)-2a\,\mathrm{e}^{-\alpha\frac{y}{k}}\right]}{1+(y^0-y^{\infty})\,\frac{\alpha}{k}\,\mathrm{e}^{-\frac{\alpha y}{k}}\left[(a+1)-2a\,\mathrm{e}^{-\frac{\alpha y}{k}}\right]} \tag{31}$$

Substituing (31) in (30) and taking into account (28), after somewhat laborious calculations we obtain

$$y^{m}_{1,2} - y^{\infty} = \frac{(a+1)^2}{8a}\,(y^0-y^{\infty}) - \frac{1}{2\alpha} \pm \frac{a+1}{2}\sqrt{\frac{y^0-y^{\infty}}{2\alpha a} + \left[\frac{a+1}{4a}\,(y^0-y^{\infty})\right]^2} \tag{32}$$

$$k_{1,2} = \alpha y/\ln\left[\frac{a+1}{2a} \pm \sqrt{\left(\frac{a+1}{2a}\right)^2 - \frac{1}{a}\,\frac{y_{1,2}-y^{\infty}}{y^0-y^{\infty}}}\,\right]^{-1} \tag{33}$$

where k denotes the optimum value of subordination number.

We shall recapitulate the above argument with a qualitative analysis of the obtained solutions for control u. Note that, according to Fig. 1a, if the whole potential of the element is to be used it should operate near the maximum point. On the other hand, the maximum point is unstable and the element is uncontrolable, therefore, the selection of the maximum point is not desirable, and the question arises as to whether u^1 or u^2 should be chosen.

In the former case at the top of the hierarchy we have a certain resource Y the ratio of which to the control u^1 equals the subordination number k_1. Because $u^1 < u^2$, we obtain $k_1 > k_2$ at the same load (i.e. productivity y_i) of particular elements. Therefore the productivity of the system $E_1 = \frac{k_1 - 1}{k_1} y_1$ will be greater than $E_2 = \frac{k_2 - 1}{k_2} y_1$, but at the same time the total number of elements of the system I will by far exceed that of the system II, for $n_1 = \frac{k_1^{p-1}}{k_1 - 1} \gg \frac{k_2^{p-1}}{k_2 - 1}$. On the other hand, the point u^1 has the following convenient properties:

1. It is a stable point
2. We have some reserve productivity, i.e. we can increase the productivity by increasing the control u.

Selection of u^2 entails the occurence of opposite effects. We shall now consider the case discussed by Mańczak in [1].

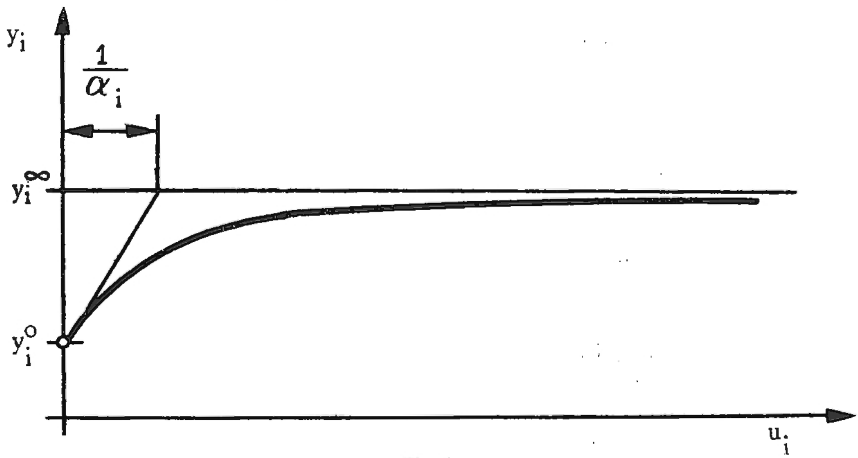


Fig. 4

Let us assume that $a_i = 0$ in (1). Then the characteristics of the elements are shown in Fig 4, and we can rewrite the curve equation as

$$y_i = y_i^\infty - (y_i^\infty - y_i^0) e^{-\alpha_i u_i} \tag{34}$$

And now, contrary to the previous case,

$$y_i^\infty > y_i^0$$

and

$$\alpha_i > 0 \tag{35}$$

$$u_i \geqslant 0$$

Let us assume moreover that in this particular case all the weight functions are identical and equal 1

$$w_1 = w_2 = \dots = w_k = 1 \tag{36}$$

We shall arrange the elements in such a way that

$$\alpha_1 \leqslant \alpha_2 \leqslant \dots \leqslant \alpha_k \tag{37}$$

Then we can formulate the problem of partial optimization.
The goal is to maximize the function

$$F(u_1, u_2, \dots u_k) = \sum_{i=1}^{k} y_i^{\infty} - \sum_{i=1}^{k} (y_i^{\infty} - y_i^0) e^{-\alpha_i u_i} \tag{38}$$

(which can represent for example the distribution of a superviser's time between his coworkers, or distribution of resources).
subject to the constraints

$$G(u_1, \dots u_k) = \sum_{i=1}^{k} u_i - Y = 0 \tag{39}$$

$$u_i \geqslant 0 \qquad i = 1, \dots k$$

We define a Lagrangian function

$$L(u_1, \dots u_k, \lambda) = F(u_1, \dots u_k) - \lambda G(u_1, \dots u_k)$$

From the necessary conditions of optimality it follows that

$$u_i = \frac{1}{\alpha_i} [\ln(y_i^{\infty} - y_i^0) - \ln \lambda] \tag{40}$$

which, substituted in (39), gives

$$\ln \lambda = \frac{\displaystyle\sum_{i=1}^{k} \frac{1}{\alpha_i} \ln(y_i^{\infty} - y_i^0) - Y}{\displaystyle\sum_{i=1}^{k} \frac{1}{\alpha_i}} \tag{41}$$

Then substituting (41) in (40) and taking into account a possibility of a boundary solution we obtain finally that if

$$Y > \sum_{i=1}^{k} \frac{1}{\alpha_i} \ln \frac{y_i^{\infty} - y_i^0}{y_k^{\infty} - y_k^0} \tag{42}$$

then the solutions are

$$u_j = \frac{1}{\alpha_j} \frac{\sum_{i=1}^{k} \frac{1}{\alpha_i} \ln \frac{y_j^\infty - y_j^0}{y_i^\infty - y_i^0}}{\sum_{i=1}^{k} \frac{1}{\alpha_i}} \qquad j = 1, 2 \ldots k \tag{43}$$

and if

$$\sum_{i=1}^{l} \frac{1}{\alpha_i} \ln \frac{y_i^\infty - y_i^0}{y_l^\infty - y_l^0} < Y < \sum_{i=1}^{l+1} \frac{1}{\alpha_i} \ln \frac{y_i^\infty - y_i^0}{y_{l+1}^\infty - y_{l+1}^0} \tag{44}$$

then the solutions are

$$u_j = \begin{cases} \dfrac{1}{\alpha_j} \dfrac{\sum_{i=1}^{l} \frac{1}{\alpha_i} \ln \frac{y_j^\infty - y_j^0}{y_i^\infty - y_i^0} + Y}{\sum_{i=1}^{l} \frac{1}{\alpha_i}}, & j = 1, 2 \ldots, l \\ 0 \qquad , & j = l+1, \ldots, k \end{cases} \tag{45}$$

In the case of identical elements the inequality (42) is always satisfied and the solutions given by (43) can be rewritten as

$$u_j = \frac{Y}{k} \qquad j = 1, 2, \ldots k \tag{46}$$

Again, like in the previous case, we can calculate the optimal productivity per element.

In general, it can be noted that in structures with a small subordination number k there are many levels of hierarchy and many controller — elements, and the number of elements engaged in the actual production is relatively small, although they are well controlled.

On the contrary, in structures with a big k, there are few levels of hierarchy and few controller — elements, and although there are many immediately productive elements they may be not sufficiently controlled; therefore the productivity of the system per element may be low. For these reasons we expect that there exists an optimal subordination number k.

It can be easily calculated for structures with the number of levels $p \to \infty$ and the total number of elements $n \to \infty$.

Like in the previous case

$$E_{(\infty)} = \frac{k-1}{k} y \tag{47}$$

$$y = y^{\infty} - (y^{\infty} - y^{0}) \, e^{-\alpha \frac{y}{k}} \tag{48}$$

Differentiating $E_{(\infty)}$ with respect to k we obtain

$$\frac{dE_{(\infty)}}{dk} = \frac{1}{k^2} y + \frac{dy}{dk} \frac{k-1}{k} = 0 \tag{49}$$

Differentiating (48) with respect to k we have

$$\frac{dy}{dk} = \frac{(y^{\infty} - y^{0}) \dfrac{\alpha}{k^2} e^{-\alpha \frac{y}{k}} \cdot y}{(y^{\infty} - y^{0}) \dfrac{\alpha}{k} e^{-\alpha \frac{y}{k}} - 1} \tag{50}$$

Substituing (50) in (49) we have

$$\frac{1}{k^2} y + \frac{k-1}{k} \frac{(y^{\infty} - y^{0}) \dfrac{\alpha}{k^2} e^{-\alpha \frac{y}{k}} \cdot y}{(y^{\infty} - y^{0}) \dfrac{\alpha}{k} e^{-\alpha \frac{y}{k}} - 1} = 0 \tag{51}$$

from which it follows that

$$e^{-\alpha \frac{y}{k}} = \frac{1}{(y^{\infty} - y^{0}) \alpha} \tag{52}$$

Then substituting (51) in (48) we obtain

$$y_{opt} = y^{\infty} - \frac{1}{\alpha} \tag{53}$$

which, substituted in (52), gives

$$k_{opt} = \frac{\alpha y^{\infty} - 1}{\ln [(y^{\infty} - y^{0}) \alpha]} \tag{54}$$

and

$$E_{(\infty) opt} = y^{\infty} - \frac{1}{\alpha} \ln [(y^{\infty} - y^{0}) \alpha] - \frac{1}{\alpha} \tag{55}$$

If we consider k_{opt} as a function of α, then k_{opt} is extremal at α_0 defined by the equation

$$\alpha_0 \, y^{\infty}(1 - \ln \alpha_0 (y^{\infty} - y^{0})) = 1 \tag{56}$$

and then

$$k_{opt}(\alpha_0) = \alpha_0 \, y^{\infty} \tag{57}$$

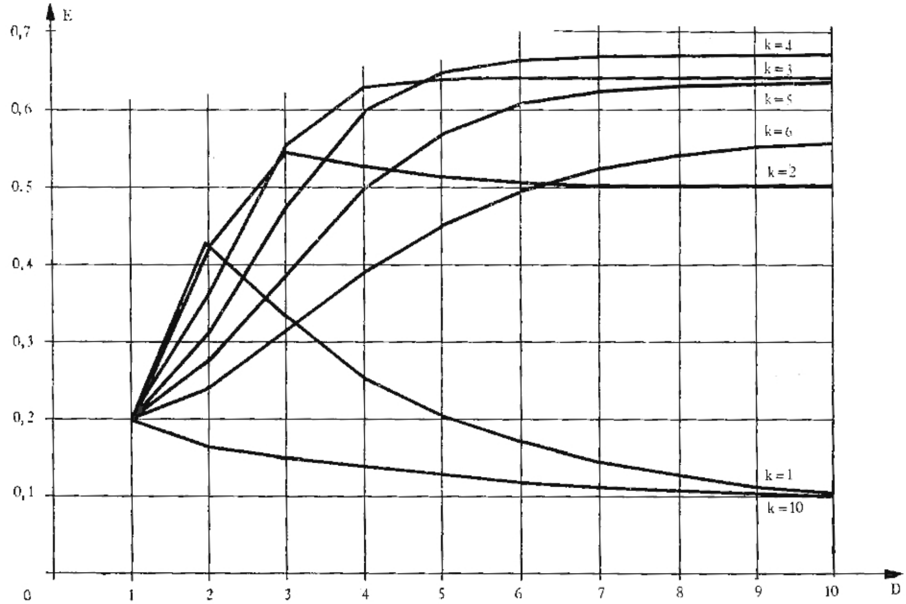


Fig. 5

LEVEL	STRUCTURE			
	I	II	III	IV
1	1	1	1	1
2	1	1	2	2
3	2	3	2	3
4	3	3	3	3
5	3	4	3	4
6	4	4	4	4
7—∞	4	4	4	4

Fig. 6

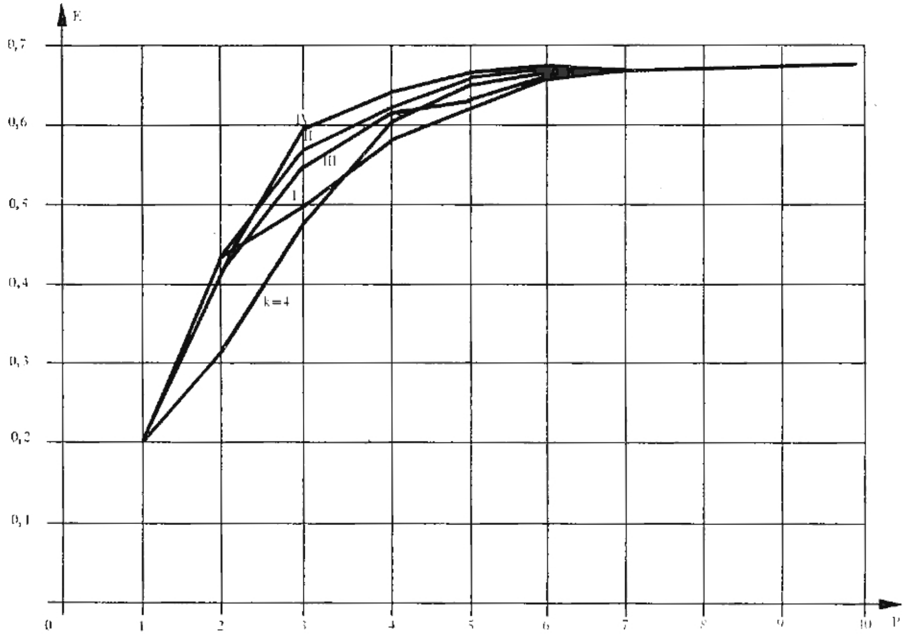
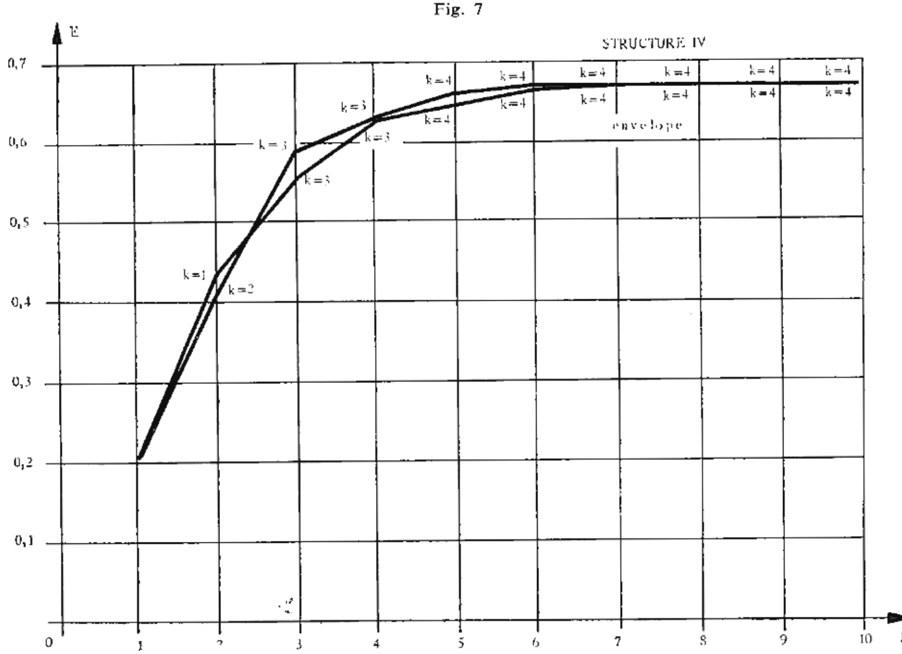

E
0,7
0,6
0,5
0,4
0,3
0,2
0,1
0
1
2
3
4
5
6
7
8
9
10
P
IV
II
III
I
k=4

Fig. 7


E
STRUCTURE IV
0,7
0,6
0,5
0,4
0,3
0,2
0,1
0
1
2
3
4
5
6
7
8
9
10
P
k=1
k=2
k=3
k=3
k=3
k=3
k=4
k=4
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k=4
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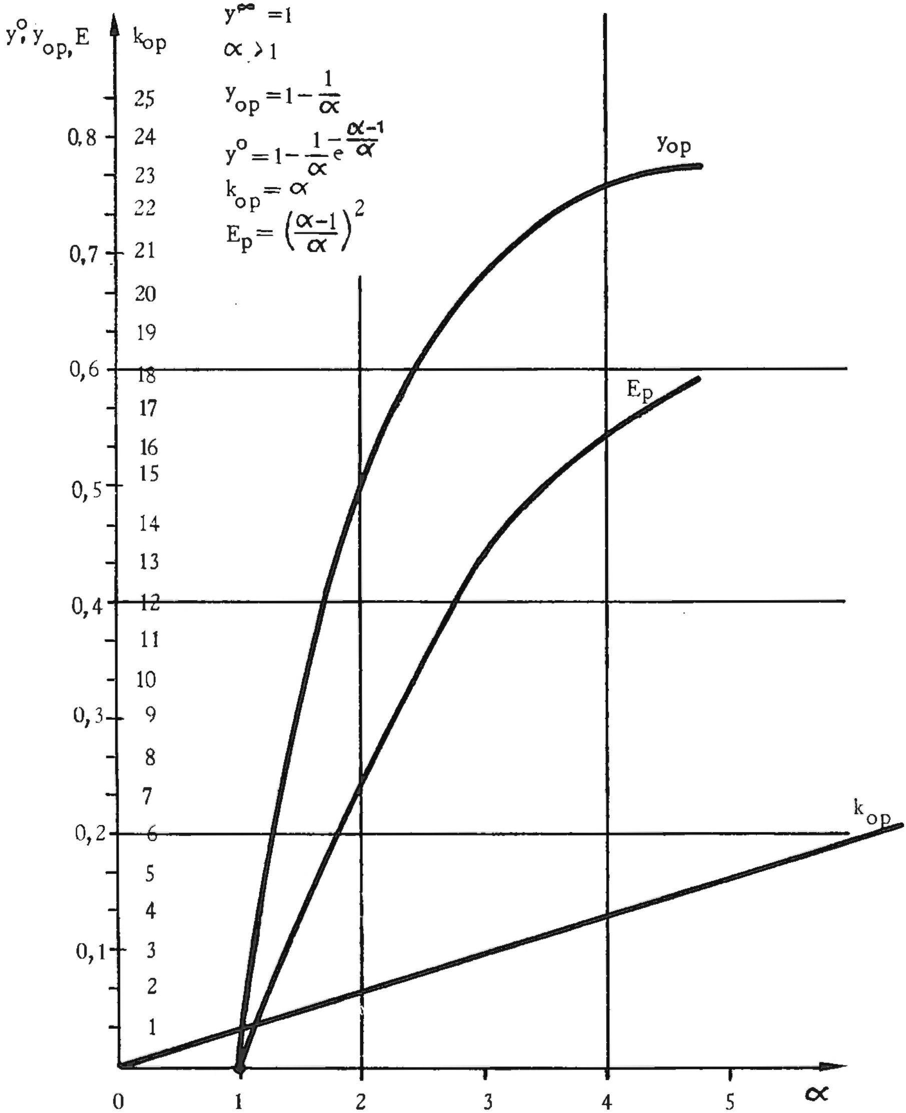
k=4
k=4
k=4
envelope

Fig. 8


$y^{\infty} = 1$
$\alpha > 1$
$y_{op} = 1 - \frac{1}{\alpha}$
$y^{o} = 1 - \frac{1}{\alpha} e^{-\frac{\alpha-1}{\alpha}}$
$k_{op} = \alpha$
$E_p = \left(\frac{\alpha-1}{\alpha}\right)^2$
y^o, y_{op}, E
k_{op}
y_{op}
E_p
k_{op}
α
0,1
0,2
0,3
0,4
0,5
0,6
0,7
0,8
1
2
3
4
5
6
7
8
9
10
11
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0
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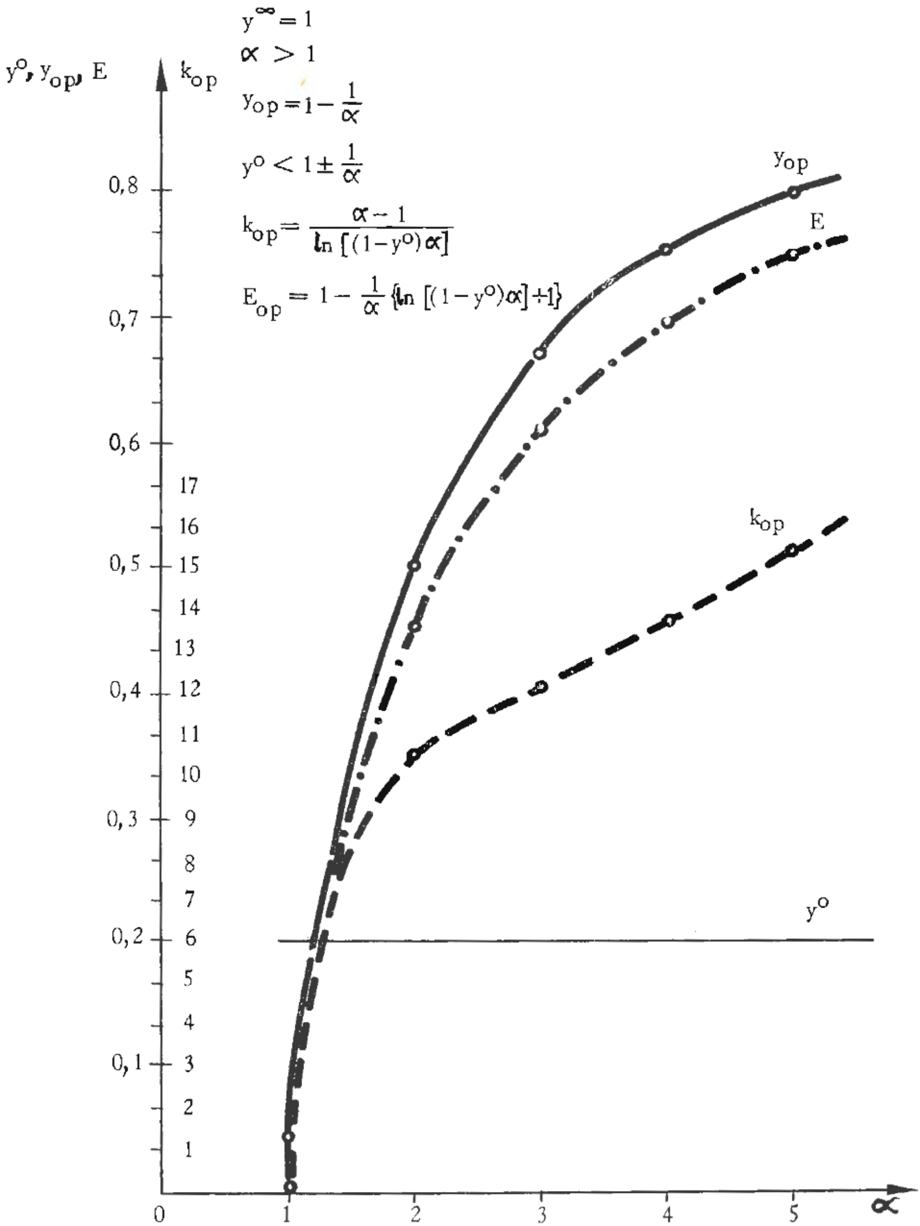
4
5
Fig. 9



$y^{\infty} = 1$
$\alpha > 1$
$y_{op} = 1 - \frac{1}{\alpha}$
$y^{o} < 1 \pm \frac{1}{\alpha}$
$k_{op} = \frac{\alpha - 1}{\ln[(1-y^{o})\alpha]}$
$E_{op} = 1 - \frac{1}{\alpha}\{\ln[(1-y^{o})\alpha] + 1\}$
y^{o}, y_{op}, E
k_{op}
0,8
0,7
0,6
0,5
0,4
0,3
0,2
0,1
0
17
16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
2
3
4
5
α
y_{op}
E
k_{op}
y^{o}

Fig. 10

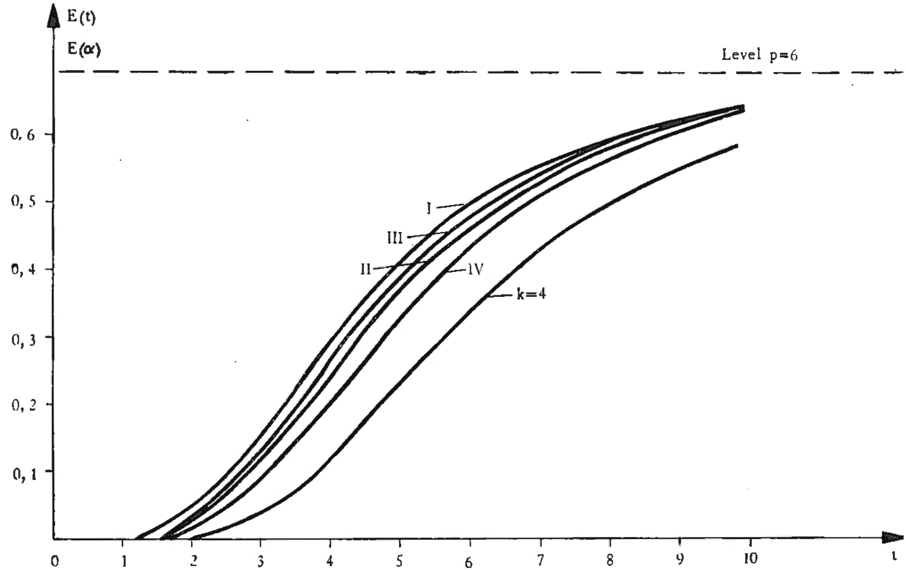


Fig. 11

The optimum value k_{opt} with respect to y^{∞} is reached at y^{∞} satisfying the equation

$$[(y_0^{\infty} - y^0)\,\alpha]^{(y_0^{\infty} - y^0)\,\alpha} = e \tag{58}$$

then

$$k_{opt}(y_0^{\infty}) = (\alpha y_0^{\infty} - 1)\,(y_0^{\infty} - y^0)\,\alpha \tag{59}$$

The above stages of the argument apply to structures with a constant subordination number. Computer simulations show that it is possible to obtain better results with structures with a variable subordination number. The computer results are shown in Fig. 5, 6, 7, 8. For particular types of elements the dependence of the productivity of the system E on the subordination number, control y_{opt}^{2} and parameters of elements i.e. controllability α and initiative y^0 is shown.

3. PROBLEMS OF THE DYNAMIC OPTIMIZATION OF HIERARCHICAL STRUCTURES

3.1. INTRODUCTION

The main shortcoming of static optimal structures is that we ignore the time period which is necessary to carry out any action. Presently we shall include the time factor in the considerations, both in characteristics of elements and

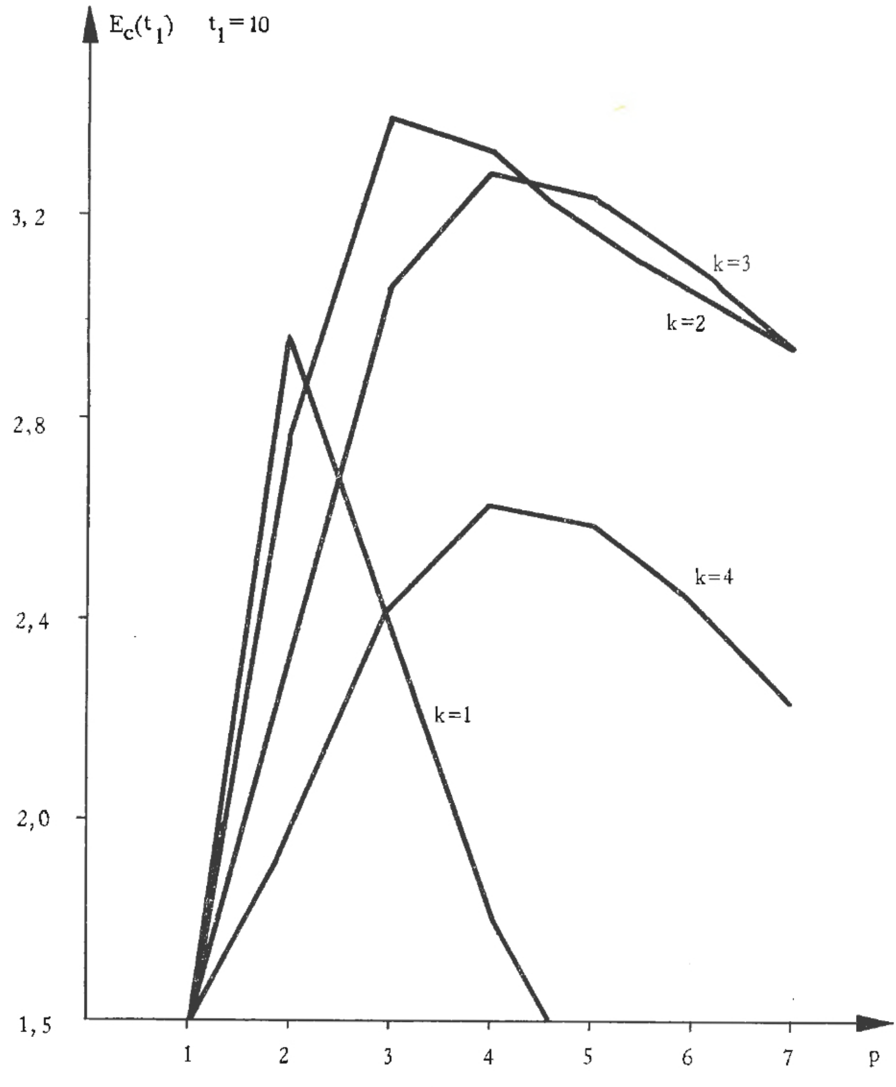


Fig. 12

in maximized functional. As it can be seen later, the chosen time horizon, i.e. the time period in which the task is to be completed, motivates the particular structure of the system. The obtained optimal subordination numbers will depend on the time horizon and in consequence they will differ from those of the optimal structures for static systems.

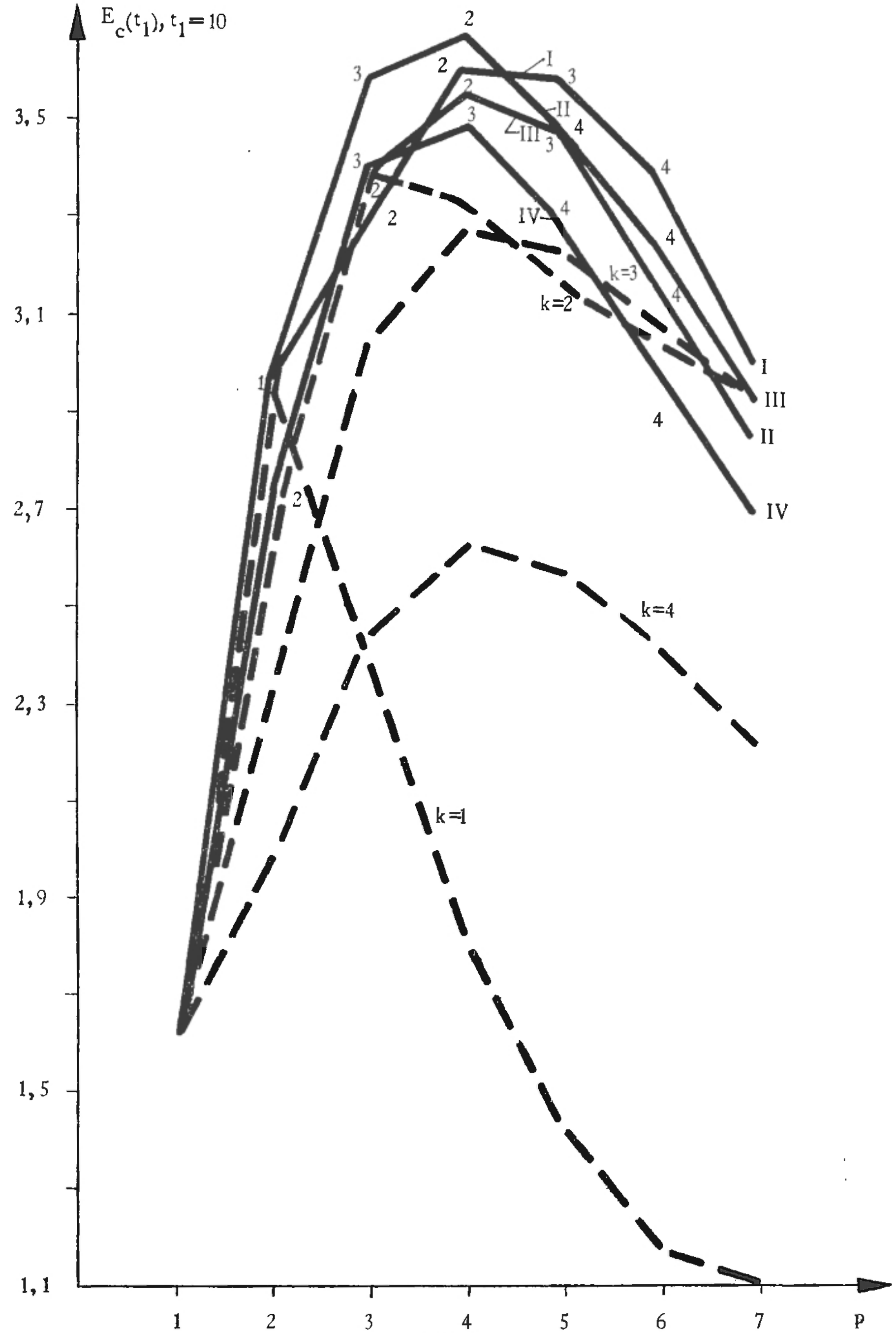


Fig. 13

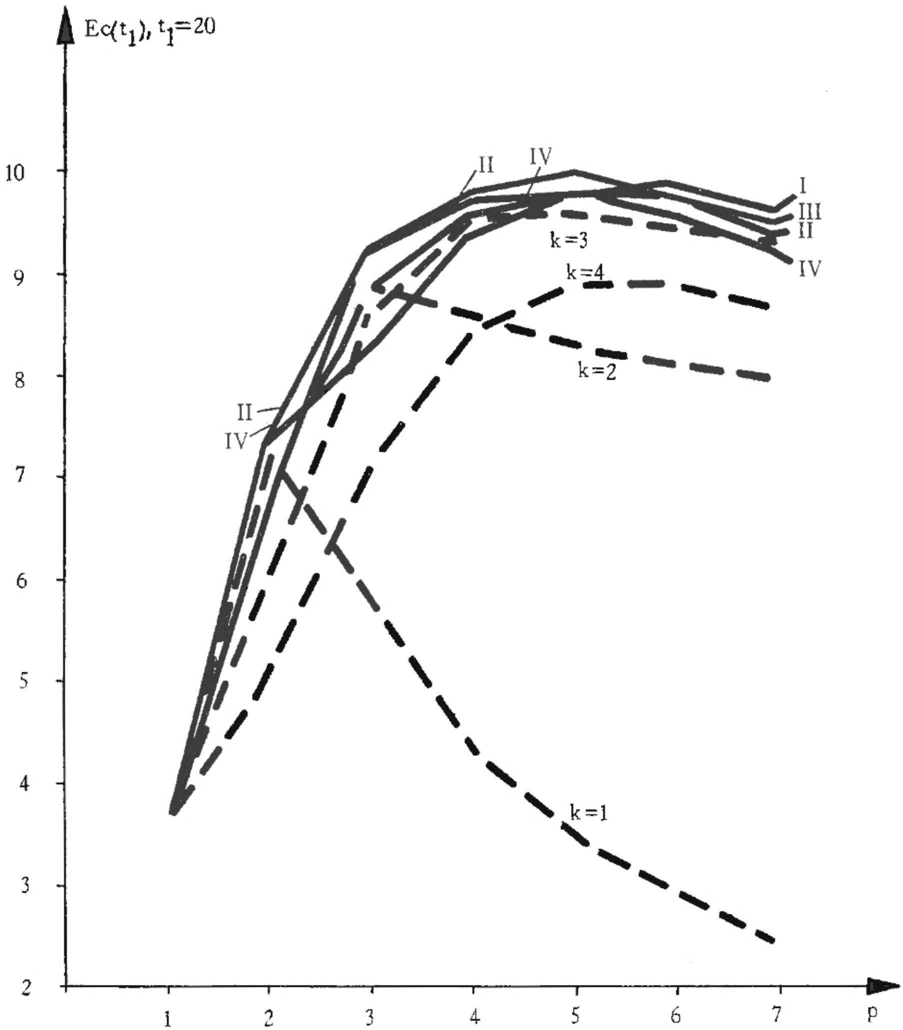


Fig. 14

3.2. PROBLEM STATEMENT

We shall confine our considerations to structures consisting of elements with nonlinear static characteristics somewhat simpler than the ones considered in the case of static optimization. We shall assume that $y_i^0 = 0$, then

$$f_i(u_i) = y_i^{\infty}(1 - e^{-\alpha_i u_i}) \tag{60}$$

Moreover, we shall assume that elements have linear dynamic characteristics of the first order

$$T_i \frac{dy_i(t)}{dt} + y_i(t) = f_i(u_i(t)) \qquad i = 1, \dots k \tag{61}$$

We shall allow for time delays in the control functions

$$T_i \frac{dy_i(t)}{dt} + y_i(t) = f_i(u_i(t-\tau_i)) \qquad i = 1, \dots k \tag{62}$$

Like in the previous case, we shall assume that the immediately productive elements are those on the lowest level. The performance index of the system will be

$$F(t_1) = \int_0^{t_1} \sum_{i \in p} y_i(t)\, dt \tag{63}$$

where I_p is the set of indexes of elements of the lowest lewel.

The problem of global dynamic optimization can be split into two mutually related subproblems.

Subproblem 1

Find the optimal controls of the system for a given structure a $\in \pi$ and a given performance index

$$F_c(s, t_1) = \max_{u(t) \in G(t)} \{F_c(y(u(t)) : s : t_1)\} \tag{64}$$

where π is the set of admissible structures

$$y(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_k(t) \end{pmatrix}$$

$$u(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_k(t) \end{pmatrix}$$

$G(t)$ is the set of admissible controls,
t_1 is the performance time

Subproblem 2

Choose the optimal structure from the set of admissible structures for which controls were optimized with respect to (64). The choisce is based on the following criterion

$$F_0(t_1) = \max_{s \in \pi} F_0(s : \hat{u}(t), t_1)() \tag{65}$$

3.3. THE SOLUTION OF SUBPROBLEM 1, CONTROL OPTIMIZATION

Let us consider the problem of optimization of control of k dynamic elements on the same level.

The goal is to find the control, maximizing the functional

$$\hat{u}(t): \max_{u(t) \in \Omega} F_0(u(t)) \tag{66}$$

where

$$F_0(u(t)) = \int_0^{t_1} \sum_{i=1}^{k} y_i(u_i(t))\, dt \tag{67}$$

Ω is the set of admissible controls

$$\Omega = \{u(t) : G_w(t) \geqslant 0\}$$

$$G_w(t) = \{G_{w,i}(t) \geqslant 0, \quad i = 1, \dots k+1\} \tag{68}$$

$$G_{w,1}(t) = y(t) - \sum_{i=1}^{k} u_i(t) = 0 \tag{69}$$

$$G_{w,i}(t) = u_i(t) \geqslant C, \quad i = 1, 2, \dots k \tag{70}$$

We shall solve the problem examining the Lagrangian functional

$$\Phi(\boldsymbol{u}, \lambda) = F(\boldsymbol{u}) + \lambda[G(\boldsymbol{u})] \tag{71}$$

where λ is a nonnegative linear functional defined on the set of values of the constraint operator $G(u)$ i.e. on the space

$$\lambda(G(u)) = \int_0^{t_1} \sum_{i=1}^{l} G_i(\boldsymbol{u}) \cdot \mu_i(t)\, dt$$

the constraint functions $g(t) \in L^p\,[0, t_1]$ and $\mu_i(t) \in L^2\,[0, t_1)$,

$$u(t) \in U = L^2[0, t_1] \times \dots \times L^2[0, t_1]$$

U is a Banach space being the k Cartesian product of $L^2[0, t_1]$ with the form

$$\|u(t)\| = \sqrt{\sum_{i=1}^{k} \|u_i(t)\|^2} \tag{72}$$

where

$$\|u_i(t)\| = \{\int_0^{t_1} [u_i(t)]^2\, dt\}^{\frac{1}{2}} \tag{73}$$

We shall base on a theorem formulated by R. Kulikowski [3].

Theorem

Let a functional $F : U \to R$ and operator $G : U \to \Gamma$ be concave and differentiable in the Gateaux sense on the space $\boldsymbol{U}$, where U and Γ are Banach spaces.

If there exist $\hat{u} \in U$ and a nonnegative linear functional λ such that for all $u \in U$

$$d_h \Phi(\hat{u}, \lambda) = 0$$

$$G(u) \geqslant 0$$

$$\lambda[G(u)] = 0$$

then the functional $F(\hat{u})$ takes on its maximum value at the point $\hat{u}$ with the condition $G(\hat{u}) \geqslant 0$ satisfied. By $d_b \Phi(\hat{u}, \lambda)$ we denote the Gateaux derivative at the point $\hat{u}$ in the direction h, where h, $\hat{u} \in U$.

For a system without delays

from equation (74), in virtue of (71), (67) and (69), it follows that

$$\hat{u}_i(t) = \frac{1}{\alpha_i} \ln \frac{y_i^\infty \alpha_i}{\hat{u}(t)} \left(1 - e^{\frac{t - t_1}{T_i}}\right) \tag{77}$$

$$i = 1, \ldots, k$$

and on the basis of (69) we have

$$-\ln \hat{\mu}(t) = \frac{y(t) - \sum_{i=1}^{k} \frac{1}{\alpha_i} \ln y_i^\infty \alpha_i \left(1 - e^{\frac{t - t_1}{T_i}}\right)}{\sum_{i=1}^{k} \frac{1}{\alpha_i}} \tag{78}$$

Substituting (78) in (77) we obtain

$$\hat{u}_j(t) = \frac{1}{\alpha_j \sum_{i=1}^{k} \frac{1}{\alpha_i}} \left\{ y(t) + \sum_{i=1}^{k} \frac{1}{\alpha_i} \ln \frac{y_j^\infty \alpha_j \left(1 - e^{\frac{t - t_1}{T_j}}\right)}{y_i^\infty \alpha_i \left(1 - e^{\frac{t - t_1}{T_i}}\right)} \right\} \qquad \begin{array}{l} j = 1, 2 \ldots k \\ t \in [0, t_1] \end{array} \tag{79}$$

Passing to the limit as $t \to t_1$ we have

$$\hat{u}_j(t_1) = \lim_{t \to t_1} \hat{u}_j(t) = \frac{1}{\alpha_j \sum_{i=1}^{k} \frac{1}{\alpha_i}} \left[y(t) + \sum_{i=1}^{k} \frac{1}{\alpha_i} \ln \frac{y_j^\infty \alpha_j T_i}{y_i^\infty \alpha_i T_j} \right] \tag{80}$$

where $y(t)$ is the output from the superior element.

We obtained the solution of subproblem 1 i.e. we obtained optimal controls.

It must be noted that we ignored the inequality constraints $u_j(t) > 0, j = 1, \dots$ $\dots k$. If the formulas (79) and (80) yield negative controls on a certain interval of time, then to satisfy the constraints the controls on the interval should be taken as equalling zero. The other controls are then calculated from (79) and (80), with the terms with the indexes of the negative controls neglected.

Proceeding similarly we can solve subproblem 1 for systems with delays

$$\hat{u}_j(t) = \frac{1}{\alpha_j \sum_{i=1}^{m_i} \frac{1}{\alpha_i}} \left[y(t) + \sum_{i=1}^{m_i} \frac{1}{\alpha_i} \ln \frac{y_j^{\infty} \alpha_j \left(1 - e^{\frac{t+\tau_j - t_1}{T_j}}\right)}{y_i^{\infty} \alpha_i \left(1 - e^{\frac{t+\tau_i - t_1}{T_i}}\right)} \right] \tag{81}$$

where $j = 1, \dots m_i$

$$t \in [t_1 - \tau_{m_{i+1}}, t - \tau_{m_i})$$

$$\tau_{m_{m+1}} \stackrel{df}{=} t_1$$

and

$$\hat{u}_j(t) = 0 \begin{cases} m_i < j \leqslant k = m_m \\ t \in [t_1 - \tau_{m_{i+1}}, t_1 - \tau_{m_i}) \end{cases} \tag{82}$$

The formulas (81) and (82) define the coordinates of the optimal control vector in the time intervals

$$[0, t_1 - \tau_{m_m}), [t_1 - \tau_{m_m}, t_1 - \tau_{m_{m-1}}), \dots [t_1 - \tau_{m_2}, t_1 - \tau_{m_1})$$

sequentially. Thus the optimal control is defined on the interval $(0, t_1 - \tau_{m_1})$, where τ_{m_1} is the least and τ_{m_m} the greatest of the delays of the elements considered in subproblem 1.

If all the delays are identical then the set of indexes m_i contains only one element $m_m = k$ and the control can be calculated from the formula (81) for the whole interval $[0, t_1 - \tau)$ at a time. The values of the control at the right endpoints of the intervals are obtained in a similar way

$$\hat{u}_j(t_1 - \tau_{m_1}) = \begin{cases} \dfrac{1}{\alpha_j \sum_{i=1}^{m_1} \frac{1}{\alpha_i}} \left[y(t_1 - \tau_{m_1}) + \sum_{i=1}^{m_1} \frac{1}{\alpha_i} \ln \frac{y_j^{\infty} \alpha_j \; T_i}{y_i^{\infty} \alpha_i \; T_j} \right] \\ \qquad\qquad\qquad\qquad j = 1, \dots, m \\ 0 \quad , \quad \text{for } m_1 < j \leqslant k \end{cases} \tag{83}$$

The inequality constraints are to be satisfied in the way described above.

If all k elements are identical, then all the coordinates of the control vector are identical and defined by

$$\hat{u}_j(t) = \frac{y(t)}{k} \qquad j = 1, \dots k \tag{84}$$

Passing to the limit as $t_1 \to \infty$ we obtain for controls the same results an in the case of static optimization.

The solution of subproblem 2. Choosing the optimal structure.

Investigation was carried out both for constant and variable subordination number.

The performance index of the system is

$$F_0(t_1) = \int_0^{t_1} F(t)\,dt = \int_0^{t_1} n_p \cdot y_p(t)\,dt \tag{85}$$

where n_p is the number of elements on the lowest level

y_p is the output from one of the elements on level P.

Controls inside the system are defined by

$$u_p(t) = \frac{y_{p-1}(t)}{k} \qquad p = 1, \dots P \tag{86}$$

In order to find the values $y_p(t)$, in (86), for all $t \in [0, t_1]$, $p = 1, \dots P$ as the response to the input $u_1(t)$ of the element being at the top of the hierarchy one has to solve the set of P differential equations

$$T\frac{dy_p(t)}{dt} + y_p(t) = y^{\infty} - y^{\infty}e^{-\alpha u_p(t)} \qquad p = 1, \dots, P \tag{87}$$

with (86) satisfied.

The productivity of the system per element is given by the formula

$$E_c(t_1) = \frac{1}{n}\,F_c(t_1) = \frac{1}{n}\int_0^{t_1} F(t)\,dt \tag{88}$$

The examination of the system and search for the optimal structure, i.e. a structure at which the functional (88) takes on its maximum value, were carried out using programs for a digital computer Odra 1304 at the Institute of Computer Science and Control Engineering.

The results obtained are shown in the enclosed diagrams.

It appeared that the obtained structures were essentially different from the optimal structure in the static case. They depend significantly on the time horizon t_1.

FINAL REMARKS

1. Global efficiency of static systems practically does not change if number of levels exceeds 5—6. Futher enlargment of the number of levels is useless.

2. Optimal subordination number k strongly depends on the coefficient of controlability α.

3. Structures with changeable subordination number k are better than structures with constant subordination number k and the higher is the level the smaller should be the subordination number k.

4. In dynamic systems number of levels is smaller than in corresponding static systems ($p=3$—4 is sufficient) and depends strongly on time horizon of work t_1.

5. The optimal control strongly depends on time constants T_i and time--delay τ_i but only in final interwals of optimization and does not depend on T_i or τ_i in initial interwals because the values of exponential functions $\exp\left(\frac{t+\tau_j-t_1}{T_j}\right)$ are close to zero. When time-horizon $t_1 \to \infty$ then the control does not depend on T_j and τ_j and the problem tends to have static character.

6. The problem of finding optimal dynamic structures is a very complicated one and can be solved only by using digital computers.

POSSIBLE EXTENSIONS

1. Various characteristics of the elements.
2. Various numbers of elements on each level.
3. Higher order (possibly nonlinear) differential equations of the elements.
4. Various global criterion functions.
5. Various local criterion function for each level.
6. Different time horizons for each particular level.
7. Feed-backs between levels.

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SUMMARY

The paper deals with the problem of optimization of hierarchical structures. The elements of structures are described, either by nonlinear static characteristics or by dynamic characteristics. Depending on the kind of characteristics the problem of choice of optimal static or dynamic system is being considered. The depandences of the number of hierarchical levels and the subordination numbers in optimal structures on the characteristics parameters of the system elements and on the optimization horizon are investigated. As the optimization criterion an average system effectiveness computed on the lowest level of system is taken.

The algorithms and programs enabling the optimization of system structure are prepared.

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