POLSKA AKADEMIA NAUK
INSTYTUT BADAŃ SYSTEMOWYCH

PROCEEDINGS OF THE 3rd ITALIAN-POLISH CONFERENCE ON APPLICATIONS OF SYSTEMS THEORY TO ECONOMY, MANAGEMENT AND TECHNOLOGY

WARSZAWA 1977

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PREFACE

The present volume comprises the papers from the 3rd Italian-Polish Conference which was held in Mądralin near Warsaw from 1980 to 11. The conference organized by the Centre of Control Systems and Computation of the Italian National Research Council in Rome and the Institute for Organization, Management and Control Sciences of Polish Academy of Sciences and the Ministry of Science, Higher Education and Technology in Warsaw was the third conference of this type.

The first one took place in Kraków (Poland) in 1972, the second in Pugnochiuso (Italy) in 1974.

This conference, like both the previous ones, gathered the leading representatives of many scientific centres from Italy as well as from Poland, representing also the different aspects of the systems analysis applied to the economic, management, data processing and technological systems.

From the past, such meetings enrich the exchange of views and experiences between the specialists from both countries what is the Institute especially useful to both sides, above the natural part, the intensification of the systems and control theory, specially in the recent decades contributed to the essential development of the quantitative methods and the particular applications of large complex systems.

The meaning of such meetings, however, goes far beyond that what is happening during the conference because the established contacts are also fruitful in form of common scientific research works and — last but not least — form the friendly links between the scientists of both countries which last for years.

The proceedings of the conference were divided into three parts:

1. Optimization and Control Theory;
2. Systems Theory in Economics;
3. Technological, Management and Information Systems.

While the first two parts are of rather homogeneous character, the third part contains the papers concerning the different types of models — for the economic, technological, management and data processing systems.

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ON A NEW FREQUENCY-RESPONSE APPROACH TO THE SYNTHESIS OF MULTIINPUT-MULTIOUTPUT LINEAR CONTROL SYSTEMS

1. INTRODUCTION

In spite of various attempts and proposals devoted to the creaction of the synthesis method of multivariable linear control systems there is still a lack of relatively simple engineering method wich could be compared to the well--known methods being applied to the single-input-single-output systems. The object of the paper is to present a new proposal concerned with such a method for multiinput — multioutput systems based on the frequency-response approach. Assuming the controller of the diagonal form the aim of synthesis is to ensure the system stability and sufficiently small steady-state control errors in each loop (or sufficiently large disturbance dampings in the system) over the prescribed frequency band. The idea of the method proposed is to divide the system into a number of noninteracting loops equivalent to the system under consideration and then to synthesize them sequentially by means of simple, well-known frequency methods. It has been proved that the control system synthesized in this way may, under some conditions, have better performance than the noninteracting loops.

For the sake of simplicity and clarity, but with no loss of generality, all considerations have been carried out for double-input-double-output system.

2. STATEMENT OF THE PROBLEM

Consider the double-input-double-output linear time-invariant plant described by the transfer function matrix

$$\boldsymbol{G}(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \tag{1}$$

Let the diagonal controller

$$\boldsymbol{R}(s) = \begin{bmatrix} R_1(s) & 0 \\ 0 & R_2(s) \end{bmatrix} \tag{2}$$

be applied to this plant. Let us assume that the disturbances act on the plant outputs additively. Then we will obtain the feedback control system as shown in Fig. 1. In the transforms domain this systems is governed by equations

$$\boldsymbol{Y}(s) = (\boldsymbol{I}+\boldsymbol{G}(s)\,\boldsymbol{R}(s))^{-1}\boldsymbol{G}(s)\,\boldsymbol{R}(s)\,\boldsymbol{Y}_0(s)+(\boldsymbol{I}+\boldsymbol{G}(s)\,\boldsymbol{R}(s))^{-1}\boldsymbol{Z}(s) \tag{3}$$

$$\boldsymbol{E}(s) = \boldsymbol{Y}_0(s)-\boldsymbol{Y}(s) = (\boldsymbol{I}+\boldsymbol{G}(s)\,\boldsymbol{R}(s))^{-1}\boldsymbol{Y}_0(s)-(\boldsymbol{I}+\boldsymbol{G}(s)\,\boldsymbol{R}(s))^{-1}\boldsymbol{Z}(s) \tag{4}$$

where $\boldsymbol{Y}_0(s)=(Y_{01}(s), Y_{02}(s))$ — the references transforms, $\boldsymbol{Z}(s)=(Z_1(s), Z_2(s))$ — the disturbances transforms, $\boldsymbol{E}(s)=(E_1(s), E_2(s))$ — the errors transforms.

Let us assume as yet that the system under consideration is stable. Then it is easy to see that in the frequency domain the performance of the system can be represented by the functions matrix

$$\boldsymbol{Q}(j\omega) = [q_{ij}(j\omega)]_{i,j=1,2} = (\boldsymbol{I}+\boldsymbol{G}(j\omega)\,\boldsymbol{R}(j\omega))^{-1} \tag{5}$$

which can be interpreteted either as the error transfer functions matrix (if $\boldsymbol{Z}(s)=\boldsymbol{0}$) or as the damping transfer functions matrix (if $\boldsymbol{Y}_0(s)=\boldsymbol{0}$ (stabilizing control)).

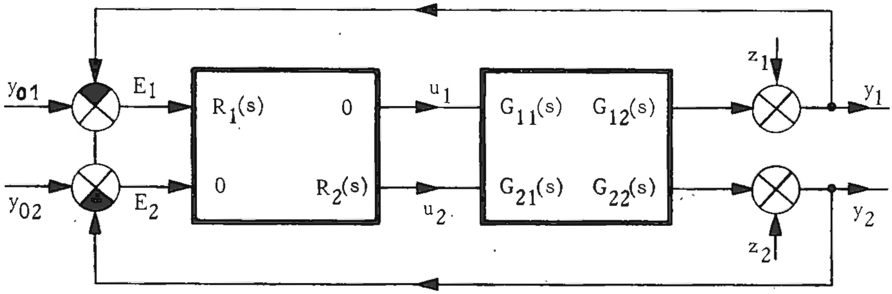

Fig. 1

This the synthesis problem could be formulated as follows: **determine the diagonal controller $\boldsymbol{R}(s)$ so as the system shall be stable and the values of performance functions $q_{ij}(j\omega)$ shall not be greater than the admissible (ie. prescribed) values $\delta_{ij} > 0$ over the given frequency band** $[0, \omega_r]$, i.e.

$$|q_{ij}(j\omega) \geqslant \delta_{ij} \quad \text{for} \quad \omega \in [0, \omega_r], \qquad i, j = 1, 2$$

As we shall see later it is not possible to satisfy these requirements for all $|q_{ij}(j\omega)|$ — functions independently and the synthesis problem will be subject to reformulation.

3. MAIN LOOPS

Consider the system in Fig. 1 in detail and represent it as in Fig. 2. We shall refer to the transfer functions $G_{11}(s)$ and $G_{22}(s)$ as the main transfer functions of the system and the transfer functions $G_{12}(s)$ and $G_{21}(s)$ as the interaction

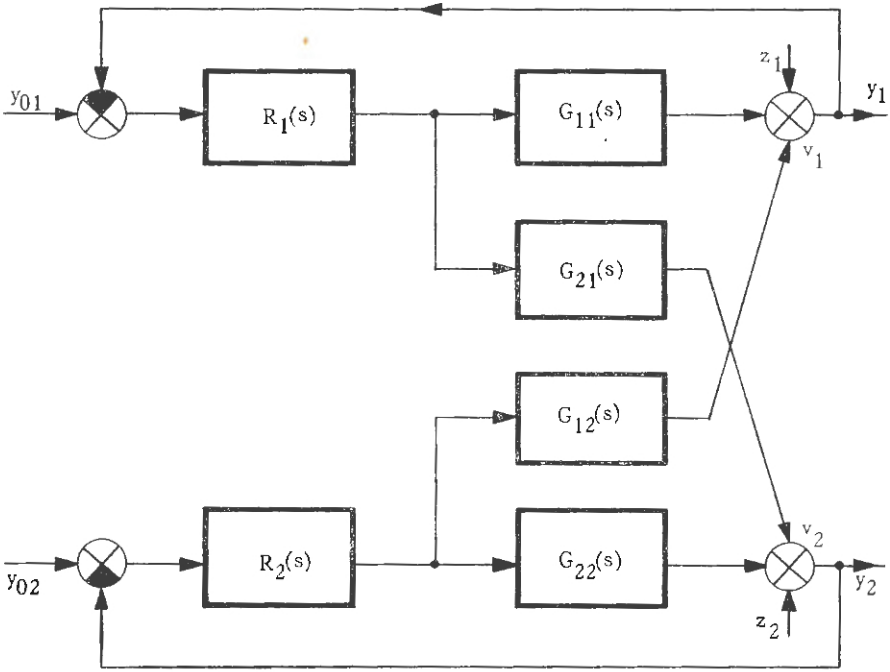

Fig. 2

transfer functions. According to this we shall call the functions $|q_{11}(j\omega)|$ and $|q_{22}(j\omega)|$ the main performance functions (with respect to (y_{01}, z_1) and (y_{02}, z_2) respectively) and the functions $|q_{12}(j\omega)|$ and $|q_{21}(j\omega)|$ the interaction performance functions (with respect to (y_{02}, z_2) and (y_{01}, z_1) respectively). It should be stressed however that, in general, the kind of either pair of the plant transfer functions is subject to choice, i.e. the inputs and outputs of the plant have been here paired arbitrarly.

If there were $G_{12}(s)=G_{21}(s)=0$ then we would obtain two noninteracting control loops presented in Fig. 3. These loops we shall refer to as **the main loops** of the system. Denoting by $W_i(s)$ the closed loopd transfer functions

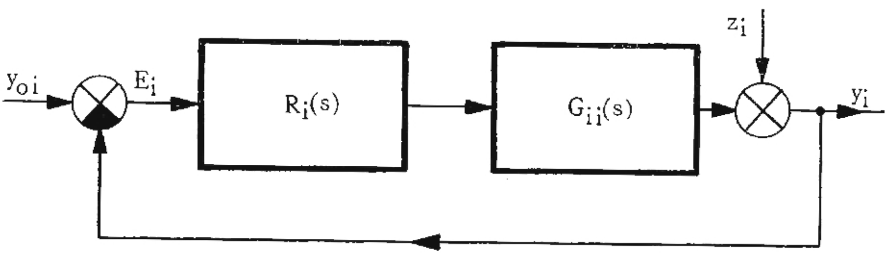

Fig. 3

and by $q_i(s)$ the error, damping or, generally, perfor ance transfer functions for these loops we obviously have

$$W_i(s) = \frac{G_{ii}(s)\,R_i(s)}{1+G_{ii}(s)\,R_i(s)}, \qquad i = 1, 2 \tag{6}$$

$$q_i(s) = \frac{1}{1+G_{ii}(s)\,R_i(s)}, \qquad i = 1, 2 \tag{7}$$

It is worth notice that the functions $q_i(s)$ can be considered as the damping transfer functions of interacting system (see Fig. 2) with respect to the disturbances z_i+v_i where v_i however are not independent variables (each loop "disturbs" other).

4. REFORMULATION OF THE PROBLEM

Determine the inverse matrix (5). We shall obtain

$$\boldsymbol{Q}(j\omega) = (\boldsymbol{I}+\boldsymbol{G}(j\omega)\,\boldsymbol{R}(j\omega))^{-1} = \frac{1}{M(j\omega)} \times$$

$$\times \begin{bmatrix} \dfrac{1}{1+G_{11}(j\omega)\,R_1(j\omega)} & \dfrac{-G_{12}(j\omega)R_2(j\omega)}{(1+G_{11}(j\omega)\,R_1(j\omega))\,(1+G_{22}(j\omega)R_2(j\omega))} \\ \dfrac{-G_{21}(j\omega)\,R_1(j\omega)}{(1+G_{11}(j\omega)\,R_1(j\omega))\,(1+G_{22}(j\omega)\,R_2(j\omega))} & \dfrac{1}{1+G_{22}(j\omega)R_2(j\omega)} \end{bmatrix} \tag{8}$$

where

$$M(j\omega) = 1 - \frac{G_{12}(j\omega)\,G_{21}(j\omega)}{G_{11}(j\omega)\,G_{22}(j\omega)}\,\frac{G_{11}(j\omega)\,R_1(j\omega)}{1+G_{11}(j\omega)\,R_1(j\omega)}\,\frac{G_{22}(j\omega)\,R_2(j\omega)}{1+G_{22}(j\omega)\,R_2(j\omega)} \tag{9}$$

Using Eqs. (6) and (7) to the entries of matrix (8) we have

$$\left.\begin{aligned} q_{11}(j\omega) &= \frac{q_1(j\omega)}{M(j\omega}, \\ q_{21}(j\omega) &= -\frac{G_{21}(j\omega)}{G_{11}(j\omega)}\,W_1(j\omega)\,\frac{q_2(j\omega)}{M(j\omega)} \\ q_{12}(j\omega) &= -\frac{G_{12}(j\omega)}{G_{22}(j\omega)}\,W_2(j\omega)\,\frac{q_1(j\omega)}{M(j\omega)} \\ q_{22}(j\omega) &= \frac{q_2(j\omega)}{M(j\omega)} \end{aligned}\right\} \tag{10}$$

and

$$M(j\omega) = 1 - \frac{G_{12}(j\omega)\,G_{21}(j\omega)}{G_{11}(j\omega)\,G_{22}(j\omega)}\,W_1(j\omega)\,W_2(j\omega) \tag{11}$$

We see that if both main loops are stable the term $M(s)$ will decide on the stability of interacting system. If one of interactions equals zero then $M(s) = 1$ and the system will be stable. The same term, as it follows from Eqs. (10), plays decisive role in the system performance. Let us try to evaluate its magnitude.

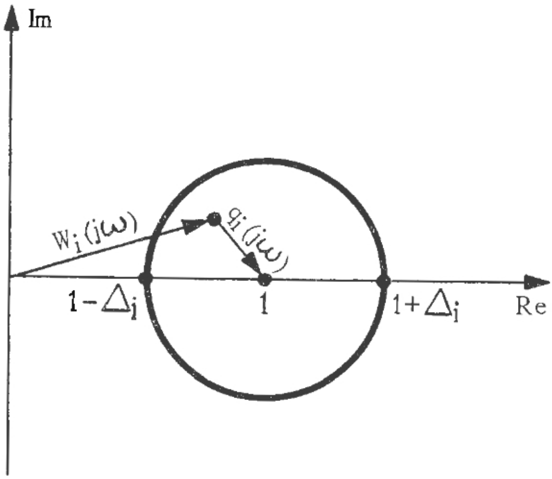

Fig. 4

Assume that both main loops are stable and they have been synthesized in such a way that

$$|q_i(j\omega)| \leqslant \Delta_i \ll 1 \qquad \text{for} \qquad \omega \in [0, \omega_r], \qquad i = 1, 2 \tag{12}$$

where $\Delta_i > 0$ — the prescribed admissible values. Then the vector

$$W_i(j\omega) = 1 - q_i(j\omega) \tag{13}$$

lies on the complex plane in the circle centered at the point $(1, j\,0)$ and of the radius $1 - \Delta_i$ as shown in Fig. 4. Now rewrite the term $M(j\omega)$ from Eq. (11) in the form

$$M(j\omega) = 1 - \frac{G_{12}(j\omega)\,G_{21}(j\omega)}{G_{11}(j\omega)\,G_{22}(j\omega)}\,W_1(j\omega)\,W_2(j\omega) =$$

$$= (1 - W_1(j\omega)\,W_2(j\omega)) + \frac{\det \mathbf{G}(j\omega)}{G_{11}(j\omega)\,G_{22}(j\omega)}\,W_1(j\omega)\,W_2(j\omega) \tag{14}$$

With regard to the assumption (12) for the first term in Eq. (14) we have

$$|1-W_1(j\omega)\,W_2(j\omega)| = |q_1(j\omega)+q_2(j\omega)-q_1(j\omega)q_2(j\omega)| \leqslant$$
$$\leqslant \Delta_1+\Delta_2 \quad \text{for} \quad \omega\in[0,\omega_r] \tag{15}$$

i.e. the vector $1-W_1(j\omega)\,W_2(j\omega)$ lies in the circle A centered at the origin and of the radius $\Delta_1+\Delta_2$ as presented in Fig. 5. The second term in Eq. (14) can be presented as follows

$$\frac{\det \mathbf{G}(j\omega)}{G_{11}(j\omega)\,G_{22}(j\omega)}\,W_1(j\omega)\,W_2(j\omega) =$$
$$= \frac{\det \mathbf{G}(j\omega)}{G_{11}(j\omega)\,G_{22}(j\omega)}\left[1-(q_1(j\omega)+q_2(j\omega)-q_1(j\omega)\,q_2(j\omega))\right] \tag{16}$$

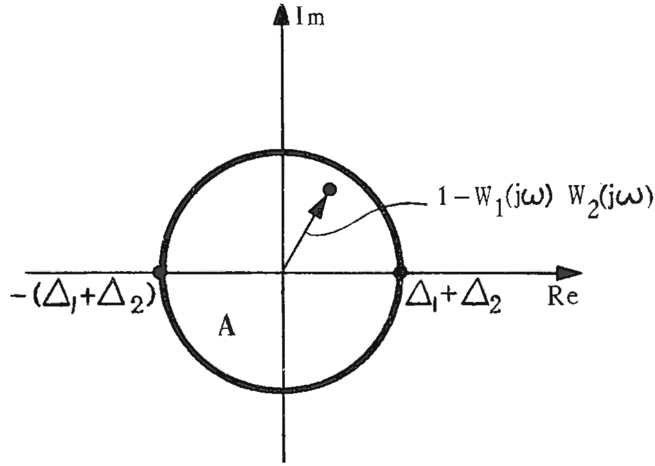

Fig. 5

Taking into acount the inequality (15) we obtain

$$\left|\frac{\det \mathbf{G}(j\omega)}{G_{11}(j\omega)\,G_{22}(j\omega)}(q_1(j\omega)+q_2(j\omega)-q_1(j\omega)\,q_2(j\omega))\right| \leqslant$$
$$\leqslant \left|\frac{\det \mathbf{G}(j\omega)}{G_{11}(j\omega)\,G_{22}(j\omega)}\right|(\Delta_1+\Delta_2) \quad \text{for} \quad \omega\in[0,\omega_r] \tag{17}$$

Now it follows from Eqs. (16) and (17) thet the vector $\frac{\det \mathbf{G}(j\omega)}{G_{11}(j\omega)\,G_{22}(j\omega)}\,W_1(j\omega)\,W_2(j\omega)$ lies in the circle B centered at the point $\frac{\det \mathbf{G}(j\omega)}{G_{11}(j\omega)\,G_{22}(j\omega)}$ and of the radius $\left|\frac{\det \mathbf{G}(j\omega)}{G_{11}(j\omega)\,G_{22}(j\omega)}\right|(\Delta_1+\Delta_2)$ as it is shown in Fig. 6.

Generally, the circles A and B in Fig. 6 may intersect or not but, what should be stressed, their locations on the complex plane depend only upon the plant dynamics and the prescribed values Δ_1 and Δ_2 (assuming that they are satisfied by controllers R_1 and R_2)

Denote

$$L = \min_{\omega \in [0,\omega_r]} \left| \frac{\det \boldsymbol{G}(j\omega)}{G_{11}(j\omega)\, G_{22}(j\omega)} \right| \tag{18}$$

$$K = \max_{\omega \in [0,\omega_r]} \left| \frac{\det \boldsymbol{G}(j\omega)}{G_{11}(j\omega)\, G_{22}(j\omega)} \right| \tag{19}$$

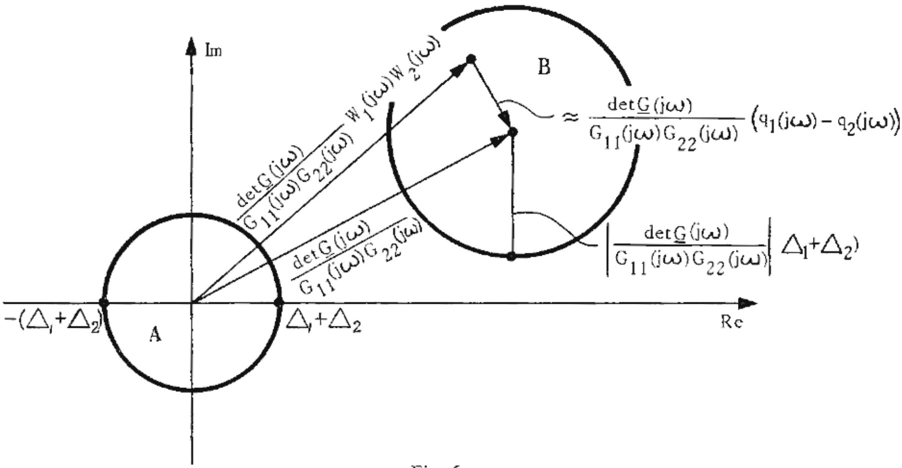

Fig. 6

Now we are already able to evaluate the magntiude of $M(j\omega)$:

i) under the assumption that the circles A and B do not intersect we have the estimation from below

$$|M(j\omega)| \leqslant L(1-(\Delta_1+\Delta_2))-(\Delta_1+\Delta_2) \quad \text{for} \quad \omega \in [0, \omega_r] \tag{20}$$

ii) in any case we have the estimation from above

$$|M(j\omega)| \geqslant K(1+(\Delta_1+\Delta_2))+(\Delta_1+\Delta_2) \quad \text{for} \quad \omega \in [0, \omega_r] \tag{21}$$

From now on we shall be interested only in the case when the circles A and B do not intersect. It is easy to show that the condition of the circles nonintersecting has the form

$$\frac{L}{L+1} > \Delta_1 + \Delta_2 \tag{22}$$

which can always be satisfied by assuming sufficiently small admissible values Δ_1 and Δ_2.

Having the evaluations (20), (21) one is able to evaluate the performance functions (10). We obtain from below and from above

$$\frac{\Delta_i}{K(1+\Delta_1+\Delta_2)+\Delta_1+\Delta_2} \leqslant |q_{ii}(j\omega)| = \left|\frac{q_i(j\omega)}{M(j\omega)}\right| \leqslant$$

$$\leqslant \frac{\Delta_i}{L(1-\Delta_1-\Delta_2)-\Delta_1-\Delta_2} \qquad i=1,2 \quad \text{for} \quad \omega \in [0, \omega_r] \tag{23}$$

and with regard to the assumption (12) we get

$$|q_{ij}(j\omega)| = \left|\frac{G_{ij}(j\omega)}{G_{jj}(j\omega)}\right| \cdot |W_j(j\omega)| \cdot \left|\frac{q_i(j\omega)}{M(j\omega)}\right| \approx \left|\frac{G_{ij}(j\omega)}{G_{jj}(j\omega)}\right| \cdot |q_{ii}(j\omega)|,$$

$$(i,j) = (1,2),\ (2,1) \quad \text{for} \quad \omega \in [0, \omega_r] \tag{24}$$

The estimations (23) shows the relations between the values of main performance functions and the admissible values Δ_i refered to the main (noninteracting) loops. On the other hand the relations (24) point out that once the requirements to the main performance functions have been fulfilled then by means of the diagonal controller we are not able to impose independent requirements on the interaction performance functions. The ratio of either main performance function to the corresponding interaction performance function depends only upon the properties of the plant and does not depend on the controller. This can be refered to as the domination of diagonal and leads to the conclusion that the ratios $\left|\frac{G_{ij}(j\omega)}{G_{jj}(j\omega)}\right|$ should be small over the interval $[0, \omega_r]$ in the sense of criterion selected. For instance, there may be required that

$$\min\left[\max_{\omega\in[0,\omega_r]}\left|\frac{G_{12}(j\omega)}{G_{22}(j\omega)}\right|, \max_{\omega\in[0,\omega_r]}\left|\frac{G_{21}(j\omega)}{G_{11}(j\omega)}\right|\right] \leqslant$$

$$\leqslant \min\left[\max_{\omega\in[0,\omega_r]}\left|\frac{G_{22}(j\omega)}{G_{12}(j\omega)}\right|, \max_{\omega\in[0,\omega_r]}\left|\frac{G_{11}(j\omega)}{G_{21}(j\omega)}\right|\right] \tag{25}$$

otherwise the inputs and the outputs of the plant can be considered as to be paired incorrectly.

Thus we can reformulate the problem of synthesis stated in sec. 1: **determine the diagonal controller $R(s)$ so as the system shall be stable and the values of main performance functions $q_{ii}(j\omega)$ shall not be greater than the admissible values $\delta_i > 0$ over the given frequency band** $[0, \omega_r]$, i.e.

$$|q_{ii}(j\omega)| \leqslant \delta_i \quad \text{for} \quad \omega \in [0, \omega_r], \quad i = 1, 2$$

5. OUTLINE OF THE SYNTHESIS PROCEDURE

The considerations above would imply as yet the following synthesis procedure:

i) pair the inputs and outputs of the plant with respect to the criterion of pairing, e.g. to the criterion (25)

ii) set the admissible values δ_1 and δ_2 according to performance requirements having in mind the relationship (24)

iii) examine the function $\left|\frac{\det G(j\omega)}{G_{11}(j\omega)\,G_{22}(j\omega)}\right|$ over the interval $[0, \omega_r]$ and find the values L and K according to eqs. (18) and (19)

iv) taking the admissible values for interacting system δ_1 and δ_2 compute the admissible values for main loops Δ_1 and Δ_2 following from the right--hand inequalities (23), i.e. from the formulas

$$\frac{\Delta_1}{L(1-\Delta_1-\Delta_2)-\Delta_1-\Delta_2} \leqslant \delta_1 \tag{26}$$

$$\frac{\Delta_2}{L(1-\Delta_1-\Delta_2)-\Delta_1-\Delta_2} \leqslant \delta_2 \tag{27}$$

v) synthesize the main loops independently in such a way as to achieve the performance values Δ_1 and Δ_2 obtained from (26) and (27) keeping in mind that the interacting system (3) has to be stable.

It should be mentioned that achieving in the main loops the performance values Δ_1 and Δ_2 is, in general, according to (23), (26) and (27), sufficient (not necesseary) condition to achieve in the interacting system the performance values δ_1 and δ_2.

The steps (i) through (iii) are clear enough and there is no need to discuss them but the points (iv) and (v) of this procedure require the discussion in detail.

It follows from eqs. (26) and (27) that, in general, we shall not obtain the unique solution for Δ_1 and Δ_2 but the set of solutions (Δ_1, Δ_2) to choose from. Denoting

$$\xi_1 = \frac{\Delta_1}{\delta_1}, \qquad \xi_2 = \frac{\Delta_2}{\delta_2} \tag{28}$$

we get this set, the closed set Ω, on the plane (ξ_1, ξ_2) constrained by the lines

$$\left.\begin{aligned} &\xi_1[(L+1)\delta_1]+\xi_2(L+1)\delta_2 = L \\ &\xi_1(L+1)\delta_1+\xi_2[(L+1)\delta_2+1] = L \\ &\xi_1 = 0, \quad \xi_2 = 0 \end{aligned}\right\} \tag{29}$$

as it is presented in Fig. 7. Rewriting the condition (22) of nonintersection of the circles A and B from Fig. 6 in the form

$$\frac{L}{L+1} \geqslant \xi_1 \delta_1 + \xi_2 \delta_2 \tag{30}$$

we can easily obtain the set of points (ξ_1, ξ_2) which this condition is satisfied for. This set $(\sum)$ is constrained by the lines

$$\left.\begin{aligned} &\xi_1 \delta_1 + \xi_2 \delta_2 = \frac{L}{L+1} \\ &\xi_1 = 0, \quad \xi_2 = 0 \end{aligned}\right\} \tag{31}$$

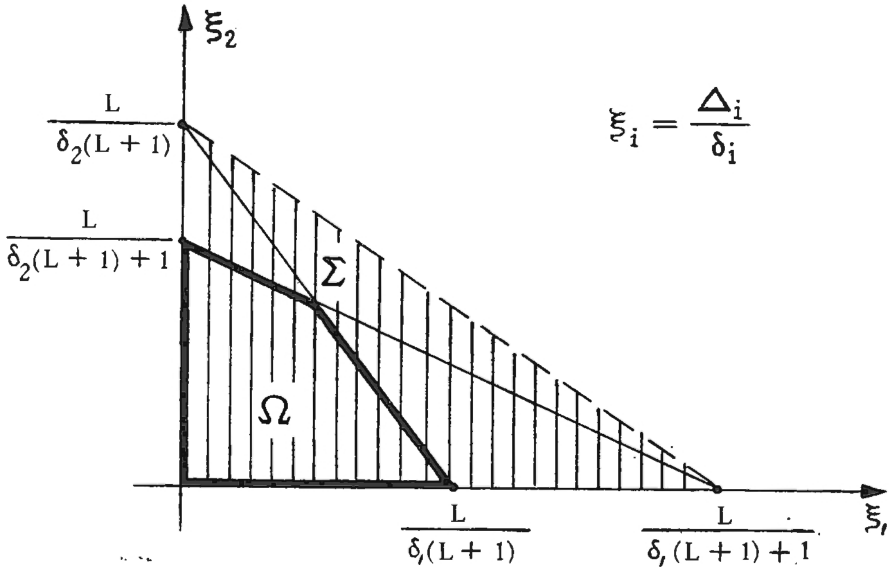

Fig. 7

and obviously $\Omega \subset \sum$, i.e. for any point $(\xi_1, \xi_2) \in \Omega$ the condition (30) is fulfilled automaticly. It is interesting to notice that if the point $\xi_1 = 1$, $\xi_2 = 1$ belongs to the set Ω i.e. if

$$(1, 1) \in \Omega \tag{32}$$

then there is a subset comprising the points $(\xi_1 \geqslant 1, \xi_2 \geqslant 1)$ for which the performance of the interacting system considered may be better than the corresponding performance of the noninteracting loops. Putting $\xi_1 = 1$, $\xi_2 = 1$ in eqs. (29) we find that this condition (32) holds if

$$\delta_1 + \delta_2 \geqslant \frac{L-1}{L+1} \tag{33}$$

In general, the greater L the more flexible and easier to be realized the procedure of synthesis. Of course, the situation is worst when $L=0$, e.g. when the matrix $\boldsymbol{G}(j\omega)$ is singular (det $\boldsymbol{G}(j\omega)=0$). In such a case the set Ω is reduced to the unrealizable point ($\xi_1=0$, $\xi_2=0$)[1].

6. STABILITY

The considerations above are valid, of course, under the assumption that after the synthesis is made the interacting system will be stable. Although the system performance is related to the lower-frequency band $[0, \omega_r]$ and the system stability — to the upper-frequency hand $[\omega_r, \infty)$ and either can be treated almost independently, it would be useful to combine both features in one approach. To do this we shall use the Mayne's method of return differences. Applied to 2×2 — system this method is as follows.

Let's denote

$$\boldsymbol{R}^1(s)=\begin{bmatrix} R_1(s) & 0 \\ 0 & 0 \end{bmatrix}, \qquad \boldsymbol{R}^2(s)=\boldsymbol{R}(s)=\begin{bmatrix} R_1(s) & 0 \\ 0 & R_2(s) \end{bmatrix} \tag{34}$$

The controller $\boldsymbol{R}^1(s)$ corresponds to the situation when the first control loop in the interacting system is closed and the second — is open. The controller $\boldsymbol{R}^2(s)$, of course, corresponds to the normal operation, i.e. when both loops are closed. Consider the free responses $Y_1^0(s)$ and $Y_2^0(s)$ refered to the systems with controllers $\boldsymbol{R}^1(s)$ and $\boldsymbol{R}^2(s)$ respectively. It is easily to check that we shall obtain the equations

$$Y_1^0(s)\,[1+G_{11}(s)\,R_1(s)] = H_1(s) \qquad \text{for} \qquad \boldsymbol{R}^1(s) \tag{35}$$

$$Y_2^0(s)\,[1+G_{22}^1(s)\,R_2(s)] = H_2(s) \qquad \text{for} \qquad \boldsymbol{R}^2(s) \tag{36}$$

where $H_1(s)$ and $H_2(s)$ represent the initial conditions and $G_{22}^1(s)$ denotes the expression

$$G_{22}^1(s) = G_{22}(s) - \frac{G_{12}(s)\,G_{21}(s)\,R_1(s)}{1+G_{11}(s)\,R_1(s)} \tag{37}$$

The expressions in brackets in eqs. (35) and (36), i.e.

$$F_1(s) = 1+G_{11}(s)\,R_1(s) \tag{38}$$

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[1] It is worth notice that the quantity $\dfrac{\det G(j\omega)}{G_{11}(j\omega)\,G_{22}(j\omega)}$ can be considered as one of the interaction measures and for $\omega=0$ it is called the indesx of structural instability (ISU). There exists the theorem saying that if all entries $G_{ij}(s)$ of the matrix $G(s)$ and the main loops are stable then the interacting closed-loop system with PID controllers is structurally and monotonously unstable if and only if $\dfrac{\det G(0)}{G_{11}(jO)\,G_{22}(jO)}<0$. This points out the significance of ISU for considerations above.

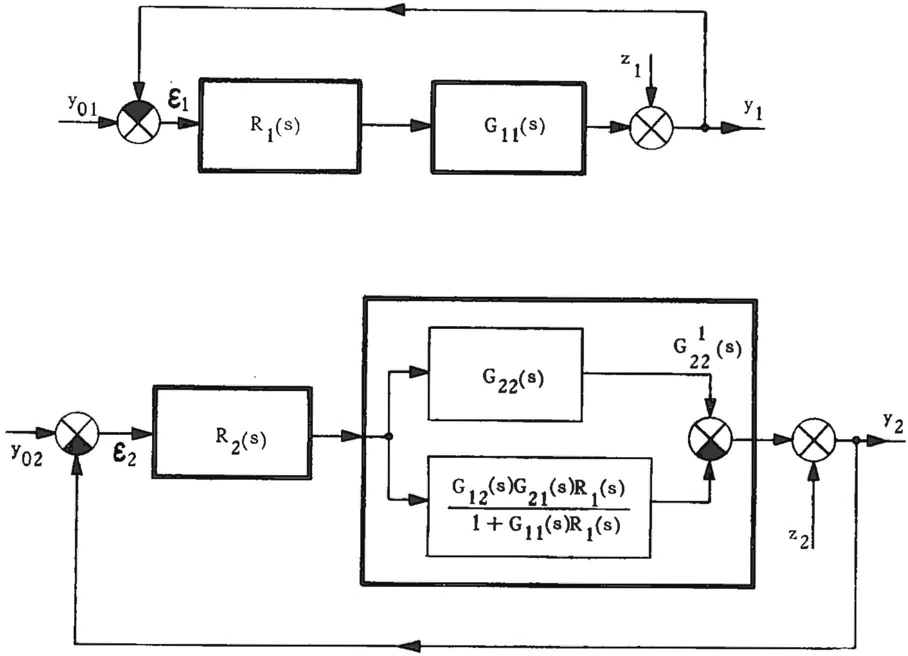

Fig. 8

and

$$F_2(s) = 1 + G^1_{22}(s)\, R_2(s) \tag{39}$$

are called the return differences. The Mayne's theorem says in this 2×2-case that the interacting system is stable if and only if both free responses $Y^0_1(s)$ and $Y^0_2(s)$ so considered will be stable, i.e. iff the functions $F_1(s)$ and $F_2(s)$ will not have the zeros in the closed right half-plane. In other words the system is stable if and only if two separate loops equivalent to the system under consideration will be stable: the first one of the OLTF (open loop fransfer function) $G_{11}(s) R_1(s)$ which is identical with the first main loop and the the second loop of OLTF $= G^1_{22}(s) R_2(s)$ where $G^1_{22}(s)$ — see (37)-takes account of the second main loop and interaction of the system (Fig. 8).

7. PROCEDURE OF THE SYNTHESIS

Now we are able to proceed to the synthesis procedure outlined in sec. 4 where the system stability was just assumed with no justification.

Following the Mayne's method and comparing (38) and (7) we see that

$$q_1(s) = \frac{1}{F_1(s)} \tag{40}$$

what leads to the synthesis of the stable first main loop, i.e. to the determination of controller R_1, according to an admissible value $\Delta_1 \geqslant |q_1(j\omega)|$. This value corresponds to a point (Δ_1, Δ_2) chosen in a way from the set Ω described by eqs. (26) and (27). Let us denote this point as (Δ_1^0, Δ_2^0) whence $\Delta_1 = \Delta_1^0$.

Subsequently, taking into account the performance function $q_{22}(j\omega)$ from (10) and the expression (11) for $j\omega = s$, it is easily to show that we have the relationship

$$q_{22}(s) = \frac{q_2(s)}{M(s)} = \frac{1}{F_2(s)} \tag{41}$$

The function $F_2(s)$ depends on both controllers $R_1(s)$ (through $G_{22}^1(s)$) and $R_2(s)$ but once the controller $R_1(s)$ was set in the previous step the function $F_2(s)$ will depend solely on the controller $R_2(s)$. This allows us to synthesize the stable second loop of OLTF $= G_{22}^1(s)R_2(s)$ with respect to the controller $R_2(s)$ and, as it turns out from eq. (41), directly according to the prescribed admissible value $\delta_2 \geqslant |q_{22}(j\omega)|$. But, in consequence, once Δ_1^0 was fixed, the insquality (27), i.e.

$$\frac{\Delta_2}{L(1-\Delta_1^0-\Delta_2)-\Delta_1^0-\Delta_2} \leqslant \delta_2 \tag{42}$$

may be satisfied in general for $\Delta_2 \leqslant \Delta_2^2$ and not only for $\Delta_2 \leqslant \Delta_2^1$ where $\Delta_2^2 \geqslant \Delta_2^1 \geqslant \Delta_2^0$ as it shown in Fig. 9. On the other hand the inequality (26), i.e.

$$\frac{\Delta_1^0}{L(1-\Delta_1-\Delta_2)-\Delta_1-\Delta_2} \leqslant \delta_1 \tag{43}$$

is in any case satisfied only for $\Delta_2 \leqslant \Delta_2^1$. In the result the point (Δ_2^0, Δ_2^2) may be found off the set Ω and this would mean that $|q_{11}(j\omega)| > \delta_1$ at least for a frequency band $[\omega_1, \omega_2] \subset [0, \omega_r]$ that contradicts the requirements assumed. It is worth of note that whichsoever point $(\Delta_1^0, \Delta_2^0) \in \Omega$ was chosen the substantial role in considerations actually will be played by the point (Δ_1^0, Δ_2^1).

In such a case we have two ways to overcome the difficulties occured. The first one is to introduce the correction with respect to the value δ_2 so that the inequality (42) shall be satisfied only for $\Delta_2 \leqslant \Delta_2^1$. Thus having Δ_1^0 we have to decrease δ_2 and the corrected value $\delta_2^1 \leqslant \delta_2$ will obviously satisfy the equation

$$\frac{\Delta_2^1}{L(1-\Delta_1^0-\Delta_2^1)-\Delta_1^0-\Delta_2^1} = \delta_2^1 \tag{44}$$

But before to compute δ_2^1 we have to compute Δ_2^1 as the second coordinate of the point (Δ_1^0, Δ_2^1). To avoid this we can proceed otherwise and compute directly the value $\delta_2^0 \leqslant \delta_2^1$ for which the inequality will hold only for $\Delta_2 \leqslant \Delta_2^0$. This value will satisfy the equation

$$\frac{\Delta_2^0}{L(1-\Delta_1^0-\Delta_2^0)-\Delta_1^0-\Delta_2^0} = \delta_2^0 \tag{45}$$

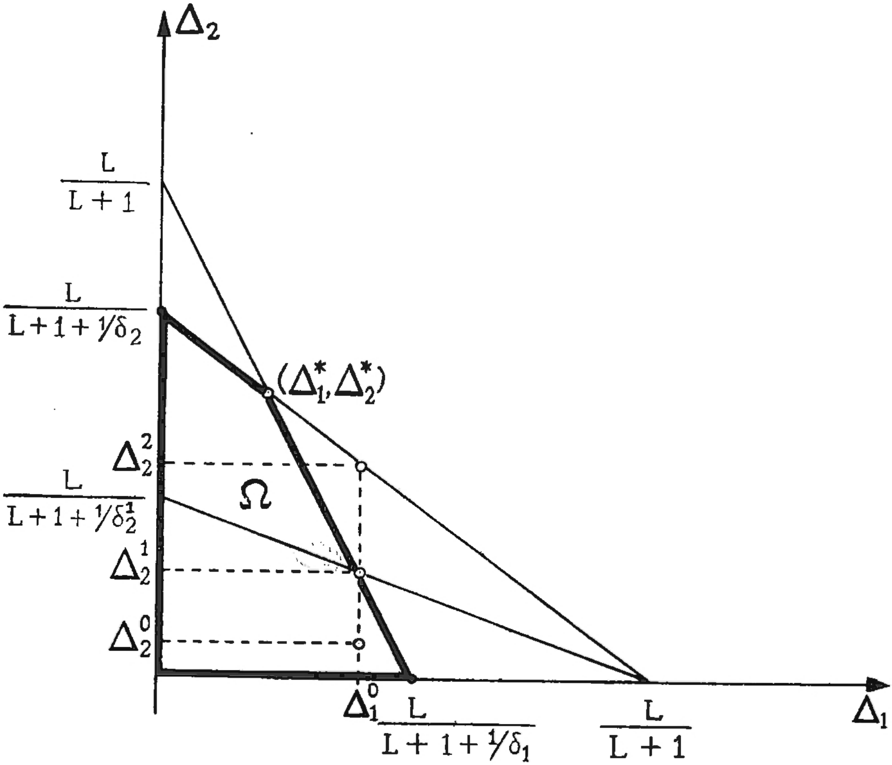

Fig. 9

However either issue presented above leads to suppresing the admissible value δ_2 compared to the required one that is unnecesseary and makes the synthesis of second loop with OLTF $= G_{22}^1(s)R(s)$ more difficult. Yet there exists another and simple possibility that allows to pass over all these troubles.

Denote by $(\varDelta_1^*, \varDelta_2^*)$ the coordinates of the "corner" of set Ω. They are the intersection of lines

$$\left.\begin{aligned} \frac{\varDelta_1}{L(1-\varDelta_1-\varDelta_2)-\varDelta_1-\varDelta_2} &= \delta_1 \\ \frac{\varDelta_2}{L(1-\varDelta_1-\varDelta_2)-\varDelta_1-\varDelta_2} &= \delta_2 \end{aligned}\right\} \tag{46}$$

Let Ω^* be the following subset of Ω

$$\Omega^* = \{(\varDelta_1, \varDelta_2) \in \Omega : \varDelta_1 \leqslant \varDelta_1^*\} \tag{47}$$

i.e. consisting of these points of Ω for which $\Delta_1 \leqslant \Delta_1^*$. It is easy to see that if the point (Δ_1^0, Δ_2^0) is chosen from the set Ω^* i.e. if $(\Delta_1^0, \Delta_2^0) \in \Omega^*$, then the inequalities (42) and (43) will be satisfied for $\Delta_2 \leqslant \Delta_2^1$) only **and** the point (Δ_1^0, Δ^1) will actually belong to the set Ω as it should be according to the idea of synthesis (compare Fig. 9 and 10). Since the values Δ_1^0 and, particularly, Δ_2^0 play auxiliary role in the procedure we can always choose them from the substet Ω^*. It is evident herewith that the greater Δ_1^0 the easier the synthesis of the first main loop. This leads to be assumed $\Delta_1^0 = \Delta_1^*$.

Furthermore, it is clear too that whatever value Δ_2^0 was chosen (at Δ_1^0 assumed) only the corresponding bonduary value Δ_2^1 is essential. All this imply finally that the "best" point (Δ_1^0, Δ_2^0) to be chosen from the set $\Omega^* \subset \Omega$ is the "corner" of set Ω, i.e. the point (Δ_1^*, Δ_2^*).

In the result of considerations above the ready-to-use procedure of synthesis consists of six steps as follows;

i) pair the inputs and outputs of the plant with respect to the criterion of pairing, e.g. to the criterion (25)

ii) set the admissible values δ_1 and δ_2 according to the performance requirements having in mind the relationship (25)

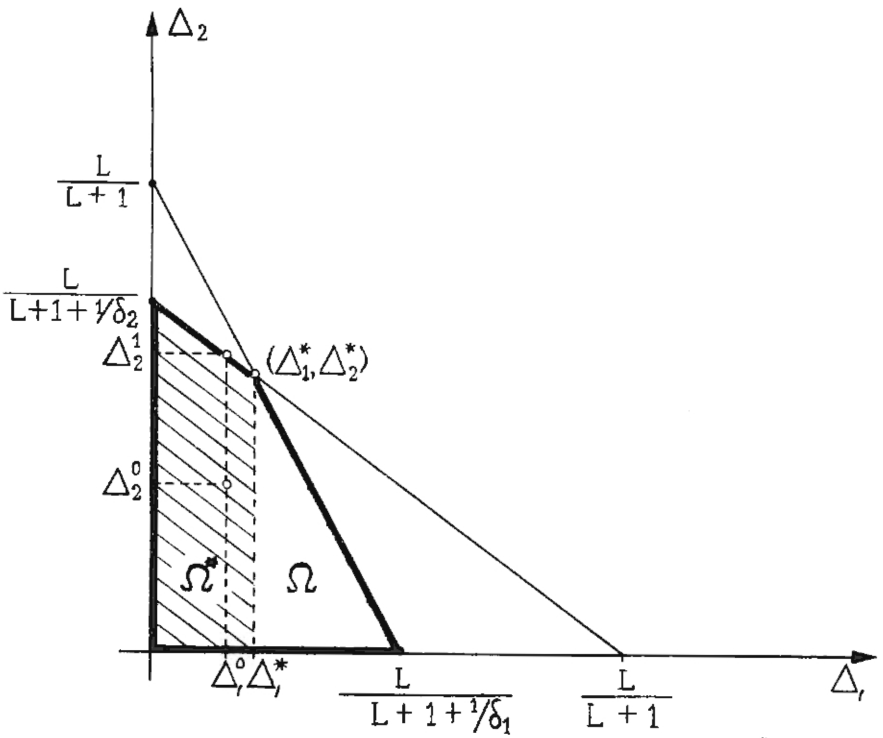

Fig. 10

iii) examine the function $\left|\frac{\det \boldsymbol{G}(j\omega)}{G_{11}(j\omega)\,G_{22}(j\omega)}\right|$ over the interval $[0, \omega_r]$ and find the value of L and, eventually, of K according to eqs. (18) and (19)

iv) given δ_1, δ_2 and L from the steps ii) and iii) compute the coordinates (Δ_1^*, Δ_2^*) of the corner of set Ω as the solution of equations system (46)

v) synthesize the first main loop of $\mathrm{OLFT} = G_{11}(s)\,R_1(s)$ in such a way as to achieve the performance value Δ_1^* obtained from the step iv)

vi) given $R_1(s)$ from the step v) and $G_{22}(s)$ from eq. (37) synthesise the second loop of $\mathrm{OLTF} = G_{22}^1(s)\,R_2(s)$ in such a way as to achieve the performance value δ_2.

These six steps guarantee to obtain the desired performance of the control system with respect to the values δ_1 and δ_2.

8. EXAMPLE

Let

$$\boldsymbol{G}(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} = \begin{bmatrix} \dfrac{2}{(s+1)^2} & \dfrac{-1}{s+1} \\ \dfrac{1}{s+1} & \dfrac{2}{(s+1)^2} \end{bmatrix} \tag{E.1}$$

be the "initial" transfer function matrix of the plant to be controlled and $[0, \omega_r] = [0, 1]$ be the frequency band to be considered.

Following the consecutive steps of synthesis procedure we obtain:

i) Inputs-outputs pairing as above yields

$$\max_{\omega \in [0,1]} \left|\frac{G_{12}(j\omega)}{G_{22}(j\omega)}\right| = \max_{\omega \in [0,1]} \left|-\frac{1}{2}(1+j\omega)\right| = 0{,}7$$

$$\max_{\omega \in [0,1]} \left|\frac{G_{21}(j\omega)}{G_{11}(j\omega)}\right| = \max_{\omega \in [0,1]} \left|\frac{1}{2}(1+j\omega)\right| = 0{,}7$$

$$\min\left[\max_{\omega \in [0,1]} \left|\frac{G_{12}(j\omega)}{G_{22}(j\omega)}\right|, \max_{\omega \in [0,1]} \left|\frac{G_{12}(j\omega)}{G_{11}(j\omega)}\right|\right] = 0{,}7 \tag{E.2}$$

Change of this pairing to the converse yields

$$\max_{\omega \in [0,1]} \left|\frac{G_{22}(j\omega)}{G_{12}(j\omega)}\right| = \max_{\omega \in [0,1]} \left|\frac{-2}{1+j\omega}\right| = 2$$

$$\max_{\omega \in [0,1]} \left|\frac{G_{11}(j\omega)}{G_{21}(j\omega)}\right| = \max_{\omega \in [0,1]} \left|\frac{2}{1+j\omega}\right| = 2$$

$$\min\left[\max_{\omega\in[0,1]}\left|\frac{G_{22}(j\omega)}{G_{21}(j\omega)}\right|, \max_{\omega\in[0,1]}\left|\frac{G_{11}(j\omega)}{G_{12}(j\omega)}\right|\right] = 2 > 0{,}7 \tag{E.3}$$

Thus the initial pairing was correct and it is subject to the next steps of procedure.

ii) Let us require to be

$$\left.\begin{aligned} &|q_{11}(j\omega)| \geqslant 0{,}2 \\ &|q_{22}(j\omega)| \geqslant 0{,}4 \quad \text{for} \quad \omega \in [0,1] \end{aligned}\right\} \tag{E.4}$$

i.e. the admissible performance values δ_1 and δ_2 are

$$\left.\begin{aligned} \delta_1 &= 0{,}2 \\ \delta_2 &= 0{,}4 \end{aligned}\right\} \tag{E.5}$$

iii) Examination of the function $\left|\dfrac{\det G(j\omega)}{G_{11}(j\omega)\,G_{22}(j\omega)}\right|$ yields

$$\max_{\omega\in[0,1]}\left|\frac{\det G(j\omega)}{G_{11}(j\omega)\,G_{22}(j\omega)}\right| = \max_{\omega\in[0,1]}\left|\frac{1}{4}\,(5-\omega^2+2j\omega)\right| = 1{,}25 \tag{E.6}$$

$$\min_{\omega\in[0,1]}\left|\frac{\det G(j\omega)}{G_{11}(j\omega)\,G_{22}(j\omega)}\right| = \min_{\omega\in[0,1]}\left|\frac{1}{4}\,(5-\omega^2+2j\omega)\right| = 1{,}1 \tag{E.7}$$

Thus we have

$$\left.\begin{aligned} L &= 1{,}10 \\ K &= 1{,}25 \end{aligned}\right\} \tag{E.8}$$

iv) Computation of the solution of equations system

$$\left.\begin{aligned} \frac{\Delta_1}{1{,}1\,(1-\Delta_1-\Delta_2)-\Delta_1-\Delta_2} &= 0{,}2 \\ \frac{\Delta_2}{1{,}1\,(1-\Delta_1-\Delta_2)-\Delta_1-\Delta_2} &= 0{,}4 \end{aligned}\right\} \tag{E.9}$$

yields the coordinates (Δ_1^*, Δ_2^*) of the corner of set Ω. We obtain

$$\left.\begin{aligned} \Delta_1^* &\approx 0{,}1 \\ \Delta_2^* &\approx 0{,}2 \end{aligned}\right\} \tag{E.10}$$

v) Synthesis of the first main loop of $\text{OLTF} = G_{11}(s)\,R_1(s) = \dfrac{2}{(s+1)^2}\,R_1(s)$ with requirement that $|q_1(j\omega)| \leqslant \Delta_1^* = 0{,}1$ for $\omega \in [0{,}1]$ yields the P-controller

$$R_1(s) = k_{p1} = 5 \tag{E.11}$$

vi) Transfer function $G_{22}^1(s)$ takes the form

$$G_{22}^1(s) = G_{22}(s) - \frac{G_{12}(s)\,G_{21}(s)\,R_1(s)}{1+G_{11}(s)\,R_1(s)} = \frac{2}{(s+1)^2} -$$

$$-\frac{\frac{1}{s+1}\cdot\frac{1}{s+1}\cdot 5}{1+\frac{2}{(s+1)^2}\cdot 5} = 7\,\frac{s^2+2s+3{,}86}{(s+1)^2(s^2+2s+11)} \qquad (E.12)$$

Synthesis of the second loop of OLTF $= G_{22}^1 R_2(s)$ with requirement that $|q_{22}(j\omega)| \leqslant \delta_2 = 0{,}4$ for $\omega \in [0{,}1]$ yields the P-controller

$$R_2(s) = k_{P2} = 3 \qquad (E.13)$$

This ends the synthesis with the result

$$R_1(s) = k_{P1} = 5\,, \qquad R_2(s) = k_{P2} = 3 \qquad (E.14)$$

9. CONCLUSIONS

As it was stated in the introduction to this paper, all consideration presented have been carried out for double-input-double-output system. This has been done just because of geometrical aspect of the method which could be clearly illustrated on (Δ_1, Δ_2) — plane. However the idea is quite general and does not depend on the number either of inputs or outputs. In principle the method is restricted to the symmetrical system when the number of inputs and outputs are equal. But even if it is not the case the asymetrical system can be symmetrized by putting appriopriate transfer functions equal to zeros. This general approach has been carried out yet and will be presented in next paper. There also will be presented the same synthesis method but refered to the control system with the "full" (nondiagonal) controller. Such controller enables to impose all admissible performance values δ_{ij} independently what is not possible in the case of diagonal controller (see eq. (24)).

It should be stressed at the and that the synthesis procedure presented is easily programable provided the subroutine of synthesis procedure for single--input-single-output control systems is available.

REFERENCES

[1] A. G. J. Mac Farlane — Relationships between recent developments in linear control theory and classical design techniques, 3rd IFAC Symposium on Multivariable Technological Systems, Manchester 1974

SUMMARY

In spite of various attempts and proposals devoted to the creation of the synthesis method of multivariable linear control systems there is still a lack of relatively simple engineering method which could be compared to the well-known methods being applied to the single-input — single-output systems. The object of the paper is to present a new proposal concerned with such a method for multiinput — multioutput systems based on the frequency response approach. Assuming the controller of the diagonal form the aim of synthesis is to ensure the system stability and sufficiently small steady-state control errors in each loop (or sufficiently large disturbance dampings in the system) over the prescribed frequency band. The idea of the method proposed is to divide the system into a number of noninteracting loops equivalent to the system under consideration and then to synthesise them independently by means of simple, well-known frequency methods. It has been proved that the control system synthesized in this way may, under some conditions, have better performance than the noninteracting system.

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