POLSKA AKADEMIA NAUK
INSTYTUT BADAŃ SYSTEMOWYCH

PROCEEDINGS OF THE 3rd ITALIAN-POLISH CONFERENCE ON APPLICATIONS OF SYSTEMS THEORY TO ECONOMY, MANAGEMENT AND TECHNOLOGY

WARSZAWA 1977

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PREFACE

The present volume comprises the papers from the 3rd Italian-Polish Conference which was held in [illegible] from [illegible]. The conference organized by the Centre of Control Systems and Computation of the Italian National Research Council in Rome and the Institute for Organization, Management and Control Sciences of Polish Academy of Sciences and the Ministry of Science, Higher Education and Technology in Warsaw was the third conference of this type.

The first one took place in Kraków (Poland) in 1972, the second in Pugnochiuso (Italy) in 1974.

This conference, like both the previous ones, gathered the leading representatives of many scientific centres from Italy as well as from Poland, representing also the different aspects of the systems analysis applied in the economy, management, data processing and technological systems.

From the past, such meetings enrich the exchange of views and experiences between the specialists from both countries [illegible] to both sides known the general state, the advancements of the systems and control theory especially in relation to [illegible] the essential development of the quantitative methods and the [illegible] of large computer systems.

The meaning of such meetings, however, goes far beyond that, what is happening during the conference because the established contacts are also fruitful in form of common scientific research works and — last but not least — form the friendly links between the scientists of both countries which last for years.

The proceedings of the conference were divided into three parts:

1. Optimization and Control Theory;
2. Systems Theory in Economics;
3. Technological, Managerial and Information Systems.

While the first two parts are of rather homogeneous character, the third part contains the papers concerning the different types of models — for the economic, technological, management and data processing systems.

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SIGNAL FLOW GRAPHS AND STRUCTURAL PROPERTIES OF LINEAR SYSTEMS

1. INTRODUCTION AND PROBLEM STATEMENT

Let us consider a dynamic, continuous, linear, time-invariant, n-dimensional system:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

with n_b inputs u_i and n_c outputs y_i.

Assuming $u_i = 0$ for $t < 0$, $i = 1, 2, \dots n$, we can apply the Laplace transformation to both sides of eq. (1) which yields:

$$\begin{aligned} sX(s) &= AX(s) + X_0 + BU(s) \\ Y(s) &= CX(s) \end{aligned} \tag{1'}$$

where $X(s) = \mathscr{L}[x(t)]$; $Y(s) = \mathscr{L}[y(t)]$; $U(s) = \mathscr{L}[u(t)]$ and where X_0 is the initial state vector.

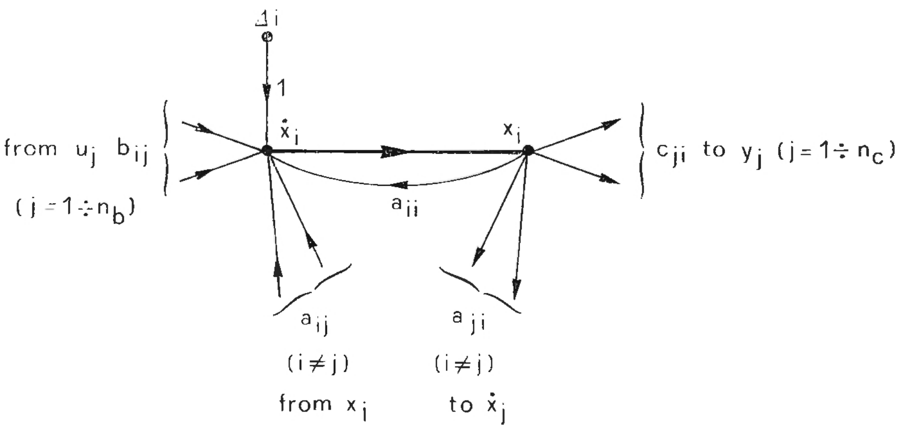


Fig. 1

We can interpret X_0 as the $\mathscr{L}$-transform of an n impulses input vector: $\Delta_i = x_i(O)\delta(t)$ (dumny or fictitious inputs), so that the actual inputs u_i (acting via the transfer constants b_{ji}) and the dummy inputs $x_i(O)\delta(t)$ (acting via unit transferences) constitute the so called generalized input.

Once we have taken into account the dummy inputs, the system is represented by the signal flow graph in fig. 1 (the graph of the figure refers to only one state component; it has to be connected to other similar graphs corresponding to other state components, in order to get the signal flow graph of the whole system).

The system we consider can be characterized by means of the usual **input-output transfer matrix:**

$$W(s) = \frac{C[adj(s \ -A)]B}{det(sI-A)} = \{W_{hk}(s)\} = \left\{\frac{\sum_{r=0}^{n-1} \gamma_{hk}^{(r)} s^r}{\sum_{r=0}^{n} \alpha_r s^r}\right\} \qquad (\alpha_n = 1) \tag{2}$$

where k is the input index $(k=1, 2, \dots n_b)$ and h is the output index $(h=1, 2, \dots n_c)$.

It is also convenient to define the **input-state transfer matrix:**

$$W^b(s) = \frac{[adj(sI-A)]B}{det(sI-A)} = \{W_{ik}^b(s)\} = \left\{\frac{\sum_{r=0}^{n-1} p_{ik}^{(r)} s^r}{\sum_{r=0}^{n} \alpha_r s^r}\right\} \qquad (\alpha_n = 1) \tag{3}$$

and the **dummy input-output transfer matrix:**

$$W^c(s) = \frac{C[adj(sI-A)]}{det(sI-A)} = \{W_{hi}^c(s)\} = \left\{\frac{\sum_{r=0}^{n-1} q_{hi}^{(r)} s^r}{\sum_{r=0}^{n} \alpha_r s^r}\right\} \qquad (\alpha_n = 1) \tag{4}$$

where i is the state index $(i=1, 2, \dots n)$.

The matrices W^b and W^c are connected to the matrix W:

$$W(s) = CW^b(s) = W^c(s)B \tag{5}$$

however, some advantages can be obtained by adopting the matrices W^b and W^c for the study of the system since they can be used for evaluating the controllability and the observability of the system.

In this paper, controllability and observability criteria for multi input-multi output systems will be presented which generalize a controllability-criterion given by the authors [1, 2] for single input-single output systems. These criteria refer to the rank of suitable matrices formed with the coefficients $p_{ik}^{(r)}$ and $q_{hi}^{(r)}$ of the matrices W^b and W^c. Various signal flow graphs which correspond to the "canonical" structures of the system are then introduced and their barnch-transferences are related to the numerator and denominator coefficients of the rational functions in W^b and W^c, i.e. to the constants $p_{ik}^{(r)}$, $q_{hi}^{(r)}$ and α_r. Suitable computation procedures will finally be suggested for evaluating the above mentioned coefficients starting from the ABC representation of the system.

2. CONTROLLABILITY AND OBSERVABILITY CONDITIONS

Consider the system $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ and assume that $\boldsymbol{x}(0)=0$. Let $\boldsymbol{u}(t)$ be an input function and denote by $\boldsymbol{x}(t)$ the corresponding trajectory in the state space for $t \geqslant 0$. The system is not completely controllable if and only if there exists a constant, nonzero vector $\boldsymbol{K}=(k_1, k_2 \dots k_n)$ such that for any input $u(t)$:

$$\boldsymbol{K}\boldsymbol{x}(t) = 0 \qquad \forall t > 0 \tag{6}$$

In terms of L-transforms, condition (6) can be set in the form:

$$\boldsymbol{K}\boldsymbol{X}(s) = 0 \qquad \forall s \tag{6'}$$

Making use of (3), (6′) can be rewritten in the form:

$$\boldsymbol{K}\boldsymbol{W}^b(s)\,\boldsymbol{U}(s) = \frac{1}{det(s\boldsymbol{I}-\boldsymbol{A})} \sum_{i=1}^{n} \sum_{k=1}^{n_b} \sum_{r=0}^{n-1} k_i\, p_{ik}^{(r)} s^r U_k(s) = 0 \tag{6''}$$

that has to hold for any component $U_k(s)$ of the input $U(s)$.

This condition can be easily restated in terms of the rank (which has to be $< n$) of the matrix:

$$\boldsymbol{P} = \{\boldsymbol{P}_1 | \boldsymbol{P}_2 | \dots | \boldsymbol{P}_{n_b}\} \tag{7}$$

whose blocks P_k are $n \times n$ matrices defined as $\|\boldsymbol{P}_k\|_{\lambda_u} \triangleq p_{\lambda_k}^{(\mu-1)}$

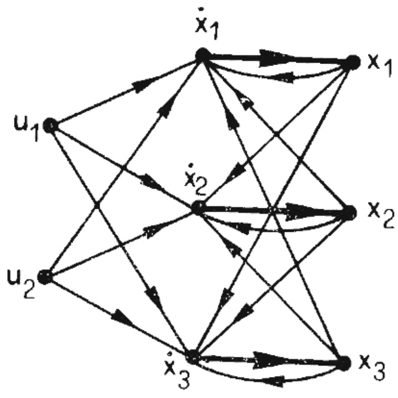
The complete controllability condition is therefore:

$$rank\,\boldsymbol{P} = n \tag{8}$$

For the single input case, $\boldsymbol{P}$ is a square $n \times n$ matrix, whose elements are the coefficients of the numerator polynomials in the input-state transfer function $\boldsymbol{W}^b(s)$.

Fig. 2 and related formulae refer to the case of a 3rd order, 2 inputs system.

At this point, the observability condition could be obtained by resorting to the duality. If we are interested in a direct proof, we have to consider the transfer function $\boldsymbol{W}^c(s)$, that connects the dummy inputs (or the initial values of the state components) to the outputs. Observability is equivalent to the existence of an one-to-one correspondence between the points of the state



$$sX = AX + BU$$

b_{ij} $\quad$ a_{ij}

$$\begin{cases} X_1(s) = \dfrac{p_{11}^{(0)} + p_{11}^{(1)} s + p_{11}^{(2)} s^2}{s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0} U_1(s) + \dfrac{p_{12}^{(0)} + p_{12}^{(1)} s + p_{12}^{(2)} s^2}{s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0} U_2(s) \\ X_2(s) = \dfrac{p_{21}^{(0)} + p_{21}^{(1)} s + p_{21}^{(2)} s^2}{s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0} U_1(s) + \dfrac{p_{22}^{(0)} + p_{22}^{(1)} s + p_{22}^{(2)} s^2}{s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0} U_2(s) \\ X_3(s) = \dfrac{p_{31}^{(0)} + p_{31}^{(1)} s + p_{31}^{(2)} s^2}{s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0} U_1(s) + \dfrac{p_{32}^{(0)} + p_{32}^{(1)} s + p_{32}^{(2)} s^2}{s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0} U_2(s) \end{cases}$$

$$P = \begin{Bmatrix} p_{11} = p_{11}^{(0)} & p_{12} = p_{11}^{(1)} & p_{13} = p_{11}^{(2)} & p_{14} = p_{12}^{(0)} & p_{15} = p_{12}^{(1)} & p_{16} = p_{12}^{()2} \\ p_{21} = p_{21}^{(0)} & p_{22} = p_{21}^{(1)} & p_{23} = p_{21}^{(2)} & p_{24} = p_{22}^{(0)} & p_{25} = p_{22}^{(1)} & p_{26} = p_{22}^{(2)} \\ p_{31} = p_{31}^{(0)} & p_{32} = p_{31}^{(1)} & p_{33} = p_{31}^{(2)} & p_{34} = p_{32}^{(0)} & p_{35} = p_{32}^{(1)} & p_{36} = p_{32}^{(2)} \end{Bmatrix}$$

Fig. 2

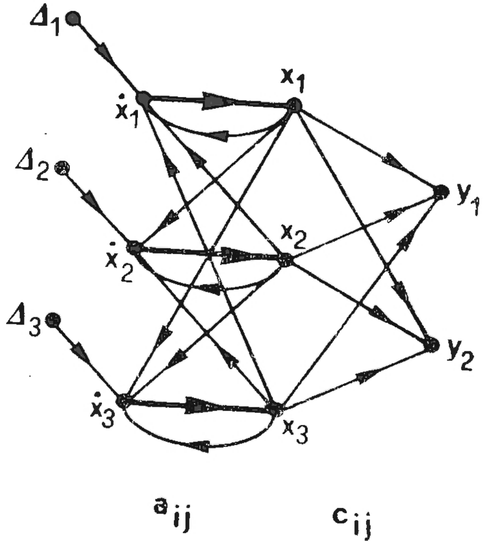
space (the set of impulsive dummy inputs) and the consequent free evolutions of the outputs (forced evolution due to the dummy inputs). A matrix:

$$Q = \{Q_1 | Q_2 | \dots Q_{nc}\} \tag{9}$$

has therefore to be considered, formed by $n \times n$ blocks Q_h defined as $\|Q_h\|_{\lambda_u} = q_{\lambda h}^{(\mu-1)}$ and the complete observability condition is stated as:

$$rank\, Q = n \tag{10}$$

For the scalar output case, Q is a square $n \times n$ matrix whose elements are the coefficients of the numerator polynomials in the dummy input-output transfer function.



$$\begin{cases} sX = AX + X_0 \\ Y = CX \end{cases}$$

$$\begin{cases} Y_1 = \dfrac{q_{11}^{(0)}+q_{11}^{(1)}s+q_{11}^{(2)}s^2}{s^3+\alpha_2 s^2+\alpha_1 s+\alpha_0}\,\Delta_1 + \dfrac{q_{12}^{(0)}+q_{12}^{(1)}s+q_{12}^{(2)}s^2}{s^3+\alpha_2 s^2+\alpha_1 s+\alpha_0}\,\Delta_2 + \dfrac{q_{13}^{(0)}+q_{13}^{(1)}s+q_{13}^{(2)}s^2}{s^3+\alpha_2 s^2+\alpha_1 s+\alpha_0}\,\Delta_3 \\[2ex] Y_2 = \dfrac{q_{21}^{(0)}+q_{21}^{(1)}s+q_{21}^{(2)}s^2}{s^3+\alpha_2 s^2+\alpha_1 s+\alpha_0}\,\Delta_1 + \dfrac{q_{22}^{(0)}+q_{22}^{(1)}s+q_{22}^{(2)}s^2}{s^3+\alpha_2 s^2+\alpha_1 s+\alpha_0}\,\Delta_2 + \dfrac{q_{23}^{(0)}+q_{23}^{(1)}s+q_{23}^{(2)}s^2}{s^3+\alpha_2 s^2+\alpha_1 s+\alpha_0}\,\Delta_2 \end{cases}$$

$$\boldsymbol{Q} = \begin{Bmatrix} q_{11}=q_{11}^{(0)} & q_{12}=q_{11}^{(1)} & q_{13}=q_{11}^{(2)} & q_{14}=q_{21}^{(0)} & q_{15}=q_{21}^{(1)} & q_{16}=q_{21}^{(2)} \\ q_{21}=q_{12}^{(0)} & q_{22}=q_{12}^{(1)} & q_{23}=q_{12}^{(2)} & q_{24}=q_{22}^{(0)} & q_{25}=q_{22}^{(1)} & q_{26}=q_{22}^{(2)} \\ q_{31}=q_{13}^{(0)} & q_{32}=q_{13}^{(1)} & q_{33}=q_{13}^{(2)} & q_{34}=q_{23}^{(0)} & q_{35}=q_{23}^{(1)} & q_{36}=q_{23}^{(2)} \end{Bmatrix}$$

Fig. 3

Fig. 3 and related formulae refer to the case of a 3rd order, 2 outputs system.

The adopted approach presents some advantages we will focus below:

1. Controllability and observability analysis is performed by using matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ whose elements are coefficients of "meaningful" transfer functions.

2. The suggested criteria are useful from a didactic point of view, because they are based on the intuitive notions of controllability and observability and on well known techniques of $\mathscr{L}$-transform.

3. When each state component exhibits an intrinsic meaning (e.g. a physical meaning or an economical meaning and a transformation into physically or economically meaningless state components can be unconvenient) the suggested approach allows us to distinguish very easily between the controllability

of the single state component and the controllability of the whole state space. If a single state component is not controllable, the numerator of the corresponding input-state transfer function is zero and $\boldsymbol{P}$ has a null row. If a single state component can be controlled but not independently by other components, the corresponding row of P is nonzero, even if it is not linearly independent on the other rows. Such difference is very important in many "a priori" problems (structural identifiability, identification experiment design, selection among models of different complexity etc.). As is well known, these problems arise if the model structure is given, i.e. when it is known if an element of $\boldsymbol{A}$, $\boldsymbol{B}$ or $\boldsymbol{C}$ matrices is equal to 0 or not, but its value, if differing from 0, is not yet known or it is not explicitely taken into account. Similar considerations can be made with reference to the observability problem.

4. It is easily evaluated the dimension of the subspace controllable by each input (from the rank of $\boldsymbol{P}_1, \boldsymbol{P}_2, \dots \boldsymbol{P}_{nb}$) and of the subspace observable from each output (from the rank of $\boldsymbol{Q}_1, \boldsymbol{Q}_2 \dots \boldsymbol{Q}_{nc}$).

3. THE ELEMENTS OF $\boldsymbol{P}$ AND $\boldsymbol{Q}$ AS BRANCH TRANSFERENCES OF SIGNAL FLOW GRAPHS

For the sake of simplicity (in particular for graphical representation problems) only single input-single output systems will be considered in this section. The specification $h=1$ and $k=1$ will be therefore omitted.

Let us consider the structure of the input-state transfer functions (right hand member in (3)). The i-th component of the state vector, x_i, can be obtained as a linear combination with coefficients $p_i^{(r)}$ $(r=0, 1, \dots n-1)$ of a function obtained from the input via the transfer function:

$$\frac{1}{det(s1-\boldsymbol{A})} = \frac{1}{\sum_{r=0}^{n} \alpha_r s^r} \quad (\alpha_n = 1) \tag{11}$$

and its first n-1 derivatives. The combination is graphically represented in fig. 4 for the case $n=3$ (with $x=0$ for $t=0$).

The system corresponding to the signal flow graph in fig. 4, which has the state vector $z=(z_1, z_2, z_3)^T$ as phase variables vector, is certainly controllable.

On the other hand the initially considered system, having $\boldsymbol{x}=(x_1, x_2, x_3)^T$ as its state vector, is controllable if and only if the $z \rightarrow \boldsymbol{x}$ transformation, represented by the matrix $\boldsymbol{P}$, is a bijection: in such a case $\boldsymbol{x}$ can be considered as a state vector also for the system in fig. 4.

Let S be the system assigned via matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$; the graph in fig. 4, having vector z as state vector, will be called controllable associated structure S_{ca}. Clearly, S_{ca} is equivalent to S if and only if S is ocntrollable too. Anyhow, a one-to-one correspondence can be set up between each S and its S_{ca}.

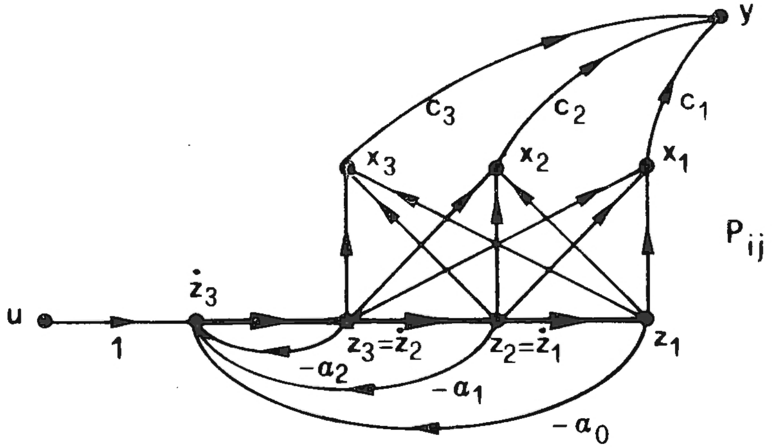


Fig. 4

Since matrices $\boldsymbol{P}$ and $\boldsymbol{C}$ and vector α formed by the ocefficients $\alpha_0 \dots \alpha_{n-1}$ supply the same information about the dynamics of S supplied by matrices $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{C}$, S_{ca} can be therefore considered as an equivalent description of S.

The consideration of S_{ca} bears the following features:

1. The information about the structural properties of the system, is disaggregated. In fact the information about stability of S is contained in a vector (α) instead that in a matrix ($\boldsymbol{A}$) and it is completely separated from the information about controllability, that is supplied by matrix $\boldsymbol{P}$ (instead than by a matrix formed by manipulating $\boldsymbol{A}$ and $\boldsymbol{B}$).

2. On the other hand, computing α and $\boldsymbol{P}$ from $\boldsymbol{A}$ and $\boldsymbol{B}$ requests only arithmetic operations (as for computing the usual controllability canonical representation) but it bears the above mentioned disaggregation advantages which can be also obtained by computing the Jordan canonical form (that implies the evaluation of the eigenvalues of the system).

3. The structure S_{ca} is redundant (in fact it implies n^2+2n coefficients) but its redundancy is not greater than the one of the assigned representation of S via $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{C}$.

4. The usual controllability canonical form can be easily obtained from structure S_{ca} as shown in fig. 5. The transfer constants connecting each z_i to the corresponding addendum of y are the coefficients $\gamma_r = \gamma_{il}$ in formula (2). Vector $\gamma = (\gamma_r)$ $(r=0, 1 \dots n-1)$ is connected to $\boldsymbol{P}$ and $\boldsymbol{C}$ via the relation:

$$\gamma = \boldsymbol{P}^T \boldsymbol{C} \tag{12}$$

Obvious duality considerations suggest the definition of the observable associated structure S_{oa}, that is represented in fig. 6 for the case of a 3rd order one-input one-output system. Structure S_{oa} is characterized by vector B,

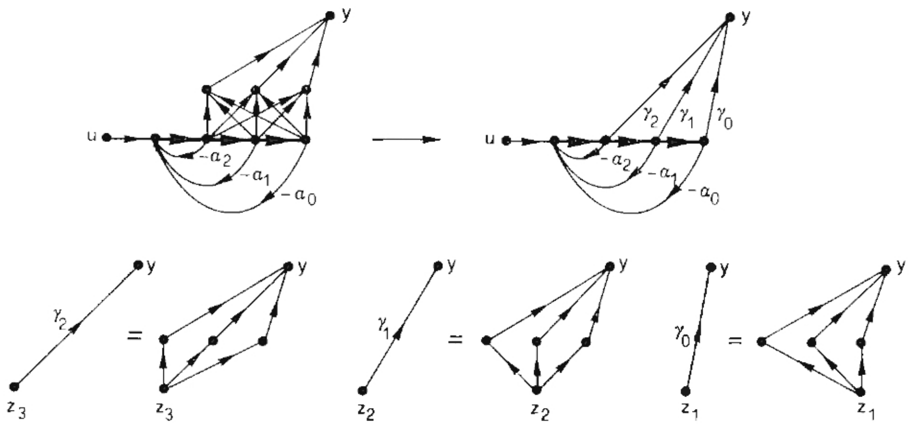


Fig. 5

matrix Q and vector α; its state variables z_i (cf. fig. 6) cannot be considered as phase variables; S_{oa} is always observable and it is therefore equivalent to S if and only if S is observable too. The main features of S_{oa} are similar to the ones of S_{ca}; in this case the information about observability is supplied in a direct way; the computation of the observability canonical form can be

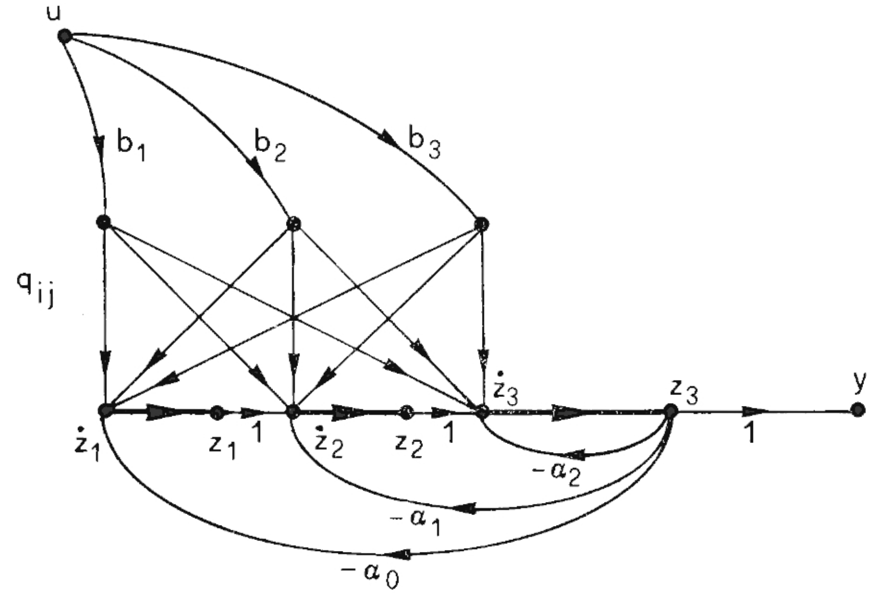


Fig. 6

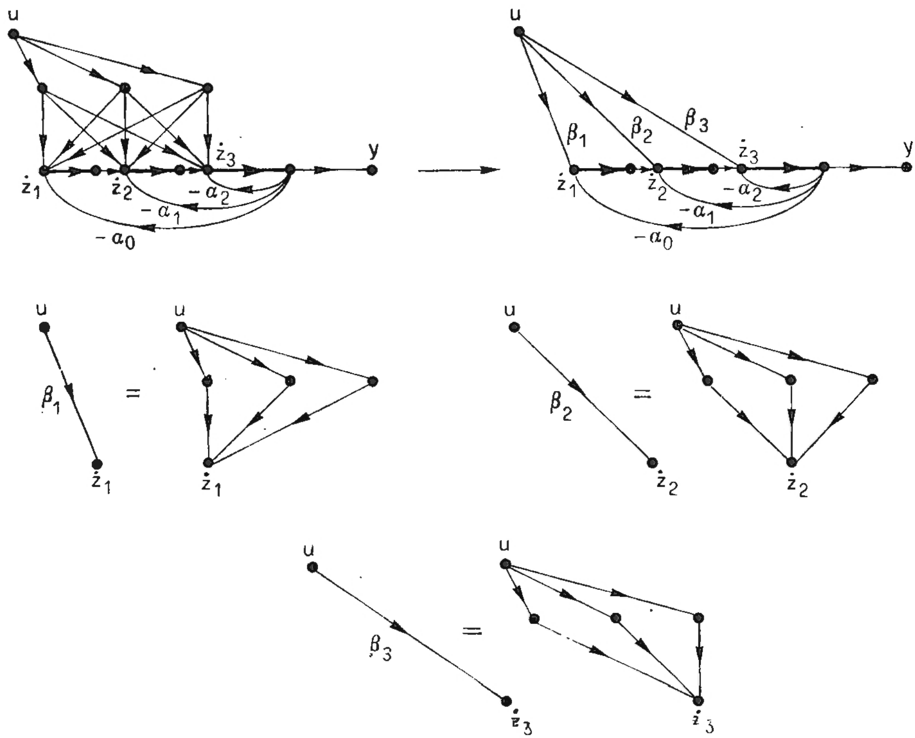


Fig. 7

implemented as shown in fig. 7. The input transference vector β of the observability canonical form is connected to $\boldsymbol{B}$ and $\boldsymbol{Q}$ via the relation:

$$\beta = \boldsymbol{QB} \tag{13}$$

Both associated structures S_{ca} and S_{oa} (cf. fig. 4 and fig. 6) can be considered as the cascade combination of a dynamic system and an instantaneous one. The dynamic system connects the input to the state z in S_{ca} and the dummy inputs (acting on each $\dot{z}_i$) to the output in S_{ca}. The structure of said systems is very simple. In fact they are formed by n cascaded integrators, with feedback branches converging only to the input of the first integrator or diverging only from the output of the last one. Correspondingly the state matrix $\boldsymbol{A}_c$ of S_{ca} is a row companion matrix and the state matrix $\boldsymbol{A}_0$ of S_{oa} is a column companion matrix.

The instantaneous system is formed by two parts, described by matrix $\boldsymbol{P}$ and vector $\boldsymbol{C}$ in S_{ca} and by vector $\boldsymbol{B}$ and matrix $\boldsymbol{Q}$ in S_{oa}.

Computation methods for evaluating α and $\boldsymbol{P}$ from $\boldsymbol{A}$ and $\boldsymbol{B}$ or α and $\boldsymbol{Q}$ from $\boldsymbol{A}$ and $\boldsymbol{C}$ will be presented in section 4.

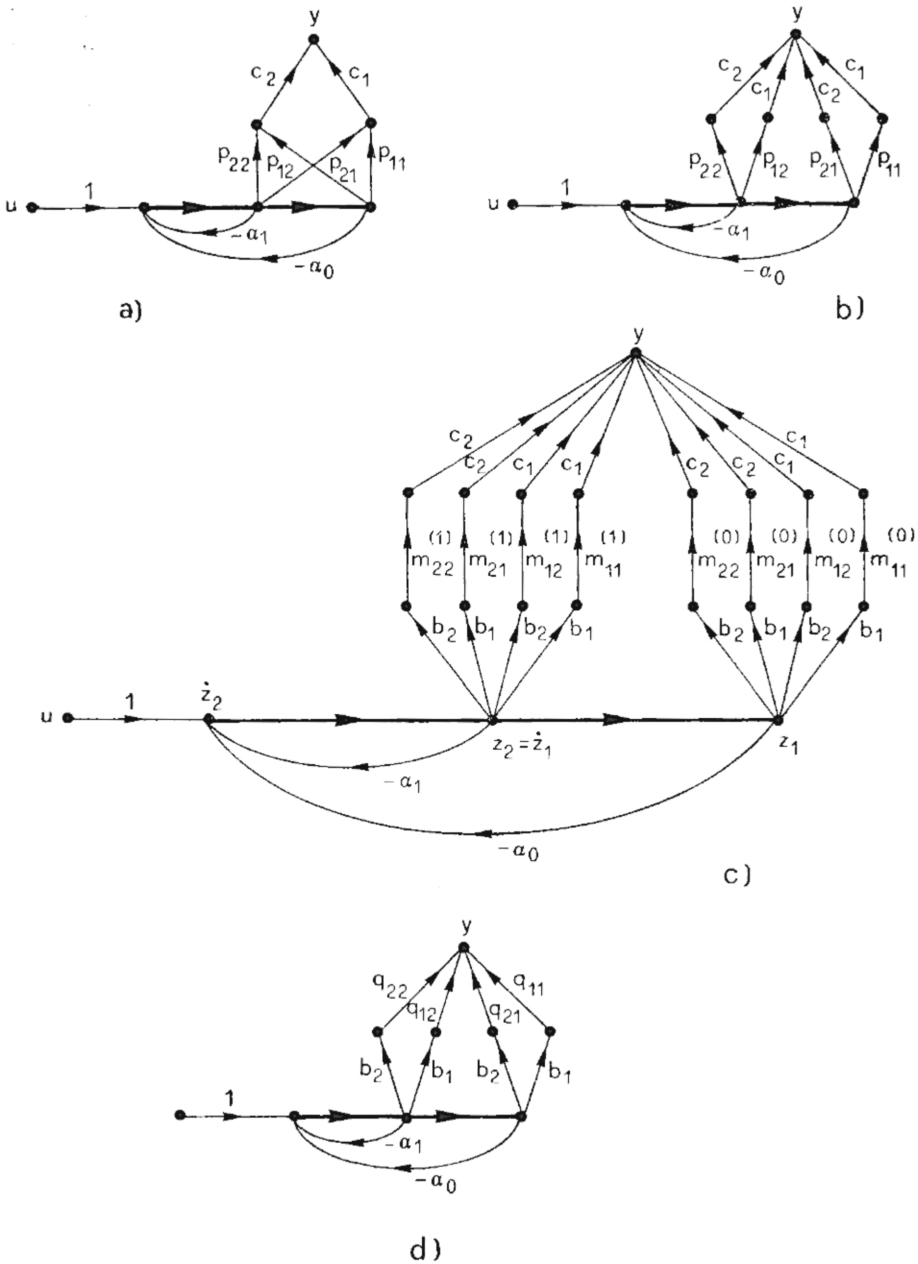

y
c2
c1
p22
p12
p21
p11
u
1
-α1
-α0
a)
b)
ż2
z2 = ż1
z1
m22 (1)
m21 (1)
m12 (1)
m11 (1)
m22 (0)
m21 (0)
m12 (0)
m11 (0)
b1
b2
c)
q22
q12
q21
q11
d)

Fig. 8

In the following part of this section further considerations will be developed about the possibility of disaggregating information regarding controllability and observability in the associated structures S_{ca} and S_{oa}.

Let us refer to formulae (3) and (4) and in particular to the numerators of $\boldsymbol{W}^b$ and $\boldsymbol{W}^c$. As is well known, adj $(s\boldsymbol{I}-\boldsymbol{A})$ can be set in the form:

$$adj\,(sI-A) = \sum_{r=0}^{n-1} M_r s^r \tag{14}$$

where:

$$\boldsymbol{M}_{n-1} = 1;\ \boldsymbol{M}_{n-2} = \boldsymbol{A}+\alpha_{n-1} 1;\ \boldsymbol{M}_{n-3} = \boldsymbol{A}^2+\alpha_{n-1}\boldsymbol{A}+\alpha_{n-2} 1;\ \ldots\ \boldsymbol{M}_0 =$$
$$= \boldsymbol{A}^{n-1}+\alpha_{n-1}\boldsymbol{A}^{n-2}+\ \ldots\ +\alpha_1 1 \tag{15}$$

(the α_i being the coefficients of det $(s1-\boldsymbol{A})$ accordingly to the previously used notation).

By equating the numerators on left and right hand sides of (3) and making use of (14) and (15), we obtain:

$$\boldsymbol{M}_r\boldsymbol{B} = \begin{bmatrix} p_1^{(r)} \\ p_2^{(r)} \\ \vdots \\ p_n^{(r)} \end{bmatrix} \qquad r = 0, 1\ \ldots\ n-1 \tag{16}$$

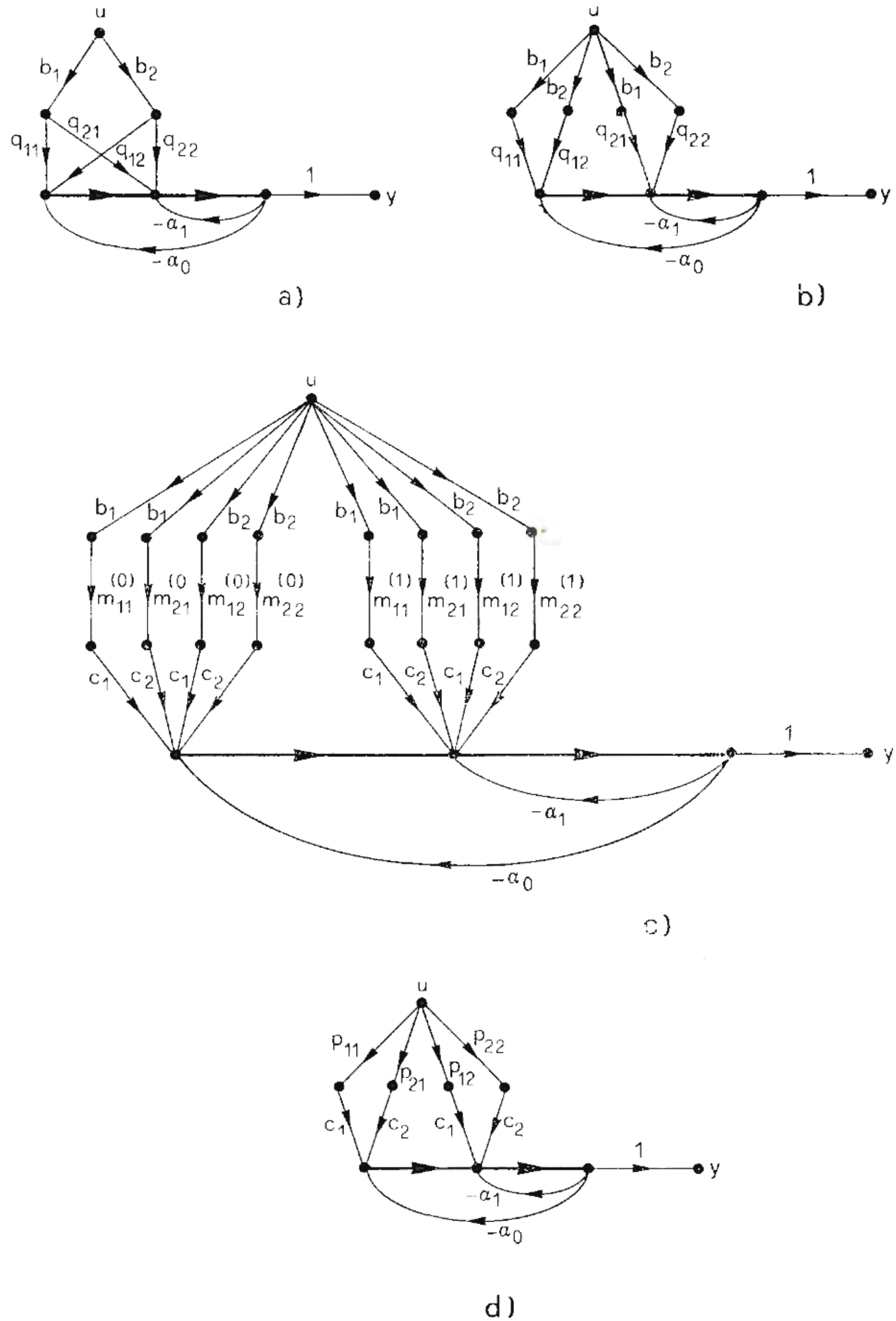
In the same way, by equating left and right hand sides of eq. (8) we obtain:

$$\boldsymbol{CM}_r = [q_1^{(r)} q_2^{(r)}\ \ldots\ q_n^{(r)}] \qquad r = 0, 1\ \ldots\ n-1 \tag{17}$$

Equation (16) allows us to substitute each $p_i^{(r)}$ branch in the signal flow graph of S_{ca} by means of a suitable combination of b_j and $m_{ij}^{(r)}$ branches. The corresponding manipulation of the graph is represented in fig. 8 for a second order system (the branch transferences $p_i^{(r)}$ are indicated as $p_{i,r+1}$). Fig. 8a) corresponds to the structure of fig. 4; after the simple manipulation of fig. 8b), each $p_i^{(r)}$ branch is decomposed according to eq. (16) and fig. 8c) is obtained. Finally, $m_{ij}^{(r)}$ and c_i branches are combined into the $q_j^{(r)}$ branches of fig. 8d) according to eq. (17) ($q_j^{(r)}$ are indicated in the figure as $q_{r+1,\,j}$).

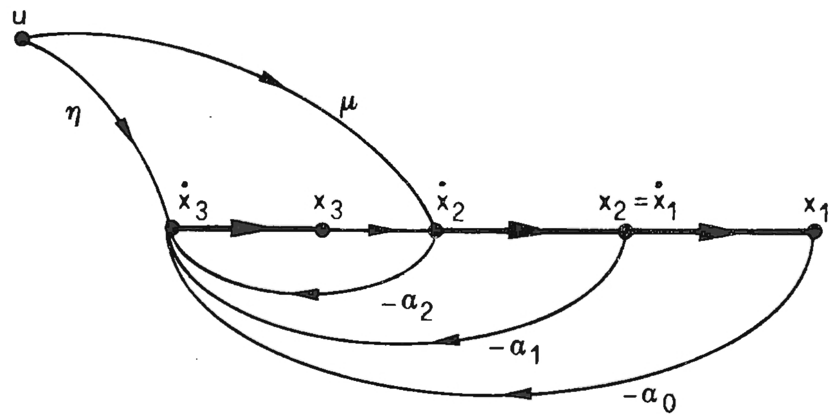
The four signal flow graphs of fig. 8 are clearly equivalent; the graphs of fig. 8a) and 8b) exhibit in a direct way the information concerning the controllability (because each element of $\boldsymbol{P}$ corresponds to the transference of a branch); the graph of fig. 8c) exhibit directly the information concerning observability (via the branch transferences $q_i^{(r)}$); in the graph of fig. 8c) the controllability analysis can be performed with reference to the subgraph formed by b_j and $m_{ij}^{(r)}$ branches and the observability analysis can be performed with reference to the subgraph formed by the same $m_{ij}^{(r)}$ branches and by the c_i branches.

The corresponding manipulations for the signal flow graph of S_{oa} are represented in fig. 9 for the same second order system.



u
b1
b2
q11
q21
q12
q22
1
y
-α1
-α0
a)
b)
m11(0)
m21(0)
m12(0)
m22(0)
m11(1)
m21(1)
m12(1)
m22(1)
c1
c2
c)
p11
p21
p12
p22
d)

Fig. 9



$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 + \mu u \\ \dot{x}_3 = -\alpha_0 x_1 - \alpha_1 x_2 - \alpha_2 x_3 + (\eta - \mu\alpha_2)\, u \end{cases}$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ \mu \\ \eta - \mu\alpha_2 \end{pmatrix}$$

$$W_1 = \frac{\eta + \mu s}{s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0}; \qquad W_2 = \frac{\frac{2}{2}}{s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0};$$

$$W_3 = \frac{-\alpha_0 \mu - \alpha_1 \mu s + (-\alpha_2 \mu + \eta)\, s^2}{s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0}$$

$$P = \begin{pmatrix} \eta & \mu & 0 \\ 0 & \eta & \mu \\ -\alpha_0 \mu & -\alpha_1 \mu & -\alpha_2 \mu + \eta \end{pmatrix}$$

$$(B\, AB\, A^2B) = \begin{pmatrix} 0 & \mu & \eta - \mu\alpha_2 \\ \mu & \eta - \mu\alpha_2 & -\alpha_1 \mu + \alpha_2^2 \mu - \alpha_2 \eta \\ \eta - \mu\alpha_2 & -\alpha_1 \mu + \alpha_2^2 \mu - \alpha_2 \eta & -\alpha_0 \mu + 2\alpha_1 \alpha_2 - \alpha_1 \eta - \alpha_2^3 \mu + \alpha_2^2 \eta \end{pmatrix}$$

Fig. 10

4. COMPUTATION PROCEDURES FOR P AND Q

4.1 GENERAL CONSIDERATIONS

This section will be devoted to computation procedures for the matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ starting by the given matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$. The features of the described procedures will be compared to the computation methods usually adopted for evaluating the controllability matrix $(\boldsymbol{A}, \boldsymbol{AB}, \boldsymbol{A}^2\boldsymbol{B} \ldots \boldsymbol{A}^{n-1}\boldsymbol{B})$ and the observability matrix $(\boldsymbol{C}^T, \boldsymbol{A}^T\boldsymbol{C}^T, (\boldsymbol{A}^T)^2\boldsymbol{C}^T \ldots (\boldsymbol{A}^T)^{n-1}\boldsymbol{C}^T)$.

A first direct procedure for evaluating $\boldsymbol{P}$ and $\boldsymbol{Q}$ can make use of (15) for computing $\boldsymbol{M}_i$ from $\boldsymbol{A}$ and of (16) and (17) for computing $\boldsymbol{P}$ from $\boldsymbol{M}_i$ and $\boldsymbol{B}$ and $\boldsymbol{Q}$ from $\boldsymbol{M}_i$ and $\boldsymbol{C}$.

This procedure can practically be considered so cumbersome as the one for computing the usual controllability and observability matrices. In both cases the powers of A are to be computed and multiplied by $\boldsymbol{B}$ or $\boldsymbol{C}$. The computation of $\boldsymbol{P}$ and $\boldsymbol{Q}$ requests also a linear combination of the powers of $\boldsymbol{A}$, in order to compute $\boldsymbol{M}_i$. This additional computation is not very heavy but it bears the advantage that $\boldsymbol{P}$ and Q can be used also for evaluating the system transference.

On the other hand it can be noted that the analytical structure of $\boldsymbol{P}$ and $\boldsymbol{Q}$ is not necessarily more complex than the one of usual controllability and observability matrices, as eq. (15), (16) and (17) could suggest.

Let us consider, for instance, the system in fig. 10. The analytical structure of $\boldsymbol{P}$ can be considered simpler than the one of $(\boldsymbol{B}, \boldsymbol{AB} \ldots A^{-1}\boldsymbol{B})$ because its elements are connected in a more direct way to the system parameters. This situation can not be considered as depending on a particular feature of the graph of fig. 10; in fact the corresponding scheme is a very common one and it is not of a canonical type (it does not correspond to the structures of fig. 4 or of figure 6).

It has to be noted, however, that the above considerations refer to the final structure of P and of the controllability matrix but not to the computation procedure. In fact the elements of P can be easily evaluated by direct inspection or by using Mason's formula that can be computed in a very simple way for the graph of fig. 10. The use of eq. (15) and (16) can result more tedious.

Therefore alternative computation methods have been studied and will be presented in 4.2 and 4.3. The first method presents the computation of the input-state and state-output transfer functions via the Mason's formula; the second one is based on the direct evaluation of the transformation from the assigned state variables to the ones of the associated structure.

4.2 METHOD BASED ON MASON'S FORMULA

Several implementation procedures of Mason's formula have been proposed in the literature (cf. for instance [3], [4] and [5]). We sketch here briefly a method already published by the authors [6], which refers to the conenction matrix of the signal flow graph connecting a source i to another node o.

Recall that to any signal flow graph C with n nodes a $n \times n$ "connection matrix" C is associated, whose elements $\|C\|_{rs}$ are 1 if G contains a branch from the r-th node to the s-th node and 0 otherwise.

C belongs to the lattice $\mathscr{L}$ of $n \times n$ matrices with elements in the boolean algebra $\{0,1\}$: obviously $M \leqslant C$, M in $\mathscr{L}$, means that $\|M\|_{rs} \leqslant \|C\|_{rs}$ $r, s = 1, 2, \dots n$.

Introduce now the subset $\mathscr{C}$ of $\mathscr{L}$

$$C = \{C'_1, C'_2, \dots\}$$

whose elements meet the following conditions:

i) $C'_j \leqslant C$

ii) each column and each row of C'_i contains at most one element equal to 1

iii) if the k-th row (column) of C'_j, $k \neq i$ ($k \neq 0$) is zero, then the k-th column (row) is also zero; if the i-th row is zero, the o-th column is zero and viceversa; the i-th column and the o-th row cannot be zero.

Each C'_j can be thought as the connection matrix of a suitable subgraph of G: such subgraph is formed, beside isolated nodes, by non touching loops and selfloops and by at most one, if any, i-o path that does not touch loops and selfloops of the subgraph.

By neglecting isolated nodes, an one to one correspondence can be set up between the elements of $\mathscr{C}$ and the numerator and denominator addenda of the Mason's formula

$$\frac{\sum (-1)^p P_k L_{j_1} \dots L_{jp}}{1 + \sum (-1)^d L_{j_1} \dots L_{jd}} \tag{18}$$

where P_k is the transference of the k-th path joining node i to node o, L_{jh} is the loop transference of the J_h loop, and summations are extended to the products $P_k L_{jl} \dots L_{jp}$ and $L_{jl} \dots L_{jq}$ which do not contain transferences of touching or/and loops paths.

Numerator and denominator addenda of Mason's formula are therefore obtained by multiplying transferences of arcs which correspond to 1-elements in each matrix C'_j. The sign of each addendum can be evaluated from the structure of matrix C'_j, following procedures presented in [6].

It is worthwile to observe that signal flow graph we are dealing with contain constant transference and l/s transference branchs, so that numerator and denominator addenda in (18) have Ks^{-k} structure, k is the number of intergators and K is the product of constant transferences b_j, a_{ij}, c_i the subgraph related to $P_k L_{jl} \dots L_{jq}$ contains.

By associating addenda with the some power s^{-k}, one gets the trasference as

$$\frac{\sum_{k=1}^{n} K_k s^{-k}}{\sum_{k=0}^{n} K'_k s^k} \tag{19}$$

and by multiplying numerator and denominator by s^n

$$\frac{\sum_{k=1}^{n} K_k s^{n-k}}{\sum_{k=0}^{n} K'_k s^{n-k}} \tag{20}$$

Coefficients K_k are the elements of P matrix (Q matrix) if the signal flow graph corresponds to the input-state relations (dummy input-output relations)*.

Starting from the graph of fig. 1, the detailed procedure goes on as follows:

1) delete the c_{ij} branchs, if input-state relations are needed for getting P matrix;

1') delete the b_{ij} arcs, and introduce branchs and nodes for the dummy inputs, if dummy input-output relations are needed for getting Q matrix;

2) for each single input-single output signal flow graph (in case 1, the outputs are the state components; in case l', the inputs are the dummy inputs) construct the connection matrix C and matrices C'_j;

3) select C'_j matrices corresponding to numerator addenda in Mason's formula and evaluate the number k of integrators

4) obtain the numerator addenda in (18) (whose sing can be determined following the procedures described in [6] and compute K_k coefficients, wich are elements of P in case 1 and elements of Q in case 1'.

4.2 METHOD FOR DIRECTLY DERIVING P MATRIX

Consider the controllability structure S_{ac} of fig. 4, wich refers to a single input-single output system

$$\dot{x} = Ax + Bu \tag{21}$$

Let

$$det(sI - A) = \alpha_0 + \alpha_1 s + \dots + \alpha_{n-1} s^{n-1} + s^n$$

and introduce the matrices

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & & \ddots & \\ & & & & 1 \\ -\alpha_0 & -\alpha_1 & \dots & & -\alpha_{n-1} \end{bmatrix}, \qquad B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

*) The integrator number k can be evaluated by properly reordering the nodes. We suggest the following procedure

1 — state variables and their first derivatives are labeled by even numbers (2, 4...$2n$) and by odd numbers (1, 3...$2n-1$) resoectively (if necessary, a node splitting has to be performed, as in fig. 6).

2 — Inputs, outputs, dummy inputs are labeled by the numbers $2n+1$, $2n+2$...

In such a way in C and in C_j matrices, integrators correspond to upper diagonal element in 1st, 3rd, ... $(2n+1)$th rows.

Since the vector z satisfies the relations

$$z = A_c z + B_c u \tag{22}$$

$$x = Pz \tag{23}$$

we obtain

$$(AP - PA_c) z + (B - PB_c) u = 0$$

Controllability of S_{ac} allows us to derive equations in the unknown matrix P

$$\begin{aligned} AP - PA_c &= 0 \\ B - PB_c &= 0 \end{aligned} \tag{24}$$

Denoting by $\mathrm{col}_k M$ the k-th column of a matrix M, (24.2) and the structure of B_c matrix imply

$$\mathrm{col}_n P = B \tag{25}$$

On the other hand, by the structure of A_c matrix, we get

$$\mathrm{col}_n(PA_c) = -\alpha_{n-1} \mathrm{col}_n P + \mathrm{col}_{n-1} P$$

$$\mathrm{col}_{n-1}(PA_c) = -\alpha_{n-2} \mathrm{col}_{n-1} P + \mathrm{col}_{n-2} P$$

$$\ldots$$

$$\mathrm{col}_2(PA_c) = -\alpha_1 \mathrm{col}_2 P + \mathrm{col}_1 P$$

$$\mathrm{col}_1(PA_c) = -\alpha_0 \mathrm{col}_1 P$$

Since the relation

$$\mathrm{col}_k(AP) = A \,\mathrm{col}_k P$$

hold for $k = 1, 2, \ldots n$, equation (24.1) gives

$$\begin{aligned} \mathrm{col}_{n-1} P &= AB + \alpha_{n-1} B \\ \mathrm{col}_{n-2} P &= A \,\mathrm{col}_{n-1} P + \alpha_{n-2} B \\ &\ldots \\ \mathrm{col}_1 P &= A \,\mathrm{col}_2 P + \alpha_1 B \end{aligned} \tag{26}$$

Since (26) is easily solved by recursion, (25) and (26) give the (unique) matrix P we are looking for.

5. CONCLUSIONS

The analysis of the dynamical proterties of a linear system described by the A, B, C matrices has been performed by resorting to the matrices P and Q such that:

1) the controllability condition is

rank $P = n$

2) the observability condition is

rank $Q = n$

3) the input output transfer matrix can be set in the form:

$$W(s) = \frac{CPK}{det(s\ -A)} = \frac{\bar{K}Q^T B}{det(s\ -A)}$$

where K and $\bar{K}$ are $nn_b \times n_b$ and $n_c \times nn_c$ polynomial matrices respectively, defined by:

$$K = \begin{Vmatrix} H & 0 & \dots & 0 \\ 0 & H & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & H \end{Vmatrix}; \qquad K = \begin{Vmatrix} H^T & 0 & \dots & 0 \\ 0 & H^T & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & H^T \end{Vmatrix}; \qquad H = \begin{Vmatrix} 1 & s & \dots & s^{n-1} \end{Vmatrix}$$

The interpretation of the elements of P and Q as branch-transferences of signal flow graphs of suitable "canonical" structures associated to the system is then suggested.

Finally some computation procedures are suggested for evaluating P and Q from the A, B, C representation of the given system.

The advantages of the use of P and Q are underlined; in particular the consideration of P and Q has been proved very convenient for the analysis of some a priori identifiability problems (cf. e.g.[7]) previously studied by the same authors in a more cumbersome way [8].

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SUMMARY

The paper deals with the evaluation of structural properties of continuous, linear, time invariant systems by resorting to suitable matrices formed by the numerator coefficients of the transfer functions connecting the inputs to the state components and suitable dummy inputs to the outputs. Previous results obtained by the authors regarding controllability of single input-single output systems are extended to the observability analysis and to the case of multiinput-multioutput systems.

The elements of the above mentioned matrices are then interpreted in terms of branch transferences of signal flow graphs, corresponding to conveniently suggested "canonical" forms.

Some methods are presented for the computation of the above mentioned matrices, with reference also to the signal flow graphs of the input-state-output relations.

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