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The International Federation of Automatic Control
The International Federation of Operational Research Societies
The International Institute for Applied Systems Analysis
The Institute of Management Sciences

SUPPORT SYSTEMS FOR DECISION AND NEGOTIATION PROCESSES

Preprints of the IFAC/IFORS/IIASA/TIMS Workshop
Warsaw, Poland
June 24-26, 1992

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VOLUME 1:

Names of first authors: A-K

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NEGOTIATION WITH THREAT STRATEGIES

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Abstract : In a two-person game with bargaining, C-optimal threat decision pairs are defined. The definition is compared with that of optimal threat decision pairs in the sense of Nash. In the case of differential games, a sufficiency condition for C-optimality of a threat strategy pair is given and illustrated by an example of collective bargaining.

Key Words : Game theory, differential games, threat strategies, C-optimality, collective bargaining.

1. Introduction

We shall be interested in the behavior of a set of "persons", called the *players*, each of whom strives to modify the state of a system or, as we shall say, the state of the *game*, in a most efficacious manner according to his own criterion. Let us first consider the case of games in which the rules assign to each player a *payoff function* of all the players' *decisions* ; that is, the *rules of the game* prescribe mappings

$$W_i : \Pi_{j=1}^{N} S_j \rightarrow \Omega_i, \quad i = 1, 2, \dots N, \qquad (1)$$

where W_i and S_i is the payoff function and decision set, respectively, for player i in

the set $J = \{J_1, J_2, \dots J_n\}$, and the Ω_i, i = 1, 2, ... N, are linear spaces.

Cooperative and Competitive Games

In the case where $\Omega_i = R^1$, i = 1, 2, ... N, we suppose, loosely speaking, that

each player desires to attain the greatest possible payoff to himself. A large part of the literature on games is concerned with two moods of play, one *cooperative* and the other *competitive*. These are due to economists Pareto (1909) and Nash (1951), respectively.

<u>Definition 1.</u> For prescribed mappings $V_i : \Pi_{j=1}^{N} S_j \rightarrow R^1$, i = 1, 2, ... N, a decision

N-tuple $s^* \in \Pi_{j=1}^{N} S_j$ is *Pareto-optimal* (or a *Pareto-equilibrium*) if and only if for

every $s \in \Pi_{j=1}^{N} S_j$ either $V_i(s) = V_i(s^*)$, i = 1, 2, ... N, or there is at least one

$i \in \{1, 2, \dots N\}$ such that $V_i(s) < V_i(s^*)$.

Later, we shall make use of the following lemma which embodies sufficiency conditions for Pareto-optimality : see Leitmann (1974)

<u>Lemma 1.</u> Decision N-tuple $s^* \in \Pi_{i=1}^{N} S_i$ is Pareto-optimal if there exists strictly positive numbers α_i, $i = 1, 2, \dots N$, such that

$$V(s) \leq V(s^*) \qquad \text{for all } s \in \Pi_{i=1}^{N} S_i \ ,$$

where $V(s) = \Sigma_{i=1}^{N} \alpha_i J_i (s)$.

If the players do not cooperate, that is if they are in strict competition, we have <u>Definition 2.</u> A decision N-tuple $s^* \in \Pi_{i=1}^{N} S_i$ is a *Nash-equilibrium* if and only if for all $i \in \{1, 2, \dots N\}$

$$V_i (s^*) \geq V_i (s_1^*, \dots s_{i-1}^*, s_i, s_{i+1}^*, \dots s_N^*)$$

for all $s_i \in S_i$.

2. C-Optimality

Now, in the more general case where the rules of the game prescribe mappings of the type (1), we have introduced in Blaquière (1974), and further discussed in Blaquière (1975), Blaquière (1976 a, b) the concept of C-optimality. This was motivated by the fact that, the concept of optimality beeing tied with the ones of preference and comparision, a *preference relation* and a *comparision relation* need be associated with each player. Here, in general, the preference cannot be defined by the natural ordering on the real line as in the cases of the above paragraph.

Let the preference relation of J_i , $i = 1, 2, \dots N$ (reflexive, not necessarily transitive) be denoted by $(\geq)_i$, $\Omega_i^2 \supset (\geq)_i$; and

let the comparision relation of J_i , $i = 1, 2, \dots N$ (reflexive and symmetric) be denoted by C_i , $(\Pi_{i=1}^{N} S_i)^2 \supset C_i$.

Then we have

<u>Definition 3.</u> A decision N-tuple $s^* \in \Pi_{i=1}^{N} S_i$ is *C-optimal for player* J_i if and only if

$$W_i (s^*) \ (\geq)_i \ W_i (s) \ \text{for all} \ s \ C_i \ s^*.$$

<u>Definition 4.</u> A decision N-tuple $s^* \in \Pi_{i=1}^{N} \ S_i$ is C-*optimal* if and only if it is optimal

for all the players, that is if and only if

$$W_i \ (s^*) \ (\geq)_i \ W_i \ (s) \ \text{ for all } \ s \ C_i \ s^*,$$

$$i = 1, 2, ... \ N.$$

Illustrative examples are given in Blaquière (1975), where it is shown that Pareto-
and Nash-optimality are special cases of C-optimality, with proper preference and
comparision relations. We will see another illustrative example in the next paragraph. In
Blaquière (1976, b) Definition 4 is used in the study of *coalitions* and for introducing the
concept of *diplomacy*.

3. C-Optimal Threat Decision Pair

In general, cooperation entails bargaining for the reason that, in most cases where
there exists a Pareto-optimal decision N-tuple, this one is not unique. A decision N-tuple
in the set of Pareto-optimal ones may be more desirable than another for some player.
Accordingly, this player will try to convince the other players to choose that cooperation
point. In practice, it appears that the efficiency of his argument will depend on his
"strengh", that is, on the efficiency of the *threats* he can put forward.

From now on, we shall consider two-person games for which a negociated solution
is envisaged. Before such a settlement can be arrived at, we will suppose that the players
exchange threats in an attempt to influence the final outcome of the game. Whether
negotiations take place and what are the results of such negotiations will depend on the
threats made. The problem of bargaining has been considered by Nash (1953), and
extended to differential games by Liu (1973). Our approach, reported in Ray and
Blaquière (1981) is different in that we define optimal threats independently of any
negotiated stages, through the concept of C-optimality.

Roughly speaking, we can think of a threat decision as a decision designed to inflict
the greatest damage possible to the opponent. In so doing, each player will have to
consider the possible reaction of his opponent. If the opponent behaves in the same way,
then both players run to risk of having considerable losses. Thus, in choosing a threat
decision, each player needs to consider the effect that it will have on the other player and
also the risk to himself associated with it. In order to make this idea more precise, let us
start with the mappings $V_i : S_1 \otimes S_2 \rightarrow R^1$, $i = 1, 2$, and with the following facts:

The selection of a decision $s_1 \in S_1$ by player J_1 has two consequences: it will

put an upper bound on the (scalar) payoff of his opponent, namely

$$V_2(s_1, s_2') = \sup_{s_2 \in S_2} [V_2(s_1, s_2)] ,$$

and a lower bound on his own (scalar) payoff, namely

$$V_1(s_1, s_1^*) = \inf_{s_2 \in S_2} [V_1(s_1, s_2)] .$$

Since $s_2' \neq s_2^*$ in most cases, it will generally be necessary for player J_2 to find a

compromise between defending his own payoff and attacking his opponent. A similar
consideration holds for player J_1.

The fact that each player is interested in a *threat-risk* pair leads us to considering
the mappings

$$W_i : S_1 \otimes S_2 \to R^2, \qquad i = 1, 2,$$

where $W_1(s_1, s_2) = W_2(s_1, s_2) := (V_1(s_1, s_2) , V_2(s_1, s_2))$ for

$(s_1, s_2) \in S_1 \otimes S_2$.

Then, the framework of C-optimality provides us with a way for defining C-optimality of
a threat decision pair; that is, we use Definition 4 with $(\geq)_i$ and C_i, $i = 1, 2$, defined by

$(x, y) (\geq)_1 (x', y') \Leftrightarrow \{x > x'$ and/or $y < y'\}$ or $\{x = x'$ and $y = y'\}$,

$(x, y) (\geq)_2 (x', y') \Leftrightarrow \{x < x'$ and/or $y > y'\}$ or $\{x = x'$ and $y = y'\}$,

$(s_1, s_2) C_1 (s_1', s_2') \Rightarrow s_2 = s_2'$,

$(s_1, s_2) C_2 (s_1', s_2') \Rightarrow s_1 = s_1'$.

In other words, a response decision s_2 for player J_2 against s_1 is optimal for J_2 if

and only if

$(V_1(s_1, s_2) , V_2(s_1, s_2)) (\geq)_2 (V_1(s_1, s_2) , V_2(s_1, s_2))$ for all $s_2 \in S_2$.

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This makes sense, because, if player J_2 selects any other decision $s_2 \in S_2$, then either his own (scalar) payoff is reduced, or the (scalar) payoff of player J_1 is increased, or both situations occur. A similar consideration holds for player J_1; that is, a response decision $\underline{s}_1$ for player J_1 against $\underline{s}_2$ is optimal for player J_1 if and only if

$$(V_1(\underline{s}_1, \underline{s}_2), V_2(\underline{s}_1, \underline{s}_2)) \; (\geq)_1 \; (V_1(s_1, \underline{s}_2), V_2(s_1, \underline{s}_2)) \quad \text{for all } s_1 \in S_1.$$

Then a threat decision pair (s^*_1, s^*_2) is C-optimal if and only if it is optimal for both J_1 and J_2.

Noting that $(x,y) \; (\geq)_1 (x',y') \Leftrightarrow (x',y') \; (\geq)_2 (x,y)$, we see that a threat decision pair (s^*_1, s^*_2) is C-optimal if and only if

$$(V_1(s^*_1, s_2), V_2(s^*_1, s_2)) \; (\geq) \; (V_1(s^*_1, s^*_2), V_2(s^*_1, s^*_2))$$

$$(\geq) \; (V_1(s_1, s^*_2), V_2(s_1, s^*_2))$$

for all $s_1 \in S_1$, and for all $s_2 \in S_2$, where $(\geq)$ is written in place of $(\geq)_1$.

As a direct consequence of the definitions of $(\geq)$ and of a Pareto-equilibrium, we have

<u>Lemma 2.</u> $(s^*_1, s^*_2) \in S_1 \otimes S_2$ is a C-optimal threat decision pair if and only if

(a) s^*_1 is a Pareto-equilibrium of $(V_1(s_1, s^*_2), -V_2(s_1, s^*_2))$; and

(b) s^*_2 is a Pareto-equilibrium of $(-V_1(s^*_1, s_2), V_2(s^*_1, s_2))$.

Lemma 1 together with Lemma 2 result in Lemma 3 and Corollary 1 which embody sufficiency conditions for C-optimality of a threat decision pair.

<u>Lemma 3.</u> Decision pair $(s^*_1, s^*_2) \in S_1 \otimes S_2$ is a C-optimal threat decision pair if there exists strictly positive numbers α_1, α_2, such that

$$V_1(s^*_1, s^*_2) - \alpha_1 V_2(s^*_1, s^*_2) \geq V_1(s_1, s^*_2) - \alpha_1 V_2(s_1, s^*_2), \text{ and}$$

$$V_1(s^*_1, s_2) - \alpha_2 V_2(s^*_1, s_2) \geq V_1(s^*_1, s^*_2) - \alpha_2 V_2(s^*_1, s^*_2),$$

for all $(s_1, s_2) \in S_1 \otimes S_2$.

Corollary 1. Decision pair $(s^*_1, s^*_2) \in S_1 \otimes S_2$ is a C-optimal threat decision pair if there exists a strictly positive number α such that the saddle-point condition

$$V_1(s^*_1, s_2) - \alpha V_2(s^*_1, s_2) \geq V_1(s^*_1, s^*_2) - \alpha V_2(s^*_1, s^*_2)$$

$$\geq V_1(s_1, s^*_2) - \alpha V_2(s_1, s^*_2)$$

is satisfied for all $(s_1, s_2) \in S_1 \otimes S_2$.

4. Nash-Optimal Threat Decision Pair

Again, consider the mappings

$$W_i : S_1 \otimes S_2 \to R^2, \qquad i = 1, 2,$$

where $W_1(s_1, s_2) = W_2(s_1, s_2) = W(s_1, s_2) := (V_1(s_1, s_2), V_2(s_1, s_2))$

for $(s_1, s_2) \in S_1 \otimes S_2$. Let us denote by $\Sigma(s_1, s_2)$ the set

$$\{(r_1, r_2) \in S_1 \otimes S_2 : [V_1(r_1, r_2) - V_1(s_1, s_2)] > 0, [V_2(r_1, r_2) - V_2(s_1, s_2)] > 0\}.$$

Then we have

Definition 5. A *Nash bargaining solution associated with* $(s_1, s_2) \in S_1 \otimes S_2$, whenever it exists, is a pair $(\sigma_1, \sigma_2) \in \Sigma(s_1, s_2)$ such that, either

(i) $[V_1(r_1, r_2) - V_1(s_1, s_2)] \, [V_2(r_1, r_2) - V_2(s_1, s_2)] \leq$

$$[V_1(\sigma_1, \sigma_2) - V_1(s_1, s_2)] \, [V_2(\sigma_1, \sigma_2) - V_2(s_1, s_2)]$$

for all $(r_1, r_2) \in \Sigma(s_1, s_2)$,

or

(ii) $(\sigma_1, \sigma_2) = (s_1, s_2)$ if (s_1, s_2) is a Pareto equilibrium of

$(V_1(r_1, r_2), V_2(r_1, r_2))$, in which case there is no (σ_1, σ_2) satisfying condition (i).

Denote by $N(s_1, s_2)$ the set of all Nash bargaining solutions associated with

$(s_1, s_2) \in S_1 \otimes S_2$. One can see easily that if $(\sigma_1, \sigma_2) \in N(s_1, s_2)$, then (σ_1, σ_2)

is a Pareto equilibrium of $(V_1(r_1, r_2), V_2(r_1, r_2))$. In the sequel, we shall assume :

(A1) $\{(s_1, s_2) : N(s_1, s_2) \neq \emptyset\} = S_1 \otimes S_2$, and

(A2) $W(S_1 \otimes S_2)$ is convex,

which ensure existence and uniqueness of a Nash bargaining solution associated with
(s_1, s_2), for all $(s_1, s_2) \in S_1 \otimes S_2$.

From Definition 5 and elementary properties of convex sets, one obtains

Lemma 4. Let (A1), (A2) hold. Then the following conditions are equivalent :
(i) (σ_1, σ_2) is the Nash bargaining solution associated with $(s_1, s_2) \in S_1 \otimes S_2$;

(ii) there exists a unique $\mu > 0$, such that
(a) $V_1(\sigma_1, \sigma_2) - V_1(s_1, s_2) = \mu [V_2(\sigma_1, \sigma_2) - V_2(s_1, s_2)]$; and

(b) $V_1(\sigma_1, \sigma_2) + \mu V_1(\sigma_1, \sigma_2) \geq V_1(r_1, r_2) + \mu V_1(r_1, r_2)$,

for all $(r_1, r_2) \in S_1 \otimes S_2$.

Let (A1), (A2) hold. Let N denote the mapping which associates with each
$(s_1, s_2) \in S_1 \otimes S_2$, the Nash bargaining solution (σ_1, σ_2). Let $Z = W$ o N; that is

$Z : S_1 \otimes S_2 \rightarrow R^2$, with $Z_i(s_1, s_2) = [V_1(N(s_1, s_2)), V_2(N(s_1, s_2))]$, i=1,2.

Definition 6. A *Nash-optimal threat decision pair* is a Nash-equilibrium of the game with
payoff functions Z_1, Z_2.

From Lemma 4 and Definition 6, one can easily deduce the following
Lemma 5. Let (A1), (A2) hold. Then (s^*_1, s^*_2) is a Nash-optimal threat decision pair if

and only if the saddle-point condition
$V_1(s^*_1, s_2) - \mu V_2(s^*_1, s_2) \geq V_1(s^*_1, s^*_2) - \mu V_2(s^*_1, s^*_2)$

$\geq V_1(s_1, s^*_2) - \mu V_2(s_1, s^*_2)$

is satisfied for all $(s_1, s_2) \in S_1 \otimes S_2$.

From Corollary 1, we see that, *under* (A1) *and* (A2), *a Nash-optimal threat decision pair is a special case of a C-optimal threat decision pair.*

5. C-Optimal Threat Strategy Pairs in a Two-Player Differential Game
Consider now a two-person differential game with state equations

$$dx(t)/dt = f(x(t), p^1(x(t)), p^2(x(t))) ; \qquad (2)$$

where $x = (x_1, x_2, \ldots x_n) \in X$, X is a domain in Euclidean space E^n, and f is Borel

measurable on $X \otimes E^{d_1} \otimes E^{d_2}$. Let P^j denote the space of all Borel measurable

functions from X into E^{d_i}, $i = 1, 2$. A *strategy* $p^i : X \to E^{d_i}$ is *admissible* if and only

if $p^i \in P^i$ and

$$p^i(x) \in U^i(x) \quad \text{for all } x \in X.$$

for given functions

$$U^i : X \to \text{set of all nonempty subsets of } E^{d_i}.$$

We suppose that the target θ is a subset of ∂X.

A strategy pair $p = (p^1, p^2)$ is *playable* at x^0 if it is admissible and generates at least

one *terminating path* $x(.) : [t_0, t_f] \to X \cup \theta$, solution of (2), such that $x(t_0) = x^0$,

$x(t_f) \in X$ for all $t \in [t_0, t_f)$, and $x(t_f) \in \theta$. Let $J(x^0)$ denote the set of all strategy pairs

playable at x^0; we assume that $J(x^0)$ is nonempty. Let $I(x^0, p)$ denote the set of all

terminating paths generated by p from x^0.

The payoffs corresponding to a path $x(.) : [t_0, t_1] \to X \cup \theta$, generated by a pair

$p \in J(x^0)$ from x^0, are given by

$$V_i(x^0, p, x(.)) := \int_{t_0}^{t_1} h_i(x(t), p(x(t))) \, dt, \quad i = 1, 2.$$

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where h_i : $X \otimes E^{d}1 \otimes E^{d}2$. $\rightarrow R^1$ are given real valued bounded Borel

measurable functions.

<u>Definition 7.</u> A pair $p^* = (p^{1^*}, p^{2^*})$ is a C-*optimal threat strategy pair* at x^0 if and only if

(i) $p^* \in J(x^0)$, and

(ii) $V_i(x^0, p^*, x^*(.)) = V_i(x^0, p^*, x^{**}(.)) := V^*_i(x^0, p^*)$, i=1,2,

for all $x^{**}(.) \in I(x^0, p^*)$,

(iii) $(V_1(x^0, p^{1^*}, p^2, x(.)), V_2(x^0, p^{1^*}, p^2, x(.))$ $(\geq)$

$$(V^*_1(x^0, p^*), V^*_2(x^0, p^*))$$

for all $(p^{1^*}, p^2) \in J(x^0)$, and all $x(.) \in I(x^0, p^{1^*}, p^2)$; and

(iv)$(V^*_1(x^0, p^*), V^*_2(x^0, p^*))$ $(\geq)$

$$(V_1(x^0, p^1, p^{2^*}, x(.)), V_2(x^0, p^1, p^{2^*}, x(.)))$$

for all $(p^1, p^{2^*}) \in J(x^0)$, and all $x(.) \in I(x^0, p^1, p^{2^*})$.

Below we state a sufficiency theorem. Before giving the theorem we need some
definitions.

<u>Definition 8.</u> A *denumerable decomposition* D of a subset X of E^n is a denumerable

collection of pairwise disjoint subsets whose union is X. We shall write $D = \{X_k :$
$k \in \mathcal{I}\}$ where $\mathcal{I}$ is a denumerable index set of the pairwise disjoint subsets.

<u>Definition 9.</u> Let X be a subset of E^n and D a denumerable decomposition of X. A

continuous $V : X \rightarrow R^1$ is continuously differentiable with respect to D if and only if

there exists a collection $\{ (D_k, V_k) : k \in \mathcal{I}\}$ such that D_k is an open set containing X_k,

$V_k : D_k \rightarrow R^1$ is continuously differentiable, and $V_k(x) = V(x)$ for $x \in X_k$.

Now we are ready to state

<u>Theorem 1.</u> A strategy pair $p^* = (p^{1^*}, p^{2^*})$, contained in $J(x^0)$, is a C-optimal threat

strategy pair at x^0 if there exists a denumerable decomposition D of X, two constants

α_1, $\alpha_2 > 0$, and two continuous functions $V_i^*: X \cup \theta \rightarrow R^1$, $i = 1, 2$, which are

continuously differentiable with respect to D, such that

(i) $\int_{t_0}^{t_1^*} h_i(x^*(t), p^*(x^*(t)))\, dt = V_i^*(x^0)$ \qquad for all $x^*(.) \in I(x^0, p^*)$,

where t_1^* is the terminating time for $x^*(.)$;

(ii) $h_1(x, u, p^{2^*}(x)) - \alpha_1 h_2(x, u, p^{2^*}(x)) +$

$$\text{grad}\,(V^*{}_1^k - \alpha_1 V^*{}_2^k)(x)\,.\,f(x, u, p^{2^*}(x)) \leq 0$$

for all $x \in X_k$, $u \in U^1(x)$, $k \in \mathfrak{L}$.

(iii) $h_1(x, p^{1^*}(x), v) - \alpha_2 h_2(x, p^{1^*}(x), v) +$

$$\text{grad}\,(V^*{}_1^k - \alpha_2 V^*{}_2^k)(x)\,.\,f(x, p^{1^*}(x), v) \geq 0$$

for all $x \in X_k$, $v \in U^2(x)$, $k \in \mathfrak{L}$.

(iv) $V_i^*(x) = 0$ for all $x \in \theta$, $i = 1, 2$;

where $\{ (D_k, V^*{}_i^k) : k \in \mathfrak{L}\}$ is a collection associated with V_i^* and $D = \{X_k : k \in \mathfrak{L}\}$

for each $i = 1, 2$.

That theorem is a straightforward consequence of Theorem 1 of Stalford and Leitmann (1973), and Lemma 3.

6. Example of C-Optimal Threat Strategies in Collective Bargaining.

Theorem 1 can be easily applied to a dynamical game model of labor-management negotiations during a period that may but need not include a strike.

Let $[0, T]$ denote the unspecified interval during which negotiations take place. At $t \in [0, T]$, let o(t) denote the offer by management of total wages per unit time, d(t) the demand by labor for total wages per unit time, and k = const the gross profit of company per unit time. The evolution of the game is governed by differential equations

do(t)/dt = u(t), \qquad $u(t) \in [0, 1]$,

dd(t)/dt = - v(t) \qquad $v(t) \in [0, 1]$.

Starting from given initial conditions, settlement is reached the first time the offer equals demand, that is, at time T such that $d(T) - o(T) = 0$.

Thus, management chooses the rate of change of the offer, and the union chooses the rate of change of the demand. In addition, the union has the option of calling, or not calling, a strike. We represent this by another control variable w for the union, where $w \in \{0, 1\}$. We take $w = 1$ to correspond to a strike and $w = 0$ to the absence of strike.

The objective of management is to minimize the final offer $o(T)$ and the profit lost during strikes, assumed given by

$$\int_0^T \{w(x(t))[k - d(t)]\} \, dt \,,$$

The union, for its part, wishes to maximize the final offer $o(T)$ and minimize the wages lost during strikes, given by

$$\int_0^T \{w(x(t))o(t)\} \, dt \,.$$

We thus take the payoffs

$$V_1(x^0, u(.), v(.), w(.), x(.)) = - o(T) - a \int_0^T \{w(x(t))[k - d(t)]\} \, dt \,,$$

$$V_2(x^0, u(.), v(.), w(.), x(.)) = o(T) - b \int_0^T \{w(x(t))o(t)\} \, dt \,.$$

for the management and union, respectively, where a, b are positive constants.

This example has been worked out by Ray (1981) from the point of view of C-optimal threat strategy pairs. The general conclusion is the following : (i) whether or not the union threatens to strike depends on whether the offer $o^*(t)$ is less or greater than a certain fraction of the potential profit $k - d^*(t)$; and (ii) if a strike is threatened, then the union will also threaten not to lower the demand as termination is approached. This example has been discussed earlier by Leitmann (1973) who characterizes rational behaviour by a saddle-point condition. It follows that Leitmann's solution is a Nash-optimal threat strategy solution.

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