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Three-dimensional thermo-visco-elasticity with the Einstein-Debye $(\theta^3 + \theta)$-law for the specific heat. Global regular solvability

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Abstract

A three-dimensional thermo-visco-elastic system for Kelvin-Voigt type material at small strain is considered. The system involves constant heat conductivity and the specific heat satisfying the Einstein-Debye $(\theta^3 + \theta)$-law. Such nonlinear law, relevant at relatively low temperatures, represents the main novelty of the paper. The existence of global regular solutions is proved without small data assumption. The crucial part of the proof is the strictly positive lower bound on the absolute temperature θ. In case of the Debye θ^3-law this still remains an unsolved problem.
The existence of local in time solution is proved by the Banach successive approximations method. The global a priori estimates are derived with the help of the theory of anisotropic Sobolev spaces with a mixed norm. Such estimates allow to extend the local solution step by step in time.
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1 Introduction

The aim. In this paper we study three-dimensional (3-D) thermo-visco-elastic system at small strains with constant heat conductivity $k > 0$, and specific heat (heat capacity) $c(\theta)$ satisfying the Einstein-Debye $(\theta^3 + \theta)$-law, $c(\theta) = c_v^1\theta^3 + c_v^2\theta$, where $\theta > 0$ is the absolute temperature and c_v^1, c_v^2 positive constants. The system describes homogeneous, isotropic, linearly responding materials in the Kelvin-Voigt rheology at relatively low temperatures $\theta \ll \theta_D$, below the Debye temperature θ_D. According to the Debye theory the specific heat c depends on θ/θ_D with θ_D as scaling factor for different materials (known for most materials, see e.g., the monograph by Kittel [16]).

The present paper continues our previous studies [23], [24], where we addressed global regular solvability of thermo-visco-elastic systems with the specific heat of the forms $c(\theta) = c_v\theta$, $c_v = \text{const} > 0$ in [23], and $c(\theta) = c_v\theta^\sigma$, $\sigma \in \left(\frac{1}{2}, 1\right]$ in [24]. Such forms of $c(\theta)$ are relevant at very low temperature below the range where the Debye law $c(\theta) = c_v\theta^3$ is appropriate.

The Einstein-Debye $(\theta^3 + \theta)$-law combining the Einstein θ-law and the Debye θ^3-law is typical for metals at low temperatures at which electron contribution becomes significant.

Prior to discussing mathematical motivations and pointing out the associated technical difficoulties for this type of problems, let us add few physical comments (for more details see section 2).

Specific heat has a weak temperature dependence at high temperatures $\theta \gg \theta_D$ above the Debye temperature θ_D, but decreases down to zero as θ approaches 0. The constant value of the specific heat of many solids is usually referred to as *Dulong-Petit law*. In 1819 Dulong and Petit [26] found experimentally that for many solids at room temperature specific heat is constant.

At this point it is important to emphasize that the global solvability of 3-D thermo-visco-elastic system with constant heat conductivity k and constant specific heat c is in spite of great effort through many decades still open in dimensions $n \geqslant 2$. In dimension $n = 1$ it was established already at the beginning of ninetieth of the last century by Slemrod [31], Dafermos [6], and Defermos and Hsiao [7]. For detailed references concerning solvability of thermo-visco-elastic systems we refer to Roubíček [27], [28], [29], author's papers [23], [24], and the recent review paper by Zvyagin and Orlov [34]. All known results on multidimensional thermo-visco-elasticity deal with a modified energy equation. Modifications involve either nonconstant specific heat or nonconstant heat conductivity. In view of the Einstein and the Debye theories it seems natural to consider thermo-visco-elastic systems with nonlinear temperature-dependent specific hat. Our primary mathematical goal in this paper was to admit the Debye θ^3-law, $c(\theta) = c_v\theta^3$. To our best knowledge such problem has not been so far addressed in mathematical literature. Unfortunately, in the case of the θ^3-law we have been faced with a serious mathematical obstacle to prove strictly positive lower bound for the absolute temperature. We have managed to prove this after adding a linear (possibly small) term $c_v^2\theta$, $c_v^2 = \text{const} > 0$. In other words, we have assumed the Einstein-Debye $(\theta^3+\theta)$-law, $c(\theta) = c_v^1\theta^3 + c_v^2\theta$. Having proved the strict positivity of θ the existence of global regular solutions to the thermo-visco-elastic system can be concluded by using similar arguments as in [24]. These arguments, based on the idea of successive improvement of energy estimates by the application of the theory of anisotripic Sobolev spaces with a mixed norm, indicate that the main role plays just the term $c_v^1\theta^3$. Therefore, all considerations could be repeated provided the lower bound for θ is established.

Finally, let us remark that apart from the mathematical issues the system under cosideration may be of some practical interest in the cryogenic engineering problems where one needs to understand and characterize the behaviour of various materials on the basis of the mathematical model and recorded materials properties.

Thermo-visco-elastic system. The system under consideration has the following form

$$u_{tt} - \nabla\cdot[A_1\varepsilon_t + A_2(\varepsilon - \theta\alpha)] = b \quad \text{in} \quad \Omega^T := \Omega\times(0,T), \tag{1.1}$$

$$(c_v^1\theta^3 + c_v^2\theta)\theta_t - k\Delta\theta = -\theta(A_2\alpha)\cdot\varepsilon_t + (A_1\varepsilon_t)\cdot\varepsilon_t + g \quad \text{in} \quad \Omega^T, \tag{1.2}$$

where

$$\varepsilon \equiv \varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T), \quad \varepsilon_t \equiv \varepsilon(u_t) = \frac{1}{2}(\nabla u_t + (\nabla u_t)^T),$$

and c_v^1, c_v^2, k are positive constants.
Here $\Omega\subset\mathbb{R}^3$ is a bounded domain occupied by a body in a fixed reference configuration, and $(0,T)$ is the time interval. The system is completed by appropriate boundary and initial conditions. We assume

$$u = 0, \quad n\cdot\nabla\theta = 0 \quad \text{on} \quad S^T := S\times(0,T), \tag{1.3}$$

$$u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1, \quad \theta|_{t=0} = \theta_0 \quad \text{in} \quad \Omega, \tag{1.4}$$

where S is the boundary of Ω and n is the unit outward normal to S.

The field $u : \Omega^T \to \mathbb{R}^3$ is the displacement, $\theta : \Omega^T \to \mathbb{R}_+ = (0, \infty)$ is the absolute temperature, the second order tensors $\varepsilon = (\varepsilon_{ij})_{i,j=1,2,3}$ and $\varepsilon_t = ((\varepsilon_t)_{ij})_{i,j=1,2,3}$ denote, respectively, the fields of the linearized strain and the strain rate.

Equation (1.1) is the linear momentum balance with the stress tensor given by a linear thermo-visco-elastic law of the Kelvin-Voigt type (cf. [10, Chapter 5.4])

$$S = A_1\varepsilon_t + A_2(\varepsilon - \theta\alpha).$$

The fourth order tensors $A_1 = ((A_1)_{ijkl})_{i,j,k,l=1,2,3}$ and $A_2 = ((A_2)_{ijkl})_{i,j,k,l=1,2,3}$ are, respectively, the linear viscosity and the elasticity tensors, defined by

$$\varepsilon \mapsto A_m\varepsilon = \lambda_m tr\varepsilon I + 2\mu_m\varepsilon, \quad m = 1, 2, \tag{1.5}$$

where λ_1, μ_1 are the viscosity constants and λ_2, μ_2 are the Lamé constants, both λ_1, μ_1 and λ_2, μ_2 with the values within the elasticity range

$$\mu_m > 0, \quad 3\lambda_m + 2\mu_m > 0, \quad m = 1, 2, \tag{1.6}$$

$I = (\delta_{ij})_{i,j=1,2,3}$ is the identity tensor, and $tr\varepsilon$ denotes the trace of ε.

The second order symmetric tensor $\alpha = (\alpha_{ij})_{i,j=1,2,3}$ with constant entries α_{ij} represents the thermal expansion. The vector field $b : \Omega^T \to \mathbb{R}^3$ is the external body force.

Equation (1.2) is the energy balance in which the linear Fourier law for the heat flux $q = -k\nabla\theta$ with constant heat conducitity $k > 0$, and the Einstein-Debye law for the specific heat, $c(\theta) = c_v^1\theta^3 + c_v^2\theta$, with constant $c_v^1, c_v^2 > 0$, have been adopted.
The first two nonlinear terms on the right-hand side of (1.2) represent heat sources created by the deformation of the material due to thermal expansion and by the viscosity. The field $g : \Omega^T \to \mathbb{R}$ is the external heat source. The boundary conditions in (1.3) mean that the body is fixed at the boundary S and is there thermally isolated. The initial conditions (1.4) prescribe displacement, velocity and temperature at $t = 0$.

We remark that since our main goal is to focus on the existence of global regular solutions we have assumed the simplest homogeneous boundary conditions (1.3). However, with some additional technical complications, other types of nonhomogeneous boundary conditions can be considered as well.

The system (1.1)–(1.2) can be derived by various arguments of thermodynamics, see e.g., [13], [21], [27], [3]. In section 2 we summarize its thermodynamic basis. As a main point we emphasise there the Debye and the Einstein-Debye laws of the specific heat.

Above and hereafter the summation convention over the repeated indices is used. Vectors (tensors of the first order), tensors of the second order (referred to simply as tensors), and tensors of higher order are denoted by bold letters. A dot designates the scalar product, irrespective of the space in question, e.g., for $\boldsymbol{u} = (u_i)_{i=1,2,3}$, $\boldsymbol{v} = (v_i)_{i=1,2,3}$, $\boldsymbol{S} = (S_{ij})_{i,j=1,2,3}$, $\boldsymbol{R} = (R_{ij})_{i,j=1,2,3}$, $\boldsymbol{A} = (A_{ijkl})_{i,j,k,l=1,2,3}$, $\boldsymbol{\varepsilon} = (\varepsilon_{ij})_{i,j=1,2,3}$, we have

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i, \quad \boldsymbol{S} \cdot \boldsymbol{R} = S_{ij} R_{ij}, \quad \boldsymbol{S}\boldsymbol{u} = (S_{ij} u_j)_{i=1,2,3},$$

$$\boldsymbol{A}\boldsymbol{\varepsilon} = (A_{ijkl}\varepsilon_{kl})_{i,j=1,2,3}, \quad (\boldsymbol{A}\boldsymbol{\varepsilon}) \cdot \boldsymbol{\varepsilon} = A_{ijkl}\varepsilon_{kl}\varepsilon_{ij},$$

where the summation convention is used.

The term field signifies a function of a material point $\boldsymbol{x} \in \mathbb{R}^3$ and time t. For convenience we use the notation $\boldsymbol{u}_t$ (instead of $\dot{\boldsymbol{u}}$) for the material time derivative of the field $\boldsymbol{u}$ (with respect to t holding $\boldsymbol{x}$ fixed). The operators ∇ and $\nabla\cdot$ denote the material gradient and the divergence (with respect to $\boldsymbol{x}$ holding t fixed). For the divergence we use the convention of the contraction over the last index, e.g.,

$$\nabla \cdot (\boldsymbol{A}\boldsymbol{\varepsilon}) = \left(\frac{\partial}{\partial x_j} (A_{ijkl}\varepsilon_{kl}) \right)_{i=1,2,3}.$$

We write

$$f_{,i} = \frac{\partial f}{\partial x_i}, \quad i = 1,2,3, \quad f_t = \frac{df}{dt}, \quad \boldsymbol{\varepsilon} = (\varepsilon_{ij})_{i,j=1,2,3},$$
$$F_{,\varepsilon}(\boldsymbol{\varepsilon},\theta) = \left(\frac{\partial F(\boldsymbol{\varepsilon},\theta)}{\partial \varepsilon_{ij}} \right)_{i,j=1,\dots,3}, \qquad F_{,\theta}(\boldsymbol{\varepsilon},\theta) = \frac{\partial F(\boldsymbol{\varepsilon},\theta)}{\partial \theta},$$

where space and time derivatives are material.
For simplicity, whenever there is no danger of confusion, we omit arguments $(\boldsymbol{\varepsilon},\theta)$ of function $f(\boldsymbol{\varepsilon},\theta)$. The specification of tensor indices is omitted as well. For vector $\boldsymbol{b} = (b_i)_{i=1,2,3}$ and tensor $\boldsymbol{B} = (B_{ij})_{i,j=1,2,3}$ we denote

$$|\boldsymbol{b}| = (b_i b_i)^{1/2}, \quad |\boldsymbol{B}| = (B_{ij}B_{ij})^{1/2}.$$

Linear elasticity and viscosity operators. For further analysis it is convenient to formulate problem (1.1)–(1.4) in terms of the linear viscosity and elasticity operators, $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$, defined by

(1.7) $\boldsymbol{u} \mapsto \boldsymbol{Q}_m \boldsymbol{u} = \nabla \cdot (\boldsymbol{A}_m \boldsymbol{\varepsilon}(\boldsymbol{u})) = \mu_m \Delta \boldsymbol{u} + (\lambda_m + \mu_m)\nabla(\nabla \cdot \boldsymbol{u}), \quad m = 1,2,$

with domains $D(\boldsymbol{Q}_m) = \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_0(\Omega)$.

For notational simplicity we introduce also the second order symmetric tensor $\boldsymbol{B} = (B_{ij})$ defined by

$$\boldsymbol{B} := -\boldsymbol{A}_2\boldsymbol{\alpha} = -((A_2)_{ijkl}\alpha_{kl}). \tag{1.8}$$

Then system (1.1)–(1.2) takes the form

$$\begin{aligned} &u_{tt} - Q_1 u_t = Q_2 u + \nabla\cdot(\theta \boldsymbol{B}) + b && \text{in } \Omega^T,\\ &(c_v^1\theta^3 + c_v^2\theta)\theta_t - k\Delta\theta = \theta\boldsymbol{B}\cdot\varepsilon_t + (\boldsymbol{A}_1\varepsilon_t)\cdot\varepsilon_t + g && \text{in } \Omega^T, \end{aligned} \tag{1.9}$$

with boundary and initial conditions (1.3), (1.4).

Assumptions and their implications. Throughout we shall assume that

(A1) $\Omega \subset \mathbb{R}^3$ is a bounded domain with the boundary S of class at least C^2; $T > 0$ is an arbitrary finite number;

(A2) $\boldsymbol{\alpha} = (\alpha_{ij})_{i,j=1,2,3}$ is a second order symmetric tensor with constant entries α_{ij};

(A3) The fourth order tensors $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are defined by (1.5) with the coefficients μ_m, λ_m, $m = 1, 2$, satisfying (1.6).

We list the implications of assumption (A3) which are used in further analysis. The conditions (1.5), (1.6) ensure the symmetry of tensors $\boldsymbol{A}_m$

$$(A_m)_{ijkl} = (A_m)_{jikl} = (A_m)_{klij}, \quad m = 1, 2, \tag{1.10}$$

and their coercivity and boundedness

$$a_{m*}|\varepsilon|^2 \le (\boldsymbol{A}_m\varepsilon)\cdot\varepsilon \le a_m^*|\varepsilon|^2, \quad m = 1, 2, \tag{1.11}$$

where

$$a_{m*} = \min\{3\lambda_m + 2\mu_m, 2\mu_m\}, \quad a_m^* = \max\{3\lambda_m + 2\mu_m, 2\mu_m\}.$$

Moreover, (1.6) ensures the following properties of operators Q_m, $m = 1, 2$:

- Q_m are strongly elliptic (property holding true under weaker assumption $\mu_m > 0$, $\lambda_m + 2\mu_m > 0$, (see [25, section 7])) and satisfy the estimate (see [20, Lemma 3.2]):

  $$c_m\|u\|_{H^2(\Omega)} \le \|Q_m u\|_{L_2(\Omega)} \quad \text{for} \quad u \in D(Q_m), \quad m = 1, 2, \tag{1.12}$$

  with positive constants c_m depending on Ω. Since clearly,

  $$\|Q_m u\|_{L_2(\Omega)} \le \bar{c}_m\|u\|_{H^2(\Omega)}, \quad \bar{c}_m > 0,$$

  it follows that the norms $\|Q_m u\|_{L_2(\Omega)}$ and $\|u\|_{H^2(\Omega)}$ are equivalent on $D(Q_m)$.

- The operators Q_m are self-adjoint on $D(Q_m)$:

(1.13)
$$(Q_m u, v)_{L_2(\Omega)} = -\mu_m(\nabla u, \nabla v)_{L_2(\Omega)} - (\lambda_m + \mu_m)(\nabla\cdot u, \nabla\cdot v)_{L_2(\Omega)} = (u, Q_m v)_{L_2(\Omega)} \quad \text{for} \quad u, v \in D(Q_m).$$

- The operators $-Q_m$ are positive on $D(Q_m)$:

$$(1.14)\qquad (-Q_m u, u)_{L_2(\Omega)} = \mu_m\|\nabla u\|^2_{L_2(\Omega)} + (\lambda_m + \mu_m)\|\nabla\cdot u\|^2_{L_2(\Omega)} \geq 0 \quad \text{for} \quad u \in D(Q_m).$$

Hence, there exist fractional powers $Q_m^{1/2}$ with the domains $D(Q_m^{1/2}) = H_0^1(\Omega)$, satisfying

$$(1.15)\qquad (Q_m^{1/2}u, Q_m^{1/2}v)_{L_2(\Omega)} = (-Q_m u, v)_{L_2(\Omega)} = (u, -Q_m v)_{L_2(\Omega)} \quad \text{for} \quad u, v \in D(Q_m).$$

Let us also notice that by (1.11) and the Korn inequality

$$(1.16)\qquad d^{1/2}\|u\|_{H^1(\Omega)} \leq \|\varepsilon(u)\|_{L_2(\Omega)} \quad \text{for} \quad u \in H_0^1(\Omega), \quad d > 0,$$

it follows that

(1.17)
$$\|Q_m^{1/2}u\|^2_{L_2(\Omega)} = \mu_m\|\nabla u\|^2_{L_2(\Omega)} + (\lambda_m + \mu_m)\|\nabla\cdot u\|^2_{L_2(\Omega)} = (A_m\varepsilon(u), \varepsilon(u))_{L_2(\Omega)} \geq a_{m*}\|\varepsilon(u)\|^2_{L_2(\Omega)} \geq a_{m*}d\|u\|^2_{H^1(\Omega)}.$$

Thus, the norms $\|Q_m^{1/2}u\|_{L_2(\Omega)}$ and $\|u\|_{H^1(\Omega)}$ are equivalent on $D(Q_m^{1/2})$.

Main result. This result is analogous to that proved in [24].

Theorem 1.1 (existence). *Let the assumptions (A1)–(A3) formulated above be satisfied, and*

$$b \in L_{10^+}(\Omega^T) \cap L_{5,12}(\Omega^T), \quad u_0 \in W^2_{5^+}(\Omega),$$
$$u_1 \in B^{2-2/5^+}_{5^+,5^+}(\Omega), \quad g \in L_{5^+}(0,T; L_\infty(\Omega)), \quad g \geq 0,$$
$$\theta_0 \in H^1(\Omega) \cap B^{2-2/5^+}_{5^+,5^+}(\Omega) \cap L_\infty(\Omega), \quad \theta_0 \geq \underline{\theta} > 0,$$

where $\underline{\theta}$ is a constant. Then there exists a global solution to problem (1.1)–(1.4) such that

$$u_t \in W^{2,1}_{5^+}(\Omega^T) \quad \text{and} \quad \theta \in W^{2,1}_{5^+}(\Omega^T),$$

where 5^+ *ia any number larger than 5 but close to 5. The spaces used above are defined in section 3. Moreover,*

$$\theta(t) \geq \underline{\theta} \exp(-at) \equiv \theta_*(t) \quad \textit{for} \quad t \leq T,$$

where a *is a positive constant given by* $a = \frac{|B|}{2a_{1*} \min\{c_v^1, c_v^2\}}$.

Plan of the paper. In section 2 we present the thermodynamic basis of system (1.1)–(1.2). In section 3 we define spaces used in this paper, in particular the anisotropic Sobolev spaces with a mixed norm. We recall the corresponding imbeddings and interpolations as well as the trace and the inverse trace theorems for the Sobolev-Slobodetskii spaces with a mixed norm. Moreover, we present auxiliary results on the solvability of linear parabolic initial-boundary value problems in such spaces. Section 4 is devoted to the proof of a global positive infimum of temperature. In section 5, applying the Banach method of successive approximations, we state the local existence of solutions such that $u_t \in W_{5+}^{2,1}(\Omega^t)$ and $\theta \in W_{5+}^{2,1}(\Omega^t)$, where $t > 0$ is sufficiently small. In the proof we can use exactly the same arguments as in [24, section 5]. In section 6 we derive a priori global estimates such that $u_t \in W_{5+}^{2,1}(\Omega^t)$ and $\theta \in W_{5+}^{2,1}(\Omega^t)$ where $t > 0$ is arbitrary finite. In this case the derivation is much shorter than in [24].
Combining the results of sections 5 and 6 in section 7 we conclude the global existence of solutions.

2 Thermodynamic basis

We recall (see [23], [24]) the thermodynamic basis of the thermo-visco-elastic system (1.1)–(1.2) with the special emphasis on the Debye θ^3-law and the Einstein-Debye $(\theta^3 + \theta)$-law of the specific heat.

The system (1.1)–(1.2) represents the local forms of the balance laws for the linear momentum and the internal energy in a referential description, with the referential mass density assumed constant, normalized to unity, $\varrho_0 = 1$:

$$\begin{aligned} &u_{tt} - \nabla \cdot S = b, \\ &e_t + \nabla \cdot q - S \cdot \varepsilon_t = g. \end{aligned} \tag{2.1}$$

Here S is the stress tensor, q is the referential heat flux, and e is the specific internal energy.

The system is governed by two thermodynamic potentials. The first one is the specific free energy $f = \hat{f}(\varepsilon, \theta)$ which by a thermodynamic requirement is strictly concave with respect to $\theta > 0$ for all ε. The second one is the dissipation potential $\mathcal{D} = \hat{\mathcal{D}}(\varepsilon_t, \nabla\theta; \varepsilon, \theta)$, which by a thermodynamic requirement is nonnegative, convex in $(\varepsilon_t, \nabla\theta)$ – variables and such that $\mathcal{D}(0, 0; \varepsilon, \theta) = 0$ for all (ε, θ). In [14], [3] $\mathcal{D}$ is referred to as pseudopotential of dissipation.

The free energy. For system (1.1)–(1.2) it has the form

$$f(\varepsilon, \theta) = f_*(\theta) + W(\varepsilon, \theta), \tag{2.2}$$

where

$$f_*(\theta) = -\frac{c_v^1}{12}\theta^4 - \frac{c_v^2}{2}\theta^2, \quad c_v^1, c_v^2 = \text{const} > 0, \tag{2.3}$$

is the thermal (caloric) energy associated with the Einstein-Debye law of the specific heat. The case $c_v^2 \equiv 0$ corresponds to the Debye law. We point out that the form (2.3) is relevant at low temperature range; see comments below.

The second term in (2.2) represents the elastic energy

$$\begin{aligned} W(\varepsilon, \theta) &= \frac{1}{2}(\varepsilon - \theta\boldsymbol{\alpha}) \cdot \boldsymbol{A}_2(\varepsilon - \theta\boldsymbol{\alpha}) - \frac{\theta^2}{2}\boldsymbol{\alpha} \cdot (\boldsymbol{A}_2\boldsymbol{\alpha}) \\ &= \frac{1}{2}\varepsilon \cdot (\boldsymbol{A}_2\varepsilon) - \theta\varepsilon \cdot (\boldsymbol{A}_2\boldsymbol{\alpha}). \end{aligned} \tag{2.4}$$

We remind that $\boldsymbol{A}_2$ stands for the fourth order elasticity tensor given by (1.5), and $\boldsymbol{\alpha}$ for the second order thermal expansion tensor. In case of isotropic material

$$\boldsymbol{\alpha} = \alpha \boldsymbol{I}, \tag{2.5}$$

where $\alpha > 0$ is the thermal dilatability coefficient, and $\boldsymbol{I} = (\delta_{ij})$ is the unit matrix. Thus, in isotropic case the thermal expansion contribution to the elastic energy (2.4) reduces to

$$-\theta\varepsilon \cdot (\boldsymbol{A}_2\boldsymbol{\alpha}) = -\theta\alpha(3\lambda_2 + 2\mu_2)tr\varepsilon. \tag{2.6}$$

According to the thermodynamic Gibbs relations, the entropy η, the internal energy e and the specific heat c are related to the free energy f by the equations

$$\eta = -f_{,\theta}, \quad e = f + \theta\eta, \quad c = e_{,\theta} = \theta\eta_{,\theta} = -\theta f_{,\theta\theta}. \tag{2.7}$$

For free energy (2.2) this yields

$$\begin{aligned} &\eta = -f_{,\theta}(\varepsilon, \theta) = \eta_*(\theta) + \varepsilon \cdot (\boldsymbol{A}_2\boldsymbol{\alpha}), \\ &e = f(\varepsilon, \theta) + \theta\eta(\varepsilon, \theta) = e_*(\theta) + \frac{1}{2}\varepsilon \cdot (\boldsymbol{A}_2\varepsilon), \\ &c = e_{,\theta}(\varepsilon, \theta) = c_*(\theta), \end{aligned} \tag{2.8}$$

where the thermal free energy f_* is given by (2.3) and the associated entropy, internal energy and specific heat are

$$
\begin{aligned}
&\eta_*(\theta) = -f_{*,\theta} = \frac{c_v^1}{3}\theta^3 + c_v^2\theta, \\
(2.9) \qquad &e_*(\theta) = f_*(\theta) + \theta\eta_*(\theta) = \frac{c_v^1}{4}\theta^4 + \frac{c_v^2}{2}\theta^2, \\
&c_*(\theta) = e_{*,\theta} = c_v^1\theta^3 + c_v^2\theta,
\end{aligned}
$$

which according to $(2.1)_2$ and $(2.8)_3$ gives rise to the term $(c_v^1\theta^3 + c_v^2\theta)\theta_t$ in energy equation (1.2).

Remarks on the theories of spcific heat. There exists extensive literature in solid state physics on the theories of specific heat (see, e.g., [2], [5], [16], [19], [30], [12]). It seems to be of interest to compile some basic facts on the four well-known models of the specific heat:

- – the classical *Dulong-Petit model* (1819) [26];
- – the quantum mechanical *Einstein model* (1907) [11];
- – the *Debye model* (1912) [8] expanding the Einstein model;
- – the *Einstein-Debye model* for metals at low temperatures.

In the Dulong-Petit model the specific heat is constant. It is known to show poor agreement with experiment except at high temperatures. The Einstein model yields good agreement with experiment at very high and very low temperatures, but not inbetween. The Debye theory provides more accourate model. The thermal energy expression from the Debye theory of specific heat is of the form (in our notation)

$$
(2.10) \qquad e_*(\theta) = \bar{c}\frac{\theta^4}{\theta_D^3}\int_0^{\theta_D/\theta} \frac{x^3}{\exp x - 1}dx,
$$

where θ_D is the *Debye temperature* and $\bar{c}$ a positive physical constant. Thus, the Debye specific heat is the function of the ratio $\xi = \theta/\theta_D$, given by

$$
(2.11) \qquad c_*(\theta) = e_{*,\theta} = \bar{c}D\left(\frac{\theta}{\theta_D}\right),
$$

where

$$
(2.12) \qquad D(\xi) = 4\xi^3\int_0^{1/\xi} \frac{1}{\exp x - 1}dx - \frac{1}{\xi(\exp 1/\xi - 1)}
$$

is known as the *Debye specific heat function.* Even though the integral in (2.10) and (2.12) cannot be evaluated in closed form, the low and high temperature limits can be assessed.

For the high temperature case where $\theta \gg \theta_D$, the value of x is very small throughout the range of integral. This justifies using the apprximation to the exponential by the exponential series exp $\cong 1 + x$. This reduces the energy expression (2.10) to (see, e.g., [30, Chapter 7])

$$e_*(\theta) = \bar{c}\frac{\theta^4}{\theta_D^3}\int_0^{\theta_D/\theta} x^2 dx = \frac{\bar{c}}{3}\frac{\theta^4}{\theta_D^3}\left(\frac{\theta_D}{\theta}\right)^3 = \frac{\bar{c}}{3}\theta. \tag{2.13}$$

Hence, in this case

$$c_*(\theta) = e_{*,\theta} = \frac{\bar{c}}{3}, \tag{2.14}$$

which yields the constant Dulong-Petit specific heat.

For low temperatures where $\theta \ll \theta_D$, the exponential in the denominator becomes very large before reaching the limit, implying that the integrand in (2.10) is very small near the upper limit. This makes it plausible to approximate the integral by increasing the limit to infinity to make use of the standard integral

$$\int_0^\infty \frac{x^3}{\exp x - 1}dx = \frac{\pi^4}{15}.$$

Then the energy becomes

$$e_*(\theta) = \frac{\bar{c}\pi^4}{15}\frac{\theta^4}{\theta_D^3}, \tag{2.15}$$

so that the corresponding specific heat is

$$c_*(\theta) = e_{*,\theta} = c_1\left(\frac{\theta}{\theta_D}\right)^3, \quad \text{where} \quad c_1 = \frac{4\pi^4}{15}\bar{c}. \tag{2.16}$$

This yields the *Debye* θ^3*-law* for the specific heat (see e.g., [2, section 4.3]). This θ^3-form of the specific heat at low temperatures is known to agree with experiment for nonmetals. For metals the electronic specific heat becomes significant at low temperatures and results in the additional linear term in θ

$$c_*(\theta) = c_1\left(\frac{\theta}{\theta_D}\right)^3 + c_2\theta, \quad c_2 = \text{const} > 0. \tag{2.17}$$

Such form of the specific heat is referred to as the *Einstein-Debye specific heat.* The θ^3 term arises from lattice vibrations, and the linear term from electrons conduction. The Einstein contribution $c_2\theta$ becomes dominating at very low temperatures.

The dissipation potential. For system (1.1)–(1.2) it has the form (see, e.g., [23])

$$\mathcal{D} = \frac{1}{2\theta}\varepsilon_t \cdot (A_1\varepsilon_t) + \frac{k}{2}\theta^2 \left|\nabla \frac{1}{\theta}\right|^2 = \frac{1}{2\theta}\varepsilon_t \cdot (A_1\varepsilon_t) + \frac{k}{2}|\nabla \ln \theta|^2, \tag{2.18}$$

where A_1 is the fourth order viscosity tensor given by (1.5), and $k > 0$ is the constant heat conductivity.

As a consequence of the second law of thermodynamics expressed by the Clausius-Duhem inequality, the stress tensor S and the heat flux q satisfy the following relations (see e.g., [21])

$$S = \frac{\partial f}{\partial \varepsilon} + \theta \frac{\partial \mathcal{D}}{\partial \varepsilon_t}, \quad q = \frac{\partial \mathcal{D}}{\partial \nabla \frac{1}{\theta}}. \tag{2.19}$$

For f defined by (2.2)–(2.4) and $\mathcal{D}$ by (2.18) the formulas (2.19) yield the standard forms of the stress tensor and the heat flux

$$S = A_2(\varepsilon - \theta\alpha) + A_1\varepsilon_t, \quad q = k\theta^2\nabla\frac{1}{\theta} = -k\nabla\theta. \tag{2.20}$$

The relations (2.20) show that the stress tensor S is composed of two terms: the nondissipative elastic term determined by f and the dissipative one determined by $\mathcal{D}$. The dissipative heat flux q is entirely determined by $\mathcal{D}$.

Inserting the relations $(2.8)_2$, $(2.9)_2$ and (2.20) into balance laws (2.1) we arrive at the thermo-visco-elastic system (1.1)–(1.2).

The system (1.1)–(1.2) complies with the *Clausius-Duhem inequality*

$$\eta_t + \nabla \cdot \frac{q}{\theta} \geq \frac{g}{\theta}. \tag{2.21}$$

To see this (formally) let us note that on account of the identity

$$e_t = (f + \theta\eta)_t = f_t + \theta_t\eta + \theta\eta_t = \theta\eta_t + \frac{\partial f}{\partial \varepsilon} \cdot \varepsilon_t, \tag{2.22}$$

along with the relation $(2.19)_1$, the energy balance $(2.1)_2$ admits the form

$$\theta\eta_t + \nabla \cdot q = \theta\frac{\partial \mathcal{D}}{\partial \varepsilon_t} \cdot \varepsilon_t + g. \tag{2.23}$$

For $\mathcal{D}$ defined by (2.18) this leads to the following equivalent form of (1.2)

$$\theta\eta_t - k\Delta\theta = (A_1\varepsilon_t)\cdot\varepsilon_t + g. \tag{2.24}$$

Let us also note that assuming $\theta > 0$ and using the relation $(2.19)_2$, equation (2.23) may be expressed as

$$\eta_t + \nabla\cdot\frac{q}{\theta} = \sigma + \frac{g}{\theta}, \tag{2.25}$$

where

$$\sigma := \frac{\partial\mathcal{D}}{\partial\nabla\frac{1}{\theta}}\cdot\nabla\frac{1}{\theta} + \frac{\partial\mathcal{D}}{\partial\varepsilon_t}\cdot\varepsilon_t = k\theta^2\left|\nabla\frac{1}{\theta}\right|^2 + \frac{1}{\theta}(A_1\varepsilon_t)\cdot\varepsilon_t \geq 0 \tag{2.26}$$

is the specific entropy production. Hence, the Clausius-Duhem inequality follows. This inequality together with the positive lower bound for temperature constitute the basis of energy estimates in the existence proof, see sections 4–6.

3 Notation and auxiliary results

For readers convenience this section recalls basic facts from [24, section 3] and adds new ones.

Notation. Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a domain in $\mathbb{R}^n$ with boundary S. Let $\Omega^T = \Omega \times (0,T)$, $S^T = S \times (0,T)$ with $T > 0$ finite. By $W_p^k(\Omega)$, $k \in \mathbb{N}\cup\{0\} \equiv \mathbb{N}_0$, $p \in [1,\infty)$, we denote the Sobolev space with the finite norm

$$\|u\|_{W_p^k(\Omega)} = \left(\sum_{|\alpha|\leq k}\int_\Omega |D_x^\alpha u|^p dx\right)^{1/p},$$

where $\alpha = (\alpha_1,\cdots,\alpha_n)$ is a multi-index, $\alpha_i \in \mathbb{N}_0$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots \alpha_n$, $D_x^\alpha = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}$. Let $H^k(\Omega) = W_2^k(\Omega)$.
Next, we introduce anisotropic Lebesgue spaces
$L_{p,p_0}(\Omega^T) = L_{p_0}(0,T;L_p(\Omega))$, $p, p_0 \in [1,\infty]$, with the finite norm

$$\|u\|_{L_{p,p_0}(\Omega^T)} = \left(\int_0^T \|u(t)\|_{L_p(\Omega)}^{p_0} dt\right)^{1/p_0}.$$

Moreover, $W_{p,p_0}^{k,k/2}(\Omega^T)$, $k, k/2 \in \mathbb{N}_0$, $p, p_0 \in [1,\infty]$ are Sobolev spaces with a mixed norm, which are the completion of $C^\infty(\Omega^T)$-functions under the finite norm

$$\|u\|_{W^{k,k/2}_{p,p_0}(\Omega^T)} = \left(\int_0^T \left(\sum_{|\alpha|+2a\le k} \int_\Omega |D^\alpha_x \partial^a_t u|^p dx \right)^{p_0/p} dt \right)^{1/p_0}.$$

By $W^{s,s/2}_{p,p_0}(\Omega^T)$, $s \in \mathbb{R}_+$, $p, p_0 \in [1,\infty]$, we denote the Sobolev-Slobodetskii space with the finite norm

$$\|u\|_{W^{s,s/2}_{p,p_0}(\Omega^T)} = \sum_{|\alpha|+2a\le[s]} \|D^\alpha_x \partial^a_t u\|_{L_{p,p_0}(\Omega^T)}$$

$$+ \left[\int_0^T \left(\int_\Omega \int_\Omega \sum_{|\alpha|+2a=[s]} \frac{|D^\alpha_x \partial^a_t u(x,t) - D^\alpha_{x'} \partial^a_t u(x',t)|^p}{|x-x'|^{n+p(s-[s])}} dx dx' \right)^{p_0/p} dt \right]^{1/p_0}$$

$$+ \left[\int_0^T \int_0^T \left(\int_\Omega \sum_{|\alpha|+2a=[s]} \frac{|D^\alpha_x \partial^a_t u(x,t) - D^\alpha_x \partial^a_{t'} u(x,t')|^p}{|t-t'|^{1+p(s/2-[s/2])}} dx \right)^{p_0/p} dt dt' \right]^{1/p_0},$$

where $a \in \mathbb{N}_0$ and $[s]$ is the integer part of s.
For s odd the one before last term in the above norm vanishes whereas for s even the two last terms vanish.

We use also the notation $L_p(\Omega^T) = L_{p,p}(\Omega^T)$, $W^{s,s/2}_p(\Omega^T) = W^{s,s/2}_{p,p}(\Omega^T)$, and so on.

By $B^l_{p,p_0}(\Omega)$, $l \in \mathbb{R}_+$, $p, p_0 \in [1,\infty)$ we denote the Besov space of functions making the following norm finite

$$\|u\|_{B^l_{p,p_0}(\Omega)} = \|u\|_{L_p(\Omega)} + \left(\sum_{i=1}^n \int_0^\infty \frac{\|\Delta^m_i(h,\Omega)\partial^k_{x_i} u\|^{p_0}_{L_p(\Omega)}}{h^{1+(l-k)p_0}} dh \right)^{1/p_0},$$

where $k \in \mathbb{N}_0$, $m \in \mathbb{N}$, $m > l-k > 0$, $\Delta^j_i(h,\Omega)u$, $j \in \mathbb{N}$, $h \in \mathbb{R}_+$, is the finite difference of the order j of the function $u(x)$ with respect to x_i, with $\Delta^1_i(h,\Omega)u = \Delta_i(h,\Omega)u = u(x_1,\dots,x_{i-1},x_i+h,x_{i+1},\dots,x_n) - u(x_1,\dots,x_n)$, $\Delta^j_i(h,\Omega)u = \Delta_i(h,\Omega)\Delta^{j-1}_i(h,\Omega)u$ and $\Delta^j_i(h,\Omega)u = 0$ for $x + jh \notin \Omega$.

From Golovkin [15] it is known that the norms of the Besov space $B^l_{p,p_0}(\Omega)$ are equivalent for different m and k satisfying the condition $m > l-k > 0$.

By $C^{\alpha,\alpha/2}(\Omega^T)$, $\alpha \in (0,1)$, we denote the anisotropic Hölder space of functions making the following norm finite

$$\|u\|_{C^{\alpha,\alpha/2}(\Omega^T)} = \sup_{\Omega^T} |u(x,t)| + \sup_{x',x'',t} \frac{|u(x',t) - u(x'',t)|}{|x'-x''|^\alpha}$$

$$+ \sup_{x,t',t''} \frac{|u(x,t') - u(x,t'')|}{|t'-t''|^{\alpha/2}}.$$

By δ we denote a small positive number, and by c a generic positive constant which changes its value from formula to formula and depends at most on the imbedding constants, constants of the considered problem, and the regularity of the boundary.

By $\varphi = \varphi(\sigma_1, \dots, \sigma_k)$, $k \in \mathbb{N}$, we denote a generic function which is a positive increasing function of its arguments $\sigma_1, \dots, \sigma_k$, and may change its form from formula to formula.

Boldface $\boldsymbol{L}$, $\boldsymbol{W}$, $\boldsymbol{B}$ are used for the corresponding spaces of vector and tensor valued functions.

Auxiliary results. We use the following interpolation lemma

Lemma 3.1. (see [1, Chapter 4, section 18]) *Let* $u \in W^{s,s/2}_{p,p_0}(\Omega^T)$, $s \in \mathbb{R}_+$, $p, p_0 \in [1,\infty]$, $\Omega \subset \mathbb{R}^3$. *Let* $\sigma \in \mathbb{R}_+ \cup \{0\}$, *and*

$$\varkappa = \frac{3}{p} + \frac{2}{p_0} - \frac{3}{q} - \frac{2}{q_0} + |\alpha| + 2a + \sigma < s.$$

Then $D^\alpha_x \partial^a_t u \in W^{\sigma,\sigma/2}_{q,q_0}(\Omega^T)$, $q \ge p$, $q_0 \ge p_0$, *and there exists* $\varepsilon \in (0,1)$ *such that*

$$\|D^\alpha_x \partial^a_t u\|_{W^{\sigma,\sigma/2}_{q,q_0}(\Omega^T)} \le \varepsilon^{s-\varkappa} \|u\|_{W^{s,s/2}_{p,p_0}(\Omega^T)} + c\varepsilon^{-\varkappa} \|u\|_{L_{p,p_0}(\Omega^T)}.$$

As a special case of Lemma 3.1 we need

Lemma 3.2. (see [1, Chapter 4, section 18]) *Let* $u \in W^s_p(\Omega)$, $s \in \mathbb{R}_+$, $p \in [1,\infty]$, $\Omega \subset \mathbb{R}^3$. *Let* $\sigma \in \mathbb{R}_+ \cup \{0\}$, *and*

$$\varkappa = \frac{3}{p} - \frac{3}{q} + |\alpha| + \sigma < s.$$

Then $D^\alpha_x u \in W^\sigma_q(\Omega)$, $q \ge p$, *and there exists* $\varepsilon \in (0,1)$ *such that*

$$\|D^\alpha_x u\|_{W^\sigma_q(\Omega)} \le \varepsilon^{s-\varkappa} \|u\|_{W^s_p(\Omega)} + c\varepsilon^{-\varkappa} \|u\|_{L_p(\Omega)}.$$

We also need the following interpolation result

Lemma 3.3. (see [1, Chapter 3, section 15]) *Assume that* $u \in W^l_{p_2}(\Omega) \cap L_{p_1}(\Omega)$, $\Omega \subset \mathbb{R}^3$, *and*

$$\frac{3}{p} - r = (1-\theta)\frac{3}{p_1} + \theta\left(\frac{3}{p_2} - l\right).$$

Then there exists a constant c *such that*

$$\sum_{|\alpha|=r} \|D^\alpha_x u\|_{L_p(\Omega)} \le c \|u\|^\theta_{W^l_{p_2}(\Omega)} \|u\|^{1-\theta}_{L_{p_1}(\Omega)}.$$

We recall from [4] the trace and the inverse trace theorems for Sobolev-Slobodetskii spaces with a mixed norm

Lemma 3.4. *(i) Let* $u \in W^{s,s/2}_{p,p_0}(\Omega^T)$, $s \in \mathbb{R}_+$, $s > 2/p_0$, $p, p_0 \in (1,\infty)$. *Then* $u(x,t_0) = u(x,t)|_{t=t_0}$ *for* $t_0 \in [0,T]$ *belongs to* $B^{s-2/p_0}_{p,p_0}(\Omega)$, *and*

$$\|u(\cdot,t_0)\|_{B^{s-2/p_0}_{p,p_0}(\Omega)} \le c\|u\|_{W^{s,s/2}_{p,p_0}(\Omega^T)},$$

where constant c does not depend on u.

(ii) For a given $\tilde{u} \in B^{s-2/p_0}_{p,p_0}(\Omega)$, $s \in \mathbb{R}_+$, $s > 2/p_0$, $p, p_0 \in (1,\infty)$, *there exists a function* $u \in W^{s,s/2}_{p,p_0}(\Omega^T)$ *such that* $u|_{t=t_0} = \tilde{u}$ *for* $t_0 \in [0,T]$, *and*

$$\|u\|_{W^{s,s/2}_{p,p_0}(\Omega^T)} \le c\|\tilde{u}\|_{B^{s-2/p_0}_{p,p_0}(\Omega)},$$

where constant c does not depend on u.

Lemma 3.5 (see [1, Chapter 3, section 10.4 and Chapter 4, section 18]). *Let* $u \in W^{s,s/2}_{p,p_0}(\Omega^T)$, $s \in \mathbb{R}_+$, $p, p_0 \in (1,\infty)$, $\Omega \subset \mathbb{R}^3$. *Let* $\sigma \in \mathbb{R}_+$, *and*

$$\varkappa = \frac{3}{p} + \frac{2}{p_0} + \sigma < s.$$

Then for $t \le T$, $u \in C^{\sigma,\sigma/2}(\Omega^t)$, *and*

$$\|u\|_{C^{\sigma,\sigma/2}(\Omega^t)} \le \varepsilon^{s-\varkappa}\|u\|_{W^{s,s/2}_{p,p_0}(\Omega^t)} + c\varepsilon^{-\varkappa}\|u\|_{L_{p,p_0}(\Omega^t)}.$$

Lemma 3.6 (imbedding between Besov spaces [1, Chapter 3, section 18]). *Let* $u \in B^{\sigma-2/r_2}_{r_1,r_2}(\Omega)$. *Then* $u \in B^{\sigma'-2/r_2'}_{r_1',r_2'}(\Omega)$, $\Omega \subset \mathbb{R}^3$, *if*

$$\frac{3}{r_1} + \frac{2}{r_2} - \frac{3}{r_1'} - \frac{2}{r_2'} + \sigma' \le \sigma,$$

where

$$r_1' \ge r_i, \quad i = 1,2, \quad \text{and} \quad \sigma \ge \sigma'.$$

Let us consider the problem

$$\begin{aligned} &u_t - Qu = f && \text{in } \Omega^T, \\ &u = 0 && \text{on } S^T, \\ &u|_{t=0} = u_0 && \text{in } \Omega, \end{aligned} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^3$ and

$$Qu = \mu\Delta u + \nu\nabla(\nabla \cdot u)$$

with $\mu > 0$, $\nu > 0$. Let us notice that Q replaces Q_1, so $\mu = \mu_1$, $\nu = \lambda_1 + \mu_1$. Hence assumption (1.6) implies that $\mu > 0$ and $\nu > 0$.

Lemma 3.7 (parabolic system in $W^{2,1}_{p,p_0}(\Omega^T)$ [17], [22], [32], [33]).
(i) Assume that $f \in L_{p,p_0}(\Omega^T)$, $u_0 \in B^{2-2/p_0}_{p,p_0}(\Omega)$, $p, p_0 \in (1,\infty)$, *and* $S \in C^2$. *If* $2-2/p_0-1/p>0$ *the compatibility condition* $u_0|_S = 0$ *is assumed. Then there exists a unique solution to problem (3.1) such that* $u \in W^{2,1}_{p,p_0}(\Omega^T)$ *and*

$$\|u\|_{W^{2,1}_{p,p_0}(\Omega^T)} \le c(\|f\|_{L_{p,p_0}(\Omega^T)} + \|u_0\|_{B^{2-2/p_0}_{p,p_0}(\Omega)}) \tag{3.2}$$

with constant c *depending on* Ω, S, p, p_0.
(ii) Assume that $f = \nabla\cdot g + b$, $g=(g_{ij})$, $b=(b_i)$, $g,b \in L_{p,p_0}(\Omega^T)$, *and* $u_0 \in B^{1-2/p_0}_{p,p_0}(\Omega)$. *Assume the compatibility condition*

$$u_0|_S = 0 \quad \text{if} \quad 1-2/p_0-1/p>0.$$

Then there exists a unique solution to (3.1) such that $u \in W^{1,1/2}_{p,p_0}(\Omega^T)$ *and*

$$\|u\|_{W^{1,1/2}_{p,p_0}(\Omega^T)} \le c(\|g\|_{L_{p,p_0}(\Omega^T)} + \|b\|_{L_{p,p_0}(\Omega^T)} + \|u_0\|_{B^{1-2/p_0}_{p,p_0}(\Omega)}) \tag{3.3}$$

with a constant c *depending on* Ω, S, p, p_0.

Let us consider the problem

$$\begin{aligned} &\alpha(x,t)\theta_t - \Delta\theta = f && \text{in } \Omega^T,\\ &\mathbf{n}\cdot\nabla\theta = 0 && \text{on } S^T,\\ &\theta|_{t=0} = \theta_0 && \text{in } \Omega. \end{aligned} \tag{3.4}$$

Lemma 3.8 (see [18, Chapter 4], [24], [33]). *Assume that* $f \in L_{p,p_0}(\Omega^T)$, $\theta_0 \in B^{2-2/p_0}_{p,p_0}(\Omega)$, $p,p_0 \in (1,\infty)$, $\Omega \in \mathbb{R}^n$, $S \in C^2$. *Assume that* $0 < \alpha_0 \le \alpha \le \alpha_* < \infty$, *where* α_0 *and* α_* *are constants,* $\alpha \in C^{\delta,\delta/2}(\Omega^T)$, $\alpha_t \in L_{3/2\mu,1/(1-\mu)}(\Omega^T)$, $\mu \in (0,1)$. *Then there exists a solution to problem (3.4) such that* $\theta \in W^{2,1}_{p,p_0}(\Omega^T)$ *and the following estimate holds*

$$\begin{aligned} \|\theta\|_{W^{2,1}_{p,p_0}(\Omega^T)} &\le \varphi(1/\alpha_0, \alpha_*, \|\alpha\|_{C^{\delta,\delta/2}(\Omega^T)}, \|\alpha_t\|_{L_{3/2\mu,1/(1-\mu)}(\Omega^T)})\cdot\\ &\quad\cdot(\|f\|_{L_{p,p_0}(\Omega^T)} + \|\theta_0\|_{B^{2-2/p_0}_{p,p_0}(\Omega)}). \end{aligned} \tag{3.5}$$

Remark 3.9. *The above result is a special case of the more general theorem due to Denk, Hieber, and Prüss [9, Theorem 2.3].*

Remark 3.10. *The constant* c *in (3.2), (3.3) and the fucntion* φ *in (3.5) do not depend on* T. *For* T *small the proof of these facts is evident.*
For T *large it can be deduced by applying the arguments of the proof of Theorem 3.1.1 in [35, Chapter 3].*

4 Lower bound for temperature

The existence of the lower positive bound on temperature ensures not only the thermodynamic correctness of the model but is also of basic importance for the proof of global estimates of the solutions. To show such property we use the ideas of the proof of Lemma 4.1 [23].

Lemma 4.1. *Assume that equation (1.2), boundary condition* $(1.3)_2$ *and initial condition* $(1.4)_3$ *hold,* $g \geq 0$, $\theta_0 \geq \underline{\theta} > 0$, *where* $\underline{\theta}$ *is a constant, as well as* k, c_v^1, c_v^2 *are positive constants. Assume that the coercivity and boundedness condition (1.11) hold for viscosity tensor* $\boldsymbol{A}_1$. *Then there exists a positive constant*

$$a \equiv \frac{|\boldsymbol{B}|}{2a_{1\cdot *}\min\{c_v^1, c_v^2\}},$$

where $\boldsymbol{B} = -\boldsymbol{A}_2\boldsymbol{\alpha}$, *and* a_{1*} *is defined in (1.11), such that*

$$\theta(t) \geq \underline{\theta}\exp(-at) \equiv \theta_*(t) \quad \text{for} \quad t \in [0, T]. \tag{4.1}$$

Proof. For $m \in \mathbb{R}_+$ we define the trunacation $\theta_m = \max\left\{\theta, \frac{1}{m}\right\}$ and $\Omega_m(t) = \left\{x \in \Omega : \theta(x,t) > \frac{1}{m}\right\}$. Multiplying (1.2) by $-\theta_m^{-\varrho}$ with $\varrho > 4$ (admissible test function) and integrating over $\Omega_m(t)$ gives

$$\begin{aligned} &-\left[c_v^1 \int\limits_{\Omega_m(t)} \theta^3\theta_{,t}\theta_m^{-\varrho}dx + c_v^2 \int\limits_{\Omega_m(t)} \theta\theta_{,t}\theta_m^{-\varrho}dx\right] + k \int\limits_{\Omega_m(t)} \theta_m^{-\varrho}\Delta\theta dx \\ &+ \int\limits_{\Omega_m(t)} (\boldsymbol{A}_1\boldsymbol{\varepsilon}_t)\cdot\boldsymbol{\varepsilon}_t\theta_m^{-\varrho}dx + \int\limits_{\Omega_m(t)} g\theta_m^{-\varrho}dx = \int\limits_{\Omega_m(t)} \theta\theta_m^{-\varrho}(\boldsymbol{A}_2\boldsymbol{\alpha})\cdot\boldsymbol{\varepsilon}_t dx. \end{aligned} \tag{4.2}$$

Now we examine the terms on the left-hand side of (4.2). The first term is equal to

$$\begin{aligned} &-\left[c_v^1 \int\limits_{\Omega_m(t)} \theta_m^3\theta_{m,t}\theta_m^{-\varrho}dx + c_v^2 \int\limits_{\Omega_m(t)} \theta_m\theta_{m,t}\theta_m^{-\varrho}dx\right] \\ &= \frac{c_v^1}{\varrho-4}\int\limits_{\Omega} \partial_t\theta_m^{4-\varrho}dx + \frac{c_v^2}{\varrho-2}\int\limits_{\Omega} \partial_t\theta_m^{2-\varrho}dx \\ &= \frac{c_v^1}{\varrho-4}\frac{d}{dt}\int\limits_{\Omega} \theta_m^{4-\varrho}dx + \frac{c_v^2}{\varrho-2}\frac{d}{dt}\int\limits_{\Omega} \theta_m^{2-\varrho}dx, \end{aligned} \tag{4.3}$$

because $\partial_t\theta_m^{4-\varrho} = \partial_t\theta_m^{2-\varrho} = 0$ for $x \in \Omega \setminus \Omega_m(t) = \{x \in \Omega : \theta_m(t) = 1/m\}$.

The second term equals

$$(4.4)\qquad k\int\limits_{\Omega_m(t)}\theta_m^{-\varrho}\Delta\theta_m dx = k\int\limits_{\Omega}\theta_m^{-\varrho}\Delta\theta_m dx = \frac{4k\varrho}{(\varrho-1)^2}\int\limits_{\Omega}\left|\nabla\left(\frac{1}{\theta_m^{\frac{\varrho-1}{2}}}\right)\right|^2 dx,$$

since $\nabla\theta_m = \nabla\theta$ for $x\in\Omega_m(t)$ and $\nabla\theta_m = \mathbf{0}$ for $x\in\Omega\setminus\Omega_m(t)$. On account of (1.11) the third term is bounded from below by

$$(4.5)\qquad a_{1*}\int\limits_{\Omega_m(t)}\frac{|\varepsilon_t|^2}{\theta_m^{\varrho}}dx.$$

The fourth term is nonnegative because $g\geq 0$.

In view of the boundedness of tensors $\boldsymbol{A}_2$ and $\boldsymbol{\alpha}$ the integral on the right-hand side of (4.2) is estimated by the Cauchy inequality

$$(4.6)\qquad \begin{aligned}\int\limits_{\Omega_m(t)}\frac{\theta}{\theta_m^{\varrho}}(\boldsymbol{A}_2\boldsymbol{\alpha})\cdot\varepsilon_t dx &= \int\limits_{\Omega_m(t)}\frac{\theta_m}{\theta_m^{\varrho/2}}(\boldsymbol{A}_2\boldsymbol{\alpha})\cdot\frac{\varepsilon_t}{\theta_m^{\varrho/2}}dx\\ &\leq\frac{\delta}{2}\int\limits_{\Omega_m(t)}\frac{|\varepsilon_t|^2}{\theta_m^{\varrho}}dx+\frac{|\boldsymbol{B}|}{2\delta}\int\limits_{\Omega_m(t)}\theta_m^{2-\varrho}dx,\qquad \boldsymbol{B}=-\boldsymbol{A}_2\boldsymbol{\alpha}.\end{aligned}$$

Setting $\delta = a_{1*}$ and incorpording (4.3)–(4.7) into (4.2) we arrive at

$$(4.7)\qquad \begin{aligned}&\frac{c_v^1}{\varrho-4}\frac{d}{dt}\int\limits_{\Omega}\frac{dx}{\theta_m^{\varrho-4}}+\frac{c_v^2}{\varrho-2}\frac{d}{dt}\int\limits_{\Omega}\frac{dx}{\theta_m^{\varrho-2}}+\frac{4k\varrho}{(\varrho-1)^2}\int\limits_{\Omega}\left|\nabla\left(\frac{1}{\theta_m^{\frac{\varrho-1}{2}}}\right)\right|^2 dx\\ &+\frac{a_{1*}}{2}\int\limits_{\Omega_m(t)}\frac{|\varepsilon_t|^2}{\theta_m^{\varrho}}dx\leq\frac{|\boldsymbol{B}|}{2a_{1*}}\int\limits_{\Omega_m(t)}\theta_m^{2-\varrho}dx\leq\frac{|\boldsymbol{B}|}{2a_{1*}}\int\limits_{\Omega}\frac{dx}{\theta_m^{\varrho-2}},\end{aligned}$$

where in the last inequality we taken into account that $\theta_m>0$ in Ω.

Let us introduce the positive quantities

$$(4.8)\qquad X_1(t)=\left(\int\limits_{\Omega}\frac{dx}{\theta_m^{\varrho-4}}\right)^{\frac{1}{\varrho-4}},\quad X_2(t)=\left(\int\limits_{\Omega}\frac{dx}{\theta_m^{\varrho-2}}\right)^{\frac{1}{\varrho-2}}.$$

By (4.8) we infer from (4.7) the inequality

$$(4.9)\qquad \frac{c_v^1}{\varrho-4}\frac{d}{dt}X_1^{\varrho-4}(t)+\frac{c_v^2}{\varrho-2}\frac{d}{dt}X_2^{\varrho-2}(t)\leq\frac{|\boldsymbol{B}|}{2a_{1*}}X_2^{\varrho-2}(t).$$

Let us set now

$$(4.10)\qquad Y(\varrho,t)=\frac{c_v^1}{\varrho-4}X_1^{\varrho-4}(t)+\frac{c_v^2}{\varrho-2}X_2^{\varrho-2}(t).$$

Then (4.9) yields

$$\frac{d}{dt}Y(\varrho,t) \le a(\varrho-2)Y(\varrho,t), \tag{4.11}$$

where $a \equiv |B|/(2a_{1*}\min\{c_v^1,c_v^2\})$. Integrating (4.11) with respect to time from 0 to t leads to

$$Y(\varrho,t) \le \exp[a(\varrho-2)t]Y(\varrho,0). \tag{4.12}$$

Hence, using the form of $Y(\varrho,t)$, we get

$$X_2(t) \le \exp(at)\cdot\left[\left(\frac{c_v^1}{c_v^2}\right)^{\frac{1}{\varrho-2}}\left(\frac{\varrho-2}{\varrho-4}\right)^{\frac{1}{\varrho-2}}\cdot X_1^{\frac{\varrho-4}{\varrho-2}}(0)+X_2(0)\right], \tag{4.13}$$

or equivalently,

$$\begin{aligned} &\|\theta_m^{-1}(t)\|_{L_{\varrho-2}(\Omega)} \\ &\le \exp(at)\left[\left(\frac{c_v^1}{c_v^2}\right)^{\frac{1}{\varrho-2}}\left(\frac{\varrho-2}{\varrho-4}\right)^{\frac{1}{\varrho-2}}\|\theta_m^{-1}(0)\|_{L_{\varrho-4}(\Omega)}^{\frac{\varrho-4}{\varrho-2}}+\|\theta_m^{-1}(0)\|_{L_{\varrho-2}(\Omega)}\right]. \end{aligned} \tag{4.14}$$

Letting $\varrho\to\infty$, (4.14) implies the bound

$$\theta_m(t) \ge \theta_m(0)\exp(-at) \quad \text{for} \quad t\in[0,T]. \tag{4.15}$$

Further, letting $m\to\infty$ and noting that for sufficiently large m, $\theta_m(0) = \max\left\{\theta_0,\frac{1}{m}\right\} \ge \underline{\theta} > 0$, we conclude the bound (4.1). □

5 Local existence

To prove local existence of solutions we use the following Banach successive approximation method:

$$u_{(n+1),tt} - \nabla\cdot\big(A_1\varepsilon(u_{(n+1),t})\big) = \nabla\cdot\big[A_2\varepsilon(u_{(n)})+B\theta_{(n)}\big]+b \quad \text{in } \Omega^T, \tag{5.1}$$

$$\begin{aligned} &[c_v^1\theta_{(n)}^3 + c_v^2\theta_{(n)}]\theta_{(n+1),t} - k\Delta\theta_{(n+1)} = \theta_{(n)}B\cdot\varepsilon(u_{(n),t}) \\ &\quad + A_1\varepsilon(u_{(n),t})\cdot\varepsilon(u_{(n),t}) + g \quad \text{in } \Omega^T, \end{aligned} \tag{5.2}$$

$$u_{(n+1)} = 0, \quad n\cdot\nabla\theta_{(n+1)} = 0 \quad \text{on } S^T, \tag{5.3}$$

$$u_{(n+1)}|_{t=0} = u_0, \quad u_{(n+1),t} = u_1, \quad \theta_{(n+1)}|_{t=0} = \theta_0 \quad \text{in } \Omega, \tag{5.4}$$

where $u_{(n)}$, $\theta_{(n)}$, $n \in \mathbb{N} \cup \{0\}$ are treated as given.
Moreover, the zero approximations $(u_{(0)}, \theta_{(0)})$ are constructed by on extension of the initial data in such a way that

$$u_{(0)}|_{t=0} = u_0, \quad u_{(0),t}|_{t=0} = u_1, \quad \theta_{(0)}|_{t=0} = \theta_0 \quad \text{in } \Omega, \tag{5.5}$$

and

$$u_{(0)} = 0, \quad n \cdot \nabla\theta_{(0)} = 0 \quad \text{on } S^T. \tag{5.6}$$

We note that problem (5.1)–(5.6) and that analysed in [24, section 5] differ only by the presence of the additional term $c_v^1\theta_{(n)}^3$ in (5.2) which has the same properties as $c_v^2\theta_{(n)}$. For this reason in order to prove the uniform boundedness of the sequence $\{u_{(n)}, \theta_{(n)}\}$ we can use exactly the same arguments as in Lemma 5.1 [24].
We have

Lemma 5.1 (Boundedness of the approximation).
Let $X_0(t) = \|u_{(0),t}\|_{W^{2,1}_{p,p_0}(\Omega^t)} + \|\theta_{(0)}\|_{W^{2,1}_{q,q_0}(\Omega^t)}$, *where* $u_{(0)}$, $\theta_{(0)}$ *are introduced by (5.5), be finite. Let* $\theta_0 \geq \underline{\theta} > 0$. *Further, let*

$$D(t) = \|u_0\|_{W^2_p(\Omega)} + u_1\|_{B^{2-2/p_0}_{p,p_0}(\Omega)} + \|\theta_0\|_{B^{2-2/q_0}_{q,q_0}(\Omega)} + \|b\|_{L_{p,p_0}(\Omega^t)} + \|g\|_{L_{q,q_0}(\Omega^t)}$$

be finite, and

$$3/p + 2/p_0 < 1, \quad 3/q + 2/q_0 < 1 + 3/p + 2/p_0.$$

Assume that there exists a constant A *and time* t *sufficiently small such that*

$$X_0(t) \leq A, \quad \varphi(t^\alpha A, D(t)) \leq A,$$

where $\alpha > 0$ *and the nonlinear function* φ *appear in the proof of Lemma 5.1 [24, equation (5.22)], and* $ct^{\delta/2}A \leq \underline{\theta}$, $\delta > 0$. *Then*

$$X_n(t) = \|u_{(n),t}\|_{W^{2,1}_{p,p_0}(\Omega^t)} + \|\theta_{(n)}\|_{W^{2,1}_{q,q_0}(\Omega^t)} \leq A \quad \text{for any } n \in \mathbb{N}. \tag{5.7}$$

To show convergence of the sequence $\{u_{(n)}, \theta_{(n)}\}$ we introduce the differences

$$U_n(t) = u_{(n)}(t) - u_{(n-1)}(t), \quad \vartheta_n(t) = \theta_{(n)}(t) - \theta_{(n-1)}(t), \tag{5.8}$$

$n \in \mathbb{N}$, which are solutions to the problem

$$\begin{aligned} &U_{n+1,tt} - \nabla\cdot(A_1\varepsilon(U_{n+1,t})) = \nabla\cdot[A_2\varepsilon(U_n) + B\vartheta_n] && \text{in } \Omega^T,\\ &U_{n+1} = 0 && \text{on } S^T,\\ &U_{n+1}|_{t=0} = 0, \quad U_{n+1,t}|_{t=0} = 0 && \text{in } \Omega, \end{aligned} \tag{5.9}$$

and

$$
\begin{aligned}
&(c_v^1\theta_{(n)}^3 + c_v^2\theta_{(n)})\vartheta_{n+1,t} - k\Delta\vartheta_{n+1} = -c_v^1(\theta_{(n)}^3 - \theta_{(n-1)}^3)\theta_{(n),t} \\
&\quad - c_v^2\vartheta_n\theta_{(n),t} + \vartheta_n B\cdot\varepsilon(u_{(n),t}) + \theta_{(n-1)}B\cdot\varepsilon(U_{(n),t}) \\
&\quad + A_1\varepsilon(U_{n,t})\cdot\varepsilon(u_{(n),t}) + A_1\varepsilon(u_{(n-1),t})\cdot\varepsilon(U_{n,t}) \quad \text{in } \Omega^T, \\
&n\cdot\nabla\vartheta_{n+1} = 0 \qquad \text{on } S^T, \\
&\vartheta_{n+1}|_{t=0} = 0 \qquad \text{in } \Omega.
\end{aligned}
\tag{5.10}
$$

Let

$$
Y_n(t) = \|U_{n,t}\|_{W^{2,1}_{p',p_0'}(\Omega^t)} + \|\vartheta_n\|_{W^{2,1}_{q',q_0'}(\Omega^t)}. \tag{5.11}
$$

Like for the uniform boundedenss we can repeat the arguments of the corresponding proof of the convergence of approximation of [24, Lemma 5.3]. This lemma required (see, [24, equation (5.30)]) several technical restrictions on the indices p, p_0, q, q_0, p', p_0', q', q_0' of the involved Sobolev spaces with a mixed norm $W^{2,1}_{p,p_0}(\Omega^t)$, $W^{2,1}_{q,q_0}(\Omega^t)$, $W^{2,1}_{p',p_0'}(\Omega^t)$, $W^{2,1}_{q',q_0'}(\Omega^t)$. As noted in [24, Corollary 5.5] these restrictions and the restrictions of Lemma 5.1 can be satisfied for the following special choice:

(5.12) $p = p_0 = 5^+$, $q = q_0 = 5^+$, $p' = p_0' = 5$, $q' = q_0' = 5$, where 5^+ is any number larger than 5 possibly close to 5.

Then we have

Lemma 5.2 (Convergence of the approximation). *Let the assumptions of Lemma 5.1 be satisfied and (5.12) holds. Then there exist a positive constant $d = d(A)$ and $a > 0$ such that*

$$
Y_{n+1}(t) \le dt^a Y_n(t). \tag{5.13}
$$

From Lemmas 5.1 and 5.2 it follows

Theorem 5.3 (Local existence). *Let the assumptions of Lemmas 5.1 and (5.2) hold. Then there exists a local solution to problem (1.1)–(1.4) such that $u_t \in W^{2,1}_{5+}(\Omega^{\tilde T})$, $\theta \in W^{2,1}_{5+}(\Omega^{\tilde T})$, where $\tilde T$ is sufficiently small.*

6 Global estimates

In this section we prove global estimates on an arbitrary finite time interval $(0, T)$ for a regular local solution. All estimates use the regularity of local solutions. By Lemma 4.1 we know that there exists the lower positive bound on temperature

$$\theta(t) \geq \theta_* := \theta_*(T) > 0 \quad \text{for} \quad t \leq T. \tag{6.1}$$

Throughout we assume that assumptions (A1)–(A3) of Theorem A hold.

Lemma 6.1 (Energy estimates). *Assume that*

$$u_0 \in H^1(\Omega), \quad u_1 \in L_2(\Omega), \quad \theta_0 \in L_4(\Omega),$$
$$b \in L_2(\Omega^t), \quad g \in L_1(\Omega^t), \quad g \geq 0, \quad t \leq T.$$

Then solutions to problem (1.1)–(1.4) satisfy the estimate

$$\begin{aligned} &\|u(t)\|^2_{H^1(\Omega)} + \|u_t(t)\|^2_{L_2(\Omega)} + \|\theta(t)\|^4_{L_4(\Omega)} \leq c(t)(\|u_0\|^2_{H^1(\Omega)} \\ &\quad + \|u_1\|^2_{L_2(\Omega)} + \|\theta_0\|^4_{L_4(\Omega)} + \|b\|^2_{L_2(\Omega^t)} + \|g\|^2_{L_1(\Omega^t)}) \equiv c_1(t), \end{aligned} \tag{6.2}$$

where $c(t)$ is an increasing positive function.

Proof. Multiplying (1.1) by u_t and integrating over Ω yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u_t\|^2_{L_2(\Omega)} + \int_\Omega (A_1\varepsilon_t)\cdot\varepsilon_t dx - \int_\Omega [\nabla\cdot(A_2\varepsilon)]\cdot u_t dx + \int_\Omega \theta B\cdot\varepsilon_t dx \\ &= \int_\Omega b\cdot u_t dx, \end{aligned} \tag{6.3}$$

where we remind (see (1.8)) that $B :== A_2\alpha$. Integrating (1.2) over Ω implies

$$\frac{c_v^1}{4}\frac{d}{dt}\int_\Omega \theta^4 dx + \frac{c_v^2}{2}\frac{d}{dt}\int_\Omega \theta^2 dx = \int_\Omega \theta B\cdot\varepsilon_t dx + \int_\Omega (A_1\varepsilon_t)\cdot\varepsilon_t dx + \int_\Omega g dx. \tag{6.4}$$

From the properties of the operator A_2 (see (1.5)) we have

$$\begin{aligned} &-\int_\Omega [\nabla\cdot(A_2\varepsilon)]\cdot u_t dx = -\int_\Omega [\mu_2\Delta u\cdot u_t + (\lambda_2+\mu_2)\nabla(\nabla\cdot u)\cdot u_t]dx \\ &= \frac{1}{2}\frac{d}{dt}[\mu_2\|\nabla u\|^2_{L_2(\Omega)} + (\lambda_2+\mu_2)\|\nabla\cdot u\|^2_{L_2(\Omega)}], \end{aligned} \tag{6.5}$$

where the boundary condition $(1.3)_1$ was used. Applying (6.5) in (6.3) gives

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}[\|u_t\|^2_{L_2(\Omega)} + \mu_2\|\nabla u\|^2_{L_2(\Omega)} + (\lambda_2+\mu_2)\|\nabla\cdot u\|^2_{L_2(\Omega)}] \\ &\quad + \int_\Omega (A_1\varepsilon_t)\cdot\varepsilon_t dx + \int_\Omega \theta B\cdot\varepsilon_t dx = \int_\Omega b\cdot u_t dx. \end{aligned} \tag{6.6}$$

By adding (6.4) and (6.6) we have

$$
\frac{d}{dt}\left[\frac{c_v^1}{4}\|\theta\|_{L_4(\Omega)}^4+\frac{c_v^2}{2}\|\theta\|_{L_2(\Omega)}^2+\frac{1}{2}\|u_t\|_{L_2(\Omega)}^2+\mu_2\|\nabla u\|_{L_2(\Omega)}^2 \right.
$$

$$
\left. +(\lambda_2+\mu_2)\|\nabla\cdot u\|_{L_2(\Omega)}^2\right]=\int\limits_\Omega b\cdot u_t dx+\int\limits_\Omega g dx. \tag{6.7}
$$

Integrating (6.7) with respect to time, using the lower bound (1.17) for the sum of the last two terms in the squared parenthesis, and eventually applying the Gronwall inequality we get (6.2) which concludes the proof. □

To derive "stronger" estimates for u and θ we apply the regularity theory of parabolic systems in Sobolev spaces with a mixed norm, stated in Lemmas 3.7 and 3.8. Let us first consider viscoelasticity system (1.1), $(1.3)_1$, $(1.4)_{1,2}$, expressed in the form

$$
\begin{aligned}
&u_{tt}-Q_1u_t=\nabla\cdot(A_2\varepsilon+\theta B)+b && \text{in } \Omega^T,\\
&u_t=0 && \text{on } S^T,\\
&u_t|_{t=0}=u_1,\quad u|_{t=0}=u_0 && \text{in } \Omega.
\end{aligned} \tag{6.8}
$$

We have

Lemma 6.2. *Assume that*

$$
\begin{aligned}
&\theta\in L_{p,r}(\Omega^t),\quad b\in L_{p,r}(\Omega^t),\\
&u_0\in W_p^1(\Omega),\quad u_1\in B_{p,r}^{1-2/r}(\Omega),\quad p,r\in(1,\infty),\quad t\le T.
\end{aligned}
$$

Then for solutions to problem (1.1)–(1.4) the following inequality holds

$$
\begin{aligned}
\|\varepsilon_{t'}\|_{L_{p,r}(\Omega^t)}&\le c(t)(\|\theta\|_{L_{p,r}(\Omega^t)}+\|b\|_{L_{p,r}(\Omega^t)}+\|u_0\|_{W_p^1(\Omega)}+\|u_1\|_{B_{p,r}^{1-2/r}(\Omega)})\\
&\equiv c_2(t,p,r)+c(t)\|\theta\|_{L_{p,r}(\Omega^t)}.
\end{aligned} \tag{6.9}
$$

Proof. Applying Lemma 3.7(ii) to problem (6.8), using the boundedness of tensors A_2, B we have

$$
\begin{aligned}
\|\varepsilon_{t'}\|_{L_{p,r}(\Omega^t)}\le c\|u_{t'}\|_{W_{p,r}^{1,1/2}(\Omega^t)}\le c(&\|\varepsilon\|_{L_{p,r}(\Omega^t)}\\
&+\|\theta\|_{L_{p,r}(\Omega^t)}+\|b\|_{L_{p,r}(\Omega^t)}+\|u_1\|_{B_{p,r}^{1-2/r}(\Omega)}).
\end{aligned}
$$

Using the Gronwall lemma to the latter inequality we conclude (6.9). □

Now, using (6.2) in (6.9) implies the estimate

$$
\|\varepsilon_{t'}\|_{L_{4,r}(\Omega^t)}\le c(t)c_1^{1/4}(t)+c_2(t,4,r)\equiv c_3(t,r),\quad r\in(1,\infty). \tag{6.10}
$$

We have also the following

Lemma 6.3. *Let $\nabla\theta \in L_{p,r}(\Omega^t)$, $b \in L_{p,r}(\Omega^t)$, $u_1 \in B^{2-2/r}_{p,r}(\Omega)$, $u_0 \in W^2_p(\Omega)$, $p, r \in (1,\infty)$, $t \le T$. Then for solutions to problem (1.1)–(1.4) the following inequality holds*

$$
\begin{aligned}
&\|\varepsilon_{t'}\|_{W^{1,1/2}_{p,r}(\Omega^t)} \le c\|u_{t'}\|_{W^{2,1}_{p,r}(\Omega^t)} \le c(t)(\|\nabla\theta\|_{L_{p,r}(\Omega^t)} \\
&\quad + \|b\|_{L_{p,r}(\Omega^t)} + \|u_0\|_{W^2_p(\Omega)} + \|u_1\|_{B^{2-2/r}_{p,r}(\Omega)}) \\
&\equiv c(t)\|\nabla\theta\|_{L_{p,r}(\Omega^t)} + c_4(t,p,r).
\end{aligned}
\tag{6.11}
$$

Proof. Applying Lemma 3.7 (i) to problem (6.8) and the boundednes of A_2, B yields

$$
\begin{aligned}
&\|\varepsilon_{t'}\|_{W^{1,1/2}_{p,r}(\Omega^t)} \le c(\nabla\varepsilon\|_{L_{p,r}(\Omega^t)} \\
&\quad + \|\nabla\theta\|_{L_{p,r}(\Omega^t)} + \|b\|_{L_{p,r}(\Omega^t)} + \|u_1\|_{B^{2-2/r}_{p,r}(\Omega)}).
\end{aligned}
$$

Hence, by the Gronwall lemma, (6.11) follows. □

On account of (6.10) we obtain "better" estimates on θ.

Lemma 6.4. *Let (6.10) for $r = 2$ holds true and the assumptions of Lemma 6.1 be satisfied. Let $\theta_0 \in L_5(\Omega)$ and $g \in L_{5/4,1}(\Omega^t)$. Then the following inequality holds*

$$
\begin{aligned}
&\|\theta(t)\|^5_{L_5(\Omega)} + \|\theta(t)\|^3_{L_3(\Omega)} + \|\nabla\theta\|^2_{L_2(\Omega^t)} \le [c_1^{1/2}(t)c_3(t,2) + c_1^{1/4}(t)c_3^2(t,2) \\
&\quad + \|g\|^{5/4}_{L_{5/4,1}(\Omega^t)} + \|\theta_0\|^5_{L_5(\Omega)}] \equiv c_5(t).
\end{aligned}
\tag{6.12}
$$

Proof. Multiplying (1.2) by θ, integrating with respect to time and using $(1.3)_2$, $(1.4)_3$ gives

$$
\begin{aligned}
&\int_\Omega \theta^5 dx + \int_\Omega \theta^3 dx + \int_{\Omega^t} |\nabla\theta|^2 dxdt' \le c\int_{\Omega^t} \theta^2|\varepsilon_{t'}|dxdt' \\
&\quad + c\int_{\Omega^t} \theta|\varepsilon_{t'}|^2 dxdt' + c\int_{\Omega^t} g\theta dxdt' + c\int_\Omega \theta_0^5 dx + c\int_\Omega \theta_0^3 dx.
\end{aligned}
\tag{6.13}
$$

The first term on the right-hand side of (6.13) is bounded by

$$
\begin{aligned}
&\int_0^t dt' \int_\Omega \theta^2|\varepsilon_{t'}|dx \le \int_0^t \|\theta\|^2_{L_4(\Omega)}\|\varepsilon_{t'}\|_{L_2(\Omega)}dt' \le c_1^{1/2}(t)\int_0^t \|\varepsilon_{t'}\|_{L_2(\Omega)}dt' \\
&\quad \le c_1^{1/2}(t)t^{1/2}c_3(t,2),
\end{aligned}
$$

and the second one by

$$\int_0^t dt' \int_\Omega \theta|\varepsilon_{t'}|^2 dx \le \int_0^t \|\theta\|_{L_4(\Omega)} \|\varepsilon_{t'}\|^2_{L_{8/3}(\Omega)} dt' \le c_1^{1/4}(t) c_3^2(t,2).$$

The third term is bounded by

$$\begin{aligned}\int_0^t \|\theta\|_{L_5(\Omega)} \|g\|_{L_{5/4}(\Omega)} dt' &\le \sup_t \|\theta\|_{L_5(\Omega)} \|g\|_{L_{5/4,1}(\Omega^t)} \\ &\le \delta\|\theta\|^5_{L_5(\Omega)} + \frac{c}{\delta}\|g\|^{5/4}_{L_{5/4,1}(\Omega^t)}, \quad \delta > 0.\end{aligned}$$

Applying the above inequalities in (6.13) we conclude (6.12). This completes the proof. □

Let us note that from (6.9) and (6.12) it follows that

$$\|\varepsilon_{t'}\|_{L_{5,r}(\Omega^t)} \le c_2(t,5,r) + c(t)c_5^{1/5}(t) \equiv c_6(t,r), \quad r \in (1,\infty). \tag{6.14}$$

We continue with further estimates for θ.

Lemma 6.5. *Let the assumptions of Lemma 6.4 be satisfied, and estimate (6.14) holds. Moreover, assume that* $\theta_0 \in L_{15}(\Omega)$, $g \in L_{36/25,12}(\Omega^t)$, $t \le T$. *Then*

$$\begin{aligned}&\|\theta(t)\|^{15}_{L_{15}(\Omega)} + \int_0^t \|\theta\|^{12}_{L_{36}(\Omega)} dt' + \int_{\Omega^t} |\nabla\theta^6|^2 dt' \\ &\le c(c_6^5(t,12) + c_6^{24}(t,12) + \|g\|^{12}_{L_{36/25,12}(\Omega^t)} + \|\theta_0\|^{15}_{L_{15}(\Omega)}) \equiv c_7(t).\end{aligned} \tag{6.15}$$

Proof. Multiplying (1.2) by $\theta^{\alpha-1}$, where $\alpha > 1$ is a finite number, integrating the result over Ω, taking into account the boundedness of tensor $\boldsymbol{B}$, $\boldsymbol{A}_1$ and the boundary condition $(1.3)_2$, we obtain

$$\begin{aligned}&\frac{c_v^1}{\alpha+3}\frac{d}{dt}\int_\Omega \theta^{\alpha+3} dx + \frac{c_v^2}{\alpha+1}\frac{d}{dt}\int_\Omega \theta^{\alpha+1} dx + \frac{4k(\alpha-1)}{\alpha^2}\int_\Omega |\nabla\theta^{\frac{\alpha}{2}}|^2 dx \\ &\le c\int_\Omega \theta^\alpha|\varepsilon_t| dx + c\int_\Omega \theta^{\alpha-1}|\varepsilon_t|^2 dx + \int_\Omega g\theta^{\alpha-1} dx.\end{aligned} \tag{6.16}$$

Integration of (6.16) with respect to time gives

$$
\begin{aligned}
&\frac{1}{\alpha+3}\int_\Omega \theta^{\alpha+3}dx+\frac{1}{\alpha+1}\int_\Omega \theta^{\alpha+1}dx+\frac{4(\alpha-1)}{\alpha^2}\int_{\Omega^t}|\nabla\theta^{\frac{\alpha}{2}}|^2dxdt'\\
(6.17)\qquad &\le c\int_{\Omega^t}\theta^\alpha|\varepsilon_{t'}|dxdt'+c\int_{\Omega^t}\theta^{\alpha-1}|\varepsilon_{t'}|^2dxdt'+\int_{\Omega^t}g\theta^{\alpha-1}dxdt'\\
&+\frac{c}{\alpha+3}\|\theta_0\|^{\alpha+3}_{L_{\alpha+3}(\Omega)}+\frac{c}{\alpha+1}\|\theta_0\|^{\alpha+1}_{L_{\alpha+1}(\Omega)}.
\end{aligned}
$$

Prior to deal with the terms on the right-hand side of (6.17) we first estimate from below the third term on the left-hand side by applying a Sobolev imbedding. Setting $u=\theta^{\alpha/2}$ this term takes the form

$$
\frac{4(\alpha-1)}{\alpha^2}\int_0^t\int_\Omega|\nabla u|^2dxdt'.
$$

Now we add to the both sides of (6.17) the term

$$
\frac{2(\alpha-1)}{\alpha^2}\int_0^t\int_\Omega u^2dxdt'.
$$

Then we have

$$
\begin{aligned}
&\frac{2(\alpha-1)}{\alpha^2}\int_0^t\int_\Omega(|\nabla u|^2+u^2)dxdt'\ge\frac{2c(\alpha-1)}{\alpha^2}\int_0^t\|u\|^2_{L_6(\Omega)}dt'\\
(6.18)\qquad &=\frac{2c(\alpha-1)}{\alpha^2}\int_0^t\|\theta\|^\alpha_{L_{3\alpha}(\Omega)}dt'.
\end{aligned}
$$

The additional term on the right-hand side of (6.17) equals

$$
\frac{2(\alpha-1)}{\alpha^2}\int_0^t|\theta^{\alpha/2}|^2dxdt'=\frac{2(\alpha-1)}{\alpha^2}\int_0^t\int_\Omega\theta^\alpha dxdt',
$$

so by applying the Hölder and the Young inequalities is bounded by

$$
\delta_1\sup_t\int_\Omega\theta^{\alpha+3}dx+c(1/\delta_1,\alpha,t),\quad \delta_1>0.
$$

Consequently, employing (6.18) in (6.17) gives
(6.19)

$$\frac{1}{\alpha+3}\|\theta\|^{\alpha+3}_{L_{\alpha+3}(\Omega)}+\frac{1}{\alpha+1}\|\theta\|^{\alpha+1}_{L_{\alpha+1}(\Omega)}+\frac{2c(\alpha-1)}{\alpha^2}\int_0^t\|\theta\|^{\alpha}_{L_{3\alpha}(\Omega)}dt'$$

$$+\frac{2(\alpha-1)}{\alpha^2}\int_{\Omega^t}|\nabla\theta^{\frac{\alpha}{2}}|^2dxdt'$$

$$\le c\int_0^t\|\theta\|^{\alpha}_{L_{\alpha\lambda_1}(\Omega)}\|\varepsilon_{t'}\|_{L_{\lambda_2}(\Omega)}dt'+c\int_0^t\|\theta\|^{\alpha-1}_{L_{(\alpha-1)\mu_1}(\Omega)}\|\varepsilon_{t'}\|^2_{L_{2\mu_2}(\Omega)}dt'$$

$$+c\int_0^t\|g\|_{L_{\nu_1}(\Omega)}\|\theta\|^{\alpha-1}_{L_{(\alpha-1)\nu_2}(\Omega)}dt'+\frac{c}{\alpha+3}\|\theta_0\|^{\alpha+3}_{L_{\alpha+3}(\Omega)}+\frac{c}{\alpha+1}\|\theta_0\|^{\alpha+1}_{L_{\alpha+1}(\Omega)},$$

where $1/\lambda_1+1/\lambda_2=1$, $1/\mu_1+1/\mu_2=1$, $1/\nu_1+1/\nu_2=1$.
On account of (6.14) we can assume that $\lambda_2=5$, so $\lambda_1=5/4$. Setting $5\alpha/4=\alpha+3$, we get $\alpha=12$. Then the first term on the right-hand side of (6.19) is bounded by

$$\delta_2\int_0^t\|\theta\|^{\alpha+3}_{L_{\alpha+3}(\Omega)}dt'+c/\delta_2c_6^{\frac{\alpha+3}{3}}(t,12),\quad \delta_2>0.$$

In the second term on the right-hand side of (6.19) we assume that $\mu_2=5/2$, $\mu_1=5/3$ and $(\alpha-1)\mu_1\le3\alpha$, so $(\alpha-1)\frac{5}{3}\le3\alpha$. We note that the latter inequality is satisfied for any $\alpha>1$. Hence, the second term is bounded by

$$\delta_3\int_0^t\|\theta\|^{\alpha}_{L_{3\alpha}(\Omega)}dt'+c/\delta_3\int_0^t\|\varepsilon_{t'}\|^{2\alpha}_{L_5(\Omega)}dt'\le\delta_3\int_0^t\|\theta\|^{\alpha}_{L_{3\alpha}(\Omega)}dt'+c/\delta_3c_6^{2\alpha}(t,12),$$

where (6.14) is used.
In the third term on the right-hand side of (6.19) we assume that $\nu_2=\frac{3\alpha}{\alpha-1}$ so $\nu_1=\frac{3\alpha}{2\alpha+1}$. Then this term is bounded by

$$\delta_4\int_0^t\|\theta\|^{\alpha}_{L_{3\alpha}(\Omega)}dt'+c/\delta_4\int_0^t\|g\|^{\alpha}_{L_{\frac{3\alpha}{2\alpha+1}}(\Omega)}dt',\quad\delta_4>0.$$

From the above considerations it follows that we can take $\alpha=12$. Employing the obtained estimates in (6.19), choosing δ_k, $k=1,\cdots,4$, appropriately, in particular assuming that $\delta_2-\delta_3$ is sufficiently small, we arrive at (6.15). This concludes the proof. □

Let us note that using (6.15) in (6.9) yields

$$\|\varepsilon_{t'}\|_{L_{15,r}(\Omega^t)} \le c_2(t,15,r) + c(t)c_7^{1/15}(t) \equiv c_8(t,r), \quad r \in (1,\infty). \tag{6.20}$$

We proceed now to prove that $\theta \in L_\infty(0,t;L_\alpha(\Omega))$ for any finite α. For this purpose we repeat and improve appropriately the arguments of the proof of Lemma 6.5.

Lemma 6.6. *Let (6.15) and (6.20) with $r=\alpha\in(1,\infty)$ hold. Moreover, assume that*

$$\theta_0 \in L_{\alpha+3}(\Omega) \quad \text{and} \quad g \in L_{\frac{3\alpha}{2\alpha+1},\alpha}(\Omega^t), \quad t \le T.$$

Then

$$\begin{aligned}
&\frac{1}{\alpha+3}\|\theta(t)\|^{\alpha+3}_{L_{\alpha+3}(\Omega)} + \frac{1}{\alpha+1}\|\theta(t)\|^{\alpha+1}_{L_{\alpha+1}(\Omega)} \\
&\quad + \frac{4k(\alpha-1)}{\alpha^2}\int_{\Omega^t}|\nabla\theta^{\alpha/2}|^2dxdt' \\
&\le c(c_7(t),c_8(t,\alpha)) + c\|g\|^\alpha_{L_{\frac{3\alpha}{2\alpha+1},\alpha}(\Omega^t)} + \frac{c}{\alpha+3}\|\theta_0\|^{\alpha+3}_{L_{\alpha+3}(\Omega)} \\
&\equiv c_9(t,\alpha), \quad \alpha<\infty.
\end{aligned} \tag{6.21}$$

Proof. Let us turn to the inequality (6.17) from the proof of Lemma 6.5. We proceed now as follows. The first term on the right-hand side of (6.17) we express in the form

$$\int_0^t\int_\Omega \theta^{\alpha-1}\theta|\varepsilon_{t'}|dxdt'.$$

On account of (6.15) and (6.20) it is estimated by

$$\begin{aligned}
&\int_0^t \|\theta\|^{\alpha-1}_{L_{(\alpha-1)\frac{15}{13}}(\Omega)}\|\theta\|_{L_{15}(\Omega)}\|\varepsilon_{t'}\|_{L_{15}(\Omega)}dt' \\
&\le c_7^{1/15}(t)\int_0^t \|\theta\|^{\alpha-1}_{L_{(\alpha-1)\frac{15}{13}}(\Omega)}\|\varepsilon_{t'}\|_{L_{15}(\Omega)}dt' \\
&\le \delta_1\int_0^t\|\theta\|^\alpha_{L_{3\alpha}(\Omega)}dt' + c(1/\delta_1,c_7^{1/15}(t))\int_0^t\|\varepsilon_{t'}\|^\alpha_{L_{15}(\Omega)}dt' \\
&\le \delta_1\int_0^t\|\theta\|^\alpha_{L_{3\alpha}(\Omega)}dt' + c(1/\delta_1,c_7^{1/15}(t),c_8^\alpha(t,\alpha)), \quad \delta_1>0,
\end{aligned}$$

where we used the relation $(\alpha-1)\frac{15}{13}\le 3\alpha$, holding true for any finite α. Similarly, the second term on the right-hand side of (6.17) is bounded by

$$\int_0^t \|\theta\|^{\alpha-1}_{L_{(\alpha-1)\frac{15}{13}}(\Omega)}\|\varepsilon_{t'}\|^2_{L_{15}(\Omega)}dt' \le \delta_2\int_0^t \|\theta\|^{\alpha}_{L_{3\alpha}(\Omega)}dt' + c(1/\delta_2)\int_0^t \|\varepsilon_{t'}\|^{2\alpha}_{L_{15}(\Omega)}dt'$$

$$\le \delta_2\int_0^t \|\theta\|^{\alpha}_{L_{3\alpha}(\Omega)}dt' + c(1/\delta_2)c_8^{2\alpha}(t,\alpha),\quad \delta_2>0.$$

Finally, the third term on the right-hand side of (6.17) is bounded by

$$\int_0^t \|\theta\|^{\alpha-1}_{L_{3\alpha}(\Omega)}\|g\|_{L_{\frac{3\alpha}{2\alpha+1}}(\Omega)}dt'$$

$$\le \delta_3\int_0^t \|\theta\|^{\alpha}_{L_{3\alpha}(\Omega)}dt' + \frac{1}{\delta_3}\int_0^t \|g\|^{\alpha}_{L_{\frac{3\alpha}{2\alpha+1}}(\Omega)}dt',\quad \delta_3>0.$$

Employing the above estimates in (6.17), and setting δ_k sufficiently small, we arrive at (6.21). This proves the lemma. □

Let us note that from (6.21) it follows in particular that

$$(6.22)\quad \|\theta\|_{L_\infty(0,t;L_\alpha(\Omega))}\le[(\alpha+3)c_9(t,\alpha)]^{\frac{1}{\alpha+3}}\equiv c_{10}(t,\alpha)\quad \text{for any } \alpha<\infty.$$

We obtain now an estimate on θ_t.

Lemma 6.7. *Let the assumptions of the previous lemmas be satisfied, in particular the lower bound (6.1) holds, $\theta_0\in H^1(\Omega)$ and $g\in L_2(\Omega^t)$, $g\ge 0$, $t\le T$. Then*

$$(6.23)\quad \begin{aligned}&\|\theta_{t'}\|^2_{L_2(\Omega^t)}+\|\theta\|^2_{L_\infty(0,t;H^1(\Omega))}\le c(1/\theta_*,c_{10}^2(t,4),c_8^2(t,4))\\&+c(1/\theta_*)\|g\|^2_{L_2(\Omega^t)}+c\|\theta_0\|^2_{H^1(\Omega)}\equiv c_{11}^2(t).\end{aligned}$$

Proof. Multiplying (1.2) by θ_t, integrating over Ω^t, $t\le T$, using boundary condition $(1.3)_2$, the boundedness of tensors $\boldsymbol{A}_1$, $\boldsymbol{B}=-\boldsymbol{A}_2\boldsymbol{\alpha}$, and the global lower bound (6.1) for θ, we get

$$(6.24)\quad \begin{aligned}&\|\theta_{t'}\|^2_{L_2(\Omega^t)}+\frac{k}{2}\|\nabla\theta(t)\|^2_{L_2(\Omega)}\le\frac{c}{\theta_*^3}\bigg[\int_{\Omega^t}\theta|\varepsilon_{t'}|\,|\theta_{t'}|dxdt'\\&+\int_{\Omega^t}|\varepsilon_{t'}|^2|\theta_{t'}|dxdt'+\int_{\Omega^t}|g|\,|\theta_{t'}|dxdt'\bigg]+\frac{k}{2}\|\theta_0\|^2_{H^1(\Omega)}.\end{aligned}$$

Therefore, by the Young inequality, we have

$$
(6.25)\qquad \begin{aligned} \|\theta_{t'}\|^2_{L_2(\Omega^t)} + \frac{k}{2}\|\nabla\theta(t)\|^2_{L_2(\Omega)} \le \frac{c}{\theta_*^6}\bigg[\int\limits_{\Omega^t} \theta^2|\varepsilon_{t'}|^2 dxdt' \\ + \int\limits_{\Omega^t} |\varepsilon_{t'}|^4 dxdt' + \int\limits_{\Omega^t} |g|^2 dxdt'\bigg] + \frac{k}{2}\|\theta_0\|^2_{H^1(\Omega)}. \end{aligned}
$$

Hence, on account of estimates (6.20) and (6.22) we conclude (6.23). This completes the proof. □

We shall apply now the elliptic regularity result. In view of estimate (6.23) we express (1.2), $(1.3)_2$ in the form of the following elliptic problem

$$
(6.26)\qquad \begin{aligned} &k\Delta\theta = (c_v^1\theta^3 + c_v^2\theta)\theta_t - \theta B\cdot\varepsilon_t - (A_1\varepsilon_t)\cdot\varepsilon_t - g \quad &&\text{in } \Omega,\ t\le T,\\ &n\cdot\nabla\theta = 0 &&\text{on } S,\ t\le T. \end{aligned}
$$

We have

Lemma 6.8. *Assume that estimates (6.20), (6.22), (6.23), and the lower bound (6.1) for θ hold. Then for problem (6.26) the following estimate is satisfied*

$$
(6.27)\qquad \begin{aligned} \|\theta\|_{L_2(0,t;W^2_\varrho(\Omega))} \le c_{10}^3\Big(t,\frac{6\varrho}{2-\varrho}\Big)c_{11}(t) + c_{10}(t,4)c_8(t,2) \\ + c_8^2(t,4) + c\|g\|_{L_2(\Omega^t)} \equiv c_{12}(t,\varrho) \end{aligned}
$$

for $\varrho < 2^-$, where 2^- stands for a number less than 2 but very close to 2.

Proof. We estimate the terms on the right-hand side of $(6.26)_1$. First, by the Hölder inequality, using (6.22) and (6.23) we have

$$
\bigg(\int\limits_0^t\int\limits_\Omega (|\theta^3\theta_{t'}|^\varrho dx)^{2/\varrho}dt'\bigg)^{1/2} \le \bigg(\int\limits_0^t \|\theta\|^6_{L_{\frac{6\varrho}{2-\varrho}}(\Omega)}\|\theta_{t'}\|^2_{L_2(\Omega)}dt'\bigg)^{1/2}
$$
$$
\le \sup_t \|\theta\|^3_{L_{\frac{6\varrho}{2-\varrho}}(\Omega)}\|\theta_{t'}\|_{L_2(\Omega^t)} \le c_{10}^3\Big(t,\frac{6\varrho}{2-\varrho}\Big)c_{11}(t),
$$

where $\varrho < 2$ but very close to 2. Similarly,

$$
\bigg(\int\limits_0^t\int\limits_\Omega (\theta\theta_{t'}|^\varrho dx)^{2/\varrho}dt'\bigg)^{1/2} \le \bigg(\int\limits_0^t \|\theta\|^2_{L_{\frac{2\varrho}{2-\varrho}}(\Omega)}\|\theta_{t'}\|^2_{L_2(\Omega)}dt'\bigg)^{1/2}
$$
$$
\le \sup_t \|\theta\|_{L_{\frac{2\varrho}{2-\varrho}}(\Omega)}\|\theta_{t'}\|_{L_2(\Omega^t)} \le c_{10}\Big(t,\frac{2\varrho}{2-\varrho}\Big)c_{11}(t) \le c_{10}\Big(t,\frac{6\varrho}{2-\varrho}\Big)c_{11}(t).
$$

Finally, using the boundedness of tensors $\boldsymbol{B}$, $\boldsymbol{A}_1$, and applying (6.20), (6.22) yield

$$\Big(\int_0^t\int_\Omega |\theta \boldsymbol{B}\cdot\varepsilon_{t'}|^2 dxdt'\Big)^{1/2} \le c\sup_t \|\theta\|_{L_4(\Omega)}\Big(\int_0^t \|\varepsilon_{t'}\|^2_{L_4(\Omega)}dt'\Big)^{1/2}$$
$$\le c_{10}(t,4)c_8(t,2),$$

and

$$\Big(\int_0^t\int_\Omega |(\boldsymbol{A}_1\varepsilon_{t'})\cdot\varepsilon_{t'}|^2 dxdt'\Big)^{1/2} \le c\Big(\int_0^t \|\varepsilon_{t'}\|^4_{L_4(\Omega)}dt'\Big)^{1/2}$$
$$= c\|\varepsilon_{t'}\|^2_{L_4(\Omega^t)} \le c_8^2(t,4).$$

On account of the above estimates we conclude (6.27) and thereby complete the proof. □

From (6.23) and (6.27) it follows that

(6.28) $$\|\theta\|_{W^{2,1}_{\varrho,2}(\Omega^t)} \le c_{11}(t) + c_{12}(t,\varrho) \equiv c_{13}(t,\varrho) \quad \text{for} \quad \varrho < 2^-.$$

Hence, by the imbedding (see Lemma 3.1) it follows that $\nabla\theta \in L_{5\varrho/3}(\Omega^t)$, $\varrho < 2^-$. Consequently, due to (6.11),

(6.29)
$$\|\varepsilon_{t'}\|_{W^{1,1/2}_{5\varrho/3}(\Omega^t)} \le c(t)(c_{13}(t,\varrho) + c_4(t,5\varrho/3,5\varrho/3)) \equiv c_{14}(t,\varrho), \quad \varrho < 2^-.$$

Further, by the imbedding, we have the estimates

(6.30) $$\|\varepsilon_{t'}\|_{L_q(\Omega^t)} \le c(c_{14}(t,\varrho)) \quad \text{for} \quad q < 10, \quad \varrho < 2^-,$$

and

(6.31) $$\|\varepsilon_{t'}\|_{L_2(0,t;L_\infty(\Omega))} \le c(c_{14}(t,\varrho)),$$

which holds for $3/2 < \varrho < 2^-$. The latter estimate plays the key role in getting $L_\infty(\Omega^T)$-norm bound for θ.

Lemma 6.9 ($L_\infty(\Omega^T)$-nporm bound on θ). *Assume that $\theta_0 \in L_\infty(\Omega)$, $g \in L_1(0,t;L_\infty(\Omega))$, $g \ge 0$, $t \le T$, and estimate (6.31) holds. Then*

(6.32) $$\|\theta\|_{L_\infty(\Omega^t)} \le c(c_{14}(t,2^-), \|g\|_{L_1(0,t;L_\infty(\Omega))}) \equiv c_{15}(t).$$

Proof. Multiplying (1.2) by θ^t, $r > 1$, integrating over Ω, and using (6.31), we get

$$
\begin{aligned}
&c_v^1 \|\theta\|^{r+3}_{L_{r+4}(\Omega)} \frac{d}{dt} \|\theta\|_{L_{r+4}(\Omega)} + c_v^2 \|\theta\|^{r+1}_{L_{r+2}(\Omega)} \frac{d}{dt} \|\theta\|_{L_{r+2}(\Omega)} \\
&+ \frac{4kr}{(r+1)^2} \int\limits_\Omega |\nabla \theta^{\frac{r+2}{2}}|^2 dx \le c[\|\varepsilon_t\|_{L_\infty(\Omega)} \|\theta\|^{r+1}_{L_{r+1}(\Omega)} \\
&+ \|\varepsilon_t\|^2_{L_\infty(\Omega)} \|\theta\|^r_{L_r(\Omega)} + \|g\|_{L_\infty(\Omega)} \|\theta\|^r_{L_r(\Omega)}].
\end{aligned} \tag{6.33}
$$

Taking into account that $\theta \ge \theta_* > 0$ we deduce from (6.33) that

$$
\begin{aligned}
&c_v^1 \|\theta\|^{r+3}_{L_{r+4}(\Omega)} \frac{d}{dt} \|\theta\|_{L_{r+4}(\Omega)} + c_v^2 \|\theta\|^{r+1}_{L_{r+2}(\Omega)} \frac{d}{dt} \|\theta\|_{L_{r+2}(\Omega)} \\
&\le c(1/\theta_*)[\|\varepsilon_t\|_{L_\infty(\Omega)} + \|\varepsilon_t\|^2_{L_\infty(\Omega)} + \|g\|_{L_\infty(\Omega)}] \|\theta\|^{r+4}_{L_{r+4}(\Omega)} \\
&\equiv \alpha(t) \|\theta\|^{r+4}_{L_{r+4}(\Omega)}.
\end{aligned} \tag{6.34}
$$

Expressing (6.34) in the form

$$
\begin{aligned}
&\frac{d}{dt} \left[\frac{c_v^1}{r+4} \|\theta^{r+4}_{L_{r+4}(\Omega)} + \frac{c_v^2}{r+2} \|\theta\|^{r+2}_{L_{r+2}(\Omega)} \right] \\
&\le \frac{\alpha(t)}{c_v^1} (r+4) \left[\frac{c_v^1}{r+4} \|\theta\|^{r+4}_{L_{r+4}(\Omega)} + \frac{c_v^2}{r+2} \|\theta\|^{r+2}_{L_{r+2}(\Omega)} \right],
\end{aligned}
$$

and introducing the notation

$$
Y(t) = \frac{c_v^1}{r+4} \|\theta(t)\|^{r+4}_{L_{r+4}(\Omega)} + \frac{c_v^2}{r+2} \|\theta(t)\|^{r+2}_{L_{r+2}(\Omega)},
$$

we have

$$
\frac{d}{dt} Y(t) \le \frac{\alpha(t)(r+4)}{c_v^1} Y(t). \tag{6.35}
$$

Integrating (6.35) with respect to time yields

$$
Y(t) \le Y(0) \exp \left(\frac{r+4}{c_v^1} \int\limits_0^t \alpha(t') dt' \right), \quad t \le T.
$$

From the above inequality we get

$$
\frac{c_v^1}{r+4} \|\theta(t)\|^{r+4}_{L_{r+4}(\Omega)} \le Y(0) \exp \left(\frac{r+4}{c_v^1} \int\limits_0^t \alpha(t') dt' \right).
$$

Hence,

$$
\begin{aligned}
\|\theta(t)\|_{L_{r+4}(\Omega)} &\le \left(\frac{r+4}{c_v^1}Y(0)\right)^{\frac{1}{r+4}} \exp\left(\frac{1}{c_v^1}\int_0^t \alpha(t')dt'\right) \\
&\le \left(\|\theta_0\|_{L_{r+4}(\Omega)} + \frac{c_v^2}{c_v^1}\left(\frac{r+4}{r+2}\right)\|\theta_0\|_{L_{r+2}(\Omega)}^{\frac{r+2}{r+4}}\right)\exp\left(\frac{1}{c_v^1}\int_0^t \alpha(t')dt'\right).
\end{aligned} \tag{6.36}
$$

Now, letting $r\to\infty$ in (6.36) we conclude (6.32). The proof is complete. □

To prove the Hölder continuity of θ we follow exactly the considerations in [24, Lemma 6.14 and Corollary 6.15] related to thermo-visco-elasticity with the specific heat $c = c_v\theta^\sigma$, $\sigma\in(1/2,1]$. Consequently, we have

Lemma 6.10 (Hölder continuity of θ). *Assume that $\theta(t)\ge\theta_*>0$ for $t\le T$. Let $M=\|\theta\|_{L_\infty(\Omega^t)}\le c_{15}(T)$ (see (6.32)), $\|\theta_0\|_{L_\infty(\Omega)}<k$, and $M-k<\delta$ for some sufficiently small $\delta>0$. Let $g\in L_\lambda(\Omega^t)$, $\varepsilon_{t'}\in L_{2\lambda}(\Omega^t)$, where $\lambda=\frac{1}{1-\frac{2}{r}(1+\varkappa)}$, $\frac{3}{r}+\frac{2}{q}=\frac{3}{2}$, q, r are positive numbers, and $\varkappa>0$. Then*

$$
\theta\in C^{\beta,\beta/2}(\Omega^t),\quad \beta\in(0,1),\quad t\le T, \tag{6.37}
$$

where β depends on θ_*, M, δ, $\varkappa$, r.

To prove global existence of solutions to problem (1.1)–(1.4) we need the existence of local solutions and a global estimate in the norms in which the local existence is proved. More precisely, we are going to obtain a global estimate for $u_t\in W^{2,1}_{5^+}(\Omega^t)$ and $\theta\in W^{2,1}_{5^+}(\Omega^t)$.

Lemma 6.11 (global a priori estimates compatible with estimates for local solution). *Assume that $b\in L_{10^+}(\Omega^t)\cap L_{5,12}(\Omega^t)$, $u_0\in W^2_{5^+}(\Omega)$, $u_1\in B^{2-2/5^+}_{5^+,5^+}(\Omega)$, $g\in L_{5^+}(0,t;L_\infty(\Omega))$, $g\ge0$, $\theta_0\in H^1(\Omega)\cap B^{2-2/5^+}_{5^+,5^+}(\Omega)\cap L_\infty(\Omega)$. Then solutions to problem (1.1)–(1.4) satisfy the estimate*

$$
\|u_t\|_{W^{2,1}_{5^+}(\Omega^t)}\le\varphi(t,1/\theta_*,d(t)) \tag{6.38}
$$

and

$$
\|\theta\|_{W^{2,1}_{5^+}(\Omega^t)}\le\varphi(t,1/\theta_*,d(t)), \tag{6.39}
$$

where

$$
\begin{aligned}
d(t) &= \|b\|_{L_{10^+}(\Omega^t)\cap L_{5,12}(\Omega^t)} + \|u_0\|_{W^2_{5^+}(\Omega)} + \|u_1\|_{B^{2-2/5^+}_{5^+,5^+}(\Omega)} + \|g\|_{L_{5^+}(0,t;L_\infty(\Omega))} \\
&\quad + \|\theta_0\|_{H^1(\Omega)\cap B^{2-2/5^+}_{5^+,5^+}(\Omega)\cap L_\infty(\Omega)},\quad t\le T.
\end{aligned}
$$

Proof. Firstly, let us note that by (6.32), $\theta \in L_\infty(\Omega^t)$, and so $\theta \in L_{p,r}(\Omega^t)$ for any $p, r \in (1,\infty)$. Hence, applying the bound (6.32) in (6.9) we obtain

$$\tag{6.40} \begin{aligned} \|\varepsilon_{t'}\|_{L_{p,r}(\Omega^t)} &\le c(t)(\|b\|_{L_{p,r}(\Omega^t)} + \|u_0\|_{W^1_p(\Omega)} + \|u_1\|_{B^{1-2/r}_{p,r}(\Omega)}) \\ &\quad + c(t)c_{15}(t) \quad \text{for} \quad p, r \in (1,\infty), \end{aligned}$$

where, by the definitions of the bounds $c_k(\cdot)$, $k = 1, \dots, 15$ (see (6.1), (6.9)–(6.12), (6.14)–(6.15), (6.20)–(6.23), (6.27)–(6.29), (6.32)),

$$\tag{6.41} \begin{aligned} c(t)c_{15}(t) &\le \varphi(t, 1/\theta_*, \|b\|_{L_{5,12}(\Omega^t)}, \|u_0\|_{W^1_5(\Omega)}, \|u_1\|_{B^{1-2/12}_{5,12}(\Omega)}, \\ &\quad \|g\|_{L_2(0,t;L_\infty(\Omega))}, \|\theta_0\|_{H^1(\Omega)\cap L_\infty(\Omega)}) \equiv c_{16}(t). \end{aligned}$$

The assumptions of Lemma 6.10 are satisfied due to (6.32) and (6.31). Indeed, by (6.31), setting $q = 2$, $r = 6$ and $\varkappa = \left(\frac{7}{5}\right)^+$ we get $\lambda = 5^+$, and so $g \in L_{5^+}(\Omega^t)$, $\varepsilon_{t'} \in L_{10^+}(\Omega^t)$. Moreover, by (6.23), $\theta_{t'} \in L_2(\Omega^t)$. Therefore, we can apply Lemma 3.8 to problem (1.2), $(1.3)_2$, $(1.4)_3$.

Let us set in (6.40) $p = r = 10^+$. Then

$$\tag{6.42} \begin{aligned} \|\varepsilon_{t'}\|_{L_{10^+}(\Omega^t)} &\le \varphi(t, 1/\theta_*, \|b\|_{L_{10^+}(\Omega^t)\cap L_{5,12}(\Omega^t)}, \\ &\quad \|u_0\|_{W^1_{10^+}(\Omega)}, \quad \|u_1\|_{B^{1-2/10^+}_{10^+,10^+}(\Omega)\cap B^{1-2/12}_{5,12}(\Omega)}, \\ &\quad \|g\|_{L_2(0,t;L_\infty(\Omega))}, \|\theta_0\|_{H^1(\Omega)\cap L_\infty(\Omega)}) \equiv c_{17}(t). \end{aligned}$$

Hence, by Lemma 3.8, $\theta \in W^{2,1}_{5^+}(\Omega^t)$ and satisfies the estimate

$$\tag{6.43} \begin{aligned} \|\theta\|_{W^{2,1}_{5^+}(\Omega^t)} &\le \varphi(1/\theta_*, \|\theta\|_{L_\infty(\Omega^t)}, \|\theta\|_{C^{\beta,\beta/2}(\Omega^t)}, \|\theta_{t'}\|_{L_2(\Omega^t)}) \\ &\quad \cdot c(\|\theta\|_{L_\infty(\Omega^t)}\|\varepsilon_{t'}\|_{L_{5^+}(\Omega^t)} + \|\varepsilon_{t'}\|^2_{L_{10^+}(\Omega^t)} + \|g\|_{L_{5^+}(\Omega^t)} \\ &\quad + \|\theta_0\|_{B^{2-2/5^+}_{5^+,5^+}(\Omega)}). \end{aligned}$$

Consequently, on account of the bounds (6.32), (6.23) and (6.42), we have

$$\tag{6.44} \begin{aligned} \|\theta\|_{W^{2,1}_{5^+}(\Omega^t)} &\le \varphi(t, 1/\theta_*, c_{15}(t), c_{17}(t)) \\ &\le \varphi(t, 1/\theta_*, \|b\|_{L_{10^+}(\Omega^t)\cap L_{5,12}(\Omega^t)}, \|u_0\|_{W^1_{10^+}(\Omega)}, \\ &\quad \|u_1\|_{B^{1-2/10^+}_{10^+,10^+}(\Omega\cap B^{1-2/12}_{5,12}(\Omega))}, \|g\|_{L_{5^+}(0,t;L_\infty(\Omega))}, \\ &\quad \|\theta_0\|_{H^1(\Omega)\cap L_\infty(\Omega)\cap B^{2-2/5^+}_{5^+,5^+}(\Omega)}) \equiv c_{18}(t). \end{aligned}$$

By the imbedding the above estimate gives

$$\tag{6.45} \|\nabla\theta\|_{L_{5^+}(\Omega^t)} \le c(c_{18}(t)).$$

Then Lemma (6.3) applied to problem (1.1), $(1.3)_1$, $(1.4)_{1,2}$ yields

$$\|u_t\|_{W^{2,1}_{5+}(\Omega^t)} \le c(c_{18}(t) + \|b\|_{L_{5+}(\Omega^t)} + \|u_0\|_{W^2_{5+}(\Omega)} + \|u_1\|_{B^{2-2/5+}_{5+,5+}(\Omega)}).$$

Summarizing the estimates on the data and using the imbeddings between the Besov spaces (see Lemma 3.6)

$$B^{2-2/5^+}_{5^+,5^+}(\Omega) \subset B^{1-2/10^+}_{10^+,10^+}(\Omega), \quad B^{2-2/5^+}_{5^+,5^+}(\Omega) \subset B^{1-2/12}_{5,12}(\Omega),$$

and the imbedding $W^2_{5^+}(\Omega) \subset W^1_{10^+}(\Omega)$, we conclude the assertion. □

7 Global existence

Proof of Theorem 1.1. Theorem 5.3 provides the local existence of solutions to problem (1.1)–(1.4) such that $u_t \in W^{2,1}_{5^+}(\Omega^t)$ and $\theta \in W^{2,1}_{5^+}(\Omega^t)$, where t is sufficiently small. By virtue of Lemma 6.11 we have global a priori estimates for problem (1.1)–(1.4) such that $u_t \in W^{2,1}_{5^+}(\Omega^t)$ and $\theta \in W^{2,1}_{5^+}(\Omega^t)$ for any t finite. These estimates are compatible with the estimates for local solutions on the time interval of the local existence. This implies a possibility of extension of the local solution for any finite time. □

References

[1] O.V. Besov, V.P. Il'in, and S.M. Nikolskij, Integral Representation of Functions and Theorems of Imbeddings, Nauka, Moscow, 1975 (in Russian).

[2] F.J.Blatt, Modern Physics, McGraw-Hill, 1992.

[3] E. Bonetti and G. Bonfanti, *Existence and uniqueness of the solution to a 3D thermoelastic system*, Electron. J. Differential Equations, (2003), pp. 1–15.

[4] Y.S. Bugrov, *Function spaces with mixed norm*, Math. USSR. Izv., **5** (1971), pp. 1145–1167.

[5] J.D.Clayton, Nonlinear Mechanics of Crystals, Solid Mechanics and its Applications 177, Springer, 2011.

[6] C.M. Dafermos, *Global smooth solutions to the initial-boundary value problem for the equations of one-dimensional nonlinear thermoviscoelasticity*, SIAM J. Math. Anal., **13** (1982), pp. 397–408.

[7] C.M. Dafermos and L. Hsiao, *Global smooth thermomechanical processes in one-dimensional nonlinear thermoviscoelasticity*, Nonlinear Anal., **6** (1982), pp. 435–454.

[8] P. Debye, *Zur Theorie der spezifischen Wärmen*, Ann. Phys. (Leipzig), **39** (1912), 789–939.

[9] R. Denk, M. Hieber, and J. Prüss, *Optimal $L^p - L^q$ estimates for parabolic boundary value problems with inhomogeneous data.* Math. Z., 257 (2007), pp. 193–224.

[10] C. Eck, J. Jarušek and M. Krbec, Unilateral Contact Problems: Variational Methods and Existence Theorems, Pure and Applied Mathematics, Chapman & Hall/CRC, Boca Raton, FL 2005.

[11] A. Einstein, *Die Plancksche Theeorie der Strahlung und die Theorie der spezifischen Wärme*, Ann. Phys. (Leipzig), **22** (1907), 180–190.

[12] M. Fabrizio, D. Giorgi, and A. Morro, *A continuum theory for first-order phase transitions based on the balance of structure order*, Math. Methods Appl. Sci., **31** (2008), pp. 627–653.

[13] G. Francfort and P.M. Suquet, *Homogenization and mechanical dissipation in thermoviscoelasticity*, Arch. Ration. Mech. Anal., **96** (1986), pp. 265–293.

[14] M. Frémond, Non-smooth Thermomechanics, Springer-Verlag, Berlin, 2002.

[15] K.K. Golovkin, *On equivalent norms for fractional spaces*, Amer. Math. Soc. Transl. Ser. 2, **81** (1969), pp. 257–280.

[16] Ch.Kittel, Introduction to Solid State Physics, 7th Ed., Wiley (1996).

[17] N.V. Krylov, *The Calderon-Zygmund theorem and its application for parabolic equations*, Algebra i Analiz, **13** (2001), pp. 1–25 (in Russian).

[18] O.A. Ladyzhenskaya, V.A. Solonnikov, and N.N. Ural'tseva, Linear and Quasilinear Equations of Parabolic Type, Nauka, Moscow, 1967 (in Russian).

[19] H.P. Meyers, Introductory Solid State Physics, 2nd Ed., Taylor&Francis, 1997.

[20] J. Nečas, Les Méthodes Directes en Théorie des Équations Elliptiques, Masson, Paris, 1967.

[21] I. Pawłow, *Three dimensional model of thermomechanical evolution of shape memory materials*, Control Cybernet., **29** (2000), pp. 341–365.

[22] I. Pawłow and W.M. Zajączkowski, *Unique solvability of a nonlinear thermoviscoelasticity system in Sobolev space with a mixed norm*, Discrete Contin. Dyn. Syst. Ser. S, **4** (2011), pp. 441–466.

[23] I. Pawłow and W.M. Zajączkowski, *Global regular solutions to a Kelvin-Voigt type thermoviscoelastic system*, SIAM J. Math. Anal., **45** (2013), pp. 1997–2045.

[24] I. Pawłow and W.M. Zajączkowski, *Global regular solutions to three-dimensional thermo-visco-elasticity with nonlinear temperature-dependent specific heat*, Commun. Pure Appl. Anal., accepted 2016.

[25] I. Pawłow and A. Żochowski, *Existence and uniqueness for a three-dimensional thermoelastic system*, Dissertationes Math., **406** (2002), p. 46.

[26] A. Petit and P. Dulong, *Sur quelques points importants de la theorie de la chaleur*, Ann. Chim. Phys. **10** (1819), 395–413.

[27] T. Roubíček, *Thermo-viscoelasticity at small strains with L^1-data*, Quart. Appl. Math., **67** (2009), pp. 47–71.

[28] T. Roubíček, *Thermodynamics of rate-independent processes in viscous solids at small strains*, SIAM J. Math. Anal., **42** (2010), pp. 256–297.

[29] T. Roubíček, *Nonlinearly coupled thermo-visco-elasticity*, Nonlinear Differ. Equ. Appl., **20** (2013), pp. 1243–1275.

[30] D.V. Schroeder, An Introduction to Thermal Physics, Addison-Wesley, 2000.

[31] M. Slemrod, *Global existence, uniqueness and asymptotic stability of classical smooth solutions in one-dimensional non-linear thermoelasticity*, Arch. Ration. Mech. Anal., **76** (1981), pp. 97–133.

[32] V.A. Solonnikov, *Estimates of solutions of the Stokes equations in S. L. Sobolev spaces with a mixed norm*, Zap. Nauchn. Sem. S. Petersburg Otdel. Mat Inst. Steklov (POMI), **288** (2002), pp. 204–231.

[33] V.A. Solonnikov, *On boundary value problems for linear parabolic systems of differential equations of general type*, Trudy MIAN **83** (1965), (in Russian).

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[34] V.G. Zvyagin and V.P. Orlov, *Existence and uniqueness results for a coupled problem in continuum thermomechanics*, Westnik WGU, Ser. Fizika. Matematika No 2 (2014), pp. 120–141.

[35] W. von Wahl, The Equations of the Navier-Stokes and Abstract Parabolic Equations, Braunschweig 1985.