Raport Badawczy
Research Report

RB/9/2012

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Warszawa 2012

THE GLOBAL SOLVABILITY OF A SIXTH ORDER CAHN-HILLIARD TYPE EQUATION VIA THE BÄCKLUND TRANSFORMATION

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ABSTRACT. We consider again the sixth order Cahn-Hilliard type equation
with a nonlinear diffusion, addressed in our previous paper in Commun. Pure
Appl. Anal. 10 (2011), 1823–1847. Such PDE arises as a model of oil-water-
surfactant mixtures. Applying the approach based on the Bäcklund transfor-
mation and the Leray-Schauder fixed point theorem we generalize the exis-
tence result of the above mentioned paper by imposing weaker assumptions on
the data. Here we prove the global unique solvability of the problem in the
Sobolev space $H^{6,1}(\Omega \times (0,T))$ under the assumption that the initial datum
is in $H^3(\Omega)$ whereas previously $H^6(\Omega)$-regularity was required. Moreover, we
admit a broader class of nonlinear terms in the free energy potential.

1. **Introduction.** In this article we reconsider an initial-boundary value problem
for a sixth order Cahn-Hilliard type equation with a nonlinear diffusion which has
been previously addressed in [18]. Our aim here is to generalize the global existence
result of [18] by admitting more general data. This is achieved with the help of
the approach based on the Bäcklund transformation and the Leray-Schauder fixed
point theorem.

The Bäcklund transformation associated with model B of phase transitions, ac-
cording to the Hohenberg-Halperin classification [12], has been proposed by Mitlin
[17]. A new equation describing the evolution of the averaged, modulated structure
of the order parameter in model B has been derived there and demonstrated to
have a great computational advantage in simulations of large scale systems.

The present study shows that the Bäcklund transformation has also theoretical
advantages. In the case of a sixth order Cahn-Hilliard type equation with a nonlinear

diffusion it allows to prove the existence of a unique global strong solution in the Sobolev space $H^{6,1}(\Omega \times (0,T))$, $\Omega \subset \mathbb{R}^3$ bounded, $T > 0$ arbitrary, under a natural assumption that the initial datum $\chi|_{t=0}$ belongs to the corresponding space of traces $H^3(\Omega)$.

In the previous result in [18] the solvability in $H^{6,1}(\Omega \times (0,T))$ has been proved under a restrictive assumption that the time derivative of the solution at the initial time moment, $\chi_t|_{t=0}$ belongs to $L_2(\Omega)$ (see Theorems A, B and Remark at the end of this section). The existence proofs of both results are based on the Leray-Schauder fixed point theorem. The difference consists in another way of deriving a priori estimates which are crucial for the Leray-Schauder argument. More precisely, for the problem under consideration the basic difficulty in getting suitable a priori estimates comes from the treatment of a nonlinear boundary condition associated with the nonlinear diffusion. The Bäcklund transformation allows to obtain stronger regularity estimates and thereby to handle efficiently the boundary nonlinearity.

Problem statement. We consider the following system of equations for the order parameter χ and the chemical potential μ:

$$\chi_t = M\Delta\mu \quad \text{in} \ \ \Omega^T := \Omega \times (0,T), \tag{1.1}$$

$$\mu = f_0'(\chi) + \frac{1}{2}\varkappa_1'(\chi)|\nabla\chi|^2 - \nabla \cdot (\varkappa_1(\chi)\nabla\chi) + \varkappa_2\Delta^2\chi \quad \text{in} \ \ \Omega^T, \tag{1.2}$$

with the initial and boundary conditions

$$\chi|_{t=0} = \chi_0 \quad \text{in} \ \ \Omega, \tag{1.3}$$

$$\mathbf{n} \cdot \nabla\chi = 0, \quad \mathbf{n} \cdot \nabla\Delta\chi = 0, \quad \mathbf{n} \cdot \nabla\mu = 0 \quad \text{on} \ \ S^T := S \times (0,T). \tag{1.4}$$

Here $\Omega \subset \mathbb{R}^3$ is a bounded domain with a smooth boundary S, $T > 0$ is the final time, M and $\varkappa_2$ are positive constants, $f_0 = f_0(\chi)$ and $\varkappa_1 = \varkappa_1(\chi)$ are given functions specified below, $\mathbf{n}$ is the unit outward vector normal to S, $\chi_t = \partial\chi/\partial t$, $f' = df(\chi)/d\chi$, the dot means the scalar product, and $\nabla\cdot$ stands for the spatial divergence.

System (1.1)–(1.4) can be equivalently formulated in the form of the following initial-boundary value problem for the sixth order Cahn-Hilliard type equation

$$\begin{aligned} \chi_t - M\varkappa_2\Delta^3\chi &= M\Delta\left[f_0'(\chi) - \frac{1}{2}\varkappa_1'(\chi)|\nabla\chi|^2 - \varkappa_1(\chi)\Delta\chi\right] &\quad &\text{in} \ \ \Omega^T, \\ \chi|_{t=0} &= \chi_0 &\quad &\text{in} \ \ \Omega, \\ \mathbf{n} \cdot \nabla\chi &= 0, \quad \mathbf{n} \cdot \nabla\Delta\chi = 0 &\quad &\text{on} \ \ S^T, \\ \mathbf{n} \cdot \nabla\Delta^2\chi &= \frac{1}{2\varkappa_2}\varkappa_1'(\chi)\mathbf{n} \cdot \nabla(|\nabla\chi|^2) &\quad &\text{on} \ \ S^T. \end{aligned} \tag{1.5}$$

We notice that the coefficient $\varkappa_1(\cdot)$ gives rise to the fifth order nonlinear boundary condition on S^T.

The Bäcklund transformation. We introduce the new variable

$$v = M \int_0^t \mu \, dt' + v_0$$

with $v_0 = v_0(x)$ satisfying the following elliptic problem

$$\Delta v_0 = \chi_0 - \chi_m \quad \text{in } \Omega,$$
$$\mathbf{n} \cdot \nabla v_0 = 0, \quad \text{on } S$$

$$\fint_\Omega v_0 dx = 0,$$

where $\chi_m := \fint_\Omega \chi_0 dx = \fint \chi(t) dx$ is the spatial mean of χ, preserved in the evolution (see (3.2) below). Then problem (1.1)–(1.4) is transformed to

$$v_t - M\varkappa_2 \Delta^3 v = M \left[f_0''(\Delta v + \chi_m) - \frac{1}{2}\varkappa_1'(\Delta v + \chi_m)|\nabla \Delta v|^2 \right.$$

$$\left. - \varkappa_1(\Delta v + \chi_m)\Delta^2 v \right] \qquad \text{in } \Omega^T, \qquad (1.6)$$

$$v|_{t=0} = v_0 \qquad \text{in } \Omega,$$

$$\mathbf{n} \cdot \nabla v = 0, \quad \mathbf{n} \cdot \nabla \Delta v = 0, \quad \mathbf{n} \cdot \nabla \Delta^2 v = 0 \qquad \text{on } S^T.$$

In [17] the quantity v is termed the dynamical field potential. It is appropriate to describe the evolution of the slow (averaged) variations of the order parameter. The quantities v, χ and μ are linked by the following relations (see Lemma 4.1 below)

$$\Delta v = \chi - \chi_m, \quad v_t = M\mu.$$

These relations allow to deduce the regularity estimates on v from the energy estimates on χ and μ, and then on their basis the regularity estimates for χ and μ. This is the main idea behind applying the Bäcklund transformation.

From the mathematical point of view the formulation (1.6) is better than (1.5) because the Laplacian of v enters as the argument of the nonlinearities on the right-hand side of equation (1.6)$_1$, and because the boundary conditions for v, up to the fifht order, are zero.

Thermodynamic background. System (1.1)–(1.4) is governed by the second order gradient free energy of the Landau-Ginzburg type

$$f = f(\chi, \nabla\chi, \nabla^2\chi) = f_0(\chi) + \frac{1}{2}\varkappa_1(\chi)|\nabla\chi|^2 + \frac{1}{2}\varkappa_2|\Delta\chi|^2, \qquad (1.7)$$

where $f_0(\chi)$ is the multiwell volumetric free energy, $\varkappa_1(\chi)$ is the first gradient coefficient which may be of arbitrary sign, and $\varkappa_2$ is the second gradient coefficient which is assumed to be a positive constant. Equation (1.1) represents the balance of mass

$$\chi_t + \nabla \cdot \mathbf{j} = 0, \qquad (1.8)$$

with the mass flux $\mathbf{j}$ given by

$$\mathbf{j} = -M\nabla\mu, \qquad (1.9)$$

where the positive constant M denotes the mobility. Equation (1.2) is the constitutive relation for the chemical potential

$$\mu = \frac{\delta f(\chi)}{\delta \chi} = f_0'(\chi) + \frac{1}{2}\varkappa_1'(\chi)|\nabla\chi|^2 - \nabla \cdot (\varkappa_1(\chi)\nabla\chi) + \varkappa_2\Delta^2\chi$$

$$= f'(\chi) - \frac{1}{2}\varkappa_1'(\chi)|\nabla\chi|^2 - \varkappa_1(\chi)\Delta\chi + \varkappa_2\Delta^2\chi, \qquad (1.10)$$

where $\delta f(\chi)/\delta\chi$ denotes the first variation of the free energy (1.7), which is defined by the condition that

$$\frac{d}{d\lambda}\int_\Omega f(\chi+\lambda\xi, \nabla\chi+\lambda\nabla\xi, \nabla^2\chi+\lambda\nabla^2\xi)dx|_{\lambda=0} =: \int_\Omega \frac{\delta f(\chi)}{\delta\chi}\xi dx$$

must hold for all test functions $\xi \in C_0^\infty(\Omega)$.

Combining (1.8)–(1.10) yields equation $(1.5)_1$ for χ, or equivalently model B of the phase transition theory according to the Hohenberg-Halperin classification [12]:

$$\chi_t = M\Delta\frac{\delta f(\chi)}{\delta\chi}.$$

In turn, equation $(1.6)_1$ for v has the structure similar to model A of the phase transition theory

$$v_t = M\frac{\delta f(\Delta v + \chi_m)}{\delta(\Delta v + \chi_m)}.$$

Examples. Problem (1.1)–(1.4) may desribe phase transitions in oil-water-surfactant mixtures. The free energy associated with such mixtures has been proposed by Gompper et al. [6–11]. It is given by (1.7) with constant $\varkappa_2 > 0$ and functions f_0, $\varkappa_1$ approximated, respectively, by a sixth and a second order polynomial:

$$f_0(\chi) = (\chi+1)^2(\chi^2 + h_0)(\chi - 1)^2, \quad \varkappa_1(\chi) = g_0 + g_2\chi^2, \qquad (1.11)$$

where h_0, g_0, g_2 are constants, $g_2 > 0$ and h_0, g_0 of arbitrary sign. Here the order parameter χ represents the local difference between the oil and water concentrations.

The problem (1.1)–(1.4) arises also as the so-called phase field crystal (PFC) model describing the crystal growth on atomic length, proposed by Elder et al. [1, 2, 4, 5]. Originally, it is based on the fourth order gradient free energy

$$f_{\text{PFC}} = f_{\text{PFC}}(\chi, \nabla^2\chi, \nabla^4\chi) = -\alpha\frac{\chi^2}{2} + \frac{\chi^4}{4} + \frac{\chi}{2}(1+\Delta)^2\chi, \qquad (1.12)$$

where χ corresponds to atomic mass density, $\alpha = a(\theta - \theta_c)$, $a > 0$ is the parameter of the system periodicity, $\theta - \theta_c$ is the quench depth with critical temperature θ_c and actual temperature θ. The chemical potential is defined by

$$\mu = \frac{\delta f_{\text{PFC}}(\chi)}{\delta\chi} = (1-\alpha)\chi + \chi^3 + 2\Delta\chi + \Delta^2\chi. \qquad (1.13)$$

It is of interest to notice that the second order gradient free energy

$$f = f(\chi, \nabla\chi, \nabla^2\chi) = (1-\alpha)\frac{\chi^2}{2} + \frac{\chi^4}{4} - |\nabla\chi|^2 + \frac{1}{2}|\Delta\chi|^2 \qquad (1.14)$$

has the same first variation as f_{PFC}, thus provides the same equation for μ as above:

$$\mu = \frac{\delta f(\chi)}{\delta\chi} = (1-\alpha)\chi + \chi^3 + 2\Delta\chi + \Delta^2\chi. \qquad (1.15)$$

One can see that free energy (1.14) is a special case of (1.7) with

$$f_0(\chi) = (1-\alpha)\frac{\chi^2}{2} + \frac{\chi^4}{4}, \quad \varkappa_1 = -2, \quad \varkappa_2 = 1. \qquad (1.16)$$

Structural assumptions and main result. Let the free energy f be given by (1.7) with the following polynomial forms of f_0 and $\varkappa_1$, comprising the oil-water-surfactant model (1.11) and the PFC model (1.16) as the particular cases:

$$f_0(\chi) = \sum_{i=0}^{2k} a_i \chi^i \quad \text{with} \quad a_i \in \mathbb{R}, \quad a_{2k} > 0, \quad k \geq 1, \tag{1.17}$$

$$\varkappa_1(\chi) = \sum_{i=0}^{2l} b_i \chi^i \quad \text{with} \quad b_i \in \mathbb{R}, \quad b_{2l} > 0, \quad l \geq 1. \tag{1.18}$$

Theorem A (Existence and uniqueness). *Let us assume that $\Omega \subset \mathbb{R}^3$ is a bounded domain with a boundary S of class C^7, $T > 0$ is a given number functions $f_0(\chi)$ and $\varkappa_1(\chi)$ are given by (1.17), (1.18), $\varkappa_2$, M are positive constants, and the initial datum is such that*

$$\chi_0 \in H^3(\Omega) \quad \text{with the spatial mean value}$$

$$\fint_\Omega \chi_0 dx = \frac{1}{|\Omega|} \int_\Omega \chi_0 dx =: \chi_m \ (|\Omega| = \operatorname{meas}\Omega), \text{ and} \tag{1.19}$$

satisfying the compatibility condition $\mathbf{n} \cdot \nabla \chi_0 = 0$ *on* S.

Then for any $T > 0$ problem (1.1)–(1.4) admits a unique strong solution (χ, μ) such that

$$\chi \in H^{6,1}(\Omega^T), \quad \mu \in L_2(0,T;H^2(\Omega)),$$

$$\chi|_{t=0} = \chi_0, \quad \text{and} \quad \fint_\Omega \chi(t) dx = \chi_m \ \text{for all} \ t \geq 0, \tag{1.20}$$

satisfying the energy estimate

$$\|\chi\|_{L_\infty(0,T;H^2(\Omega))} + \|\nabla\mu\|_{L_2(0,T;L_2(\Omega))} \leq c_1 \tag{1.21}$$

with $c_1 = \varphi(\|\chi_0\|_{H^2(\Omega)}, |\chi_m|)$, and the regularity estimate

$$\|\chi\|_{H^{6,1}(\Omega^T)} + \|\mu\|_{L_2(0,T;H^2(\Omega))} \leq c_2 \tag{1.22}$$

with $c_2 = \varphi(c_1, T) + c\|\chi_0\|_{H^3(\Omega)}$, where $\varphi(\cdot)$ is a positive, increasing function of its arguments.

For a direct comparison we recall the previous result from [18] which was concerned with the particular model (1.11) of the oil-water-surfactant mixture, and required a restrictive regularity assumption on the initial datum χ_0.

Theorem B (see [18]; Theorem 1.1 and Corollary 1). *Let us assume that $\Omega \subset \mathbb{R}^3$ is a bounded domain with a boundary S of class C^6, $T > 0$ is a given number, function $f_0(\chi)$ is a sixth order polynomial satisfying the condition*

$$f_0(\chi) \geq c\chi^6 - \bar{c} \quad \text{for all} \quad \chi \in \mathbb{R}, \tag{1.23}$$

with constants $c > 0$ and $\bar{c} \geq 0$; function $\varkappa_1(\chi)$ is given by (1.11)$_2$ with constants $g_0 \in \mathbb{R}$ and $g_2 > 0$; $\varkappa_2$, M are positive constants, and the initial datum is such that

$$\chi_0 \in H^3(\Omega) \quad \text{with} \quad \fint_\Omega \chi_0 dx = \chi_m. \tag{1.24}$$

Moreover, $\chi_t(0)$, computed from equation (1.5)$_1$, satisfies

$$\chi_t(0) = M\varkappa_2\Delta^3\chi_0 + M\Delta\left[f_0'(\chi_0) - \frac{1}{2}\varkappa_1'(\chi_0)|\nabla\chi_0|^2 - \varkappa_1(\chi_0)\Delta\chi_0\right] \in L_2(\Omega), \tag{1.25}$$

where (1.25) is treated as the elliptic problem for χ_0 with the following boundary conditions on S:

$$\mathbf{n} \cdot \nabla\chi_0 = 0, \quad \mathbf{n} \cdot \nabla\Delta\chi_0 = 0, \quad \varkappa_2\mathbf{n} \cdot \nabla\Delta^2\chi_0 = \frac{1}{2}\varkappa_1'(\chi_0)\mathbf{n} \cdot \nabla(|\nabla\chi_0|^2). \quad (1.26)$$

Then for any $T > 0$ problem (1.1)–(1.4) has a unique strong solution (χ, μ) satisfying (1.20), energy estimate (1.21) and the regularity estimate

$$\|\chi\|_{H^{6,1}(\Omega^T)} + \|\mu\|_{L_2(0,T;H^2(\Omega))} \leq c \quad (1.27)$$

with $c = \varphi(\|\chi_0\|_{H^3(\Omega)}, \|\chi_t(0)\|_{L_2(\Omega)}, c_1, T)$, where $\varphi(\cdot)$ is a positive, increasing function of its arguments.

Remark. 1. The proofs of Theorems A and B are based on the Leray-Schauder fixed point theorem applied to the parabolic sixth order problem (1.5).

2. One can see that Theorem A improves B in two respects: (i) by imposing a weaker assumption on the initial datum χ_0; (ii) by admitting functions f_0 and $\varkappa_1$ as polynomials of an arbitrary order.

3. The assumption $\chi_0 \in H^3(\Omega)$ in Theorem A is natural for the solvability of the parabolic sixth order problem in the Sobolev space $H^{6,1}(\Omega^T)$. The assumption $\chi_t(0) \in L_2(\Omega)$ in Theorem B is ensured, for example, for $\chi_0 \in H^6(\Omega)$. In [18] the analysis was based on the direct application of the parabolic theory to the sixth order problem (1.5). The main difficulty was concerned with the treatment of the nonlinear boundary condition $(1.5)_4$. To get an estimate on χ_t the equation $(1.5)_1$ was differentiated with respect to time. This gave rise to restrictive assumptions (1.25), (1.26). The difficulty can be avoided by means of the Bäcklund transformation which replaces problem (1.1)–(1.4) by (1.6). Thanks to the Bäcklund relations $\Delta v = \chi - \chi_m$, $v_t = M\mu$ (see Lemma 4.1) we obtain estimates on v and v_t directly from the energy estimates on χ and μ. Next, by the elliptic regularity, we deduce additional spatial estimates on v. Finally, having a priori estimates for $\nabla v \in H^{6,1}(\Omega^T)$ we apply the parabolic theory to obtain estimates for $\Delta v \in H^{6,1}(\Omega^T)$ which eventually provide the desired estimates (1.22).

4. Finally, we notice that since, by energy estimates, $\chi \in L_\infty(\Omega^T)$, it is straightforward to admit functions f_0 and $\varkappa_1$ as polynomials of an arbitrary order (see (1.17), (1.18)). We mention also that a viscous version of the sixth order Cahn-Hilliard type equation with f_0, $\varkappa_1$ given by (1.17), (1.18) has been studied in [19].

Relation to other results on sixth order Cahn-Hilliard type equations. As mentioned above problem (1.1)–(1.4) has been previously studied in [18]. In a more general setting admitting the logarithmic volumetric free energy f_0 and a possible viscous term $\beta\chi_t$, $\beta \geq 0$ in (1.2), system (1.1)–(1.4) has been addressed in [21] from the point of view of the existence of weak solutions. The existence of strong global solutions to a class of sixth order viscous Cahn-Hilliard type equations admitting the terms $\beta\chi_t - \gamma\Delta\chi_t$, $\beta, \gamma > 0$, in (1.2), has been recently proved in [19].

A sixth order convective Cahn-Hilliard type equation arising as a model for the faceting of a growing crystalline surface, derived by Savina et al. [20], has been recently studied in one- and two space dimensions by Korzec et al. [13–15].

Plan of the paper. In Section 2 notation and some auxiliary results are introduced. In Section 3 the basic energy estimates for problem (1.1)–(1.4) are recalled. In Section 4 the Bäcklund transformation is introduced and the corresponding transformation relations are presented. In Section 5 a priori estimates for the transformed

problem are derived. In Section 6 the existence proof by the Leray-Schauder fixed point theorem is presented. It proceeds much the same way as in [18], the only differences being in technical modifications.

2. **Notation and auxiliary results.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be an open bounded subset with a smooth boundary S, and $\Omega^T = \Omega \times (0, T)$, $T \in \mathbb{R}_+ \equiv (0, \infty)$.
We deal with the following spaces:
$W_p^k(\Omega)$, $k \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$, $p \in [1, \infty)$ – the Sobolev space on Ω endowed with the standard norm $\|\cdot\|_{W_p^k(\Omega)}$;
$W_2^k(\Omega) \equiv H^k(\Omega)$, $k \in \mathbb{N}_0$; $H^0(\Omega) \equiv L_2(\Omega)$;
$W_p^{kl,l}(\Omega^T) = L_p(0, T; W_p^{kl}(\Omega)) \cap W_p^l(0, T; L_p(\Omega))$, $k, l \in \mathbb{N}_0$, $p \in [1, \infty)$ – the Sobolev space on Ω^T with the finite norm

$$\|u\|_{W_p^{kl,l}(\Omega^T)} = \left(\sum_{|\alpha| + ka \leq kl} \int_{\Omega^T} |D_x^\alpha \partial_t^a u|^p dx dt \right)^{1/p},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is the multiindex, $\alpha_i \in \mathbb{N}_0$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $a \in \mathbb{N}_0$;
$W_2^{kl,l}(\Omega^T) \equiv H^{kl,l}(\Omega^T)$;
$W_p^{ks,s}(\Omega^T) = L_p(0, T; W_p^{ks}(\Omega)) \cap W_p^s(0, T; L_p(\Omega))$, $k \in \mathbb{N}$, $s \in \mathbb{R}_+$, $p \in [1, \infty)$ – the Sobolev-Slobodecki space on Ω^T with the finite norm

$$\|u\|_{W_p^{ks,s}(\Omega^T)} = \left(\sum_{|\alpha| + ka \leq [ks]} \int_{\Omega^T} |D_x^\alpha \partial_t^a u|^p dx dt \right.$$
$$+ \sum_{|\alpha| = [ks]} \int_0^T \int_\Omega \int_\Omega \frac{|D_x^\alpha u(x,t) - D_{x'}^\alpha u(x',t)|^p}{|x - x'|^{n + p(ks - [ks])}} dx dx' dt$$
$$+ \left. \int_\Omega \int_0^T \int_0^T \frac{|\partial_t^{[s]} u(x,t) - \partial_{t'}^{[s]} u(x,t')|^p}{|t - t'|^{1 + p(s - [s])}} dt dt' dx \right)^{1/p},$$

where $[s]$ is the integer part of s;
$W_2^{ks,s}(\Omega^T) \equiv H^{ks,s}(\Omega^T)$.

By c we denote a generic positive constant which changes its value from formula to formula and depends at most on imbedding constants, parameters of the problem and the regularity of the boundary.

By $\varphi = \varphi(\sigma_1, \ldots, \sigma_k)$, $k \in \mathbb{N}$, we denote a generic function which is positive, increasing function of its arguments $\sigma_1, \ldots, \sigma_k$, and may change its form from formula to formula.
Moreover, ε will denote arbitrarily small positive constant.

Imbeddings in Sobolev-Slobodecki spaces. Following [22, 23] we introduce the fractional derivative norms. For $\mu \in (0, 1)$ and $p \in [1, \infty)$ let

$$[u]_{\mu, p, \Omega^T, x} = \left(\int_0^T \int_\Omega \int_\Omega \frac{|u(x,t) - u(x',t)|^p}{|x - x'|^{n + p\mu}} dx dx' dx \right)^{1/p} \equiv \|\partial_x^\mu u\|_{L_p(\Omega^T)},$$

$$[u]_{\mu, \infty, \Omega^T, x} = \sup_{t \in (0,T)} \sup_{x, x' \in \Omega} \frac{|u(x,t) - u(x',t)|}{|x - x'|^\mu} \equiv \|\partial_x^\mu u\|_{L_\infty(\Omega^T)},$$

and

$$[u]_{\mu,p,\Omega^T,t} = \left(\int_\Omega \int_0^T \int_0^T \frac{|u(x,t) - u(x,t')|^p}{|t-t'|^{1+p\mu}} dt dt' dx \right)^{1/p} \equiv \|\partial_t^\mu u\|_{L_p(\Omega^T)},$$

$$[u]_{\mu,\infty,\Omega^T,t} = \sup_{x\in\Omega} \sup_{t,t'\in(0,T)} \frac{|u(x,t) - u(x,t')|}{|t-t'|^\mu} \equiv \|\partial_t^\mu u\|_{L_\infty(\Omega^T)}.$$

For simplicity we denote the fractional derivatives by $\partial_x^m u$ and $\partial_t^\mu u$.
We shall use the following known results.

Theorem 2.1 (see [3]; Chap. 3, Sect. 10). *Let* $u \in W_p^{ks,s}(\Omega^T)$, $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $s \in \mathbb{R}_+$, $p \in [1,\infty]$. *Let*

$$\varkappa = \left(\frac{n+k}{p} - \frac{n+k}{q} + |\alpha| + ka \right) \frac{1}{ks} \leq 1,$$

where $q \in [1,\infty]$, $\alpha = (\alpha_1,\ldots,\alpha_n)$ *is the multiindex,* $\alpha_i \in \mathbb{N}_0$, $i = 1,\ldots,n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $a \in \mathbb{N}_0$. *then*

$$D_x^\alpha \partial_t^a u \in L_q(\Omega^T), \quad D_x^\alpha = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n},$$

and the interpolation holds

$$\|D_x^\alpha \partial_t^a u\|_{L_q(\Omega^T)} \leq \varepsilon^{1-\varkappa} \left(\|\partial_t^s u\|_{L_p(\Omega^T)} + \sum_{i=1}^n \|\partial_{x_i}^{ks} u\|_{L_p(\Omega^T)} \right)$$
$$+ c\varepsilon^{-\varkappa} \|u\|_{L_p(\Omega^T)} \tag{2.1}$$

where $\varepsilon \in \mathbb{R}_+$ *and* $q \geq p$.
In the case $q = \infty$, *(2.1) holds provided* $\varkappa < 1$.

Theorem 2.2 (Direct boundary trace theorem [22]). *Let us assume that:*
(1) $\Omega \subset \mathbb{R}^n$ *be a domain and* S *be either a boundary of* Ω *or a subdomain of* Ω *with* $\dim S = n - 1$.
(2) $u \in W_p^{ks,s}(\Omega^T)$, $k \in \mathbb{N}$, $s \in \mathbb{R}_+$, $p \in [1,\infty)$, $S \in C^{ks}$.
Then there exists a function $\tilde{u} = u|_{S^T}$ *such that* $\tilde{u} \in W_p^{ks-1/p,s-1/kp}(S^T)$, *and*

$$\|\tilde{u}\|_{W_p^{ks-1/p,s-1/kp}(S^T)} \leq c\|u\|_{W_p^{ks,s}(\Omega)},$$

where constant c *does not depend on* u.

Theorem 2.3 (Direct initial trace theorem [22]). *Let* $u \in W_p^{ks,s}(\Omega^T)$, $k \in \mathbb{N}$, $s \in \mathbb{R}_+$, $s > 1/p$, $p \in (1,\infty)$. *Then* $\tilde{u} = u|_{t=t_0}$, *where* $t_0 \in [0,T]$, *belongs to* $W_p^{ks-k/p}(\Omega)$, *and*

$$\|\tilde{u}\|_{W_p^{ks-k/p}(\Omega)} \leq c\|u\|_{W_p^{ks,s}(\Omega^T)},$$

where c *does not depend on* u.

Auxiliary linear problems. Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be an open, bounded subset of $\mathbb{R}^n$, with a smooth boundary S. Let us consider the problem

$$\Delta\chi = f \qquad \text{in } \Omega,$$
$$n \cdot \nabla\chi = 0 \qquad \text{on } S,$$
$$\fint_\Omega \chi dx = \chi_m \tag{2.2}$$

where the spatial mean χ_m of χ is a given constant. We recall

Lemma 2.4 (see e.g., [18]). *Let us assume that $f \in H^r(\Omega)$, $S \in C^{r+2}$, $r \in \mathbb{N}_0$, and the compatibility condition $\int_\Omega f \, dx = 0$ holds. Then there exists a unique solution $\chi \in H^{r+2}(\Omega)$ to (2.2) such that*

$$\|\chi\|_{H^{r+2}(\Omega)} \leq c(\|f\|_{H^r(\Omega)} + |\chi_m|), \tag{2.3}$$

where c depends at most on r and S.

Next, let us consider the sixth order elliptic problem

$$\begin{aligned}
\Delta^3 v &= f & \text{in } \Omega, \\
\mathbf{n} \cdot \nabla v &= 0, \quad \mathbf{n} \cdot \nabla \Delta v = 0, \quad \mathbf{n} \cdot \nabla \Delta^2 v = 0 & \text{on } S, \\
\fint_\Omega v \, dx &= v_m,
\end{aligned} \tag{2.4}$$

where the spatial mean v_m of v is given. We have

Lemma 2.5. *Let us assume that $f \in H^r(\Omega)$, $S \in C^{r+6}$, $r \in \mathbb{N}_0$, and the compatibility condition $\int_\Omega f \, dx = 0$ holds. Then there exists a unique solution $v \in H^{r+6}(\Omega)$ to (2.4) such that*

$$\|v\|_{H^{r+6}(\Omega)} \leq c(\|f\|_{H^r(\Omega)} + |v_m|), \tag{2.5}$$

where c depends at most on r and S.

Proof. By the elliptic estimate [16, Vol. I, Chap. 2, Sec. 5] we have

$$\|v\|_{H^{6+r}(\Omega)} \leq c(\|f\|_{H^r(\Omega)} + \|v\|_{L_2(\Omega)}). \tag{2.6}$$

To conclude (2.5) we have to estimate the norm $\|v\|_{L_2(\Omega)}$. To this end we multiply (2.4)$_1$ by v, integrate over Ω, use boundary conditions (2.4)$_2$, and the fact that $\int_\Omega f \, dx = 0$. This leads to

$$\int_\Omega |\nabla \Delta v|^2 dx = -\int_\Omega (v - v_m) f \, dx \leq \varepsilon \|\nabla v\|_{L_2(\Omega)}^2 + c(1/\varepsilon)\|f\|_{L_2(\Omega)}^2. \tag{2.7}$$

Further, by the Poincaré inequality, the fact that $\int_\Omega \Delta v \, dx = 0$, and (2.7) it holds

$$\|\Delta v\|_{L_2(\Omega)}^2 \leq c\|\nabla \Delta v\|_{L_2(\Omega)}^2 \leq \varepsilon \|\nabla v\|_{L_2(\Omega)}^2 + c(1/\varepsilon)\|f\|_{L_2(\Omega)}^2. \tag{2.8}$$

Let us consider now the auxiliary artificial problem (with $\Delta \chi \equiv g \in L_2(\Omega)$ on the right-hand side of (2.9)$_1$ treated as given)

$$\begin{aligned}
\Delta v &= \Delta v \equiv g & \text{in } \Omega, \\
\mathbf{n} \cdot \nabla v &= 0 & \text{on } S, \\
\fint_\Omega v \, dx &= v_m.
\end{aligned} \tag{2.9}$$

Multiplying (2.9)$_1$ by v, integrating over Ω, and using that $\fint_\Omega g \, dx = 0$, we obtain

$$\|\nabla v\|_{L_2(\Omega)}^2 \leq c\|g\|_{L_2(\Omega)}^2 = c\|\Delta v\|_{L_2(\Omega)}^2. \tag{2.10}$$

Using (2.10) in (2.8) and assuming that ε is sufficiently small, we deduce

$$\|\nabla v\|_{L_2(\Omega)}^2 \leq c\|f\|_{L_2(\Omega)}^2. \tag{2.11}$$

Now, thanks to (2.11), it follows that

$$\|v\|_{L_2(\Omega)}^2 = \int_\Omega (v - v_m + v_m)^2 dx \le c\|f\|_{L_2(\Omega)}^2 + cv_m^2. \tag{2.12}$$

Hence, using (2.12) in (2.6) yields the desired estimate (2.5).   □

Finally, we recall the solvability result for the sixth order linear parabolic problem which is used in the proof of Theorem A.

Lemma 2.6 (see [16; Vol. II, Chap. 4, 23]). *Let us consider the linear initial-boundary value problem*

$$\chi_t - \Delta^3 \chi = F \qquad\qquad in \ \Omega^T = \Omega \times (0, T),$$
$$\chi(0) = \chi_0 \qquad\qquad in \ \Omega, \tag{2.13}$$
$$\mathbf{n} \cdot \nabla\chi = 0, \ \ \mathbf{n} \cdot \nabla\Delta\chi = 0, \ \ \mathbf{n} \cdot \nabla\Delta^2\chi = G \qquad on \ S^T = S \times (0, T),$$

where $\Omega \subset \mathbb{R}^n$, $n \ge 1$, is a domain with a boundary of class C^6. Assume that

$$F \in L_2(\Omega^T), \quad G \in H^{1/2, 1/12}(S^T), \quad \chi_0 \in H^3(\Omega). \tag{2.14}$$

Moreover, let the following compatibility condition holds on S

$$\mathbf{n} \cdot \nabla\chi_0 = 0. \tag{2.15}$$

Then for any $T > 0$ problem (2.13) has a unique solution $\chi \in H^{6,1}(\Omega^T)$ satisfying the estimate

$$\|\chi\|_{H^{6,1}(\Omega^T)} \le c(\|F\|_{L_2(\Omega^T)} + \|G\|_{H^{1/2, 1/12}(S^T)} + \|\chi_0\|_{H^3(\Omega)}) \tag{2.16}$$

with a constant c independent of T.

3. Energy estimates. We record first the basic properties of problem (1.1)–(1.4), referring for details to [18, 19]. From (1.1) and the third condition in (1.4) it follows that

$$\frac{d}{dt} \int_\Omega \chi dx = 0, \tag{3.1}$$

which shows that the spatial mean of χ is preserved,

$$\fint_\Omega \chi(t) dx = \fint_\Omega \chi_0 dx =: \chi_m \quad \text{for all } t > 0. \tag{3.2}$$

Next, we notice that problem (1.1)–(1.4) has a variational structure. For sufficiently regular solutions (χ, μ) the following energy equality is satisfied

$$\frac{d}{dt} \int_\Omega \left(f_0(\chi) + \frac{1}{2}\varkappa_1(\chi)|\nabla\chi|^2 + \frac{1}{2}\varkappa_2|\Delta\chi|^2 \right) dx + M \int_\Omega |\nabla\mu|^2 dx = 0. \tag{3.3}$$

Formally, (3.3) results by multiplying (1.1) by μ, (1.2) by χ_t, taking the difference of the obtained relations, integrating with respect to space variables, using the no-flux conditions (1.4), and performing suitable integrations by parts.

To deduce estimates from (3.3) let us notice that on account of assumption (1.17) and (1.18) there exist positive constants c_{f_0} and $c_{\varkappa_1}$ such that

$$f_0(\chi) \ge \frac{1}{2}a_{2k}\chi^{2k} - c_{f_0}, \quad \varkappa_1(\chi) \ge \frac{1}{2}b_{2l}\chi^{2l} - c_{\varkappa_1}. \tag{3.4}$$

Lemma 3.1 (Energy estimate) [19; Lemma 4.1]). *Let us assume that f_0, $\varkappa_1$ are given by (1.17), (1.18), and $\chi_0 \in H^2(\Omega)$, $\fint \chi_0 dx = \chi_m$. Then for a sufficiently regular solution (χ, μ) to system (1.1)–(1.4) the following estimate holds:*

$$
a_{2k}\|\chi\|^{2k}_{L_{2k}(\Omega)} + b_{2l}\|\chi^l \nabla\chi\|^2_{L_2(\Omega)} + \varkappa_2\|\chi\|^2_{H^2(\Omega)}
$$
$$
+ \|\nabla\mu\|^2_{L_2(\Omega^t)} \leq c_1 \quad \text{for all } t > 0,
$$
(3.5)

with $c_1 = \varphi(\|\chi_0\|_{H^2(\Omega)}, |\chi_m|, \varkappa_2, c_{\varkappa_1}, c_{f_0}, a_{2k})$.

Corollary 1. *In the sequel we shall use (3.5) in the following simplified form*

$$
\|\chi\|_{L_\infty(0,t;H^2(\Omega))} + \|\nabla\mu\|_{L_2(\Omega^t)} \leq \varphi(c_1) \quad \text{for } t > 0.
$$
(3.6)

Corollary 2. *On account of boundary coditions $(1.4)_{1,2}$, integrating of (1.2) gives*

$$
\int_\Omega \mu dx = \int_\Omega \left(f_0'(\chi) + \frac{1}{2}\varkappa_1'(\chi)|\nabla\chi|^2 \right) dx.
$$
(3.7)

Hence, by (1.17), (1.18) and (3.6), it follows that

$$
\mathrm{esssup}_{t' \in [0,t]} \left| \int_\Omega \mu dx \right| \leq \varphi(c_1).
$$
(3.8)

Moreover, by the Poincaré inequality, (3.6) and (3.8) imply that

$$
\|\mu\|_{L_2(0,t;H^1(\Omega))} \leq \varphi(c_1)(t^{1/2} + 1) \quad \text{for } t > 0.
$$
(3.9)

4. The Bäcklund transformation. As a preparatory step before applying the transformation we introduce the translation of the unknown function

$$
\tilde{\chi} = \chi - \chi_m \quad \text{with} \quad \chi_m = \fint_\Omega \chi_0 dx.
$$
(4.1)

Then problem (1.1)–(1.4) is reduced to

$$
\tilde{\chi}_t = M\Delta\tilde{\mu} \qquad \text{in } \Omega^T,
$$

$$
\tilde{\mu} = f_0'(\tilde{\chi} + \chi_m) + \frac{1}{2}\varkappa_1'(\tilde{\chi} + \chi_m)|\nabla\tilde{\chi}|^2
$$
$$
- \nabla \cdot (\varkappa_1(\tilde{\chi} + \chi_m)\nabla\tilde{\chi}) + \varkappa_2\Delta^2\tilde{\chi}
$$
$$
\equiv \frac{\delta f(\tilde{\chi} + \chi_m)}{\delta(\tilde{\chi} + \chi_m)} = \frac{\delta f(\chi)}{\delta\chi} = \mu \qquad \text{in } \Omega^T,
$$
(4.2)

$$
\tilde{\chi}|_{t=0} = \tilde{\chi}_0 := \chi_0 - \chi_m \qquad \text{in } \Omega,
$$

$$
\mathbf{n} \cdot \nabla\tilde{\chi} = 0, \quad \mathbf{n} \cdot \nabla\Delta\tilde{\chi} = 0, \quad \mathbf{n} \cdot \nabla\tilde{\mu} = 0 \qquad \text{on } S^T,
$$

where

$$
\fint_\Omega \tilde{\chi}_0 dx = 0.
$$

We remark that the artificial notation $\tilde{\mu} = \mu$ is introduced just to remain in an agreement with the notation $\tilde{\chi}$.

Now, following [17], we introduce the new scalar variable, termed there the dynamical field potential,

$$
v = M \int_0^t \tilde{\mu} dt' + v_0 \quad (\tilde{\mu} \equiv \mu)
$$
(4.3)

where $v_0 = v_0(x)$ is a solution to the elliptic problem

$$\Delta v_0 = \tilde{\chi}_0 \qquad \text{in } \Omega,$$
$$\mathbf{n} \cdot \nabla v_0 = 0 \qquad \text{on } S,$$
$$\fint_\Omega v_0 dx = 0, \tag{4.4}$$

with $\tilde{\chi}_0 = \chi_0 - \chi_m$ satisfying the compatibility condition $\int_\Omega \tilde{\chi}_0 dx = 0$.

Lemma 4.1 (Transformation relations). *(i) Let $(\tilde{\chi}, \tilde{\mu})$ satisfy problem (4.2) and v be defined by (4.3), (4.4). Then*

$$\Delta v = \tilde{\chi}, \quad v_t = M\tilde{\mu} \quad \text{in } \Omega^T, \tag{4.5}$$

equivalently,

$$\Delta v = \chi - \chi_m, \quad v_t = M\mu \quad \text{in } \Omega^T,$$

and

$$\mathbf{n} \cdot \nabla v = 0 \qquad \text{on } S^T,$$
$$\mathbf{n} \cdot \nabla \Delta v = 0 \qquad \text{on } S^T,$$
$$\mathbf{n} \cdot \nabla \Delta^2 v = 0 \qquad \text{on } S^T, \tag{4.6}$$
$$v|_{t=0} = v_0 \qquad \text{in } \Omega,$$

where v_0 is defined by (4.4).
(ii) Conversely, let us assume that relations (4.5), (4.6) hold. Then

$$\tilde{\chi}_t = M\Delta\tilde{\mu} \quad \text{in } \Omega^T, \tag{4.7}$$

equivalently,

$$\chi_t = M\Delta\mu \quad \text{in } \Omega^T,$$

and

$$v = M \int_0^t \tilde{\mu} dt' + v_0 \quad \text{in } \Omega^T \tag{4.8}$$

with $v_0 = v_0(x)$ satisfying (4.4), and

$$\mathbf{n} \cdot \nabla \tilde{\chi} = 0 \qquad \text{on } S^T,$$
$$\mathbf{n} \cdot \nabla \Delta \tilde{\chi} = 0 \qquad \text{on } S^T, \tag{4.9}$$
$$\mathbf{n} \cdot \nabla \tilde{\mu} = 0 \qquad \text{on } S^T.$$

Proof. (i) By $(4.2)_1$, $(4.4)_1$ and (4.3) we have

$$\tilde{\chi}(t) = \int_0^t \tilde{\chi}_{t'} dt' + \tilde{\chi}_0 = M \int_0^t \Delta\tilde{\mu} dt' + \Delta v_0 = \Delta v \tag{4.10}$$

which gives the first equality in (4.5). The second one results immediately from (4.3). Further, by $(4.5)_1$ the boundary conditions $(4.2)_4$ imply that

$$\mathbf{n} \cdot \nabla \Delta v = \mathbf{n} \cdot \nabla \tilde{\chi} = 0 \qquad \text{on } S^T,$$
$$\mathbf{n} \cdot \nabla \Delta^2 v = \mathbf{n} \cdot \nabla \Delta \tilde{\chi} = 0 \qquad \text{on } S^T, \tag{4.11}$$

and, recalling (4.3), $(4.4)_2$,

$$\mathbf{n} \cdot \nabla v = M \int_0^t \mathbf{n} \cdot \nabla \tilde{\mu} dt' + \mathbf{n} \cdot \nabla v_0 = 0 \quad \text{on} \quad S^T.$$

Hence, $(4.6)_1$–$(4.6)_3$ hold. The equality $(4.6)_4$ is immediate.
(ii) By relations (4.5) it follows that

$$\tilde{\chi}_t = \Delta v_t = M \Delta \tilde{\mu}$$

which yields (4.7). Further, by $(4.5)_1$ and $(4.6)_{2,3}$ we have

$$\begin{aligned}
\mathbf{n} \cdot \nabla \tilde{\chi} &= \mathbf{n} \cdot \nabla \Delta v = 0 & \text{on} \quad S^T, \\
\mathbf{n} \cdot \nabla \Delta \tilde{\chi} &= \mathbf{n} \cdot \nabla \Delta^2 v = 0 & \text{on} \quad S^T.
\end{aligned} \tag{4.12}$$

Moreover, by $(4.5)_2$ and $(4.6)_1$,

$$M\mathbf{n} \cdot \nabla \tilde{\mu} = \mathbf{n} \cdot \nabla v_t = 0 \quad \text{on} \quad S^T.$$

This proves (4.9). Finally, from $(4.5)_2$ and $(4.6)_4$ we have

$$v(t) = \int_0^t v_{t'} dt' + v_0 = M \int_0^t \tilde{\mu} dt' + v_0 \tag{4.13}$$

which yields (4.8) and completes the proof.   □

Consequently, in view of Lemma 4.1 we conclude that if (χ, μ) satisfy problem (1.1)–(1.4) then v, defined by $4.3)$, satisfies

$$\begin{aligned}
v_t &= M\mu & \text{in} \quad \Omega^T, \\
\mu &= \frac{\delta f(\Delta v + \chi_m)}{\delta(\Delta v + \chi_m)} = f_0'(\Delta v + \chi_m) - \frac{1}{2}\varkappa_1'(\Delta v + \chi_m)|\nabla \Delta v|^2 \\
&\quad - \varkappa_1(\Delta v + \chi_m)\Delta^2 v + \varkappa_2 \Delta^3 v & \text{in} \quad \Omega^T, \\
v|_{t=0} &= v_0, \quad \fint_\Omega v_0 dx = 0 & \text{in} \quad \Omega, \\
\mathbf{n} \cdot \nabla v &= 0, \quad \mathbf{n} \cdot \nabla \Delta v = 0, \quad \mathbf{n} \cdot \nabla \Delta^2 v = 0 & \text{on} \quad S^T.
\end{aligned} \tag{4.14}$$

Thus, after inserting $(4.14)_2$ into $(4.14)_1$, system (4.14) reduces to

$$\begin{aligned}
v_t - M\varkappa_2 \Delta^3 v &= M\left[f_0'(\Delta v + \chi_m) - \frac{1}{2}\varkappa_1'(\Delta v + \chi_m)|\nabla \Delta v|^2 \right. \\
&\quad \left. - \varkappa_1(\Delta v + \chi_m)\Delta^2 v \right] \equiv K & \text{in} \quad \Omega^T, \\
v|_{t=0} &= v_0 & \text{in} \quad \Omega, \\
\mathbf{n} \cdot \nabla v &= 0, \quad \mathbf{n} \cdot \nabla \Delta v = 0, \quad \mathbf{n} \cdot \nabla \Delta^2 v = 0 & \text{on} \quad S^T,
\end{aligned} \tag{4.15}$$

with v_0 defined by (4.4).

Conversely, if v satisfies (4.15), then (χ, μ) defined by (4.5) satisfy (1.1)–(1.4). In this sense problems (1.1)–(1.4) and (4.15) are equivalent.

5. **A priori estimates for the transformed problem.** We study here the implications of energy estimates presented in Section 3 on the transformed problem (4.15).

Lemma 5.1 (First regularity estimate). *Let the assumptions of Lemma 3.1 be satisfied. Moreover, let the boundary S of the domain $\Omega \subset \mathbb{R}^3$ be of class C^6. Then a solution v to (4.15) satisfies the estimate*

$$\|v\|_{L_\infty(0,t;H^4(\Omega))} + \|v\|_{L_2(0,t;H^6(\Omega))} + \|v_{t'}\|_{L_2(0,t;H^1(\Omega))}$$
$$\leq \varphi(c_1, t), \quad t > 0, \tag{5.1}$$

with $c_1 = \varphi(\|\chi_0\|_{H^2(\Omega)}, |\chi_m|, \varkappa_2, c_{\varkappa_1}, c_{f_0}, a_{2k})$ introduced in Lemma 3.1.

Proof. Let us note that by the second relation in (4.5)

$$\frac{d}{dt} \int_\Omega v \, dx = M \int_\Omega \bar{\mu} \, dx = M \int_\Omega \mu \, dx.$$

Hence, since by $(4.4)_3$,

$$\int_\Omega v(0) \, dx = \int_\Omega v_0 \, dx = 0,$$

we have

$$\fint_\Omega v(t) \, dx = \frac{M}{|\Omega|} \int_{\Omega^t} \mu \, dx \, dt' =: v_m(t). \tag{5.2}$$

Now, let us consider the following elliptic problem which results from the first relation in (4.5), the first boundary condition in $(4.15)_3$, and (5.2):

$$\Delta v = \bar{\chi} \qquad\qquad \text{in } \Omega,$$
$$\mathbf{n} \cdot \nabla v = 0 \qquad\qquad \text{on } S,$$
$$\fint_\Omega v \, dx = v_m(t), \quad t > 0. \tag{5.3}$$

Due to the elliptic regularity estimate (2.3),

$$\|v\|_{H^4(\Omega)} \leq c(\|\bar{\chi}\|_{H^2(\Omega)} + |v_m(t)|), \tag{5.4}$$

where, by (3.6),

$$\|\bar{\chi}\|_{L_\infty(0,t;H^2(\Omega))} \leq \varphi(c_1),$$

and, by (3.8),

$$|v_m(t)| \leq \frac{M}{|\Omega|} \varphi(c_1) t, \quad t > 0. \tag{5.5}$$

Thus, we conclude that

$$\|v\|_{L_\infty(0,t;H^4(\Omega))} \leq \varphi(c_1, t). \tag{5.6}$$

Further, on account of the relation $M\bar{\mu} = v_t$, it follows from (3.9) that

$$\|v_{t'}\|_{L_2(0,t;H^1(\Omega))} \leq \varphi(c_1, t). \tag{5.7}$$

Thanks to (5.6) and (5.7) we can treat (4.15) as a sixth order elliptic problem

$$M\varkappa_2 \Delta^3 v = v_t - K \qquad\qquad \text{in } \Omega$$
$$\mathbf{n} \cdot \nabla v = 0, \quad \mathbf{n} \cdot \nabla \Delta v = 0, \quad \mathbf{n} \cdot \nabla \Delta^2 v = 0 \qquad \text{on } S,$$
$$\fint_\Omega v \, dx = v_m(t), \quad t > 0, \tag{5.8}$$

where we recall that

$$K = M\left[f_0'(\Delta v + \chi_m) - \frac{1}{2}\varkappa_1'(\Delta v + \chi_m)|\nabla\Delta v|^2 - \varkappa_1(\Delta v + \chi_m)\Delta^2 v\right].$$

By virtue of (5.6), using assumptions (1.17), (1.18) we have

$$\|K\|_{L_2(0,t;L_2(\Omega))} \leq \varphi(c_1, t). \tag{5.9}$$

Hence, by the elliptic estimate (2.5), we conclude from (5.9), (5.7) and (5.5) that

$$\begin{aligned}\|v\|_{L_2(0,t;H^6(\Omega))} &\leq c(\|K\|_{L_2(0,t;L_2(\Omega))} \\ &+ \|v_{t'}\|_{L_2(0,t;L_2(\Omega))} + \|v_m(t')\|_{L_2(0,t)}) \leq \varphi(c_1, t).\end{aligned} \tag{5.10}$$

The estimates (5.6), (5.7) and (5.10) yield (5.1). □

With the help of (5.1) one can easily deduce more spatial regularity of v.

Lemma 5.2 (Second regularity estimate). *Let the assumptions of Lemma 3.2 be satisfied. Moreover, let the boundary S be of class C^7. Then a solution v to problem (4.15) satisfies in addition to (5.1) the estimate*

$$\|v\|_{L_2(0,t;H^7(\Omega))} \leq \varphi(c_1, t), \quad t > 0. \tag{5.11}$$

Proof. Let us consider again the elliptic problem (5.8). Taking into account that by (5.1), $v \in H^{6,1}(\Omega^t)$ and $\Delta v \in L_\infty(\Omega^t)$, it follows that

$$\begin{aligned}\|K\|_{L_2(0,t;H^1(\Omega))} &\leq c(\|\nabla\Delta v\|_{L_2(\Omega^t)} + \||\nabla\Delta v|\,|\nabla\Delta v|^2\|_{L_2(\Omega^t)} \\ &+ \||\nabla\Delta v|\,|\nabla^2\Delta v|\|_{L_2(\Omega^t)} + \||\nabla\Delta v|\,\Delta^2 v\|_{L_2(\Omega^t)} \\ &+ \|\nabla\Delta^2 v\|_{L_2(\Omega^t)}) \equiv \sum_{k=1}^{5} I_k,\end{aligned} \tag{5.12}$$

where, by the imbeddings $\|\nabla^3 v\|_{L_6(\Omega^t)} + \|\nabla^4 v\|_{L_3(\Omega^t)} \leq c\|v\|_{H^{6,1}(\Omega^t)}$, we have

$$\begin{aligned}I_1 + I_5 &\leq \varphi(c_1, t), \qquad I_2 \leq \|\nabla\Delta v\|_{L_6(\Omega^t)}^3 \leq \varphi(c_1, t), \\ I_3 + I_4 &\leq \|\nabla\Delta v\|_{L_6(\Omega^t)}\|\nabla^2\Delta v\|_{L_3(\Omega^t)} \\ &\leq \varphi(c_1, t)\|\nabla^2\Delta v\|_{L_3(\Omega^t)} \leq \varphi(c_1, t).\end{aligned}$$

Hence,

$$\|K\|_{L_2(0,t;H^1(\Omega))} \leq \varphi(c_1, t). \tag{5.13}$$

Consequently, by the elliptic regularity (2.5), estimates (5.13), (5.7) and (5.5) imply (5.11). □

Our goal now is to obtain an estimate for $v \in L_2(0, T; H^8(\Omega))$. For this purpose we consider the following problem (formally resulting by acting the Δ-operator on

(4.15)):

$$\Delta v_t - M \varkappa_2 \Delta^3 \Delta v = M \Delta \left[f_0'(\Delta v + \chi_m) \right.$$

$$\left. - \frac{1}{2} \varkappa_1'(\Delta v + \chi_m)|\nabla \Delta v|^2 - \varkappa_1(\Delta v + \chi_m)\Delta^2 v \right] = \Delta K \equiv F \qquad \text{in } \Omega^T,$$

$$\Delta v|_{t=0} = \Delta v_0 = \chi_0 - \chi_m \qquad \text{in } \Omega,$$

$$\mathbf{n} \cdot \nabla \Delta v = 0, \quad \mathbf{n} \cdot \nabla \Delta^2 v = 0 \qquad \text{on } S^T,$$

$$\mathbf{n} \cdot \nabla \Delta^3 v = \frac{1}{2\varkappa_2} \varkappa_1'(\Delta v + \chi_m)\mathbf{n} \cdot (|\nabla \Delta v|^2) \equiv G \qquad \text{on } S^T.$$

$$(5.14)$$

The third nonlinear boundary condition on S^T arises in compatibility with equation $(5.14)_1$ and the three homogeneous boundary conditions $(4.15)_3$. In fact, we have

$$M \varkappa_2 \mathbf{n} \cdot \nabla \Delta^3 v = \mathbf{n} \cdot \nabla v_t - M \mathbf{n} \cdot \nabla \left[f_0'(\Delta v + \chi_m) \right.$$

$$\left. - \frac{1}{2} \varkappa_1'(\Delta v + \chi_m)|\nabla \Delta v|^2 - \varkappa_1(\Delta v + \chi_m)\Delta^2 v \right] \qquad (5.15)$$

$$= \frac{M}{2} \varkappa_1'(\Delta v + \chi_m)\mathbf{n} \cdot \nabla(|\nabla \Delta v|^2).$$

Since by (4.5), $\Delta v = \chi - \chi_m = \tilde{\chi}$, $v_t = \mu$, we see that (5.14), treated as a parabolic problem for Δv, and the original problem (1.5) are equivalent. Moreover, by (5.15), the boundary conditions $(5.14)_{3,4}$ imply that

$$\mathbf{n} \cdot \nabla v_t = \mathbf{n} \cdot \nabla \mu = 0 \qquad \text{on } S^T.$$

By virtue of the parabolic theory (see Lemma 2.6), if $F \in L_2(\Omega^T)$, $G \in H^{1/2,1/12}(S^T)$, $\Delta v_0 \in H^3(\Omega)$ and satisfies the compatibility condition $\mathbf{n} \cdot \nabla \Delta v_0 = 0$ on S, then the solution $\Delta v = \tilde{\chi}$ to problem (5.14) satisfies $\Delta v \in H^{6,1}(\Omega^T)$, and

$$\|\Delta v\|_{H^{6,1}(\Omega^T)} \le c(\|F\|_{L_2(\Omega^T)} + \|G\|_{H^{1/2,1/12}(S^T)} + \|\Delta v_0\|_{H^3(\Omega)}). \qquad (5.16)$$

Now, using (5.16) we prove the following.

Lemma 5.3 (Third regularity estimate). *Let the assumptions of Lemma 3.1 hold, $S \in C^7$, and $\Delta v_0 = \chi_0 - \chi_m \in H^3(\Omega)$ satisfies the compatibility condition $\mathbf{n} \cdot \nabla \Delta v_0 = \mathbf{n} \cdot \nabla \chi_0 = 0$ on S. Then a solution Δv to (5.14) satisfies the estimate*

$$\|\Delta v\|_{H^{6,1}(\Omega^T)} \le \varphi(c_1, T) + c\|\Delta v_0\|_{H^3(\Omega)} \equiv c_2, \qquad (5.17)$$

with $c_1 = \varphi(\|\chi_0\|_{H^2(\Omega)}, |\chi_m|, \varkappa_2, c_{\varkappa_1}, c_{f_0}, a_{2k})$ as in Lemma 3.1, and $\Delta v_0 = \chi_0 - \chi_m$.

Proof. We estimate the first two terms on the right-hand side of (5.16). From Lemmas 5.1 and 5.2 we have

$$v \in L_\infty(0, T; H^4(\Omega)) \cap L_2(0, T; H^7(\Omega)) \cap H^1(0, T; H^1(\Omega)).$$

Hence,

$$u \equiv \nabla v \in H^{6,1}(\Omega^T), \text{ and}$$

$$A \equiv \|u\|_{H^{6,1}(\Omega^T)} \le \varphi(c_1, T). \qquad (5.18)$$

The following imbeddings will be used

$$\|\nabla u\|_{L_\infty(\Omega^T)} + \|\nabla^2 u\|_{L_{18}(\Omega^T)} + \|\nabla^3 u\|_{L_6(\Omega^T)} + \|\nabla^4 u\|_{L_{\frac{18}{5}}(\Omega^T)} \le cA. \qquad (5.19)$$

We have

$$\|F\|_{L_2(\Omega^T)} \leq c\|\Delta f_0'(\Delta v + \chi_m)\|_{L_2(\Omega^T)}$$
$$+ c\|\Delta(\varkappa_1'(\Delta v + \chi_m)|\nabla \Delta v|^2\|_{L_2(\Omega^T)} \tag{5.20}$$
$$+ c\|\Delta(\varkappa(\Delta v + \chi_m)\Delta^2 v)\|_{L_2(\Omega^T)} \equiv F_1 + F_2 + F_3.$$

The terms F_k are estimated as follows

$$F_1 \leq \|f_0''''(\Delta v + \chi_m)|\nabla \Delta v|^2\|_{L_2(\Omega^T)} + \|f_0''(\Delta v + \chi_m)\nabla^2 \Delta v\|_{L_2(\Omega^T)}$$
$$\leq c(\|\Delta^2 v\|_{L_2(\Omega^T)} + \| |\nabla \Delta v|^2\|_{L_2(\Omega^T)} \equiv c(F_1^1 + F_1^2),$$

where

$$F_1^1 \leq \|\nabla \Delta u\|_{L_2(\Omega^T)} \leq cA,$$
$$F_1^2 \leq \| |\Delta u|^2\|_{L_2(\Omega^T)} = \|\Delta u\|_{L_4(\Omega^T)}^2 \leq cA^2.$$

Further,

$$F_2 \leq c\|\varkappa_1'''|\nabla \Delta v|^4\|_{L_2(\Omega^T)} + c\|\varkappa_1''|\nabla \Delta v|^2 \nabla^2 \Delta v\|_{L_2(\Omega^T)}$$
$$+ c\|\varkappa_1'|\nabla^2 \Delta v|^2 + \varkappa_1'|\nabla \Delta v| |\nabla^3 \Delta v| \|_{L_2(\Omega^T)}$$
$$\leq c(\| |\nabla^3 v|^4\|_{L_2(\Omega^T)} + \| |\nabla^4 v| |\nabla^3 v|^2\|_{L_2(\Omega)} + \| |\nabla^4 v|^2\|_{L_2(\Omega^T)}$$
$$+ \| |\nabla^3 v| |\nabla^5 v| \|_{L_2(\Omega^T)})$$
$$\leq c(\| |\nabla^2 u|^4\|_{L_2(\Omega^T)} + \| |\nabla^3 u| |\nabla^2 u|^2\|_{L_2(\Omega^T)} + \|\nabla^3 u\|_{L_4(\Omega^T)}$$
$$+ \| |\nabla^2 u| |\nabla^4 u| \|_{L_2(\Omega^T)}) \equiv c\sum_{l=1}^{4} F_2^l,$$

where

$$F_2^1 = \|\nabla^2 u\|_{L_8(\Omega^T)}^4 \leq cA^4,$$
$$F_2^2 \leq \|\nabla^3 u\|_{L_{\frac{18}{3}}(\Omega^T)}\|\nabla^2 u\|_{L_{18}(\Omega^T)}^2 \leq cA^3,$$
$$F_2^3 \leq cA^2, \quad F_2^1 \leq \|\nabla^2 u\|_{L_{18}(\Omega^T)}\|\nabla^4 u\|_{L_{\frac{9}{4}}(\Omega^T)} \leq cA^2.$$

Finally,

$$F_3 \leq c(\|\varkappa_1''|\nabla \Delta v|^2 \Delta^2 v\|_{L_2(\Omega^T)} + \|\varkappa_1'|\nabla^2 \Delta v\|_{L_2(\Omega^T)}$$
$$+ \|\varkappa_1'|\nabla \Delta v| |\nabla \Delta^2 v| \|_{L2(\Omega^T)} + \|\varkappa_1 \nabla^2 \Delta^2 v\|_{L_2(\Omega^T)})$$
$$\leq c(\| |\nabla^2 u|^2 \nabla^3 u\|_{L_2(\Omega^T)} + \| |\nabla \Delta u| |\nabla^3 u| \|_{L_2(\Omega^T)}$$
$$+ \| |\Delta u| |\nabla^4 u| \|_{L_2(\Omega^T)} + \|\nabla^5 u\|_{L_2(\Omega^T)}) \equiv c\sum_{l=1}^{4} F_3^l,$$

where

$$F_3^1 \leq F_2^2 \leq cA^3, \quad F_3^2 \leq F_2^3 \leq cA^2, \quad F_3^3 \leq F_2^4 \leq cA^2, \quad F_3^4 \leq cA.$$

Summarizing, we conclude that

$$\|F\|_{L_2(\Omega^T)} \leq \varphi(A) \leq \varphi(c_1, T). \tag{5.21}$$

To estimate the boundary term on the right-hand side of (5.16) we introduce a smooth extension of the outward unit normal n to S onto a neighbourhood of S. Then, by the direct boundary trace theorem (see Theorem 2.2),

$$\|G\|_{H^{1/2,1/12}(S^T)} \leq c\|G\|_{H^{1,1/6}(\Omega^T)}, \tag{5.22}$$

where $H^{1/2,1/12}(S^T)$ is the space of traces of functions from $H^{1,1/6}(\Omega^T)$. We have

$$
\begin{aligned}
\|G\|_{H^{1,1/6}(\Omega^T)} &= \frac{1}{2\varkappa_2}\|\varkappa_1'(\Delta v + \chi_m)\mathbf{n}\cdot\nabla(|\nabla\Delta v|^2)\|_{L_2(0,T;H^1(\Omega))} \\
&\quad + \frac{1}{2\varkappa_2}\|\varkappa_1'(\Delta v + \chi_m)\mathbf{n}\cdot\nabla(|\nabla\Delta v|^2)\|_{H^{1/6}(0,T;L_2(\Omega))} \\
&\equiv I_1 + I_2.
\end{aligned}
\tag{5.23}
$$

Using the imbeddings (5.19) we get

$$
I_1 \leq c\|\varkappa_1''|\nabla\Delta v|^2\nabla^2\Delta v\|_{L_2(\Omega^T)} + c\|\varkappa_1'|\nabla^2\Delta v|^2\|_{L_2(\Omega^T)}
$$

$$
+ c\|\varkappa_1'|\nabla\Delta v|\,|\nabla^3\Delta v|\,\|_{L_2(\Omega^T)} + c\|\varkappa_1'|\nabla\Delta v|\,|\nabla^2\Delta v|\,\|_{L_2(\Omega^T)} \equiv \sum_{l=1}^{4} I_1^l,
$$

where

$$
\begin{aligned}
I_1^1 &\leq c\|\,|\Delta u|^2|\nabla\Delta u|\,\|_{L_2(\Omega^T)} \leq c\|\nabla^3 u\|_{L_{18}(\Omega^T)}\|\nabla^2 u\|_{L_{18}(\Omega^T)}^2 \leq cA^3, \\
I_1^2 &\leq c\|\,|\nabla\Delta u|^2\|_{L_2(\Omega^T)} \leq c\|\nabla\Delta u\|_{L_4(\Omega^T)}^2 \leq cA^2, \\
I_1^3 &\leq c\|\,|\Delta u|\,|\nabla^4 u|\,\|_{L_2(\Omega^T)} \leq c\|\,\|\Delta u\|_{L_6(\Omega)}\|\nabla^4\|_{L_3(\Omega)}\|_{L_2(0,T)} \\
&\leq c\|\Delta u\|_{L_\infty(0,T;H^1(\Omega))}\|\nabla^4 u\|_{L_2(0,T;L_3(\Omega))} \leq \varphi(c_1,T)A, \\
I_1^4 &\leq c\|\,|\Delta u|\nabla^3 u\|_{L_2(\Omega^T)} \leq c\|\,\|\Delta u\|_{L_6(\Omega)}\|\nabla^3 u\|_{L_3(\Omega)}\|_{L_2(0,T)} \\
&\leq c\|\Delta u\|_{L_\infty(0,T;H^1(\Omega))}\|\nabla^3 u\|_{L_2(0,T;L_3(\Omega))} \leq \varphi(c_1,T)A.
\end{aligned}
$$

In I_1^3 and I_1^4 we used the bound $\|v\|_{L_\infty(0,T;H^4(\Omega))} \leq \varphi(c_1,T)$ and the imbedding

$$
\|\nabla^4 u\|_{L_2(0,T;L_3(\Omega))} \leq c\|u\|_{H^{6,1}(\Omega^T)}.
$$

Consequently,

$$
I_1 \leq \varphi(c_1,T). \tag{5.24}
$$

For the term I_2 in (5.23) we have

$$
\begin{aligned}
I_2 &\leq c\|\varkappa_1''|\partial_t^{1/6}\Delta v|\,|\nabla\Delta v|\,|\nabla^2\Delta v|\,\|_{L_2(\Omega^T)} \\
&\quad + c\|\varkappa_1'|\partial_t^{1/6}\nabla\Delta v|\,|\nabla^2\Delta v|\,\|_{L_2(\Omega^T)} + c\|\varkappa_1'|\nabla\Delta v|\,|\partial_t^{1/6}\nabla^2\Delta v|\,\|_{L_2(\Omega^T)} \\
&\equiv \sum_{l=1}^{3} I_2^l.
\end{aligned}
$$

Again, applying the imbeddings (5.19), and

$$
\begin{aligned}
\|\partial_t^{1/6}\nabla u\|_{L_{18}(\Omega^T)} &+ \|\partial_t^{1/6}\nabla^2 u\|_{L_6(\Omega^T)} \\
&+ \|\partial_t^{1/6}\nabla^3 u\|_{L_2(0,T;L_3(\Omega))} \leq c\|u\|_{H^{6,1}(\Omega^T)},
\end{aligned}
\tag{5.25}
$$

where the fractional derivative notation is used (see Section 2), we obtain

$$
\begin{aligned}
I_2^1 &\leq c\|\,|\partial_t^{1/6}\nabla u|\,|\Delta u|\,|\nabla\Delta u|\,\|_{L_2(\Omega^T)} \\
&\leq c\|\partial_t^{1/6}\nabla u\|_{L_{18}(\Omega^T)}\|\nabla^2 u\|_{L_{18}(\Omega^T)}\|\nabla^3 u\|_{L_{18}(\Omega^T)} \leq cA^3,
\end{aligned}
$$

$$
I_2^2 \leq c\|\,|\partial_t^{1/6}\Delta u|\,|\nabla\Delta u|\,\|_{L_2(\Omega^T)} \leq c\|\partial_t^{1/6}\nabla^2 u\|_{L_4(\Omega^T)}\|\nabla^3 u\|_{L_4(\Omega^T)} \leq cA^2,
$$

$$
\begin{aligned}
I_2^3 &\leq c\|\,|\Delta u|\,|\partial_t^{1/6}\nabla\Delta u|\,\|_{L_2(\Omega^T)} \leq c\|\,\|\Delta u\|_{L_6(\Omega)}\|\partial_t^{1/6}\nabla^3 u\|_{L_3(\Omega)}\|_{L_2(0,T)} \\
&\leq c\|\Delta u\|_{L_\infty(0,T;H^1(\Omega))}\|\partial_t^{1/6}\nabla^3 u\|_{L_2(0,T;L_3(\Omega))} \leq \varphi(c_1,T)A.
\end{aligned}
$$

Thus,

$$I_2 \leq \varphi(c_1, T). \qquad (5.26)$$

By combining (5.22)–(5.24) and (5.26) it follows that

$$\|G\|_{H^{1/2,1/12}(S^T)} \leq \varphi(c_1, T). \qquad (5.27)$$

Now, using (5.21) and (5.27) in (5.16) yields the desired estimate (5.17). □

Corollary 3. *In view of the relations* $\Delta v = \chi - \chi_m$, $v_t = \mu$, $\Delta v_0 = \chi_0 - \chi_m$ *(see (4.5), (4.4)$_1$) the estimates (5.1) and (5.17) imply*

$$\begin{aligned}
\|\chi\|_{H^{0,1}(\Omega^T)} + \|\mu\|_{L_2(0,T;H^2(\Omega))} &\leq c_2, \\
\|\mu\|_{L_2(0,T;H^1(\Omega))} &\leq \varphi(c_1, T),
\end{aligned} \qquad (5.28)$$

where $c_2 = \varphi(c_1, T) + c\|\chi_0\|_{H^3(\Omega)}$.

6. Existence proof. We remind that due to the relations $\Delta v = \chi - \chi_m = \tilde{\chi}$, $v_t = \mu$, problem (5.14) is equivalent to the original one (1.5), which we recall here for convenience

$$\begin{aligned}
\chi_t - M\varkappa_2 \Delta^3 \chi &= M\Delta\left[f_0'(\chi) - \frac{1}{2}\varkappa_1'(\chi)|\nabla\chi|^2 - \varkappa_1(\chi)\Delta\chi \right] && \text{in } \Omega^T, \\
\chi|_{t=0} &= \chi_0 && \text{in } \Omega, \\
\mathbf{n} \cdot \nabla\chi &= 0, \quad \mathbf{n} \cdot \nabla\Delta\chi = 0 && \text{on } S^T, \\
\mathbf{n} \cdot \nabla\Delta^2\chi &= \frac{1}{2\varkappa_2}\varkappa_1'(\chi)\mathbf{n} \cdot \nabla(|\nabla\chi|^2) && \text{on } S^T.
\end{aligned} \qquad (6.1)$$

The existence of a strong solution to (6.1) is proved with the help of the Leray-Schauder fixed point theorem by the same reasoning as in [18]. There are some technical changes due to a more general form of functions f_0, $\varkappa_1$ compared to that in [18]. The main difference concerns a priori estimate for a fixed point. We apply the Leray-Schauder fixed point theorem in the following formulation

Theorem 6.1 (Leray-Schauder). *Let* X *be a Banach space. Assume that* $\Phi : [0,1] \times X \to X$ *is a map with the following properties:*
(i) for any fixed $\tau \in [0,1]$ *the map is completely contiuous;*
(ii) For every bounded subset B *of* X, *the family of maps* $\Phi(\cdot, \xi) : [0,1] \to X$, $\xi \in B$, *is uniformly equicontinuous;*
(iii) $\Phi(0, \cdot)$ *has precisely one fixed point in* X;
(iv) There is a bounded subsed B *of* X *such that any fixed point in* X *of* $\Phi(\tau, \cdot)$ *is contained in* B *for every* $\tau \in [0,1]$.
The $\Phi(1, \cdot)$ *has at least one fixed point.*

We choose as the solution space the Sobolev-Slobodecki space

$$X = H^{6s,s}(\Omega^T), \quad s \in (0,1) \quad \Omega \subset \mathbb{R}^3. \qquad (6.2)$$

The parameter $s \in (0,1)$ will be specified below in Lemma 6.2.

The solution map

$$\Phi(\tau, \cdot) : H^{6s,s}(\Omega^T) \ni \tilde{\chi} \to \chi \in H^{6,1}(\Omega^T) \subset H^{6s,s}(\Omega^T), \quad \tau \in [0,1], \qquad (6.3)$$

is defined by means of the following initial-boundary value problem

$$
\begin{aligned}
\chi_t - M\varkappa_2\Delta^3\chi &= \tau M\Delta\left[f_0'(\bar\chi) - \frac{1}{2}\varkappa_1'(\bar\chi)|\nabla\bar\chi|^2 \quad -\varkappa_1(\bar\chi)\Delta\bar\chi\right] \\
&\equiv \tau F(\bar\chi) && \text{in } \Omega^T, \\
\chi|_{t=0} &= \tau\chi_0 && \text{in } \Omega, && (6.4) \\
\mathbf{n}\cdot\nabla\chi &= 0, \quad \mathbf{n}\cdot\nabla\Delta\chi = 0 && \text{on } S^T, \\
\mathbf{n}\cdot\nabla\Delta^2\chi &= \tau\frac{1}{2\varkappa_2}\varkappa_1'(\bar\chi)\mathbf{n}\cdot\nabla(|\nabla\bar\chi|^2) \equiv \tau G(\bar\chi) && \text{on } S^T.
\end{aligned}
$$

Clearly, χ defined as a fixed point of $\Phi(1,\cdot)$ is a solution to problem (6.1).

We prove first that the map $\Phi(\tau,\cdot)$ is well-defined.

Lemma 6.2. *Let the solution map $\Phi(\tau,\cdot)$ be defined by (6.3)–(6.4) and the solution space be $H^{6s,s}(\Omega^T)$ with $s \in \left[\frac{5}{6},1\right)$. Then for any $\bar\chi \in H^{6s,s}(\Omega^T)$ and $\chi_0 \in H^3(\Omega)$ satisfying the compatibility condition*

$$
\mathbf{n}\cdot\nabla\chi_0 = 0 \quad \text{on } S, \tag{6.5}
$$

there exists a unique solution $\chi \in H^{6,1}(\Omega^T)$ to problem (6.4) such that

$$
\|\chi\|_{H^{6,1}(\Omega^T)} \le \varphi(\|\bar\chi\|_{H^{6s,s}(\Omega^T)}, \|\chi_0\|_{H^3(\Omega)}). \tag{6.6}
$$

Proof. Let $\bar\chi \in H^{6s,s}(\Omega^T)$ where $s \in \left[\frac{5}{6},1\right)$, and let

$$
\bar A \equiv \|\bar\chi\|_{H^{6s,s}(\Omega^T)}.
$$

By the parabolic theory (see Lemma 2.6) problem (6.4) has the unique solution $\chi \in H^{6,1}(\Omega^T)$ provided that $F(\bar\chi) \in L_2(\Omega^T)$, $G(\bar\chi) \in H^{1/2,1/12}(S^T)$, $\chi_0 \in H^3(\Omega)$, and the compatibility condition (6.5) holds. Then

$$
\|\chi\|_{H^{6,1}(\Omega^T)} \le c\tau(\|F(\bar\chi)\|_{L_2(\Omega^T)} + \|G(\bar\chi)\|_{H^{1/2s,1/12}(S^T)} + \|\chi_0\|_{H^3(\Omega)}). \tag{6.7}
$$

The terms on the right-hand side of (6.7) can be estimated as in Lemma 5.3. More precisely, since $\Delta v = \chi - \chi_m$ and $u = \nabla v$, we can repeat the estimates in (5.20) and (5.23) using the imbeddings

$$
\begin{aligned}
&\|\bar\chi\|_{L_\infty(\Omega^T)} + \|\nabla\bar\chi\|_{L_{18}(\Omega^T)} + \|\nabla^2\bar\chi\|_{L_6(\Omega^T)} \\
&\quad + \|\nabla^3\bar\chi\|_{L_{\frac{18}{5}}(\Omega^T)} + \|\nabla^3\bar\chi\|_{L_2(0,T;L_3(\Omega))} \le c\bar A,
\end{aligned} \tag{6.8}
$$

and

$$
\|\partial_t^{1/6}\bar\chi\|_{L_{18}(\Omega^T)} + \|\partial_t^{1/6}\nabla\bar\chi\|_{L_6(\Omega^T)} + \|\partial_t^{1/6}\nabla^2\bar\chi\|_{L_2(0,T;L_3(\Omega))} \le c\bar A,
$$

which hold true for $s \in \left[\frac{5}{6},1\right)$. Then we conclude that

$$
\|F(\bar\chi)\|_{L_2(\Omega^T)} + \|G(\bar\chi)\|_{H^{1/2,1/12}(S^T)} \le \varphi(\bar A). \tag{6.9}
$$

Consequently, by (6.7),

$$
\begin{aligned}
\|\chi\|_{H^{6,1}(\Omega^T)} &\le \varphi(\bar A) + c\|\chi_0\|_{H^3(\Omega)} \\
&\le \varphi(\|\bar\chi\|_{H^{6,s,s}(\Omega^T)}, \|\chi_0\|_{H^3(\Omega)})
\end{aligned} \tag{6.10}
$$

for any $\tau \in [0,1]$. This proves the assertion. □

From (6.3) it follows immediately

Corollary 4. *For $s \in \left[\frac{5}{6},1\right)$ the map $\Phi(\tau,\cdot): H^{6s,s}(\Omega^T) \to H^{6,1}(\Omega^T)$ is compact because the imbedding $H^{6,1}(\Omega^T) \subset H^{6s,s}(\Omega^T)$ is compact.*

Thus, to show the complete continuity of the map $\Phi(\tau, \cdot)$ it remains to prove its continuity.

For a fixed $\tau \in [0, 1]$, let $\chi_1 = \Phi(\tau, \bar{\chi}_1)$ and $\chi_2 = \Phi(\tau, \bar{\chi}_2)$ be two solutions of problem (6.4) corresponding to $\bar{\chi}_1$ and $\bar{\chi}_2$ from a bounded subset of $H^{6s,s}(\Omega^T)$, such that

$$\|\bar{\chi}_k\|_{H^{6s,s}(\Omega^T)} \leq \bar{B}, \quad k = 1, 2. \tag{6.11}$$

Introducing the differences

$$K = \chi_1 - \chi_2, \quad \bar{K} - \bar{\chi}_1 - \bar{\chi}_2,$$

we see that K satisfies the following problem

$$
\begin{aligned}
K_t - M\varkappa_2\Delta^3 K &= \tau[F(\bar{\chi}_1) - F(\bar{\chi}_2)] \equiv \tau\bar{F}(\bar{\chi}_1, \bar{\chi}_2, \bar{K}) && \text{in } \Omega^T, \\
K|_{t=0} &= 0 && \text{in } \Omega, \\
\mathbf{n} \cdot \nabla K &= 0, \quad \mathbf{n} \cdot \nabla\Delta K = 0 && \text{on } S^T, \\
\mathbf{n} \cdot \nabla\Delta^2 K &= \tau[G(\bar{\chi}_1) - G(\bar{\chi}_2)] \equiv \tau\bar{G}(\bar{\chi}_1, \bar{\chi}_2, \bar{K}) && \text{on } S^T.
\end{aligned}
\tag{6.12}
$$

In the same manner as in the proof of Lemma 4.3 from [18] we conclude after straightforward calculations the following

Lemma 6.3 (Continuity of Φ). *For any $\bar{\chi}, \bar{\chi}_2 \in H^{6s,s}(\Omega^T)$, $s \in \left[\frac{5}{6}, 1\right)$, satisfying (6.11), and for any $\tau \in [0, 1]$, the unique solution $K \in H^{6,1}(\Omega^T)$ to problem (6.12) satisfies the estimate*

$$\|K\|_{H^{6,1}(\Omega^T)} \leq \tau\varphi(\bar{B})\|\bar{K}\|_{H^{6s,s}(\Omega^T)}. \tag{6.13}$$

Corollary 5. *The continuity of the map Φ with respect to τ is evident (see Lemma 4.4 from [18]).*

Corollary 6. *By virtue of the linear parabolic theory (see Lemma 2.6) problem (6.4) with $\tau = 0$ has the unique solution $\chi = 0$.*

Corollary 7. *It follows from Lemma 5.3 that there exists a bounded subset $\mathcal{B}$ of $H^{6s,s}(\Omega^T)$, given by*

$$
\begin{aligned}
\mathcal{B} \equiv \{\chi \in H^{6,1}(\Omega^T) : \ \|\chi\|_{H^{6,1}(\Omega^T)} + \|\mu\|_{L_2(0,T;H^2(\Omega))} \\
\leq c_2 \equiv \varphi(c_1, T) + c\|\chi_0\|_{H^3(\Omega)}\},
\end{aligned}
\tag{6.14}
$$

where $M\Delta\mu = \chi_t$ such that any fixed point of $\Phi(1, \cdot)$ is contained in $\mathcal{B}$. It is clear that the same property holds for any $\tau \in [0, 1]$. Moreover, by energy estimate in Lemma 3.1,

$$\|\chi\|_{L_\infty(0,T;H^2(\Omega))} + \|\nabla\mu\|_{L_2(0,T;L_2(\Omega)} \leq c_1, \tag{6.15}$$

where $c_1 = \varphi(\|\chi_0\|_{H^2(\Omega)}, |\chi_m|, \varkappa_2, c_{\varkappa_1}, c_{f_0}, a_{2k})$.

From Lemmas 6.2, 6.3 and Corollaries 4–7 we infer that the map Φ satisfies the assumptions of the Leray-Schauder fixed point theorem. Hence, there exists at least one fixed point of the map $\Phi(1, \cdot)$ in the space $H^{6s,s}(\Omega^T)$, $s \in \left[\frac{5}{6}, 1\right)$. By the regularity properties (6.6) of this map it follows that the fixed point belongs to the space $H^{6,1}(\Omega^T)$. Clearly, in view of the definition of the map $\Phi(1, \cdot)$ this means that problem (6.1) has a solution $\chi \in H^{6,1}(\Omega^T)$ satisfying estimates (6.14) and (6.15). This proves the existence part of Theorem A.

The uniqueness, holding true for any solution satisfying $\chi \in L_\infty(0, T; H^2(\Omega))$, can be proved by standard arguments in the same way as in [18].

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