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# On a class of Cahn-Hilliard models with nonlinear diffusion

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## Abstract

In the present work, we address a class of Cahn-Hilliard equations characterized by a nonlinear diffusive dynamics and possibly containing an additional sixth order term. This model describes the separation properties of oil-water mixtures, when a substance enforcing the mixing of the phases (a surfactant) is added. However, the model is also closely connected with other Cahn-Hilliard-type equations relevant in different types of applications. We first discuss the existence of a weak solution to the sixth-order model in the case when the configuration potential of the system is of singular (e.g., logarithmic) type. Then, we study the behavior of the solutions in the case when the sixth order term is let tend to 0, proving convergence to solutions of the fourth order system in a special case. The fourth order system is then investigated by a direct approach and existence of a weak solution is shown under very general conditions by means of a fixed point argument. Finally, additional properties of the solutions, like uniqueness and parabolic regularization are discussed, both for the sixth order and for the fourth order model, under more restrictive assumptions on the nonlinear diffusion term.

**Key words:** Cahn-Hilliard equation, nonlinear diffusion, variational formulation, existence theorem.

**AMS (MOS) subject classification:**

## 1 Introduction

This paper is devoted to the mathematical analysis of the following class of parabolic systems:

$$u_t - \Delta w = 0, \quad (1.1)$$

$$w = \delta \Delta^2 u - a(u) \Delta u - \frac{a'(u)}{2} |\nabla u|^2 + f(u) + \varepsilon u_t, \quad (1.2)$$

on  $(0, T) \times \Omega$ ,  $\Omega$  being a bounded smooth subset of  $\mathbb{R}^3$  and  $T > 0$  an assigned final time. The system is coupled with the initial and boundary conditions

$$u|_{t=0} = u_0, \quad \text{in } \Omega, \quad (1.3)$$

$$\partial_n u = \partial_n w = \delta \partial_n \Delta u = 0, \quad \text{on } \partial\Omega, \quad \text{for } t \in (0, T) \quad (1.4)$$

and represents a variant of the Cahn-Hilliard model for phase separation in binary materials. The function  $f$  stands for the derivative of a *singular* potential  $F$  of a *double obstacle* type. Namely,  $F$  is assumed to be  $+\infty$  outside a bounded interval (assumed equal to  $[-1, 1]$  for simplicity), where the extrema correspond to the pure states. A physically significant example is given by the so-called logarithmic potential

$$F(r) = (1 - r) \log(1 - r) + (1 + r) \log(1 + r) - \frac{\lambda}{2} r^2, \quad \lambda \geq 0. \quad (1.5)$$

As in this example, we will assume  $F$  be at least  $\lambda$ -convex, i.e., convex up to a quadratic perturbation. In this way, we can also allow for singular potentials having more than two minima in the interval  $[-1, 1]$  (as it happens in the case of the oil-water-surfactant models described below, where the third minimum appears in relation to the so-called “microemulsion” phase).

We assume the coefficients  $\delta, \varepsilon$  be  $\geq 0$ , with the case  $\delta > 0$  giving rise to a *sixth order* model and the case  $\varepsilon > 0$  related to possible viscosity effects that are likely to appear in several models of Cahn-Hilliard type (see, e.g., [19]).

The main novelty of system (1.1)-(1.2) is related to the presence of the nonlinear function  $a$  in (1.2), which is supposed smooth, bounded, and strongly positive (i.e., everywhere larger than some constant  $\underline{a} > 0$ ). Mathematically, the latter is an unavoidable assumption as we are mainly interested in the behavior of the problem when  $\delta$  is let tend to 0 and in the properties of the (fourth order) limit system  $\delta = 0$ . On the other hand, at least in the physical context of the sixth order model, it would also be meaningful to admit  $a$  to take negative values, as it may happen in presence of the “microemulsion” phase (see [16, 17]). We will not deal with this situation, but we just point out that, as far as  $\delta > 0$  is fixed, this should create no additional mathematical difficulties since the nonlinear diffusion term is then dominated by the sixth order term.

From the analytical point of view, as a basic observation we can notice that this class of systems has an evident variational structure. Indeed, (formally) testing (1.1) by  $w$ , (1.2) by  $u$ , taking the difference of the obtained relations, integrating w.r.t. space variables, using the *no-flux* conditions (1.4), and performing suitable integrations by parts, one readily gets the *a-priori* bound

$$\frac{d}{dt} \mathcal{E}_\delta(u) + \|\nabla w\|_{L^2(\Omega)}^2 + \varepsilon \|u_t\|_{L^2(\Omega)}^2 = 0, \quad (1.6)$$

which has the form of an *energy equality* for the *energy functional*

$$\mathcal{E}_\delta(u) = \int_{\Omega} \left( \frac{\delta}{2} |\Delta u|^2 + \frac{a(u)}{2} |\nabla u|^2 + F(u) \right), \quad (1.7)$$

where the interface (gradient) part contains the nonlinear function  $a$ . In other words, the system (1.1)-(1.2) arises as the  $(H^1)$ -gradient flow problem for the functional  $\mathcal{E}_\delta$ .

Once setting the class of equations, we describe some physical problems behind it.

A sixth order Cahn-Hilliard type equation arises in the modelling of phase transitions in ternary oil-water-surfactant mixtures. In a series of papers, Gompper et al. (see, e.g., [16, 17, 18] and other references in [27]) have proposed a phenomenological Landau-Ginzburg theory for such mixtures. The theory is based on free energy functional (1.7) with constant  $\delta > 0$  (in general, however, this coefficient can depend on  $u$ , see [31]), and with  $F(u)$ ,  $a(u)$  approximated, respectively, by a sixth and a second order polynomial:

$$F(u) = (u + 1)^2(u^2 + h_0)(u - 1)^2, \quad a(u) = g_0 + g_2 u^2, \quad (1.8)$$

where  $h_0, g_0, g_2$  are constants,  $g_2 > 0$  and  $h_0, g_0$  of arbitrary sign;  $u$  represents the local difference between oil and water concentrations. A well-posedness of the corresponding sixth order model given by (1.1)-(1.4) in the case  $\delta > 0$  and  $\varepsilon = 0$  has been recently proved in [27].

Sixth order Cahn-Hilliard type equations also arise in other physical contexts. We mention a two-dimensional model for the faceting of a growing crystalline surface, derived by Savina et al. [30]. In this model, on the contrary to (1.1)-(1.2), the order parameter  $u$  is not a conserved quantity due to the presence of a force-like term related to the deposition rate. Recently this problem has been addressed mathematically by Korzec et al. [20, 21].

In the case  $\delta = 0$ ,  $a(u) = \text{const} > 0$ , the functional (1.7) represents the classical Cahn-Hilliard free energy [10, 11]. In [22], the original Cahn-Hilliard free energy derivation has been extended by accounting for composition dependence of the gradient energy coefficient  $a(u)$ . For a face-centered cubic crystal the following expressions for  $a(u)$  have been derived, depending on the level of approximation of the nearest-neighbor interactions:

$$a(u) = a_0 + a_1 u + a_2 u^2, \quad (1.9)$$

where  $a_0 > 0$ ,  $a_1, a_2 \in \mathbb{R}$  in the case of four-body interactions,  $a_2 = 0$  in the case of three-body interactions, and  $a_1 = a_2 = 0$  in the case of pairwise interactions.

A specific free energy with composition dependent gradient energy coefficient  $a(u)$  also arises in modelling of phase separation in polymers [13]. This energy, known as Flory-Huggins-deGennes one, has the form (1.7) with  $\delta = 0$ ,  $F(u)$  being the logarithmic potential (1.5), and singular coefficient

$$a(u) = \frac{1}{(1-u)(1+u)}. \quad (1.10)$$

We mention also that various formulations of phase-field models with gradient energy coefficient dependent on the order parameter (and possibly on other fields) appear, e.g., in [1, 9].

Our objective in this paper is threefold. First, we would like to extend the result of [27] both to the viscous problem ( $\epsilon > 0$ ) and to the case when the configuration potential is *singular* (e.g., of the form (1.5)). While the first extension is almost straightforward, considering constraint (singular) terms in fourth order equations ((1.2), in the specific case) gives rise to regularity problems since it is not possible, up to our knowledge, to estimate all the terms of equation (1.2) in  $L^p$ -spaces. For this reason, the nonlinear term  $f(u)$  has to be intended in a weaker form, namely, as a selection of a nonlinear, and possibly multivalued, mapping acting from  $V = H^1(\Omega)$  to  $V'$ . This involves some monotone operator technique that is developed in a specific section of the paper.

As a second step, we investigate the behavior of the solutions to the sixth order system as the parameter  $\delta$  is let tend to 0. In particular, we would like to show that, at least up to subsequences, we can obtain in the limit suitably defined solutions to the fourth order system obtained setting  $\delta = 0$  in (1.2). Unfortunately, we are able to prove this fact only under additional conditions. The reason is that the natural estimate required to control second space derivatives of  $u$ , i.e., testing (1.2) by  $-\Delta u$ , is compatible with the nonlinear term in  $\nabla u$  only under additional assumptions on  $a$  (e.g., if  $a$  is concave). This nontrivial fact depends on an integration by parts formula devised by Dal Passo, Garcke and Grün in [12] in the frame of the thin-film equation and whose use is necessary to control the nonlinear gradient term. It is however likely that the use of more refined integration by parts techniques may permit to control the nonlinear gradient term under more general conditions on  $a$ .

Since we are able to take the limit  $\delta \searrow 0$  only in special cases, in the subsequent part of the paper we address the fourth order problem by using a direct approach. In this way, we can obtain existence of a weak solution under general conditions on  $a$  (we notice that, however, uniqueness is no longer guaranteed for  $\delta = 0$ ). The proof of existence is based on an "ad hoc" regularization of the equations by means of a system of phase-field type. This kind of approach has been proved to be effective also in the frame of other types of Cahn-Hilliard equations (see, e.g., [4]). Local existence for the regularized system is then shown by means of the Schauder theorem, and, finally, the regularization is removed by means of suitable a-priori estimates and compactness methods. This procedure involves some technicalities since parabolic spaces of Hölder type have to be used for the fixed point argument; indeed, the use of Sobolev techniques seems not suitable due to the nonlinearity in the highest order term, which prevents from having compactness of the fixed point map with respect to Sobolev norms. A further difficulty is related with the necessity of estimating the second order space derivatives of  $u$  in presence of the nonlinear term in the gradient. This is obtained by introducing a proper transformed variable, and rewriting (1.2) in terms of it. Proceeding in this way, we can get rid of that nonlinearity, but at the same time, we can still exploit the good monotonicity properties of  $f$ . We note here that a different method based on entropy estimates could also be used to estimate  $\Delta u$  without making the change of variable, which seems however a simpler technique.

Finally, in the last section of the paper, we discuss further property of weak solutions. More precisely, we address the problems of uniqueness (only for the 4th order system, since in the case  $\delta > 0$

it is always guaranteed) and of parabolic time-regularization of solutions (both for the 6th and for the 4th order system). The validity of such properties seems to depend on additional assumption on  $a$ . In particular, if  $a$  is a convex function and  $1/a$  is concave, then the whole functional  $\mathcal{E}_\delta$  turns out to be  $\lambda$ -convex (see, e.g., [14] for convexity conditions related to more general energy functionals). In particular, the nonlinear part (1.2) is essentially monotone, and this fact permits to obtain additional estimates. As a final result, we can also show that, both in the 6th and in the *viscous* 4th order case, all weak solutions satisfy the energy equality (1.6), at least in an integrated form (and not just an energy inequality). Actually, this property can constitute the starting point for of the long-time investigation of the dynamical process associated to system (1.1)-(1.2) from the point of view of global attractors. We will address these issues in a forthcoming paper. On the other hand, the question whether the energy equality holds in the nonviscous 4th order case seems to be more delicate, and, actually, we could not give a positive answer to it.

The plan of the paper is as follows. In the next Section 2, we will report our notation and hypotheses, together with some general tools that will be used in the proofs. Section 3 will contain the analysis of the sixth order model. The limit  $\delta \searrow 0$  will then be analyzed in Section 4. Section 5 will be devoted to the analysis of the fourth order model. Finally, in Section 6 uniqueness and regularization properties of the solutions will be discussed, as well as the validity of the energy equality.

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## 2 Notations and technical tools

Let  $\Omega$  be a smooth bounded domain of  $\mathbb{R}^3$  of boundary  $\Gamma$ ,  $T > 0$  a given final time, and let  $Q := (0, T) \times \Omega$ . Let  $H := L^2(\Omega)$ , endowed with the standard scalar product  $(\cdot, \cdot)$  and norm  $\|\cdot\|$ . Let also  $V := H^1(\Omega)$ . We note by  $\langle \cdot, \cdot \rangle$  the duality between  $V'$  and  $V$  and by  $\|\cdot\|_X$  the norm in the generic Banach space  $X$ .

We make the following assumptions on the nonlinear terms in (1.1)-(1.2):

$$a \in C_b^2(\mathbb{R}; \mathbb{R}), \quad \exists \underline{a}, \bar{a} > 0 : \underline{a} \leq a(r) \leq \bar{a} \quad \forall r \in \mathbb{R}; \quad (2.1)$$

$$\exists a_-, a_+ \in [\underline{a}, \bar{a}] : a(r) \equiv a_- \quad \forall r \leq -2, \quad a(r) \equiv a_+ \quad \forall r \geq 2; \quad (2.2)$$

$$f \in C^1((-1, 1); \mathbb{R}), \quad f(0) = 0, \quad \exists \lambda \geq 0 : f'(r) \geq -\lambda \quad \forall r \in (-1, 1); \quad (2.3)$$

$$\lim_{|r| \rightarrow 1} f(r)r = \lim_{|r| \rightarrow 1} \frac{f'(r)}{|f(r)|} = +\infty. \quad (2.4)$$

The latter condition in (2.4) is just a technical hypotheses which is actually verified in all significant cases. We also notice that, since any weak solution will take values only in the physical interval  $[-1, 1]$ , the behavior of  $a$  is also significant only in that interval. We have extended it outside  $[-1, 1]$  only for the purpose of properly constructing the approximating problem. In (2.1),  $C_b^2$  denotes the space of functions that are continuous and globally bounded together with their derivatives up to the second order. Concerning  $f$ , (2.3) states that it can be written in the form

$$f(r) = f_0(r) - \lambda r, \quad (2.5)$$

i.e., as the difference between a (dominating) monotone part  $f_0$  and a linear perturbation. By (2.3)-(2.4), we can also set, for  $r \in (-1, 1)$ ,

$$F_0(r) := \int_0^r f_0(s) ds \quad \text{and} \quad F(r) := F_0(r) - \frac{\lambda}{2} r^2, \quad (2.6)$$

so that  $F' = f$ . Notice that  $F_0$  may be bounded in  $(-1, 1)$  (e.g., this occurs in the case of the logarithmic potential (1.5)). If this is the case, we extend it by continuity to  $[-1, 1]$ . Then,  $F_0$  is set to be  $+\infty$  either outside  $(-1, 1)$  (if it is unbounded in  $(-1, 1)$ ) or outside  $[-1, 1]$  (if it is bounded in  $(-1, 1)$ ). This standard procedure permits to penalize the non-physical values of the variable  $u$  and to intend  $f_0$  as the subdifferential of the (extended) convex function  $F_0 : \mathbb{R} \rightarrow [0, +\infty]$ .

That said, we define a number of operators. First, we set

$$A : V \rightarrow V', \quad \langle Av, z \rangle := \int_{\Omega} \nabla v \cdot \nabla z, \quad \text{for } v, z \in V. \quad (2.7)$$

Then, we define

$$W := \{z \in H^2(\Omega) : \partial_n z = 0 \text{ on } \Gamma\} \quad (2.8)$$

and recall that (a suitable restriction of)  $A$  can be seen as an unbounded linear operator on  $H$  having domain  $W$ . The space  $W$  is endowed with the natural  $H^2$ -norm. We then introduce

$$\mathcal{A} : W \rightarrow H, \quad \mathcal{A}(z) := -a(z)\Delta z - \frac{a'(z)}{2}|\nabla z|^2. \quad (2.9)$$

It is a standard issue to check that, indeed,  $\mathcal{A}$  takes its values inside  $H$ .

## 2.1 Weak subdifferential operators

To state the weak formulation of the 6th order system, we need to introduce a proper relaxed form of the maximal monotone operator associated to the function  $f_0$  and acting in the duality between  $V'$  and  $V$  (rather than in the scalar product of  $H$ ). Actually, it is well known (see, e.g., [8, Ex. 2.1.3, p. 21]) that  $f_0$  can be interpreted as a maximal monotone operator on  $H$  by setting, for  $v, \xi \in H$ ,

$$\xi = f_0(v) \text{ in } H \iff \xi(x) = f_0(v(x)) \text{ a.e. in } \Omega. \quad (2.10)$$

If no danger of confusion occurs, the new operator on  $H$  will be still noted by the letter  $f_0$ . Correspondingly,  $f_0$  is the  $H$ -subdifferential of the convex functional

$$\mathcal{F}_0 : H \mapsto [0, +\infty], \quad \mathcal{F}_0(v) := \int_{\Omega} F_0(v(x)), \quad (2.11)$$

where the integral might possibly be  $+\infty$  (this happens, e.g., when  $|v| > 1$  on a set of strictly positive Lebesgue measure).

The weak form of  $f_0$  can be introduced by setting

$$\xi \in f_{0,w}(v) \iff \langle \xi, z - v \rangle \leq \mathcal{F}_0(z) - \mathcal{F}_0(v) \text{ for any } z \in V. \quad (2.12)$$

Actually, this is nothing else than the definition of the subdifferential of (the restriction to  $V$  of)  $\mathcal{F}_0$  w.r.t. the duality pairing between  $V'$  and  $V$ . In general,  $f_{0,w}$  can be a *multivalued* operator; namely,  $f_{0,w}$  is a *subset* of  $V'$  that may contain more than one element. It is not difficult to prove (see, e.g., [6, Prop. 2.5]) that, if  $v \in V$  and  $f_0(v) \in H$ , then

$$\{f_0(v)\} \subset f_{0,w}(v). \quad (2.13)$$

Moreover,

$$\text{if } v \in V \text{ and } \xi \in f_{0,w}(v) \cap H, \text{ then } \xi = f_0(v) \text{ a.e. in } \Omega. \quad (2.14)$$

In general, the inclusion in (2.13) is strict and, for instance, it can happen that  $f_0(v) \notin H$  (i.e.,  $v$  does not belong to the  $H$ -domain of  $f_0$ ), while  $f_{0,w}(v)$  is nonempty. Nevertheless, we still have some “automatic” gain of regularity for any element of  $f_{0,w}(v)$ :

**Proposition 2.1.** *Let  $v \in V$ ,  $\xi \in f_{0,w}(v)$ . Then,  $\xi$  can be seen as an element of the space  $\mathcal{M}(\bar{\Omega}) = C^0(\bar{\Omega})'$  of the bounded real-valued Borel measures on  $\bar{\Omega}$ . More precisely, there exists  $T \in \mathcal{M}(\bar{\Omega})$ , such that*

$$\langle \xi, z \rangle = \int_{\bar{\Omega}} z \, dT \quad \text{for any } z \in V \cap C^0(\bar{\Omega}). \quad (2.15)$$

**PROOF.** Let  $z \in C^0(\bar{\Omega}) \cap V$  such that  $-1/2 \leq z(x) \leq 1/2$  for all  $x \in \bar{\Omega}$ . Then, by definition (2.12), it is easy to see that

$$\langle \xi, z \rangle \leq \langle \xi, v \rangle + \mathcal{F}_0(z) - \mathcal{F}_0(v) \leq |\langle \xi, v \rangle| + |\Omega|(F_0(-1/2) + F_0(1/2)). \quad (2.16)$$

This actually shows that the linear functional  $z \mapsto \langle \xi, z \rangle$  defined on  $C^0(\bar{\Omega}) \cap V$  (that is a dense subspace of  $C^0(\bar{\Omega})$ , recall that  $\Omega$  is smooth) is continuous with respect to the sup-norm. Thus, by the Riesz representation theorem, it can be represented over  $C^0(\bar{\Omega})$  by a measure  $T \in \mathcal{M}(\bar{\Omega})$ . ■

Actually, we can give a general definition, saying that a functional  $\xi \in V'$  belongs to the space  $V' \cap \mathcal{M}(\bar{\Omega})$  provided that  $\xi$  is continuous w.r.t. the sup-norm on  $\bar{\Omega}$ . In this case, we can use (2.15) and say that the measure  $T$  represents  $\xi$  on  $\mathcal{M}(\bar{\Omega})$ . We now recall a result [7, Thm. 3] that will be exploited in the sequel.

**Theorem 2.2.** *Let  $v \in V$ ,  $\xi \in f_{0,w}(v)$ . Then, denoting by  $\xi_a + \xi_s = \xi$  the Lebesgue decomposition of  $\xi$ , with  $\xi_a$  ( $\xi_s$ ) standing for the absolute continuous (singular, respectively) part of  $\xi$ , we have*

$$\xi_a v \in L^1(\Omega), \quad (2.17)$$

$$\xi_a(x) = f_0(v(x)) \quad \text{for a.e. } x \in \Omega, \quad (2.18)$$

$$\langle \xi, v \rangle - \int_{\Omega} \xi_a v \, dx = \sup \left\{ \int_{\bar{\Omega}} z \, d\xi_s, z \in C^0(\bar{\Omega}), z(\bar{\Omega}) \subset [-1, 1] \right\}. \quad (2.19)$$

Actually, in [7] a slightly different result is proved, where  $V$  is replaced by  $H_0^1(\Omega)$  and, correspondingly,  $\mathcal{M}(\bar{\Omega})$  is replaced by  $\mathcal{M}(\Omega)$  (i.e., the dual of  $C_c^0(\Omega)$ ). Nevertheless, thanks to the smoothness of  $\Omega$ , one can easily realize that the approximation procedure used in the proof of the theorem can be extended to cover the present situation. The only difference is given by the fact that the singular part  $\xi_s$  may be supported also on the boundary.

Let us now recall that, given a pair  $X, Y$  of Banach spaces, a sequence of (multivalued) operators  $T_n : X \rightarrow 2^Y$  is said to G-converge (strongly) to  $T$  iff

$$\forall (x, y) \in T, \exists (x_n, y_n) \in T_n \text{ such that } (x_n, y_n) \rightarrow (x, y) \text{ strongly in } X \times Y. \quad (2.20)$$

We would like to apply this condition to an approximation of the monotone function  $f_0$  that we now construct. Namely, for  $\sigma \in (0, 1)$  (intended to go to 0 in the limit), we would like to have a family  $\{f_\sigma\}$  of monotone functions such that

$$f_\sigma \in C^1(\mathbb{R}), \quad f'_\sigma \in L^\infty(\mathbb{R}), \quad f_\sigma(0) = 0, \quad (2.21)$$

$$f_\sigma \rightarrow f_0 \quad \text{uniformly on compact subsets of } (-1, 1). \quad (2.22)$$

Moreover, noting

$$F_\sigma(r) := \int_0^r f_\sigma(s) \, ds, \quad \text{for } r \in \mathbb{R}, \quad (2.23)$$

we ask that

$$F_\sigma(r) \geq \lambda r^2 - c, \quad (2.24)$$

for some  $c \geq 0$  independent of  $\sigma$  and for all  $r \in \mathbb{R}$ ,  $\sigma \in (0, 1)$ , where  $\lambda$  is as in (2.3) (note that the analogue of the above property holds for  $F$  thanks to the first (2.4)). Moreover, we ask the monotonicity condition

$$F_{\sigma_1}(r) \leq F_{\sigma_2}(r) \quad \text{if } \sigma_2 \leq \sigma_1 \quad \text{and for all } r \in \mathbb{R}. \quad (2.25)$$

Finally, on account of the last assumption (2.4), we require that

$$\forall m > 0, \exists C_m > 0 : f'_\sigma(r) - m|f_\sigma(r)| \geq -C_m, \quad \forall r \in [-2, 2] \quad (2.26)$$

with  $C_m$  being independent of  $\sigma$ . Notice that it is sufficient to ask the above property for  $r \in [-2, 2]$ . The details of the construction of a family  $\{f_\sigma\}$  fulfilling (2.21)-(2.26) are standard and hence we leave them to the reader.

Thanks to the monotonicity property (2.25), we can apply [2, Thm. 3.20], which gives that

$$f_\sigma \text{ G-converges to } f_0 \quad \text{in } H \times H, \quad (2.27)$$

$$f_\sigma \text{ G-converges to } f_{0,w} \quad \text{in } V \times V'. \quad (2.28)$$

A notable consequence of G-convergence is the following property, whose proof can be obtained by slightly modifying [5, Prop. 1.1, p. 42]:



**Lemma 2.3.** *Let  $X$  be an Hilbert space,  $B_\sigma, B$  be maximal monotone operators in  $X \times X'$  such that*

$$B_\sigma \text{ G-converges to } B \text{ in } X \times X', \quad (2.29)$$

*as  $\sigma \searrow 0$ . Let also, for any  $\sigma > 0$ ,  $v_\sigma \in X$ ,  $\xi_\sigma \in X'$  such that  $\xi_\sigma \in B_\sigma(v_\sigma)$ . Finally, let us assume that, for some  $v \in X$ ,  $\xi \in X'$ , there holds*

$$v_\sigma \rightarrow v \text{ weakly in } X, \quad \xi_\sigma \rightarrow \xi \text{ weakly in } X', \quad (2.30)$$

$$\limsup_{\sigma \searrow 0} \langle \xi_\sigma, v_\sigma \rangle_X \leq \langle \xi, v \rangle_X. \quad (2.31)$$

*Then,  $\xi \in B(v)$ .*

Next, we present an integration by parts formula:

**Lemma 2.4.** *Let  $u \in W \cap H^3(\Omega)$ ,  $\xi \in V'$  such that  $\xi \in f_{0,w}(u)$ . Then, we have that*

$$\langle \xi, Au \rangle \geq 0. \quad (2.32)$$

**PROOF.** Let us first note that the duality above surely make sense in the assigned regularity setting. Actually, it is  $Au \in V$ . We then consider the elliptic problem

$$u_\sigma \in V, \quad u_\sigma + A^2 u_\sigma + f_\sigma(u_\sigma) = u + A^2 u + \xi \quad \text{in } V'. \quad (2.33)$$

Since  $f_\sigma$  is Lipschitz continuous and the above right hand side lies in  $V'$ , it is not difficult to show that the above problem admits a unique solution  $u_\sigma \in W \cap H^3(\Omega)$ .

Moreover, the standard a priori estimates for  $u_\sigma$  lead to the following convergence relations, which hold, for some  $v \in V$  and  $\zeta \in V'$ , up to the extraction of (non-relabelled) subsequences (in fact uniqueness guarantees them for the whole  $\sigma \searrow 0$ ):

$$u_\sigma \rightarrow v \text{ weakly in } H^3(\Omega) \text{ and strongly in } W, \quad (2.34)$$

$$A^2 u_\sigma \rightarrow A^2 v \text{ weakly in } V', \quad (2.35)$$

$$f_\sigma(u_\sigma) \rightarrow \zeta \text{ weakly in } V'. \quad (2.36)$$

As a byproduct, the limit functions satisfy  $v + A^2 v + \zeta = u + A^2 u + \xi$  in  $V'$ . Moreover, we deduce from (2.33)

$$(f_\sigma(u_\sigma), u_\sigma) = \langle u + A^2 u + \xi - u_\sigma - A^2 u_\sigma, u_\sigma \rangle, \quad (2.37)$$

whence

$$\lim_{\sigma \rightarrow 0} (f_\sigma(u_\sigma), u_\sigma) = \langle u + A^2 u + \xi - v - A^2 v, v \rangle = \langle \zeta, v \rangle. \quad (2.38)$$

Then, on account of (2.34), (2.36), (2.28) and Lemma 2.3, we readily obtain that  $\zeta \in f_{0,w}(v)$ . By uniqueness,  $v = u$  and  $\zeta = \xi$ .

Let us finally verify the required property. Actually, for  $\sigma > 0$ , we have

$$0 \leq (f_\sigma(u_\sigma), Au_\sigma) = \langle u + A^2 u + \xi - u_\sigma - A^2 u_\sigma, Au_\sigma \rangle, \quad (2.39)$$

so that, passing to the limsup and exploiting semicontinuity of norms with respect to weak convergence, we infer

$$0 \leq \langle u + A^2 u + \xi - u - A^2 u, Au \rangle = \langle \xi, Au \rangle, \quad (2.40)$$

as desired. ■

Finally, we recall a further integration by parts formula that extends the classical result [8, Lemma 3.3, p. 73] (see, e.g., [28, Lemma 4.1] for a proof):

**Lemma 2.5.** *Let  $T > 0$  and let  $\mathcal{J} : H \rightarrow [0, +\infty]$  a convex, lower semicontinuous and proper functional. Let  $u \in H^1(0, T; V') \cap L^2(0, T; V)$ ,  $\eta \in L^2(0, T; V)$  and let  $\eta(t) \in \partial \mathcal{J}(u(t))$  for a.e.  $t \in (0, T)$ , where  $\partial \mathcal{J}$  is the  $H$ -subdifferential of  $\mathcal{J}$ . Moreover, let us suppose the coercivity property*

$$\exists k_1 > 0, k_2 \geq 0 \text{ such that } \mathcal{J}(v) \geq k_1 \|v\|^2 - k_2 \quad \forall v \in H. \quad (2.41)$$

Then, the function  $t \mapsto \mathcal{J}(u(t))$  is absolutely continuous in  $[0, T]$  and

$$\frac{d}{dt} \mathcal{J}(u(t)) = \langle u_t(t), \eta(t) \rangle \quad \text{for a.e. } t \in (0, T). \quad (2.42)$$

In particular, integrating in time, we have

$$\int_s^t \langle u_t(r), \eta(r) \rangle dr = \mathcal{J}(u(t)) - \mathcal{J}(u(s)) \quad \forall s, t \in [0, T]. \quad (2.43)$$

### 3 The 6th order problem

We start by introducing the concept of weak solution to the sixth order problem.

**Definition 3.1.** Let  $\delta > 0$  and  $\varepsilon \geq 0$ . A (global in time) weak solution to the 6th order problem is a triple  $(u, w, \xi)$ , with

$$u \in H^1(0, T; V') \cap L^\infty(0, T; W) \cap L^2(0, T; H^3(\Omega)), \quad \varepsilon u \in H^1(0, T; H), \quad (3.1)$$

$$F(u) \in L^\infty(0, T; L^1(\Omega)), \quad (3.2)$$

$$\xi \in L^2(0, T; V'), \quad (3.3)$$

$$w \in L^2(0, T; V), \quad (3.4)$$

such that the following relations hold a.e. in  $(0, T)$ :

$$u_t + Aw = 0, \quad \text{in } V', \quad (3.5)$$

$$w = \delta A^2 u + \mathcal{A}(u) + \xi - \lambda u + \varepsilon u_t, \quad \text{in } V', \quad (3.6)$$

$$\xi \in f_{0,w}(u) \quad (3.7)$$

and such that, in addition,

$$u|_{t=0} = u_0, \quad \text{a.e. in } \Omega. \quad (3.8)$$

We can then state the main result of this section:

**Theorem 3.2.** Let us assume (2.1)-(2.4). Let  $\varepsilon \geq 0$  and  $\delta > 0$ . Moreover, let us suppose that

$$u_0 \in W, \quad F(u_0) \in L^1(\Omega), \quad (u_0)_\Omega \in (-1, 1), \quad (3.9)$$

where  $(u_0)_\Omega$  is the spatial mean of  $u_0$ . Then, the sixth order problem admits one and only one weak solution.

The proof of the theorem will be carried out in several steps, presented as separate subsequences.

**Remark 3.3.** We observe that the last condition in (3.9) does not simply follow from the requirement  $F(u_0) \in L^1(\Omega)$ . Indeed, in the case when  $F$  is bounded over  $[-1, 1]$  (as for the logarithmic potential (1.5)), that condition excludes initial data being a.e. equal to  $-1$  or to  $+1$ , which we are not able to admit.

#### 3.1 Approximation and local existence

First of all, we introduce a suitably approximated statement. The monotone function  $f_0$  is regularized by taking a family  $\{f_\sigma\}$ ,  $\sigma \in (0, 1)$ , defined as in Subsection 2.1. Next, we regularize  $u_0$  taking  $u_{0,\sigma}$  as the solution to the elliptic problem

$$u_{0,\sigma} + \sigma A u_{0,\sigma} = u_0, \quad (3.10)$$

and we clearly have, by Hilbert elliptic regularity results,

$$u_{0,\sigma} \in D(A^2) \quad \forall \sigma \in (0, 1). \quad (3.11)$$

**Approximate problem.** For  $\sigma \in (0, 1)$ , we consider the problem

$$u_t + Aw = 0, \quad (3.12)$$

$$w = \delta A^2 u + \mathcal{A}(u) + f_\sigma(u) - \lambda u + (\varepsilon + \sigma)u_t, \quad (3.13)$$

$$u|_{t=0} = u_{0,\sigma}, \quad \text{a.e. in } \Omega. \quad (3.14)$$

We shall now show that it admits at least one local in time weak solution. Namely, there holds the following

**Lemma 3.4.** *Let us assume (2.1)-(2.4). Then, for any  $\sigma \in (0, 1)$ , there exist  $T_0 \in (0, T]$  (possibly depending on  $\sigma$ ) and a couple  $(u, w)$  with*

$$u \in H^1(0, T_0; H) \cap L^\infty(0, T_0; W) \cap L^2(0, T_0; D(A^2)), \quad (3.15)$$

$$w \in L^2(0, T_0; W), \quad (3.16)$$

such that (3.12)-(3.13) hold a.e. in  $(0, T_0)$  and the initial condition (3.14) is satisfied.

**PROOF.** The theorem will be proved by using the Schauder fixed point theorem. We take

$$B_R = \{v \in L^2(0, T_0; W) \cap L^4(0, T_0; W^{1,4}(\Omega)) : \|v\|_{L^2(0, T_0; W)} + \|v\|_{L^4(0, T_0; W^{1,4}(\Omega))} \leq R\}, \quad (3.17)$$

for  $T_0$  and  $R$  to be chosen below. Then, we take  $\bar{u} \in B_R$  and consider the problem given by (3.14) and

$$u_t + Aw = 0, \quad \text{in } H, \quad (3.18)$$

$$w = \delta A^2 u + \mathcal{A}(\bar{u}) + f_\sigma(u) - \lambda u + (\varepsilon + \sigma)u_t, \quad \text{in } H. \quad (3.19)$$

Then, once  $\bar{u}$  is fixed, the above system is linear in  $(u, w)$  up to a Lipschitz perturbation. Consequently, there exists a unique couple  $(u, w)$  solving the problem given by (3.18)-(3.19) and (3.14) in a proper sense and fulfilling the natural regularity conditions. To be precise, the forthcoming estimates will make clear that the regularity satisfied by this solution is exactly (3.15)-(3.16). We then note as  $\mathcal{K}$  the map such that  $\mathcal{K} : \bar{u} \mapsto u$ . To conclude the proof we will have to show the following three properties:

- (i)  $\mathcal{K}$  takes its values in  $B_R$ ;
- (ii)  $\mathcal{K}$  is continuous w.r.t. the  $L^2(0, T_0; W)$  and the  $L^4(0, T_0; W^{1,4}(\Omega))$  norms;
- (iii)  $\mathcal{K}$  is a compact map.

To prove these facts, we first notice that

$$\|\mathcal{A}(\bar{u})\|_{L^2(0, T_0; H)}^2 \leq c(\|\bar{u}\|_{L^2(0, T_0; W)}^2 + \|\bar{u}\|_{L^4(0, T_0; W^{1,4}(\Omega))}^4) \leq Q(R). \quad (3.20)$$

Here and below,  $Q$  denotes a computable function, possibly depending on  $\sigma$ , defined for any nonnegative value of its argument(s) and increasingly monotone in (each of) its argument(s).

We now perform a couple of a-priori estimates. To start, we test (3.18) by  $w$  and (3.19) by  $u_t$  (energy estimate). This gives

$$\begin{aligned} & \frac{d}{dt} \left( \frac{\delta}{2} \|Au\|^2 + \int_{\Omega} \left( F_\sigma(u) - \frac{\lambda}{2} u^2 \right) \right) + (\varepsilon + \sigma) \|u_t\|^2 + \|\nabla w\|^2 \\ &= -(\mathcal{A}(\bar{u}), u_t) \leq \frac{\sigma}{2} \|u_t\|^2 + \frac{1}{2\sigma} \|\mathcal{A}(\bar{u})\|^2 \end{aligned} \quad (3.21)$$

and the latter term can be estimated using (3.20). Next, we observe that, thanks to (2.24), it is

$$\frac{\delta}{2} \|Au\|^2 + \int_{\Omega} \left( F_\sigma(u) - \frac{\lambda}{2} u^2 \right) \geq \eta \|u\|_W^2 - c, \quad (3.22)$$

for some  $\eta > 0$ ,  $c \geq 0$  independent of  $\sigma$  and for all  $u$  in  $W$ . Thus, (3.21) provides the bounds

$$\|u\|_{L^\infty(0, T_0; W)} + \|u_t\|_{L^2(0, T_0; H)} + \|\nabla w\|_{L^2(0, T_0; H)} \leq Q(R, T_0, \|u_{0,\sigma}\|_W). \quad (3.23)$$

Next, testing (3.19) by  $A^2u$  and performing some standard computations (in particular, the terms  $(\mathcal{A}(\bar{u}), A^2u)$  and  $(f_\sigma(u), A^2u)$  are controlled by using (3.20), Hölder's and Young's inequalities, and the Lipschitz continuity of  $f_\sigma$ ), we obtain the further bound

$$\|A^2u\|_{L^2(0, T_0; H)} \leq Q(R, T_0, \|u_{0, \sigma}\|_W). \quad (3.24)$$

Hence, estimates (3.23) and (3.24) and a standard application of the Aubin-Lions lemma permit to see that the range of  $\mathcal{K}$  is relatively compact both in  $L^2(0, T_0; W)$  and in  $L^4(0, T_0; W^{1,4}(\Omega))$ . Thus, (iii) follows.

Concerning (i), we can now simply observe that, by (3.23),

$$\|u\|_{L^2(0, T_0; W)} \leq T_0^{1/2} \|u\|_{L^\infty(0, T_0; W)} \leq T_0^{1/2} Q(R, T_0, \|u_{0, \sigma}\|_W). \quad (3.25)$$

whence the right hand side can be made smaller than  $R$  if  $T_0$  is chosen small enough. A similar estimate works also for the  $L^4(0, T_0; W^{1,4}(\Omega))$ -norm since  $W \subset W^{1,4}(\Omega)$  continuously. Thus, also (i) is proved.

Finally, to prove condition (ii), we first observe that, if  $\{\bar{u}_n\} \subset B_R$  converges strongly to  $\bar{u}$  in  $L^2(0, T_0; W) \cap L^4(0, T_0; W^{1,4}(\Omega))$ , then, using proper weak compactness theorems, it is not difficult to prove that

$$\mathcal{A}(\bar{u}_n) \rightharpoonup \mathcal{A}(\bar{u}) \quad \text{weakly in } L^2(0, T_0; H). \quad (3.26)$$

Consequently, if  $u_n$  (respectively  $u$ ) is the solution to (3.18)-(3.19) corresponding to  $\bar{u}_n$  (respectively  $\bar{u}$ ), then estimates (3.23)-(3.24) hold for the sequence  $\{u_n\}$  with a function  $Q$  independent of  $n$ . Hence, standard weak compactness arguments together with the Lipschitz continuity of  $f_\sigma$  permit to prove that

$$u_n = \mathcal{K}(\bar{u}_n) \rightarrow u = \mathcal{K}(\bar{u}) \quad \text{strongly in } L^2(0, T_0; W) \cap L^4(0, T_0; W^{1,4}(\Omega)), \quad (3.27)$$

i.e., condition (ii). The proof of the lemma is concluded.  $\blacksquare$

## 3.2 A priori estimates

In this section we will show that the local solutions constructed in the previous section satisfy uniform estimates with respect both to the approximation parameter  $\sigma$  and to the time  $T_0$ . By standard extension methods this will yield a global in time solution (i.e., defined over the whole of  $(0, T)$ ) in the limit. However, to avoid technical complications, we will directly assume that the approximating solutions are already defined over  $(0, T)$ . Of course, to justify this, we will have to take care that all the constants appearing in the forthcoming estimates be independent of  $T_0$ . To be precise, in the sequel we will note by  $c > 0$  a computable positive constant (whose value can vary on occurrence) independent of all approximation parameters (in particular of  $T_0$  and  $\sigma$ ) and also of the parameters  $\varepsilon$  and  $\delta$ .

**Energy estimate.** First, integrating (3.12) in space and recalling (3.10), we obtain the *mass conservation* property

$$(u(t))_\Omega = (u_{0, \sigma})_\Omega = (u_0)_\Omega. \quad (3.28)$$

Next, we can test (3.12) by  $w$ , (3.13) by  $u_t$  and take the difference, arriving at

$$\frac{d}{dt} \mathcal{E}_{\sigma, \delta}(u) + \|\nabla w\|^2 + (\varepsilon + \sigma) \|u_t\|^2 = 0, \quad (3.29)$$

where the ‘‘approximate energy’’  $\mathcal{E}_{\sigma, \delta}(u)$  is defined as

$$\mathcal{E}_{\sigma, \delta}(u) = \int_\Omega \left( \frac{\delta}{2} |Au|^2 + \frac{a(u)}{2} |\nabla u|^2 + F_\sigma(u) - \frac{\lambda}{2} u^2 \right). \quad (3.30)$$

Actually, it is clear that the high regularity of approximate solutions (cf. (3.15)-(3.16)) allows the integration by parts necessary to write (3.29) (at least) almost everywhere in time. Indeed, all single terms in (3.13) lie in  $L^2(0, T; H)$  and the same holds for the test function  $u_t$ .

Then, we integrate (3.29) in time and notice that, by (2.24),

$$\mathcal{E}_{\sigma,\delta}(u) \geq \eta(\delta\|u\|_W^2 + \|u\|_V^2) - c \quad \forall t \in (0, T). \quad (3.31)$$

Consequently, (3.29) provides the bounds

$$\|u\|_{L^\infty(0,T;V)} + \delta^{1/2}\|u\|_{L^\infty(0,T;W)} + (\varepsilon + \sigma)^{1/2}\|u_t\|_{L^2(0,T;H)} \leq c, \quad (3.32)$$

$$\|\nabla w\|_{L^2(0,T;H)} \leq c, \quad (3.33)$$

$$\|F_\sigma(u)\|_{L^\infty(0,T;L^1(\Omega))} \leq c, \quad (3.34)$$

where it is worth stressing once more that the above constants  $c$  neither depend explicitly on  $\delta$  nor on  $\varepsilon$ .

**Second estimate.** We test (3.13) by  $u - u_\Omega$ ,  $u_\Omega$  denoting the (constant in time) spatial mean of  $u$ . Integrating by parts the term  $\mathcal{A}(u)$ , we obtain

$$\begin{aligned} \delta\|Au\|^2 + \int_\Omega a(u)|\nabla u|^2 + \int_\Omega f_\sigma(u)(u - u_\Omega) \\ \leq (w + \lambda u - (\varepsilon + \sigma)u_t, u - u_\Omega) - \int_\Omega \frac{a'(u)}{2}|\nabla u|^2(u - u_\Omega) \end{aligned} \quad (3.35)$$

and we have to estimate some terms. First of all, we observe that there exists a constant  $c$ , depending on the (assigned once  $u_0$  is fixed) value of  $u_\Omega$ , but *independent* of  $\sigma$ , such that

$$\int_\Omega f_\sigma(u)(u - u_\Omega) \geq \frac{1}{2}\|f_\sigma(u)\|_{L^1(\Omega)} - c. \quad (3.36)$$

To prove this inequality, one basically uses the monotonicity of  $f_\sigma$  and the fact that  $f_\sigma(0) = 0$  (cf. [25, Appendix] or [15, Third a priori estimate] for the details). Next, by (2.2), the function  $r \mapsto a'(r)(r - u_\Omega)$  is uniformly bounded, whence

$$- \int_\Omega \frac{a'(u)}{2}|\nabla u|^2(u - u_\Omega) \leq c\|\nabla u\|^2. \quad (3.37)$$

Finally, by the Poincaré-Wirtinger inequality,

$$\begin{aligned} (w + \lambda u - (\varepsilon + \sigma)u_t, u - u_\Omega) &= (w - w_\Omega + \lambda(u - u_\Omega) - (\varepsilon + \sigma)u_t, u - u_\Omega) \\ &\leq c\|\nabla w\|\|\nabla u\| + c\|\nabla u\|^2 + c(\varepsilon + \sigma)\|u_t\|\|\nabla u\| \\ &\leq c(\|\nabla w\| + (\varepsilon + \sigma)\|u_t\| + 1), \end{aligned} \quad (3.38)$$

the latter inequality following from estimate (3.32).

Thus, squaring (3.35), using (3.36)-(3.38), and integrating in time, we arrive after recalling (3.32), (3.33) at

$$\|f_\sigma(u)\|_{L^2(0,T;L^1(\Omega))} \leq c. \quad (3.39)$$

Next, integrating (3.13) with respect to space variables (and, in particular, integrating by parts the term  $\mathcal{A}(u)$ ), using (3.39), and recalling (3.33), we obtain (still for  $c$  independent of  $\varepsilon$  and  $\delta$ )

$$\|w\|_{L^2(0,T;V)} \leq c. \quad (3.40)$$

**Third estimate.** We test (3.13) by  $Au$ . Using the monotonicity of  $f_\sigma$  and (2.1), it is not difficult to arrive at

$$\frac{\varepsilon + \sigma}{2} \frac{d}{dt} \|\nabla u\|^2 + \delta \|\nabla Au\|^2 + \frac{\alpha}{2} \|Au\|^2 \leq (\nabla w + \lambda \nabla u, \nabla u) + c\|\nabla u\|_{L^4(\Omega)}^4. \quad (3.41)$$

Using the continuous embedding  $H^{3/4}(\Omega) \subset L^4(\Omega)$  and interpolation, and recalling estimate (3.32), the last term is treated as follows:

$$c\|\nabla u\|_{L^4(\Omega)}^4 \leq c\|u\|_{H^2(\Omega)}^{3/2} \|u\|_V^{5/2} \leq \frac{\delta}{2} \|\nabla Au\|^2 + c(\delta). \quad (3.42)$$

Note that the latter constant  $c(\delta)$  is expected to explode as  $\delta \searrow 0$  but, on the other hand, is independent of  $\sigma$ . Next, noting that

$$(\nabla w + \lambda \nabla u, \nabla u) \leq c(\|\nabla u\|^2 + \|\nabla w\|^2), \quad (3.43)$$

from (3.41) we readily deduce

$$\|u\|_{L^2(0,T;H^3(\Omega))} \leq c(\delta). \quad (3.44)$$

A similar (and even simpler) argument permits to check that it is also

$$\|\mathcal{A}(u)\|_{L^2(0,T;H)} \leq c(\delta). \quad (3.45)$$

Thus, using (3.32), (3.40), (3.44)-(3.45) and comparing terms in (3.13), we arrive at

$$\|f_\sigma(u)\|_{L^2(0,T;V')} \leq c(\delta). \quad (3.46)$$

### 3.3 Limit $\sigma \searrow 0$

We now use the machinery introduced in Subsection 2.1 to take the limit  $\sigma \searrow 0$  in (3.12)-(3.13). For convenience, we then rename as  $(u_\sigma, w_\sigma)$  the solution. Then, recalling estimates (3.32)-(3.34), (3.40) and (3.44)-(3.46), and using the Aubin-Lions compactness lemma, we deduce

$$u_\sigma \rightarrow u \quad \text{strongly in } C^0([0, T]; H^{2-\epsilon}(\Omega)) \cap L^2(0, T; H^{3-\epsilon}(\Omega)), \quad (3.47)$$

$$u_\sigma \rightarrow u \quad \text{weakly star in } H^1(0, T; V') \cap L^\infty(0, T; W) \cap L^2(0, T; H^3(\Omega)), \quad (3.48)$$

$$(\varepsilon + \sigma)u_{\sigma,t} \rightarrow \varepsilon u_t \quad \text{weakly in } L^2(0, T; H), \quad (3.49)$$

$$w_\sigma \rightarrow w \quad \text{weakly in } L^2(0, T; V), \quad (3.50)$$

$$f_\sigma(u_\sigma) \rightarrow \xi \quad \text{weakly in } L^2(0, T; V'), \quad (3.51)$$

for suitable limit functions  $u, w, \xi$ , where  $\varepsilon > 0$  is arbitrarily small. It is readily checked that the above relations ((3.47) in particular) are strong enough to guarantee that

$$\mathcal{A}(u_\sigma) \rightarrow \mathcal{A}(u), \quad \text{strongly in } L^2(0, T; H). \quad (3.52)$$

This allows us to take the limit  $\sigma \searrow 0$  in (3.12)-(3.14) (rewritten for  $u_\sigma, w_\sigma$ ) and get

$$u_t + Aw = 0, \quad \text{in } V', \quad (3.53)$$

$$w = \delta A^2 u + \mathcal{A}(u) + \xi - \lambda u + \varepsilon u_t, \quad \text{in } V', \quad (3.54)$$

$$u|_{t=0} = u_0 \quad \text{a.e. in } \Omega. \quad (3.55)$$

To identify  $\xi$ , we observe that, thanks to (3.47), (3.51), and Lemma 2.3 applied with the choices of  $X = V$ ,  $X' = V'$ ,  $\mathcal{B}_\sigma = f_\sigma$ ,  $\mathcal{B} = f_{0,w}$ ,  $v_\sigma = u_\sigma$ ,  $v = u$  and  $\xi_\sigma = f_\sigma(u_\sigma)$ , it follows that

$$\xi \in f_{0,w}(u). \quad (3.56)$$

Namely,  $\xi$  is identified with respect to the weak (duality) expression of the function  $f_0$ . This concludes the proof of Theorem 3.2 for what concerns existence.

### 3.4 Uniqueness

To conclude the proof of Theorem 3.2, it remains to prove uniqueness. With this purpose, we write both (3.5) and (3.6) for a couple of solutions  $(u_1, w_1, \xi_1)$ ,  $(u_2, w_2, \xi_2)$ , and take the difference. This gives

$$u_t + Aw = 0, \quad \text{in } V', \quad (3.57)$$

$$\begin{aligned} w = & \delta A^2 u - a(u_1)\Delta u - (a(u_1) - a(u_2))\Delta u_2 - \frac{a'(u_1)}{2}(|\nabla u_1|^2 - |\nabla u_2|^2) \\ & - \frac{a'(u_1) - a'(u_2)}{2}|\nabla u_2|^2 + \xi_1 - \xi_2 - \lambda u + \varepsilon u_t, \quad \text{in } V', \end{aligned} \quad (3.58)$$

where we have set  $(u, w, \xi) := (u_1, w_1, \xi_1) - (u_2, w_2, \xi_2)$ . Then, we test (3.57) by  $A^{-1}u$ , (3.58) by  $u$ , and take the difference. Notice that, indeed,  $u$  has zero mean value by (3.28). Thus, the operator  $A^{-1}$  makes sense since  $A$  is bijective from  $W_0$  to  $H_0$ , the subscript 0 indicating the zero-mean condition.

A straightforward computation involving use of standard embedding properties of Sobolev spaces then gives

$$\begin{aligned} & \left( -a(u_1)\Delta u - (a(u_1) - a(u_2))\Delta u_2 - \frac{a'(u_1)}{2}(|\nabla u_1|^2 - |\nabla u_2|^2) - \frac{a'(u_1) - a'(u_2)}{2}|\nabla u_2|^2, u \right) \\ & \leq Q(\|u_1\|_{L^\infty(0,T;W)}, \|u_2\|_{L^\infty(0,T;W)})\|u\|_W\|u\| \end{aligned} \quad (3.59)$$

and we notice that the norms inside the function  $Q$  are controlled thanks to (3.1). Thus, also on account of the monotonicity of  $f_{0,w}$ , we arrive at

$$\begin{aligned} & \frac{d}{dt} \left( \frac{1}{2}\|u\|_{V'}^2 + \frac{\varepsilon}{2}\|u\|^2 \right) + \delta\|Au\|^2 \leq c\|u\|_W\|u\| + \lambda\|u\|^2 \\ & \leq c\|u\|_W^{2/3}\|u\|^{2/3} \leq \frac{\delta}{2}\|Au\|^2 + c(\delta)\|u\|_{V'}^2, \end{aligned} \quad (3.60)$$

where, to deduce the last two inequalities, we used the interpolation inequality  $\|u\| \leq \|u\|_W^{2/3}\|u\|_{V'}^{1/3}$  together with the fact that the function  $\|\cdot\|_{V'} + \|A\cdot\|$  is an equivalent norm on  $W$ . Thus, the thesis of Theorem 3.2 follows by applying Gronwall's lemma to (3.60).

## 4 From the 6th order to the 4th order model

In this section, we analyze the behavior of solutions to the 6th order problem as  $\delta$  tends to 0. To start, we specify the concept of weak solution in the 4th order case:

**Definition 4.1.** Let  $\delta = 0$  and  $\varepsilon \geq 0$ . A (global in time) weak solution to the 4th order problem is a couple  $(u, w)$ , with

$$u \in H^1(0, T; V') \cap L^\infty(0, T; V) \cap L^2(0, T; W), \quad \varepsilon u \in H^1(0, T; H), \quad (4.1)$$

$$F(u) \in L^\infty(0, T; L^1(\Omega)), \quad (4.2)$$

$$f_0(u) \in L^2(0, T; H), \quad (4.3)$$

$$w \in L^2(0, T; V), \quad (4.4)$$

such that the following relations hold a.e. in  $(0, T)$ :

$$u_t + Aw = 0, \quad \text{in } V', \quad (4.5)$$

$$w = \mathcal{A}(u) + f(u) + \varepsilon u_t, \quad \text{in } H, \quad (4.6)$$

together with the initial condition (3.8).

**Theorem 4.2.** Let us assume (2.1)-(2.4) together with

$$a \text{ is concave on } [-1, 1]. \quad (4.7)$$

Let also  $\varepsilon \geq 0$  and let, for all  $\delta \in (0, 1)$ ,  $u_{0,\delta}$  be an initial datum satisfying (3.9). Moreover, let us suppose

$$u_{0,\delta} \rightarrow u_0 \text{ strongly in } V, \quad \mathcal{E}_\delta(u_{0,\delta}) \rightarrow \mathcal{E}_0(u_0), \quad \text{where } (u_0)_\Omega \in (-1, 1). \quad (4.8)$$

Let, for any  $\delta \in (0, 1)$ ,  $(u_\delta, w_\delta, \xi_\delta)$  be a weak solution to the 6th order system in the sense of Definition 3.1. Then, we have that, up to a (nonrelabelled) subsequence of  $\delta \searrow 0$ ,

$$u_\delta \rightarrow u \text{ weakly star in } H^1(0, T; V') \cap L^\infty(0, T; V) \cap L^2(0, T; W), \quad (4.9)$$

$$\varepsilon u_\delta \rightarrow \varepsilon u \text{ weakly in } H^1(0, T; H), \quad (4.10)$$

$$w_\delta \rightarrow w \text{ weakly in } L^2(0, T; V), \quad (4.11)$$

$$\delta u_\delta \rightarrow 0 \text{ strongly in } L^2(0, T; H^3(\Omega)), \quad (4.12)$$

$$\xi_\delta \rightarrow f_0(u) \text{ weakly in } L^2(0, T; V'), \quad (4.13)$$

and  $(u, w)$  is a weak solution to the 4th order problem.

PROOF. The first part of the proof consists in repeating the “Energy estimate” and the “Second estimate” of the previous section. In fact, we could avoid this procedure since we already noted that the constants appearing in those estimates were independent of  $\delta$ . However, we choose to perform once more the estimates working directly on the 6th order problem (rather than on its approximation) for various reasons. First, this will show that the estimates do not depend on the chosen regularization scheme. Second, the procedure has an independent interest since we will see that the use of “weak” subdifferential operators still permits to rely on suitable integration by parts formulas and on monotonicity methods. Of course, many passages, which were trivial in the “strong” setting, need now a precise justification. Finally, in this way we are able to prove, as a byproduct, that any solution to the 6th order system satisfies an energy equality (and not just an inequality). Actually, this property may be useful for addressing the long-time behavior of the system.

**Energy estimate.** As before, we would like to test (3.5) by  $w_\delta$ , (3.6) by  $u_{\delta,t}$ , and take the difference. To justify this procedure, we start observing that  $w_\delta \in L^2(0, T; V)$  by (3.4). Actually, since (3.5) is in fact a relation in  $L^2(0, T; V')$ , use of  $w_\delta$  as a test function makes sense. The problem, instead, arises when working on (3.6) and, to justify the estimate, we can just consider the (more difficult) case  $\varepsilon = 0$ . Actually, we notice that, computing explicitly  $\nabla \mathcal{A}(u_\delta)$  and using (3.1), we can easily prove that

$$\mathcal{A}(u_\delta) \in L^2(0, T; V), \quad (4.14)$$

whence, being  $u_{\delta,t} \in L^2(0, T; V')$  (recall we assumed  $\varepsilon = 0$ ), it is not difficult to prove that

$$\langle u_{\delta,t}, \mathcal{A}(u_\delta) \rangle = \frac{1}{2} \frac{d}{dt} \int_\Omega a(u_\delta) |\nabla u_\delta|^2, \quad \text{a.e. in } (0, T). \quad (4.15)$$

Thus, the crucial point consists in showing that

$$\langle u_{\delta,t}, \delta A^2 u_\delta + \xi_\delta \rangle = \frac{d}{dt} \int_\Omega \left( \frac{\delta}{2} |A u_\delta|^2 + F(u_\delta) \right), \quad \text{a.e. in } (0, T). \quad (4.16)$$

To prove this, we observe that a comparison of terms in (3.6) permits to verify that, for any  $\delta \in (0, 1)$ ,  $\delta A^2 u_\delta + \xi_\delta \in L^2(0, T; V)$ . Thus, the duality on the left hand side makes sense. Moreover, as we set

$$\mathcal{J}_\delta(v) := \int_\Omega \left( \frac{\delta}{2} |A v|^2 + F(v) \right), \quad (4.17)$$

then a direct computation permits to check that

$$\delta A^2 u_\delta + \xi_\delta \in \partial \mathcal{J}_\delta(u_\delta) \quad \text{a.e. in } (0, T). \quad (4.18)$$

Indeed, by definition of  $H$ -subdifferential, this corresponds to the relation

$$\langle \delta A^2 u_\delta + \xi_\delta, v - u_\delta \rangle \leq \mathcal{J}_\delta(v) - \mathcal{J}_\delta(u_\delta) \quad \forall v \in H, \quad (4.19)$$

and it is sufficient to check it for  $v \in V$  since for  $v \in V \setminus H$  the right hand side is  $+\infty$  and consequently the relation is trivial. However, for  $v \in V$ , (4.19) follows by definition of the relaxed operator  $\mathcal{J}_{0,w}$ . Thanks to (4.18), (4.16) is then a direct consequence of inequality (2.42) of Lemma 2.5.

Thus, the above procedure permits to see that (any) weak solution  $(u_\delta, w_\delta, \xi_\delta)$  to the 6th order problem satisfies the energy equality

$$\frac{d}{dt} \mathcal{E}_\delta(u(t)) + \|\nabla w(t)\|^2 + \varepsilon \|u_t(t)\|^2 = 0 \quad (4.20)$$

for almost all  $t \in [0, T]$ . As a consequence, we get back the first two convergence relations in (4.9) as well as (4.10). Moreover, we have

$$\|\nabla w_\delta\|_{L^2(0, T; H)} \leq c. \quad (4.21)$$

**Second estimate.** Next, to get (4.11) and (4.13), we essentially need to repeat the “Second estimate” of the previous section. Indeed, we see that  $u_\delta - (u_\delta)_\Omega$  is an admissible test function in (3.5). However, we now have to obtain an estimate of  $\xi_\delta$  from the duality product

$$\langle \xi_\delta, u_\delta - (u_\delta)_\Omega \rangle. \quad (4.22)$$



Actually, if  $\xi_\delta = \xi_{\delta,a} + \xi_{\delta,s}$  is the Lebesgue decomposition of the measure  $\xi_\delta$  given in Theorem 2.2, then, noting that for all  $t \in [0, T]$  it is  $u_\delta(t) \in W \subset C^0(\bar{\Omega})$ , we can write

$$\langle \xi_\delta(t), u_\delta(t) - (u_\delta)_\Omega \rangle = \int_\Omega \xi_{\delta,a}(t)(u_\delta(t) - (u_\delta)_\Omega) \, dx + \int_\Omega (u_\delta(t) - (u_\delta)_\Omega) \, d\xi_{\delta,s}(t). \quad (4.23)$$

Next, we notice that, as a direct consequence of assumption (4.8),

$$\exists \mu \in (0, 1) : \quad -1 + \mu \leq (u_{0,\delta})_\Omega \leq 1 - \mu, \quad \forall \delta \in (0, 1), \quad (4.24)$$

where  $\mu$  is independent of  $\delta$ . In other words, the spatial means  $(u_{0,\delta})_\Omega$  are uniformly separated from  $\pm 1$ . Then, recalling (2.18) and proceeding as in (3.36), we have

$$\int_\Omega \xi_{\delta,a}(t)(u_\delta(t) - (u_\delta)_\Omega) \, dx \geq \frac{1}{2} \|\xi_{\delta,a}(t)\|_{L^1(\Omega)} - c, \quad (4.25)$$

where  $c$  does not depend on  $\delta$ .

On the other hand, let us note as  $d\xi_{\delta,s} = \phi_{\delta,s} d|\xi_{\delta,s}|$  the polar decomposition of  $\xi_{\delta,s}$ , where  $|\xi_{\delta,s}|$  is the total variation of  $\xi_{\delta,s}$  (cf., e.g., [29, Chap. 6]). Then, introducing the bounded linear functional  $\mathcal{S}_\delta : C^0(\bar{\Omega}) \rightarrow \mathbb{R}$  given by

$$\mathcal{S}_\delta(z) := \int_{\bar{\Omega}} z \, d\xi_{\delta,s} \quad (4.26)$$

using, e.g., [29, Thm. 6.19], and recalling (2.19), we can estimate the norm of  $\mathcal{S}$  as follows:

$$\begin{aligned} |\xi_{\delta,s}|(\bar{\Omega}) &= \int_{\bar{\Omega}} d|\xi_{\delta,s}| = \|\mathcal{S}_\delta\|_{\mathcal{M}(\bar{\Omega})} \\ &= \sup \left\{ \int_{\bar{\Omega}} z \, d\xi_{\delta,s}, \, z \in C^0(\bar{\Omega}), \, z(\bar{\Omega}) \subset [-1, 1] \right\} \\ &= \langle \xi_\delta, u_\delta \rangle - \int_\Omega \xi_{\delta,a} u_\delta = \int_{\bar{\Omega}} u_\delta \, d\xi_{\delta,s} = \int_{\bar{\Omega}} u_\delta \phi_{\delta,s} \, d|\xi_{\delta,s}|, \end{aligned} \quad (4.27)$$

where we also used that  $u_\delta \in C^0(\bar{\Omega})$ . Comparing terms, it then follows

$$u_\delta = \phi_{\delta,s}, \quad |\xi_{\delta,s}| \text{ - a.e. in } \bar{\Omega}. \quad (4.28)$$

Then, since is clear that

$$u_\delta = \pm 1 \quad \implies \quad \frac{u_\delta - (u_\delta)_\Omega}{|u_\delta - (u_\delta)_\Omega|} = \pm 1, \quad (4.29)$$

coming back to (4.27) we deduce

$$\int_{\bar{\Omega}} d|\xi_{\delta,s}| = \int_{\bar{\Omega}} \phi_{\delta,s} \frac{u_\delta - (u_\delta)_\Omega}{|u_\delta - (u_\delta)_\Omega|} \, d|\xi_{\delta,s}| \leq c \int_{\bar{\Omega}} \phi_{\delta,s} (u_\delta - (u_\delta)_\Omega) \, d|\xi_{\delta,s}| = c \int_{\bar{\Omega}} (u_\delta - (u_\delta)_\Omega) \, d\xi_{\delta,s}. \quad (4.30)$$

Here we used again in an essential way the uniform separation property (4.24).

Collecting (4.23)-(4.30), we then have

$$\langle \xi_\delta, u_\delta - (u_\delta)_\Omega \rangle \geq \frac{1}{2} \|\xi_{\delta,a}(t)\|_{L^1(\Omega)} + \eta \int_{\bar{\Omega}} d|\xi_{\delta,s}| - c, \quad (4.31)$$

for some  $c \geq 0$ ,  $\eta > 0$  independent of  $\delta$ . On the other hand, mimicking (3.35)-(3.38), we obtain

$$\delta \|A u_\delta\|^2 + \underline{a} \|\nabla u_\delta\|^2 + \langle \xi_\delta, u_\delta - (u_\delta)_\Omega \rangle \leq c(\|\nabla w_\delta\| + \varepsilon \|u_{\delta,t}\| + 1), \quad (4.32)$$

whence squaring, integrating in time, and using (4.10) and (4.21), we obtain that the function

$$t \mapsto \|\xi_{\delta,a}(t)\|_{L^1(\Omega)} + \int_\Omega d|\xi_{\delta,s}(t)| \quad \text{is bounded in } L^2(0, T), \text{ independently of } \delta. \quad (4.33)$$

Integrating now (3.6) in space, we deduce

$$\int_{\Omega} w_{\delta} = \frac{1}{2} \int_{\Omega} a'(u_{\delta}) |\nabla u_{\delta}|^2 + \int_{\Omega} \xi_{\delta} - \lambda (u_{\delta})_{\Omega}, \quad (4.34)$$

whence

$$\left| \int_{\Omega} w_{\delta} \right| \leq c \left( \|\nabla u_{\delta}\|^2 + \|\xi_{\delta, a}(t)\|_{L^1(\Omega)} + \int_{\Omega} d|\xi_{\delta, s}(t)| + 1 \right). \quad (4.35)$$

Thus, squaring, integrating in time, and recalling (4.21) and (4.33), we finally obtain (4.11).

**Key estimate.** To take the limit  $\delta > 0$ , we have to provide a bound on  $\mathcal{A}(u_{\delta})$  independent of  $\delta$ . This will be obtained by means of the following integration by parts formula due to Dal Passo, Garcke and Grün ([12, Lemma 2.3]):

**Lemma 4.3.** *Let  $h \in W^{2,\infty}(\mathbb{R})$  and  $z \in W$ . Then,*

$$\begin{aligned} \int_{\Omega} h'(z) |\nabla z|^2 \Delta z &= -\frac{1}{3} \int_{\Omega} h''(z) |\nabla z|^4 \\ &+ \frac{2}{3} \int_{\Omega} h(z) (|D^2 z|^2 - |\Delta z|^2) + \frac{2}{3} \int_{\Gamma} h(z) (Dn \nabla z, \nabla z), \end{aligned} \quad (4.36)$$

where  $n$  is the outward unit vector on  $\partial\Gamma$ .

We then test (3.6) by  $Au_{\delta}$  in the duality between  $V'$  and  $V$ . This gives the relation

$$\frac{\varepsilon}{2} \frac{d}{dt} \|\nabla u_{\delta}\|^2 + \delta \|\nabla Au_{\delta}\|^2 + (\mathcal{A}(u_{\delta}), Au_{\delta}) + (\xi_{\delta}, Au_{\delta}) = (\nabla w_{\delta}, \nabla u_{\delta}) + \lambda \|\nabla u_{\delta}\|^2 \quad (4.37)$$

and some terms have to be estimated. First, we note that

$$(\mathcal{A}(u_{\delta}), Au_{\delta}) = \left( a(u_{\delta}) \Delta u_{\delta} + \frac{a'(u_{\delta})}{2} |\nabla u_{\delta}|^2, \Delta u_{\delta} \right). \quad (4.38)$$

Thus, using Lemma 4.3 with the choice of  $h(\cdot) = a'(\cdot)/2$ , we obtain

$$\begin{aligned} (\mathcal{A}(u_{\delta}), Au_{\delta}) &= \int_{\Omega} a(u_{\delta}) |\Delta u_{\delta}|^2 + \frac{1}{3} \int_{\Omega} a(u_{\delta}) (|D^2 u_{\delta}|^2 - |\Delta u_{\delta}|^2) \\ &- \frac{1}{6} \int_{\Omega} a''(u_{\delta}) |\nabla u_{\delta}|^4 + \frac{1}{3} \int_{\Gamma} a(u_{\delta}) (Dn \nabla u_{\delta}, \nabla u_{\delta}). \end{aligned} \quad (4.39)$$

Let us now point out that

$$\frac{1}{3} \left| \int_{\Gamma} a(u_{\delta}) (Dn \nabla u_{\delta}, \nabla u_{\delta}) \right| \leq c \|\nabla u_{\delta}\|_{L^2(\Gamma)}^2 \leq \varepsilon \|Au_{\delta}\|_W^2 + c_{\varepsilon} \|u_{\delta}\|^2, \quad (4.40)$$

for small  $\varepsilon$  to be chosen below, the last inequality following from the continuity of the trace operator (applied to  $\nabla u$ ) from  $H^s(\Omega)$  into  $L^2(\Gamma)$  for  $s \in (1/2, 1)$  and the compactness of the embedding  $W \subset H^{1+s}(\Omega)$  for  $s$  in the same range.

Thus, using the concavity assumption (4.7) on  $a$ , we get

$$(\mathcal{A}(u_{\delta}), Au_{\delta}) \geq \eta \|Au_{\delta}\|^2 - c, \quad (4.41)$$

for proper strictly positive constants  $\eta$  and  $c$ , both independent of  $\delta$ . Next, we observe that, by (3.56) and Lemma 2.4, it is  $(\xi_{\delta}, Au_{\delta}) \geq 0$ . Finally, we have

$$-(\nabla w_{\delta}, \nabla u_{\delta}) \leq c \|\nabla w_{\delta}\| \|\nabla u_{\delta}\|, \quad (4.42)$$

and the right hand side is readily estimated thanks to (4.10) and (4.11).

Thus, on account of (4.41), integrating (4.37) in time, we readily obtain the last of (4.9) as well as (4.12). Moreover, since  $-1 \leq u_{\delta} \leq 1$  almost everywhere, we have for free

$$\|u_{\delta}\|_{L^{\infty}((0,T) \times \Omega)} \leq 1. \quad (4.43)$$

Thus, by the Gagliardo-Nirenberg inequality (cf., e.g., [26])

$$\|\nabla z\|_{L^4(\Omega)} \leq c_\Omega \|z\|_W^{1/2} \|z\|_{L^\infty(\Omega)}^{1/2} + \|z\| \quad \forall z \in W, \quad (4.44)$$

we have also

$$u_\delta \rightharpoonup u \quad \text{weakly in } L^4(0, T; W^{1,4}(\Omega)). \quad (4.45)$$

This readily entails

$$\mathcal{A}(u_\delta) \rightharpoonup \mathcal{A}(u) \quad \text{weakly in } L^2(0, T; H). \quad (4.46)$$

Thus, a comparison of terms in (3.6) gives also

$$\xi_\delta \rightharpoonup \xi \quad \text{weakly in } L^2(0, T; V'). \quad (4.47)$$

Then, we can take the limit  $\delta \searrow 0$  in (3.5) and get (4.5). On the other hand, if we take the limit of (3.6), we obtain

$$w = \mathcal{A}(u) + \xi - \lambda u + \varepsilon u_t \quad (4.48)$$

and we have to identify  $\xi$ . Actually, (4.47), the strong convergence  $u_\delta \rightarrow u$  in  $L^2(0, T; V)$  (following from (4.9) and the Aubin-Lions lemma) and Lemma 2.3 permit to show that

$$\xi \in f_{0,w}(u) \quad \text{a.e. in } (0, T). \quad (4.49)$$

On the other hand, a comparison argument in (4.48) permits to see that  $\xi \in L^2(0, T; H)$ , whence, thanks to (2.14), we obtain that  $\xi(t) = f_0(u(t)) \in H$  for a.e.  $t \in (0, T)$ . This concludes the proof of Theorem 4.2.  $\blacksquare$

## 5 Analysis of the fourth order problem

In this section, we will prove existence of a weak solution to Problem (1.1)-(1.4) in the fourth order case  $\delta = 0$  by means of a direct approach not relying on the 6th order approximation. This will allow us to consider a general function  $\alpha$  (without the concavity assumption (4.7)). More precisely, we have the following

**Theorem 5.1.** *Let assumptions (2.1)-(2.4) hold, let  $\varepsilon \geq 0$  and let*

$$u_0 \in V, \quad F(u_0) \in L^1(\Omega), \quad (u_0)_\Omega \in (-1, 1). \quad (5.1)$$

*Then, there exists at least one weak solution to the 4th order problem, in the sense of Definition 4.1.*

The rest of the section is devoted to the proof of the above result, which is divided into several steps.

**Phase-field approximation.** For  $\sigma \in (0, 1)$ , we consider the system

$$u_t + \sigma w_t + A w = 0, \quad (5.2)$$

$$w = \mathcal{A}(u) + f_\sigma(u) - \lambda u + (\varepsilon + \sigma)u_t. \quad (5.3)$$

This will be endowed with the initial conditions

$$u|_{t=0} = u_{0,\sigma}, \quad w|_{t=0} = 0. \quad (5.4)$$

Similarly as before (compare with (3.10)), we have set

$$u_{0,\sigma} + \sigma A^2 u_{0,\sigma} = u_0 \quad (5.5)$$

and, by standard elliptic regularity, we have that

$$u_{0,\sigma} \in H^5(\Omega) \subset C^{3+\alpha}(\bar{\Omega}) \quad \text{for } \alpha \in (0, 1/2), \quad \partial_n u_{0,\sigma} = \partial_n A u_{0,\sigma} = 0, \quad \text{on } \Gamma. \quad (5.6)$$

Moreover, of course,  $u_{0,\sigma} \rightarrow u_0$  in a suitable sense as  $\sigma \searrow 0$ .

**Fixed point argument.** We now prove existence of a local solution to the phase-field approximation by a further Schauder fixed point argument. Namely, we introduce the system

$$u_t + \sigma w_t + Aw = 0, \quad (5.7)$$

$$\bar{w} = -a(\bar{u})\Delta u - \frac{a'(\bar{u})}{2}|\nabla \bar{u}|^2 + f_\sigma(\bar{u}) - \lambda \bar{u} + (\varepsilon + \sigma)u_t, \quad \partial_n u = 0 \text{ on } \Gamma, \quad (5.8)$$

which we still endow with the condition (5.4). Here,  $f_\sigma$  is chosen as in (2.21).

Next, we set

$$\mathcal{U} := \{u \in C^{0,1+\alpha}([0, T_0] \times \bar{\Omega}) : u|_{t=0} = u_{0,\sigma}, \|u\|_{C^{0,1+\alpha}} \leq 2R\}, \quad (5.9)$$

where  $R := \max\{1, \|u_{0,\sigma}\|_{C^{1+\alpha}(\bar{\Omega})}\}$  and  $T_0$  will be chosen at the end of the argument. It is clear that  $R$  is in fact depending on  $\sigma$  (so that the same will happen for  $T_0$ ). This dependence is however not emphasized here. For the definition of the parabolic Hölder spaces used in this proof we refer the reader to [23, Chap. 5], whose notation is adopted. Moreover, in the sequel, in place of  $C^{0,\alpha}([0, T_0] \times \bar{\Omega})$  (and similar spaces) we will just write  $C^{0,\alpha}$ , for brevity. We then also define

$$\mathcal{W} := \{w \in C^{0,\alpha} : w|_{t=0} = 0, \|w\|_{C^{0,\alpha}} \leq R\}, \quad (5.10)$$

where  $R$  is, for simplicity, the same number as in (5.9).

Then, choosing  $(\bar{u}, \bar{w})$  in  $\mathcal{U} \times \mathcal{W}$  and inserting it in (5.8), we observe that, by the Lipschitz regularity of  $a$  (cf. (2.1)) and standard multiplication properties of Hölder spaces, there exists a computable monotone function  $Q$ , also depending on  $\sigma$ , but independent of the time  $T_0$ , such that

$$\|a(\bar{u})\|_{C^{0,\alpha}} + \|a'(\bar{u})|\nabla \bar{u}|^2\|_{C^{0,\alpha}} + \|f_\sigma(\bar{u})\|_{C^{0,\alpha}} \leq Q(R). \quad (5.11)$$

Thanks to [23, Thm. 5.1.21], then there exists one and only one solution  $u$  to (5.8) with the first initial condition (5.4). This solution satisfies

$$\|u\|_{C^{1,2+\alpha}} \leq Q(R). \quad (5.12)$$

Substituting then  $u_t$  in (5.7) and applying the same theorem of [23] to this equation with the second initial condition (5.4), we then obtain one and only one solution  $w$ , with

$$\|w\|_{C^{1,2+\alpha}} \leq Q(R). \quad (5.13)$$

We then note as  $T$  the map such that  $T : (\bar{u}, \bar{w}) \mapsto (u, w)$ . As before, we need to show that:

- (i)  $T$  takes its values in  $\mathcal{U} \times \mathcal{W}$ ;
- (ii)  $T$  is continuous w.r.t. the  $C^{0,1+\alpha} \times C^{0,\alpha}$  norm of  $\mathcal{U} \times \mathcal{W}$ ;
- (iii)  $T$  is a compact map.

First of all let us prove (i). We just refer to the component  $u$ , the argument for  $w$  being analogous and in fact simpler. We start observing that, if  $u \in \Pi_1(T(\mathcal{U} \times \mathcal{W}))$ , ( $\Pi_1$  denoting projection on the first component), then

$$\|u(t)\|_{C^\alpha(\bar{\Omega})} \leq \|u_0\|_{C^\alpha(\bar{\Omega})} + \int_0^t \|u_t(s)\|_{C^\alpha(\bar{\Omega})} ds \leq R + T_0 Q(R), \quad \forall t \in [0, T_0], \quad (5.14)$$

which is smaller than  $2R$  if  $T_0$  is chosen suitably.

Next, using the continuous embedding (cf. [23, Lemma 5.1.1])

$$C^{1,2+\alpha} \subset C^{1/2}([0, T_0]; C^{1+\alpha}(\bar{\Omega})) \cap C^{\alpha/2}([0, T_0]; C^2(\bar{\Omega})), \quad (5.15)$$

we obtain that, analogously,

$$\|\nabla u(t)\|_{C^\alpha(\bar{\Omega})} \leq \|\nabla u_0\|_{C^\alpha(\bar{\Omega})} + T_0^{1/2} \|u\|_{C^{1/2}([0, T_0]; C^{1+\alpha}(\bar{\Omega}))} \leq R + T_0^{1/2} Q(R). \quad (5.16)$$

Hence, passing to the supremum for  $t \in [0, T_0]$ , we see that the norm of  $u$  in  $C^{0,1+\alpha}$  can be made smaller than  $2R$  if  $T_0$  is small enough. Thus, (i) is proved.

Let us now come to (iii). As before, we just deal with the component  $u$ . Namely, on account of (5.12), we have to show that the space  $C^{1,2+\alpha}$  is compactly embedded into  $C^{0,1+\alpha}$ . Actually, by (5.15) and using standard compact inclusion properties of Hölder spaces, this relation is proved easily. Hence, we have (iii).

Finally, we have to prove (ii). This property is however straightforward. Actually, taking  $(\bar{u}_n, \bar{w}_n) \rightarrow (\bar{u}, \bar{w})$  in  $\mathcal{U} \times \mathcal{W}$ , we have that the corresponding solutions  $(u_n, w_n) = T(\bar{u}_n, \bar{w}_n)$  are bounded in the sense of (5.12)-(5.13) uniformly in  $n$ . Consequently, a standard weak compactness argument, together to the uniqueness property for the initial value problems associated to (5.7) and to (5.8), permit to see that the whole sequence  $(u_n, w_n)$  converges to a unique limit point  $(u, w)$  solving (5.7)-(5.8) w.r.t. the limit data  $(\bar{u}, \bar{w})$ . Moreover, by the compactness property proved in (iii), this convergence holds w.r.t. the original topology of  $\mathcal{U} \times \mathcal{W}$ . This proves that  $(u, w) = T(\bar{u}, \bar{w})$ , i.e., (ii) holds.

**A priori estimates.** For any  $\sigma > 0$ , we have obtained a local (i.e., with a final time  $T_0$  depending on  $\sigma$ ) solution to (5.2)-(5.3) with the initial conditions (5.4). To emphasize the  $\sigma$ -dependence, we will note it by  $(u_\sigma, w_\sigma)$  in the sequel. To let  $\sigma \searrow 0$ , we now devise some a-priori estimates uniform both w.r.t.  $\sigma$  and w.r.t.  $T_0$ . As before, this will give a global solution in the limit and, to avoid technicalities, we can directly work on the time interval  $[0, T]$ . Notice that the high regularity of  $(u_\sigma, w_\sigma)$  gives sense to all the calculations performed below (in particular, to all the integrations by parts). That said, we repeat the “Energy estimate”, exactly as in the previous sections. This now gives

$$\|u_\sigma\|_{L^\infty(0,T;V)} + \|F_\sigma(u_\sigma)\|_{L^\infty(0,T;L^1(\Omega))} \leq c, \quad (5.17)$$

$$(\sigma + \varepsilon)^{1/2} \|u_{\sigma,t}\|_{L^2(0,T;H)} \leq c, \quad (5.18)$$

$$\sigma^{1/2} \|w_\sigma\|_{L^\infty(0,T;H)} + \|\nabla w_\sigma\|_{L^2(0,T;H)} \leq c. \quad (5.19)$$

Next, working as in the “Second estimate” of Subsection 3.2, we obtain the analogue of (3.39) and (3.40).

To estimate  $f_\sigma(u_\sigma)$  in  $H$ , we now test (5.3) by  $f_\sigma(u_\sigma)$ , to get

$$\begin{aligned} & \frac{\varepsilon + \sigma}{2} \frac{d}{dt} \int_{\Omega} F_\sigma(u_\sigma) + \int_{\Omega} \left( a(u_\sigma) f'_\sigma(u_\sigma) + \frac{a'(u_\sigma)}{2} f_\sigma(u_\sigma) \right) |\nabla u_\sigma|^2 + \|f_\sigma(u_\sigma)\|^2 \\ & = (w_\sigma + \lambda u_\sigma, f_\sigma(u_\sigma)), \end{aligned} \quad (5.20)$$

and it is a standard matter to estimate the right hand side by using the last term on the left hand side, Hölder's and Young's inequalities, and properties (5.17) and (3.40). Now, we notice that, thanks to (2.26),

$$a(r) f'_\sigma(r) + \frac{a'(r)}{2} f_\sigma(r) \geq \underline{a} f'_\sigma(r) - c |f_\sigma(r)| \geq \frac{\underline{a}}{2} f'_\sigma(r) - c, \quad (5.21)$$

with the last  $c$  being independent of  $\sigma$ . Applying this to the second term on the left hand side of (5.20), we arrive at

$$\|f_\sigma(u_\sigma)\|_{L^2(0,T;H)} \leq c. \quad (5.22)$$

The key point is represented by the next estimate, which is used to control the second space derivatives of  $u$ . To do this, we have to operate a change of variable, first. Namely, we set

$$\phi(s) := \int_0^s a^{1/2}(r) dr, \quad z_\sigma := \phi(u_\sigma) \quad (5.23)$$

and notice that, by (2.1)-(2.2),  $\phi$  is monotone and Lipschitz together with its inverse. Then, by (5.17),

$$\|z_\sigma\|_{L^\infty(0,T;V)} \leq c \quad (5.24)$$

and it is straightforward to realize that (5.3) can be rewritten as

$$w_\sigma = -\phi'(u_\sigma) \Delta z_\sigma + f_\sigma \circ \phi^{-1}(z_\sigma) - \lambda u_\sigma + (\varepsilon + \sigma) u_{\sigma,t}, \quad \partial_n u_\sigma = 0 \text{ on } \Gamma. \quad (5.25)$$

By the Hölder continuity of  $u_\sigma$  up to its second space derivatives and the Lipschitz continuity of  $a$  and  $a'$  (cf. (2.1)-(2.2)),  $-\Delta z_\sigma$  is also Hölder continuous in space. Thus, we can use it as a test function in (5.25). Using the monotonicity of  $f_\sigma$  and  $\phi^{-1}$ , and recalling (5.22), we then easily obtain

$$\|z_\sigma\|_{L^2(0,T;W)} \leq c. \quad (5.26)$$

**Passage to the limit.** As a consequence of (5.17)-(5.19), (3.39)-(3.40) and (5.22), we have

$$u_\sigma \rightarrow u \quad \text{weakly star in } H^1(0,T;V') \cap L^\infty(0,T;V), \quad (5.27)$$

$$(\sigma + \varepsilon)u_{\sigma,t} \rightarrow \varepsilon u_t \quad \text{weakly in } L^2(0,T;H), \quad (5.28)$$

$$f_\sigma(u_\sigma) \rightarrow \bar{f} \quad \text{weakly in } L^2(0,T;H), \quad (5.29)$$

$$w_\sigma \rightarrow w \quad \text{weakly in } L^2(0,T;V), \quad (5.30)$$

$$u_{\sigma,t} + \sigma w_{\sigma,t} \rightarrow u_t \quad \text{weakly in } L^2(0,T;V'), \quad (5.31)$$

for suitable limit functions  $u$ ,  $w$  and  $\bar{f}$ . Here and below, all convergence relations have to be intended to hold up to (nonrelabelled) subsequences of  $\sigma \searrow 0$ . Now, by the Aubin-Lions lemma, we have

$$u_\sigma \rightarrow u \quad \text{strongly in } C^0([0,T];H) \quad \text{and a.e. in } Q. \quad (5.32)$$

Then, (5.29) and a standard monotonicity argument (cf. [5, Prop. 1.1]) imply that  $\bar{f} = f(u)$  a.e. in  $Q$ . Furthermore, by (2.1)-(2.2) and the generalized Lebesgue's theorem, we have

$$a(u_\sigma) \rightarrow a(u), \quad a'(u_\sigma) \rightarrow a'(u), \quad \text{strongly in } L^q(Q) \quad \text{for all } q \in [1, +\infty). \quad (5.33)$$

Analogously, recalling (5.24),  $z_\sigma = \phi(u_\sigma) \rightarrow \phi(u) =: z$ , strongly in  $L^q(Q)$  for all  $q \in [1, 6)$ . Actually, the latter relation holds also weakly in  $L^2(0,T;W)$  thanks to the bound (5.26). Moreover, by (5.24), (5.26) and interpolation, we obtain

$$\|\nabla z_\sigma\|_{L^{10/3}(Q)} \leq c, \quad (5.34)$$

whence, clearly, it is also

$$\|\nabla u_\sigma\|_{L^{10/3}(Q)} \leq c. \quad (5.35)$$

As a consequence, being

$$-\Delta u_\sigma = -\frac{1}{a^{1/2}(u_\sigma)} \Delta z_\sigma + \frac{a'(u_\sigma)}{2a(u_\sigma)} |\nabla u_\sigma|^2, \quad (5.36)$$

we also have that

$$\Delta u_\sigma \rightarrow \Delta u \quad \text{weakly in } L^{5/3}(Q). \quad (5.37)$$

Combining this with (5.27) and using the generalized Aubin-Lions lemma (cf., e.g., [32]), we then arrive at

$$u_\sigma \rightarrow u \quad \text{strongly in } L^{5/3}(0,T;W^{2-\epsilon,5/3}(\Omega)) \cap C^0([0,T];H^{1-\epsilon}(\Omega)), \quad \forall \epsilon > 0, \quad (5.38)$$

whence, by standard interpolation and embedding properties of Sobolev spaces, we obtain

$$\nabla u_\sigma \rightarrow \nabla u \quad \text{strongly in } L^q(Q) \quad \text{for some } q > 2. \quad (5.39)$$

Consequently, recalling (5.33),

$$a'(u_\sigma) |\nabla u_\sigma|^2 \rightarrow a'(u) |\nabla u|^2, \quad \text{say, weakly in } L^1(Q). \quad (5.40)$$

This is sufficient to take the limit  $\sigma \searrow 0$  in (5.3) and get back (4.6). To conclude the proof, it only remains to show the regularity (4.1) for what concerns the second space derivatives of  $u$ . Actually, by (5.26) and the Gagliardo-Nirenberg inequality (4.44),

$$z \in L^2(0,T;W) \cap L^\infty(Q) \subset L^4(0,T;W^{1,4}(\Omega)). \quad (5.41)$$

Thus, it is also  $u \in L^4(0,T;W^{1,4}(\Omega))$  and, consequently, a comparison of terms in (4.6) permits to see that  $\Delta u \in L^2(0,T;H)$ , whence (4.1) follows from elliptic regularity. The proof of Theorem 5.1 is concluded.

## 6 Further properties of weak solutions

### 6.1 Uniqueness for the 4th order problem

We will now prove that, in the case of a convex gradient part of the free energy  $\mathcal{E}_\delta$ , the solution is unique also in the 4th order case. Actually, also from a merely thermodynamical point of view, the convexity condition is a rather natural requirement. Indeed, it corresponds to asking the second differential of  $\mathcal{E}_\delta$  to be positive definite, to ensure that the stationary solutions are dynamically stable (cf., e.g., [33] for more details).

**Theorem 6.1.** *Let the assumptions of Theorem 5.1 hold and assume that, in addition,*

$$a''(r) \geq 0, \quad \left(\frac{1}{a}\right)''(r) \leq -\kappa, \quad \forall r \in [-1, 1], \quad (6.1)$$

where  $\kappa > 0$  if  $\varepsilon = 0$  and  $\kappa \geq 0$  if  $\varepsilon > 0$ . Then, the 4th order problem admits a unique weak solution.

PROOF. Let us denote by  $J$  the gradient part of the energy, i.e.,

$$J : V \rightarrow [0, +\infty), \quad J(u) := \int_{\Omega} \frac{a(u)}{2} |\nabla u|^2. \quad (6.2)$$

Then, we clearly have

$$\langle J'(u), v \rangle = \int_{\Omega} \left( a(u) \nabla u \cdot \nabla v + \frac{a'(u)}{2} |\nabla u|^2 v \right). \quad (6.3)$$

and we can correspondingly compute the second derivative of  $J$  as

$$\langle J''(u)v, z \rangle = \int_{\Omega} \left( \frac{a''(u) |\nabla u|^2 v z}{2} + a'(u) v \nabla u \cdot \nabla z + a'(u) z \nabla u \cdot \nabla v + a(u) \nabla v \cdot \nabla z \right). \quad (6.4)$$

To be more precise, we have that  $J'(u) \in V'$  and  $J''(u) \in \mathcal{L}(V, V')$  at least for  $u \in W$  (this may instead not be true if it is just  $u \in V$ , due to the quadratic terms in the gradient). This is however the case for the 4th order system since for any weak solution we have that  $u(t) \in W$  at least for a.e.  $t \in (0, T)$ .

From (6.4), we then have in particular

$$\begin{aligned} \langle J''(u)v, v \rangle &= \int_{\Omega} \left( \frac{a''(u) |\nabla u|^2 v^2}{2} + 2a'(u) v \nabla u \cdot \nabla v + a(u) |\nabla v|^2 \right) \\ &\geq \int_{\Omega} \left( a(u) - \frac{2a'(u)^2}{a''(u)} \right) |\nabla v|^2, \end{aligned} \quad (6.5)$$

whence the functional  $J$  is convex, at least when restricted to functions  $u$  such that

$$u \in W, \quad u(\Omega) \subset [-1, 1], \quad (6.6)$$

provided that  $a$  satisfies

$$a(r)a''(r) - 2a'(r)^2 \geq 0 \quad \forall r \in [-1, 1]. \quad (6.7)$$

Noting that

$$\left(\frac{1}{a}\right)'' = \frac{2(a')^2 - aa''}{a^3}, \quad (6.8)$$

we have that  $J$  is (strictly) convex if  $1/a$  is (strictly) concave, i.e., (6.1) holds (cf. also [14, Sec. 3] for related results). Note that, in deducing the last inequality in (6.5), we worked as if it was  $a'' > 0$ . However, if  $a''(r) = 0$  for some  $r$ , then also  $a'(r)$  has to be 0 due to (6.7). So, this means that in the set  $\{u = r\}$  the first two summands in the right hand side of the first line of (6.5) identically vanish.

That said, let us write both (3.5) and (3.6) for a couple of solutions  $(u_1, w_1)$ ,  $(u_2, w_2)$ , and take the difference. Setting  $(u, w) := (u_1, w_1) - (u_2, w_2)$ , we obtain

$$u_t + Aw = 0, \quad (6.9)$$

$$w = J'(u_1) - J'(u_2) + f(u_1) - f(u_2) + \varepsilon u_t. \quad (6.10)$$

Then, we can test (6.9) by  $A^{-1}u$ , (6.10) by  $u$ , and take the difference. Indeed,  $u = u_1 - u_2$  has zero mean value by (3.28). We obtain

$$\frac{1}{2} \frac{d}{dt} \left( \|u\|_{\mathcal{V}'}^2 + \varepsilon \|u\|^2 \right) + \langle J'(u_1) - J'(u_2), u \rangle + (f(u_1) - f(u_2), u) = 0 \quad (6.11)$$

and, using the convexity of  $J$  coming from (6.1) and the  $\lambda$ -monotonicity of  $f$  (see (2.3)), we have, for some function  $\xi$  belonging to  $W$  a.e. in time and taking its values in  $[-1, 1]$ ,

$$\frac{1}{2} \frac{d}{dt} \left( \|u\|_{\mathcal{V}'}^2 + \varepsilon \|u\|^2 \right) + \kappa \|\nabla u\|^2 \leq \frac{1}{2} \frac{d}{dt} \left( \|u\|_{\mathcal{V}'}^2 + \varepsilon \|u\|^2 \right) + \langle J''(\xi)u, u \rangle \leq \lambda \|u\|^2. \quad (6.12)$$

Thus, in the case  $\varepsilon > 0$  (where it may be  $\kappa = 0$ ), we can just use Gronwall's Lemma. Instead, if  $\varepsilon = 0$  (so that we assumed  $\kappa > 0$ ), by the Poincaré-Wirtinger inequality we have

$$\lambda \|u\|^2 \leq \frac{\kappa}{2} \|\nabla u\|^2 + c \|u\|_{\mathcal{V}'}, \quad (6.13)$$

and the thesis follows again by applying Gronwall's lemma to (6.12).  $\blacksquare$

## 6.2 Additional regularity

We prove here parabolic regularization properties of the solutions to the 4th order system holding in the case of a convex energy functional. An analogous result would hold also for the 6th order system under general conditions on  $a$  since the bilaplacian in that case dominates the lower order terms (we omit the details).

**Theorem 6.2.** *Let the assumptions of Theorem 6.1 hold. Then, the solution satisfies the additional regularity property*

$$\|u\|_{L^\infty(\tau, T; W)} + \|u\|_{L^\infty(\tau, T; W^{1,4}(\Omega))} \leq Q(\tau^{-1}) \quad \forall \tau > 0, \quad (6.14)$$

where  $Q$  is a computable monotone function whose expression depends on the data of the problem and, in particular, on  $u_0$ .

**PROOF.** The proof is based on a further a-priori estimate, which has unfortunately a formal character in the present regularity setting. To justify it, one should proceed by regularization. For instance, a natural choice would be that of refining the fixed point argument leading to existence of a weak solution (cf. Sec. 5) by showing (e.g., using a bootstrap regularity argument) that, at least locally in time, the solution lies in higher order Hölder spaces. We leave the details to the reader.

That said, we test (3.5) by  $u_t$  and subtract the result from the time derivative of (3.6) tested by  $u_t$ . We obtain

$$\frac{1}{2} \frac{d}{dt} \|\nabla w\|^2 + \frac{\varepsilon}{2} \frac{d}{dt} \|u_t\|^2 + \langle J''(u)u_t, u_t \rangle + \int_{\Omega} f'(u)u_t^2 \leq 0. \quad (6.15)$$

Then, by convexity of  $J$ ,

$$\langle J''(u)u_t, u_t \rangle \geq \kappa \|u_t\|_{L^2(0, T; V)}^2. \quad (6.16)$$

On the other hand, the  $\lambda$ -monotonicity of  $f$  gives

$$\int_{\Omega} f'(u)u_t^2 \geq -\lambda \|u_t\|_{L^2(0, T; H)}^2 \quad (6.17)$$

and, if  $\varepsilon = 0$  (so that  $\kappa > 0$ ), we have as before

$$-\lambda \|u_t\|_{L^2(0, T; H)}^2 \geq -\frac{\kappa}{2} \|u_t\|_{L^2(0, T; V)}^2 - c \|u_t\|_{L^2(0, T; V')}. \quad (6.18)$$

Thus, recalling the first of (3.1) and applying the *uniform* Gronwall lemma (cf. [34, Lemma I.1.1]), it is not difficult to infer

$$\|\nabla w\|_{L^\infty(\tau, T; H)} + \varepsilon^{1/2} \|u_t\|_{L^\infty(\tau, T; H)} + \kappa \|u_t\|_{L^2(\tau, T; V)} \leq Q(\tau^{-1}) \quad \forall \tau > 0. \quad (6.19)$$



Next, testing (3.6) by  $u - u_\Omega$  and proceeding as in the “Second estimate” of Subsection 3.2, but taking now the essential supremum as time varies in  $[\tau, T]$ , we arrive at

$$\|w\|_{L^\infty(\tau, T; V)} + \|f(u)\|_{L^\infty(\tau, T; L^1(\Omega))} \leq Q(\tau^{-1}) \quad \forall \tau > 0. \quad (6.20)$$

Thus, thanks to (6.20), we can test (4.6) by  $-\Delta z$ , with  $z = \phi(u)$  (cf. (5.23)). Proceeding similarly with Section 5 (but taking now the supremum over  $[\tau, T]$  rather than integrating in time), we easily get (6.14), which concludes the proof.  $\blacksquare$

### 6.3 Energy equality

As noted in Section 4, any weak solution to the 6th order system satisfies the energy equality (4.20). We will now see that the same property holds also in the viscous 4th order case (i.e., if  $\delta = 0$  and  $\varepsilon > 0$ ). More precisely, we can prove the

**Proposition 6.3.** *Let the assumptions of Theorem 5.1 hold and let  $\varepsilon > 0$ . Then, any weak solution to the 4th order system satisfies the integrated energy equality*

$$\mathcal{E}_0(u(t)) = \mathcal{E}_0(u_0) - \int_0^t (\|\nabla w(s)\|^2 - \varepsilon \|u_t(s)\|^2) ds \quad \forall t \in [0, T]. \quad (6.21)$$

PROOF. As before, we proceed by testing (4.5) by  $w$ , (4.6) by  $u_t$  and taking the difference. As  $u_t \in L^2(0, T; H)$  and  $f_0(u) \in L^2(0, T; H)$  (cf. (4.1) and (4.3)), then the integration by parts

$$(f(u), u_t) = \frac{d}{dt} \int_\Omega F(u), \quad \text{a.e. in } (0, T) \quad (6.22)$$

is straightforward (it follows directly from [8, Lemma 3.3, p. 73]). Thus, the only difficulty stands in showing that

$$\int_0^t (\mathcal{A}(u(s)), u_t(s)) ds = \int_\Omega \frac{\mathcal{A}(u(t))}{2} |\nabla u(t)|^2 - \int_\Omega \frac{\mathcal{A}(u_0)}{2} |\nabla u_0|^2. \quad (6.23)$$

To prove this fact it is convenient to regularize  $u$  by setting, a.e. in  $(0, T)$ ,

$$u_\sigma - \sigma A u_\sigma = u, \quad \text{for } \sigma \in (0, 1), \quad (6.24)$$

so that (cf. (4.1) and recall that  $\varepsilon > 0$ ), for all  $\sigma \in (0, 1)$ , we have

$$u_\sigma \in H^1(0, T; W) \cap L^2(0, T; H^4(\Omega)). \quad (6.25)$$

Then, clearly, (6.23) holds for the smooth functions  $u_\sigma$ . On the other hand, using a-priori bounds and standard semicontinuity methods, it is easy to show that, as  $\sigma \searrow 0$ ,

$$u_\sigma \rightarrow u \quad \text{strongly in } H^1(0, T; H) \cap L^2(0, T; W). \quad (6.26)$$

In particular, proceeding as in the “Passage to the limit” in Sec. 5, we obtain that

$$\mathcal{A}(u_\sigma) \rightarrow \mathcal{A}(u) \quad \text{at least weakly in } L^2(0, T; H). \quad (6.27)$$

Moreover, thanks to the continuous embedding  $H^1(0, T; H) \cap L^2(0, T; W) \subset C^0([0, T]; V)$ , we also have that

$$u_\sigma \rightarrow u \quad \text{strongly in } C^0([0, T]; V), \quad (6.28)$$

and it is apparent that (6.26)-(6.28) allow to take the limit  $\sigma \searrow 0$  in (6.23) for all  $t \in [0, T]$ .  $\blacksquare$

It is worth noting that the energy equality obtained above has a key relevance in the investigation of the long-time behavior of the system. In particular, given  $m \in (-1, 1)$  (the spatial mean of the initial datum, which is a conserved quantity due to (3.28)), we can define the *phase space*

$$\mathcal{X}_{\delta, m} := \{u \in V : \delta u \in W, F(u) \in L^1(\Omega), u_\Omega = m\} \quad (6.29)$$

and view the system (both for  $\delta > 0$  and for  $\delta = 0$ ) as a (generalized) dynamical process in  $\mathcal{X}_{\delta, m}$ . Then, (6.21) (or its 6th order analogue) stands at the basis of the so-called *energy method* (cf. [3, 24]) for proving existence of the *global attractor* with respect to the natural (i.e., strong) phase space topology. This issue will be analyzed in a forthcoming work.

**Remark 6.4.** Whether the equality (6.21) still holds in the nonviscous case  $\varepsilon = 0$  seems to be a nontrivial question. The answer would be positive in case one could prove the integration by parts formula

$$\int_0^t \langle u_t, A(u) + f(u) \rangle = \int_{\Omega} \left( \frac{a(u(t))}{2} |\nabla u(t)|^2 + F(u(t)) \right) - \int_{\Omega} \left( \frac{a(u_0)}{2} |\nabla u_0|^2 + F(u_0) \right), \quad (6.30)$$

under the conditions

$$u \in H^1(0, T; V') \cap L^2(0, T; W) \cap L^\infty(Q), \quad A(u) + f(u) \in L^2(0, T; V), \quad (6.31)$$

which are satisfied by our solution (in particular the latter (6.31) follows by a comparison of terms in (4.6), where it is now  $\varepsilon = 0$ ). Actually, if (6.31) holds, then both hands sides of (6.30) make sense. However, devising an approximation argument suitable for proving (6.30) could be a rather delicate problem.

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the 1990s, the number of people in the world who are living in poverty has increased from 1.2 billion to 1.6 billion (World Bank 2000).

There are a number of reasons for this increase. One of the main reasons is the rapid population growth in the developing world. The population of the world is expected to reach 8 billion by the year 2025, with the majority of the increase occurring in the developing world (United Nations 2000). This rapid population growth has led to a corresponding increase in the number of people living in poverty, as the resources available per person have decreased.

Another reason for the increase in poverty is the unequal distribution of income and resources. In many developing countries, a small number of people own a large proportion of the land and other resources, while the majority of the population has very little. This unequal distribution has led to a concentration of wealth and power in the hands of a few, while the majority of the population remains poor and vulnerable.

There are also a number of structural factors that have contributed to the increase in poverty. These include the rapid pace of technological change, which has led to the displacement of workers in many industries, and the increasing globalization of the world economy, which has led to the loss of jobs in many developing countries.

Despite these challenges, there are a number of ways in which the world can reduce the number of people living in poverty. One of the most important is to invest in education and training, so that people can acquire the skills and knowledge needed to find better-paying jobs. Another important way is to improve the distribution of income and resources, so that everyone has access to the opportunities and resources they need to succeed.

Finally, it is important to address the structural factors that have contributed to poverty, such as the rapid pace of technological change and the increasing globalization of the world economy. This can be done by investing in research and development, so that new technologies can be developed that create jobs, and by strengthening the institutions of the world economy, so that it is more equitable and sustainable.

By taking these steps, the world can reduce the number of people living in poverty and create a more equitable and sustainable future for all.

