Stationary solutions to a strain-gradient type thermoviscoelastic system

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Abstract. In this paper we study a strain-gradient type thermoviscoelastic system. We focus on the stationary states and their dynamical stability. The adiabatic stationary state is formulated as a nonlinear eigenvalue problem with non-local terms associated with the total energy conservation. One of the purposes of this paper is to extend the results obtained in Suzuki-Tasaki [34]. We reveal a unified structure, called semi-dualities, of the thermoviscoelastic system of viscosity-capillarity type with temperature-dependent viscous and elastic moduli. We describe a physical background and outline the thermodynamic derivation of the system. Based on the semi-dual structure we construct a series of general results concerning the stationary states and their stability. The application of these results together with the bifurcation theory allows to analyze the total set of the stationary solutions in more detail.

1 INTRODUCTION

1.1 THERMOVISCOELASTIC SYSTEM

In this paper we address the question of the stability of stationary states to the following thermoviscoelastic system

$$\begin{cases} u_{tt} - \nabla \cdot ((\nu_1 + \nu_2\theta)B\epsilon_t) + \kappa_1 Q^2 u + \kappa_2 \nabla \cdot (\theta A\epsilon(Qu)) = \nabla \cdot H_{,\epsilon}, \\ -\theta H_{,\theta\theta}\theta_t - k\triangle\theta = \theta H_{,\theta\epsilon} \cdot \epsilon_t - \kappa_2 \theta \left(A\epsilon(Qu)\right) \cdot \epsilon_t \\ \quad + (\nu_1 + \nu_2\theta)B\epsilon_t \cdot \epsilon_t \qquad\qquad \text{in } \Omega \times (0,T) \end{cases} \tag{1.1}$$

with the boundary and initial conditions

$$\begin{cases} u = Qu = 0, & \dfrac{\partial \theta}{\partial n} = 0 & \text{on } \partial\Omega \times (0, T), \\ u|_{t=0} = u_0, & u_t|_{t=0} = u_1, \quad \theta|_{t=0} = \theta_0 & \text{in } \Omega. \end{cases} \qquad (1.2)$$

The system arises as a model of structural phase transitions in viscoelastic solids. Above $\Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$) denotes a bounded domain with a smooth boundary $\partial\Omega$, occupied by a body in a fixed reference configuration, n denotes the outer unit normal vector on $\partial\Omega$, and $T > 0$. The unknowns $u = (u_i)_{i=1,\cdots,d} = (u_i(x,t))_{i=1,\cdots,d}$ and $\theta = \theta(x,t) > 0$ are real-valued, denoting the displacement vector and the absolute temperature, respectively. The small-strain tensor $\epsilon = \epsilon(u)$ is defined by $\epsilon(u) = (\nabla u + {}^t(\nabla u))/2$, where ${}^t M$ denotes the transposed matrix of $M = (M_{ij})$. The tensor $\epsilon_t = \epsilon(u_t)$ stands for the strain rate. The physical coefficients $\kappa_i \geq 0$, $\nu_i \geq 0$, and $k > 0$ are constants such that $\kappa_1 + \kappa_2 > 0$ and $\nu_1 + \nu_2 \geq 0$, $i = 1, 2$, denoting the strain-gradient coefficients, the viscosity coefficients, and the heat conductivity, respectively.

The function $H = H(\epsilon, \theta)$ denotes the volumetric free energy density which in accord with thermodynamic thermal stability (the postulate that the specific heat is strictly positive) satisfies

$$\mathbb{R}_+ \ni \theta \mapsto H(\epsilon, \theta) \in \mathbb{R} \quad \text{is strictly concave.} \qquad (1.3)$$

To model phase transitions the map $\epsilon \mapsto H(\epsilon, \theta)$ is admitted to be nonconvex (multiwell) in some range of temperatures θ. More precisely, like in Falk [13] and Falk-Konopka [14] models of shape memory alloys, we assume that $H = H(\epsilon, \theta)$ splits into

$$H(\epsilon, \theta) = f_\bullet(\theta) + W(\epsilon, \theta), \qquad (1.4)$$

where $f_\bullet = f_\bullet(\theta)$ is the thermal energy and $W = W(\epsilon, \theta)$ is the elastic energy of the form

$$W(\epsilon, \theta) = W_1(\epsilon) + \theta W_2(\epsilon). \qquad (1.5)$$

We assume $f_\bullet \in C^2(\mathbb{R}_+, \mathbb{R})$ and $W_i \in C^2(\text{Sym}(d, \mathbb{R}), \mathbb{R})$, where $\text{Sym}(d, \mathbb{R})$ denotes the set of all symmetric second order tensors in $\mathbb{R}^d$. The fourth order tensors $A = (A_{ijkl})$ and $B = (B_{ijkl})$ stand for the standard elasticity and viscosity tensors defined by

$$A\epsilon = \lambda_A \text{tr}\epsilon I + 2\mu_A \epsilon,$$
$$B\epsilon = \lambda_B \text{tr}\epsilon I + 2\mu_B \epsilon,$$

where $I = (\delta_{ij})$, μ_A, λ_A are the Lamé constants and μ_B, λ_B the viscosity constants, satisfying

$$\mu_i > 0, \quad d\lambda_i + 2\mu_i > 0, \qquad i = A, B.$$

This assumption ensures the coercivity and boundedness of tensor A

$$a_\bullet |\epsilon|^2 \leq (A\epsilon) \cdot \epsilon \leq a^* |\epsilon|^2, \qquad (1.6)$$

where $a_* = \min(d\lambda_A + 2\mu_A, 2\mu_A)$, $a^* = \max(d\lambda_A + 2\mu_A, 2\mu_A)$, and the analogous properties of tensor B.

The operator Q, defined by

$$Qu := Q_A u = \nabla \cdot A\epsilon(u) = \mu_A \triangle u + (\lambda_A + \mu_A)\nabla(\nabla \cdot u), \qquad (1.7)$$

stands for the second order operator of linearized elasticity. By the assumptions on μ_A, λ_A this operator is strongly elliptic.

Throughout the paper vectors (tensors of the first order) and tensors are denoted by bold letters. The summation convention over the repeated indices is used. A dot designates the inner product, irrespective of the space in question: $a \cdot b = a_i b_i$ is the inner product of vectors $a = (a_i)$ and $b = (b_i)$, $M \cdot M' = M_{ij} M'_{ij}$ is the inner product of second order tensors $M = (M_{ij})$ and $M' = (M'_{ij})$, $A^m \cdot B^m = A^m_{i_1 \cdots i_m} B^m_{i_1 \cdots i_m}$ is the inner product of m-th order tensors $A^m = (A^m_{i_1 \cdots i_m})$ and $B^m = (B^m_{i_1 \cdots i_m})$. Moreover, for tensors $a = (a_i)$, $M = (M_{ij})$, $H = (H_{ijk})$, $A = (A_{ijkl})$ we denote:

$$Ma = (M_{ij}a_j), \quad aH = (a_i H_{ijk}),$$
$$Ha = (H_{ijk}a_k), \quad MH = (M_{ij}H_{ijk}), \quad HM = (H_{ijk}M_{jk}),$$
$$aA = (a_i A_{ijkl}), \quad Aa = (A_{ijkl}a_l),$$
$$MA = (M_{ij}A_{ijkl}), \quad AM = (A_{ijkl}M_{kl}),$$
$$HA = (H_{ijk}A_{ijkl}), \quad AH = (A_{ijkl}H_{jkl}).$$

To simplify the notation we write:

$$f_{,i} = \frac{\partial f}{\partial x_i}, \quad i = 1, \cdots, d, \quad f_t = \frac{\partial f}{\partial t},$$

where space and time derivatives are material; ∇ and $\nabla \cdot$ denote the material gradient and the divergence operators. For the divergence we use the contraction over the last index, e.g., $\nabla \cdot S = \left(\frac{\partial S_{ij}}{\partial x_j}\right)$ for $S = (S_{ij})$. Moreover, we write $f_{,A} = \frac{\partial f}{\partial A}$ for the partial derivative of a function f with respect to the variable A (scalar or tensor). For f scalar valued and $A^m = (A^m_{i_1 \cdots i_m})$ a tensor of order m, $f_{,A^m}$ is a tensor of order m with components $f_{,A^m_{i_1 \cdots i_m}}$. In particular,

$$H_{,\epsilon} = \left(\frac{\partial H}{\partial \epsilon_{ij}}\right), \quad H_{,\theta} = \frac{\partial H}{\partial \theta}.$$

The spaces notation is standard and follows [26, 33].

1.2 THERMODYNAMIC BASIS

Equations (1.1) represent the local forms of the linear momentum and internal energy balances with constant mass density $\rho = 1$ and absent external body forces and heat sources:

$$\begin{aligned} u_{tt} - \nabla \cdot S &= 0, \\ e_t + \nabla \cdot q - S \cdot \epsilon_t &= 0, \end{aligned} \qquad (1.8)$$

where S is the stress tensor, e is the specific internal energy, and q is the energy flux. This system is associated with the following two potentials: the free energy of the Landau-Ginzburg type

$$f(\epsilon(u), \nabla\epsilon(u), \theta) = H(\epsilon(u), \theta) + \frac{1}{2}(\kappa_1 + \kappa_2\theta)|Qu|^2, \qquad (1.9)$$

and the dissipation potential

$$\mathcal{D}\left(\epsilon_t, \nabla\frac{1}{\theta}, \theta\right) = \frac{1}{2}\frac{\nu_1 + \nu_2\theta}{\theta}B\epsilon_t \cdot \epsilon_t + \frac{k}{2}\left|\theta\nabla\frac{1}{\theta}\right|^2. \qquad (1.10)$$

In accord with the thermodynamic relations

$$f = e - \theta\eta, \quad \eta = -f_{,\theta}, \qquad (1.11)$$

where η denotes the specific entropy, the corresponding forms of e and η are

$$\begin{aligned}
e &= (H(\epsilon, \theta) - \theta H_{,\theta}(\epsilon, \theta)) + \frac{1}{2}\kappa_1|Qu|^2 \\
&= (f_*(\theta) - \theta f'_*(\theta)) + W_1(\epsilon) + \frac{1}{2}\kappa_1|Qu|^2, \\
\eta &= -H_{,\theta}(\epsilon, \theta) - \frac{1}{2}\kappa_2|Qu|^2 \\
&= -f'_*(\theta) - W_2(\epsilon) - \frac{1}{2}\kappa_2|Qu|^2.
\end{aligned} \qquad (1.12)$$

The second law of thermodynamics implies the following constitutive relations for S and q (see (2.21) and (2.22)):

$$\begin{aligned}
S &= \frac{\delta e}{\delta \epsilon} - \theta\frac{\delta\eta}{\delta\epsilon} + S^d, \\
q &= q^d - \epsilon_t e_{,D\epsilon},
\end{aligned} \qquad (1.13)$$

where

$$\frac{\delta e}{\delta\epsilon} = e_{,\epsilon} - \nabla \cdot e_{,D\epsilon}$$

denotes the first variation of e, and S^d, q^d are the dissipative parts of the stress tensor and the energy flux, given by

$$S^d = \theta\mathcal{D}_{,\epsilon_t}, \quad q^d = \mathcal{D}_{,D(1/\theta)}. \qquad (1.14)$$

For particular potentials (1.9), (1.10) relations (1.13)-(1.14) take the form

$$\begin{aligned}
S &= H_{,\epsilon}(\epsilon, \theta) + (\nu_1 + \nu_2\theta)B\epsilon(u_t) - (\kappa_1 + \kappa_2\theta)A\epsilon(Qu), \\
q &= -k\nabla\theta - \kappa_1\epsilon_t(AQu).
\end{aligned} \qquad (1.15)$$

Combining balance equations (1.8) with relations (1.12)$_1$ and (1.15) gives system (1.1).

It should be pointed out that system (1.1) is a special example of a thermodynamically consistent scheme for the so-called elastic materials of higher grade in which the constitutive quantities are permitted to depend not only on the first gradient of the deformation, the strain, but also on its higher gradients (see [9]). In our case the quantities f, e and η depend on the first strain gradient and S depends on the second strain gradient, expressed by the term $\nabla \cdot e_{,D\epsilon} - \theta \nabla \cdot \eta_{,D\epsilon}$ in $(1.13)_1$ (and correspondingly the third term in $(1.15)_1$).

Another property to be pointed out is that the energy flux q contains in addition to the usual dissipative flux q^d, the extra nonstationary flux, $-\epsilon_t e_{,D\epsilon}$ in $(1.13)_2$ (and correspondingly the second term in $(1.15)_2$).

System of balance laws (1.8) with constitutive equations (1.13) (in particular (1.15)) complies with the following entropy inequality in the local form (see (2.24)):

$$\eta_t + \nabla \cdot \left(\frac{q^d}{\theta} - \epsilon_t \eta_{,D\epsilon} \right) = \sigma \geq 0, \qquad (1.16)$$

where

$$\sigma = \frac{S^d}{\theta} \cdot \epsilon_t + q^d \cdot \nabla \frac{1}{\theta}$$

$$= \mathcal{D}_{,\epsilon_t} \cdot \epsilon_t + \mathcal{D}_{,D(1/\theta)} \cdot \nabla \frac{1}{\theta}$$

is the entropy production.

There are various thermodynamic approaches to higher grade materials. We mention thermomechanical theories by Dunn-Serrin [9] and Aifantis [2], and various frameworks with internal variables or additional degrees of freedom (see e.g. reviews in [21] and [40]).

Relations (1.13) have been derived in [22] by exploiting the Müller-Liu entropy inequality. The derivation is outlined in Section 2.

1.3 Relations to other models

For appropriate volumetric free energy $H = H(\epsilon, \theta)$ and $\kappa_1 > 0$, $\kappa_2 = \nu_2 = \nu_1 = 0$, $d = 1$, system (1.1)-(1.2) may represent the one-dimensional Falk model for martensitic transformations in shape memory alloys [11, 12], and in the case $\kappa_1 > 0$, $\nu_1 > 0$, $\kappa_2 = \nu_2 = 0$, $d = 3$, the thermoviscoelastic system of viscosity-capillarity type, studied in [23, 25, 26, 27, 28, 41, 43, 44].

System (1.1)-(1.2) extends that considered in the above mentioned references by admitting temperature-dependent viscosity and capillarity effects, reflected by the terms with coefficients ν_2 and κ_2, respectively. Such effects become of importance in viscoelastic materials with temperature-dependent viscous and elastic moduli, for example materials of Korteweg type (see e.g. [9]) or polymeric materials (see e.g. [4, 8, 29] and the references therein).

Concerning mathematical results on thermoviscoelastic systems with temperature-dependent moduli we mention [29] where 1-D Kelvin-Voigt type system (included in (1.1) for $\kappa_1 = \kappa_2 = 0$, $d = 1$) with nonlinear viscosity has been studied.

To the authors best knowledge system (1.1)-(1.2) has not been so far addressed in the literature. In particular, its global solvability in the case $\kappa_2 \neq 0$ and $\nu_2 \neq 0$ still remains an open question.

1.4 STATIONARY STATES, VARIATIONAL STRUCTURES, AND STABILITY CONCEPTS

The main results of the present paper concern observing a variational struture, called semi-dualities, of system (1.1)-(1.2), and consequently concluding dynamical stability of stationary solutions. To prove this we use the techniques elaborated previously in [33, 35, 34, 36, 37, 38].

On account of the conservative structure (1.8) with no external forces and heat sources, homogeneous boundary conditions $(1.2)_1$, and the local entropy inequality (1.16), system (1.1)-(1.2) satisfies the total energy conservation

$$\frac{\mathrm{d}}{\mathrm{d}t}E = 0, \tag{1.17}$$

with the total energy

$$\begin{aligned}
E = E(u, u_t, \theta) &= \int_\Omega \left(\frac{1}{2}|u_t|^2 + e \right) \mathrm{d}x \\
&= \frac{1}{2}\|u_t\|_{L^2}^2 + \frac{\kappa_1}{2}\|Qu\|_{L^2}^2 + \int_\Omega H(\epsilon(u),\theta) - H_{,\theta}(\epsilon(u),\theta)\theta \mathrm{d}x,
\end{aligned} \tag{1.18}$$

and the entropy inequality in the integral form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \eta \mathrm{d}x = \int_\Omega \sigma \mathrm{d}x \geq 0. \tag{1.19}$$

The latter inequality implies that the functional

$$\begin{aligned}
F = F(u, \theta) &= \int_\Omega (-\eta)\, \mathrm{d}x \\
&= \frac{\kappa_2}{2}\|Qu\|_{L^2}^2 + \int_\Omega H_{,\theta}(\epsilon(u),\theta)\mathrm{d}x
\end{aligned} \tag{1.20}$$

serves as a Lyapunov function for system (1.1)-(1.2), that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}F \leq 0 \tag{1.21}$$

for all sufficiently regular solutions.

The stationary state of (1.1)-(1.2) is detected by putting $u_t = 0$ and $\theta_t = 0$. Then it follows that the stationary temperature is a constant, denoted by $\bar{\theta}$, satisfying

$$b = \frac{\kappa_1}{2}\|Qu\|_{L^2}^2 + \int_\Omega H(\epsilon(u),\bar{\theta}) - H_{,\theta}(\epsilon(u),\bar{\theta})\bar{\theta}\mathrm{d}x, \qquad \bar{\theta} > 0, \tag{1.22}$$

where $u = u(x)$ stands for the stationary displacement, and

$$b = E(u_0, u_1, \theta_0) \tag{1.23}$$

is the total energy determined by the initial data. The stationary displacement u is a solution to the problem

$$\begin{cases} (\kappa_1 + \kappa_2 \overline{\theta}) \, Q^2 u = \nabla \cdot H_{,\epsilon}(\epsilon(u), \overline{\theta}) & \text{in } \Omega, \\ u = Qu = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.24}$$

The stationary state for system (1.1)-(1.2) is thus determined by (1.22) and (1.24) along with the positivity of temperature $\overline{\theta}$, namely

$$\begin{cases} (\kappa_1 + \kappa_2 \overline{\theta}) \, Q^2 u = \nabla \cdot H_{,\epsilon}(\epsilon(u), \overline{\theta}) & \text{in } \Omega, \\ u = Qu = 0 & \text{on } \partial\Omega, \\ b = \dfrac{\kappa_1}{2} \|Qu\|_{L^2}^2 + \displaystyle\int_\Omega H(\epsilon(u), \overline{\theta}) - H_{,\theta}(\epsilon(u), \overline{\theta}) \overline{\theta} \mathrm{d}x, & \overline{\theta} > 0, \end{cases} \tag{1.25}$$

where $(u, \overline{\theta}) \in H_0^1 \cap H^2(\Omega) \times \mathbb{R}_+$ is the stationary solution and $b = E(u_0, u_1, \theta_0)$ the total energy conserved in system (1.1)-(1.2).

We show that there are two variational structures characterized by $\overline{\theta}$ and b, respectively. The first one follows by noting $(1.25)_1$ is the Euler-Lagrange equation to the functional

$$J_{\overline{\theta}}(u) = \frac{\kappa_1 + \kappa_2 \overline{\theta}}{2} \|Qu\|_{L^2}^2 + \int_\Omega H(\epsilon(u), \overline{\theta}) \mathrm{d}x \tag{1.26}$$

defined for $u \in H_0^1 \cap H^2(\Omega)$. Thus we obtain the $\overline{\theta}$-formulation

$$\begin{cases} \delta J_{\overline{\theta}}(u) = 0, \quad \overline{\theta} > 0, \\ b = \dfrac{\kappa_1}{2} \|Qu\|_{L^2}^2 + \displaystyle\int_\Omega H(\epsilon(u), \overline{\theta}) - H_{,\theta}(\epsilon(u), \overline{\theta}) \overline{\theta} \mathrm{d}x \end{cases} \tag{1.27}$$

of the stationary problem (1.25), where the first variation $\delta J_{\overline{\theta}}(u)$ of $J_{\overline{\theta}}(u)$ is defined by

$$\left(\delta J_{\overline{\theta}}(u), w \right)_{L^2} = \frac{\mathrm{d}}{\mathrm{d}s} J_{\overline{\theta}}(u + sw) \bigg|_{s=0}, \quad w \in H_0^1 \cap H^2(\Omega).$$

On the basis of this variational structure, several stabilities of critical points of $J_{\overline{\theta}}$ can be defined as follows.

Definition 1 ($\overline{\theta}$-stabilities) (i) A critical point $\overline{u} \in H_0^1 \cap H^2(\Omega)$ of $J_{\overline{\theta}}$ is said to be $\overline{\theta}$-*linearized stable* if the quadratic form

$$Q_{\overline{\theta}, \overline{u}}(w, w) = \frac{\mathrm{d}^2}{\mathrm{d}s^2} J_{\overline{\theta}}(\overline{u} + sw) \bigg|_{s=0}$$

is positive definite, i.e., $Q_{\overline{\theta}, \overline{u}}(w, w) > 0$ for any $w \in H_0^1 \cap H^2(\Omega) \setminus \{0\}$.

(ii) A critical point $\overline{u} \in H_0^1 \cap H^2(\Omega)$ of $J_{\overline{\theta}}$ is said to be $\overline{\theta}$-infinitesimally stable if there is $\varepsilon_0 > 0$ such that any $\varepsilon_1 \in (0, \varepsilon_0/4]$ admits $\delta_0 > 0$ such that $\|\nabla(u - \overline{u})\|_{H^1} < \varepsilon_0$ and $J_{\overline{\theta}}(u) - J_{\overline{\theta}}(\overline{u}) < \delta_0$ imply $\|\nabla(u - \overline{u})\|_{H^1} < \varepsilon_1$.

(iii) A critical point $\overline{u} \in H_0^1 \cap H^2(\Omega)$ of $J_{\overline{\theta}}$ is said to be $\overline{\theta}$-locally minimal if it is a local minimizer of $J_{\overline{\theta}}$ on $H_0^1 \cap H^2(\Omega)$, i.e., there is $\overline{\varepsilon}_0 > 0$ such that $\|\nabla(u - \overline{u})\|_{H^1} \leq \overline{\varepsilon}_0$ implies $J_{\overline{\theta}}(u) - J_{\overline{\theta}}(\overline{u}) \geq 0$.

We note that the linearized stability is the strongest stability while the local minimality is the weakest stability.

The second variational structure, characterized by b, follows from the fact that the stationary temperature $\overline{\theta} > 0$ can be uniquely determined by b and u in a nonempty open set V_b in $H_0^1 \cap H^2(\Omega)$ through the energy conservation $b = E(u, 0, \overline{\theta})$. To see this we recall the assumption (1.3) and the form (1.4) of $H = H(\epsilon, \theta)$. Then the third equation in (1.25), which is the balance of energy, can be expressed equivalently as

$$e_*(\overline{\theta}) := f_*(\overline{\theta}) - \overline{\theta} f_*'(\overline{\theta}) = \frac{1}{|\Omega|} \left(b - \frac{\kappa_1}{2} \|Qu\|_{L^2}^2 - \int_\Omega W_1(\epsilon(u)) \mathrm{d}x \right). \quad (1.28)$$

where e_* is the thermal part of the internal energy. By assumption (1.3) f_* is strictly concave on $\mathbb{R}_+$, and since

$$e_*' = (f_* - \theta f_*')' = -\theta f_*'' > 0 \quad \text{for } \theta > 0,$$

e_* is a strictly monotone increasing function on $\mathbb{R}_+$. Thus a unique inverse function $e_*^{-1} \in C^1((s^-, s^+), \mathbb{R}_+)$ can be defined, where $(s^-, s^+) = e_*(\mathbb{R}_+)$, and equation (1.28) is rewritten as

$$\overline{\theta} = e_*^{-1} \left(\frac{b - \mathcal{I}_1(u)}{|\Omega|} \right) =: \Theta(b, u), \quad (1.29)$$

$$\mathcal{I}_1(u) = \frac{\kappa_1}{2} \|Qu\|_{L^2}^2 + \int_\Omega W_1(\epsilon(u)) \mathrm{d}x.$$

Then the stationary problem (1.25) is reformulated as the nonlinear eigenvalue problem with non-local terms

$$\begin{cases} (\kappa_1 + \kappa_2 \Theta(b, u)) Q^2 u = \nabla \cdot H_{,\epsilon}(\epsilon(u), \Theta(b, u)) & \text{in } \Omega, \\ u = Qu = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.30)$$

Let

$$\mathcal{I}_2(u) = \frac{\kappa_2}{2} \|Qu\|_{L^2}^2 + \int_\Omega W_2(\epsilon(u)) \mathrm{d}x.$$

Then it follows from the first equation of (1.25) that

$$\delta \mathcal{I}_1(u) = -\overline{\theta} \delta \mathcal{I}_2(u). \quad (1.31)$$

Further, let

$$\Phi(s) = \int_{\bar{s}}^{s} \frac{\mathrm{d}\tau}{e_*^{-1}(\tau)} \tag{1.32}$$

for a constant $\bar{s} \in (s^-, s^+)$. Then, by (1.29), we see that $\Phi \in C^2((s^-, s^+), \mathbb{R})$ is strictly monotone increasing. Moreover, we note that (1.31) is the Euler-Lagrange equation to the C^2-functional

$$\mathcal{J}_b(u) = -\Phi\left(\frac{b - \mathcal{I}_1(u)}{|\Omega|}\right) + \frac{1}{|\Omega|}\mathcal{I}_2(u). \tag{1.33}$$

This functional is defined for $u \in V_b$, where

$$V_b = \left\{ u \in H_0^1 \cap H^2(\Omega) \;\middle|\; \frac{b - \mathcal{I}_1(u)}{|\Omega|} \in (s^-, s^+) \right\} \tag{1.34}$$

is a non-empty open set in $H_0^1 \cap H^2(\Omega)$. Thus we obtain the b-formulation

$$\delta \mathcal{J}_b(u) = 0 \tag{1.35}$$

of the stationary problem (1.25), where $b = E(u_0, u_1, \theta_0)$ and $\theta_0 > 0$.

Similarly to $\bar{\theta}$-stabilities, b-stabilities of critical points of $\mathcal{J}_b$ can be defined.

Definition 2 (b-stabilities) (i) A critical point $\overline{u} \in V_b$ of $\mathcal{J}_b$ is said to be *b-linearized stable* if the quadratic form

$$\mathcal{Q}_{b,\overline{u}}(w, w) = \frac{\mathrm{d}^2}{\mathrm{d}s^2} \mathcal{J}_b(\overline{u} + sw)\bigg|_{s=0}$$

is positive definite, i.e., $\mathcal{Q}_{b,\overline{u}}(w, w) > 0$ for any $w \in H_0^1 \cap H^2(\Omega) \setminus \{0\}$.

(ii) A critical point $\overline{u} \in V_b$ of $\mathcal{J}_b$ is said to be *b-infinitesimally stable* if there is $\varepsilon_0 > 0$ such that any $\varepsilon_1 \in (0, \varepsilon_0/4]$ admits $\delta_0 > 0$ such that $\|\nabla(u - \overline{u})\|_{H^1} < \varepsilon_0$ and $\mathcal{J}_b(u) - \mathcal{J}_b(\overline{u}) < \delta_0$ imply $\|\nabla(u - \overline{u})\|_{H^1} < \varepsilon_1$.

(iii) A critical point $\overline{u} \in V_b$ of $\mathcal{J}_b$ is said to be *b-locally minimal* if it is a local minimizer of $\mathcal{J}_b$ on V_b, i.e., there is $\bar{\varepsilon}_0 > 0$ such that $\|\nabla(u - \overline{u})\|_{H^1} \leq \bar{\varepsilon}_0$ implies $\mathcal{J}_b(u) - \mathcal{J}_b(\overline{u}) \geq 0$.

1.5 SEMI-DUALITY

Assumption (1.3) implies the inequality

$$H(\epsilon, \theta) - H(\epsilon, \bar{\theta}) \geq H_{,\theta}(\epsilon, \theta)(\theta - \bar{\theta}).$$

Therefore, for any nonstationary state (u, θ) we have

$$
\begin{aligned}
b &= \frac{1}{2}\|u_t\|_{L^2}^2 + \frac{\kappa_1}{2}\|Qu\|_{L^2}^2 + \int_\Omega H(\epsilon(u), \theta) - \theta H_{,\theta}(\epsilon(u), \theta) \mathrm{d}x \\
&\geq \frac{\kappa_1}{2}\|Qu\|_{L^2}^2 + \int_\Omega H(\epsilon(u), \theta) - \theta H_{,\theta}(\epsilon(u), \theta) \mathrm{d}x \\
&\geq \frac{\kappa_1}{2}\|Qu\|_{L^2}^2 + \int_\Omega H(\epsilon(u), \overline{\theta}) - \overline{\theta} H_{,\theta}(\epsilon(u), \theta) \mathrm{d}x \\
&= \frac{\kappa_1 + \kappa_2 \overline{\theta}}{2}\|Qu\|_{L^2}^2 + \int_\Omega H(\epsilon(u), \overline{\theta}) \mathrm{d}x - \frac{\kappa_2 \overline{\theta}}{2}\|Qu\|_{L^2}^2 - \int_\Omega \overline{\theta} H_{,\theta}(\epsilon(u), \theta) \mathrm{d}x \\
&= J_{\overline{\theta}}(u) - \overline{\theta} F(u, \theta).
\end{aligned}
$$

This shows the relation

$$
\overline{\theta} F(u, \theta) + b \geq J_{\overline{\theta}}(u) \tag{1.36}
$$

between the Lyapunov functional F and the variational functional $J_{\overline{\theta}}$, called the *semi-unfolding-minimality*. A system is said to have the property of *semi-dual variation* (or simply, *semi-duality*) if its Lyapunov functional satisfies the semi-unfolding-minimality relation [33].

In a similar manner one can show the semi-duality property for the variational functional $\mathcal{J}_b$.

The semi-duality properties play the key role in the proof of the results on dynamical stability of stationary solutions to system (1.1)-(1.2). The results are precisely formulated in Section 3. Here we describe them in short.

1.6   Summary of results

By virtue of the semi-duality structure we prove that the dynamical stability of stationary solutions may be derived both from the $\overline{\theta}$-infinitesimal stability and from the b-infinitesimal stability. Next, we show that any $\overline{\theta}$-stable critical point is b-stable. This fact can be regarded as the stabilization of all stationary solutions by the non-local terms. Concerning the existence of stationary solutions, we obtain a global minimizer of the functional $\mathcal{J}_b$ by a standard variational method. Then our interest turns to its stability. The infinitesimal stability does not hold in general, but we are able to show the infinitesimal stability of any local minimizer provided that H is real analytic with respect to ϵ. Table 1 shows the established interrelations between the stabilities. They extend the results of [34] to the system (1.1). Concluding, we can claim that there is a dynamically stable stationary solution if H is real analytic with respect to ϵ. In particular, when the total energy b is so small that the trivial solution $u = 0$ is unstable, there exists a dynamically stable nontrivial stationary solution.

The application of the established results together with the bifurcation theory is also presented in Section 5, where the total set of the stationary solutions is analyzed in more detail. These results are stated in seven theorems which in an early version have been firstly established in [38]. We regard the total energy b (resp. the absolute temperature $\overline{\theta}$) as the bifurcation parameter, and
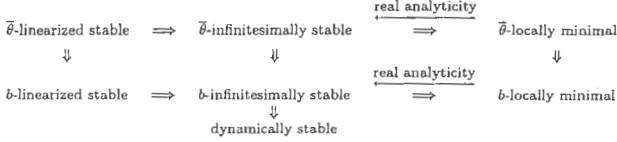
		real analyticity	
$\bar{\theta}$-linearized stable $\implies$	$\bar{\theta}$-infinitesimally stable	$\Longleftarrow$ $\implies$	$\bar{\theta}$-locally minimal
$\Downarrow$	$\Downarrow$	real analyticity	$\Downarrow$
b-linearized stable $\implies$	b-infinitesimally stable	$\Longleftarrow$ $\implies$	b-locally minimal
	$\Downarrow$		
	dynamically stable		

Table 1 Relations of stabilities

consider the total set of the stationary solutions (b, u) (resp. $(\bar{\theta}, u)$). We prove the upper bound $\bar{\theta}^* < +\infty$ of the temperature $\bar{\theta}$ for the existence of the non-trivial solution and the a priori estimate $\|u\|_{H^2} \le C(\bar{\theta}_*)$ for the solution $(\bar{\theta}, u)$ satisfying $\bar{\theta} \ge \bar{\theta}_*$. Moreover, in the one-dimensional case, $d = 1$ and $\Omega = (0, l)$ with $l > 0$, we show the bifurcation points from the trivial branch, the super- and sub-critical conditions, and the nonexistence of secondary bifurcation. We also prove that any local minimizer of the functional $\mathcal{J}_b$ has a definite sign. Then we can describe the bifurcation diagram and conclude that the total set of the stationary solutions is composed of the trivial branch and the nontrivial branches which may intersect only the trivial branch. In the sub-critical case, the bifurcated branch has a turning point in the $(\bar{\theta}, u)$-space. Using the estab-lished interrelations between the stabilities, we can observe that the bifurcated branch has a turning point also in the (b, u)-space. Consequently, the existence of the hetero-clinic orbits and the hysteretic cycle may be suggested by the bifurcation diagram.

1.7 PLAN OF THE PAPER

In Section 2 we outline the thermodynamic derivation of constitutive relations (1.13) and their specialized forms (1.15). In Section 3 we formulate five theorems on the stabilities which are the main results of the paper. The proofs of these theorems are presented in Section 4. In Section 5 the total set of the stationary solutions is analyzed in more detail by using the main results together with the bifurcation theory.

2 OUTLINE OF THE CONSTITUTIVE THEORY

2.1 DERIVATION

We recall from [22] the main steps of the derivation of constitutive relations for strain-gradient thermoviscoelastic systems. These relations include (1.13) and (1.15) as the particular cases. The derivation is based on exploiting the second law of thermodynamics in the form of the entropy inequality with multipliers,

known as the Müller-Liu entropy inequality [20, 19].

Assume that $\Omega \in \mathbb{R}^3$ is a bounded domain with a smooth boundary, occupied by a solid body in a fixed reference configuration, with constant mass density normalized to unity, $\rho_0 = 1$. The procedure consists of three main steps.

In the first step we consider the usual local forms of balance laws of linear momentum and internal energy, with body forces b and heat sources g:

$$\begin{aligned} u_{tt} - \nabla \cdot S &= b, \\ \tilde{e}_t + \nabla \cdot q - S \cdot \epsilon_t &= g, \end{aligned} \tag{2.1}$$

and the constitutive equations for the stress tensor S, the internal energy $\tilde{e}$, and the energy flux q:

$$S = \hat{S}(Y), \quad \tilde{e} = \hat{\tilde{e}}(Y), \quad q = \hat{q}(Y), \tag{2.2}$$

where

$$Y = \left(\epsilon, D\epsilon, \cdots, D^{M_0}\epsilon, \eta, D\eta, \cdots, D^{K_0}\eta, \epsilon_t\right), \quad M_0 \geq 2, \quad K_0 \geq 1,$$

denotes the set of constitutive variables. This set accounts for long range inter-actions in the material by the presence of higher strain gradients, and for the material viscosity by the strain rate. Thermal variables are expressed in Y by the entropy η and its gradients. Such choice of thermal variables is convenient for the exploitation of the entropy inequality. The notation $\tilde{e}$ instead of e is used to indicate that the internal energy is considered as a function of the entropy.

In the second step of the procedure we postulate the Müller-Liu entropy inequality. Applied to system (2.1) with constitutive equations (2.2) it asserts the existence of the entropy flux $\Psi = \hat{\Psi}(Y)$ and the multiplier $\lambda = \hat{\lambda}(Y)$ such that the inequality

$$\eta_t + \nabla \cdot \Psi - \lambda \left(\tilde{e}_t + \nabla \cdot q - S \cdot \epsilon_t\right) \geq 0 \tag{2.3}$$

is satisfied for all fields u and η. The following has been proved in [22].

Theorem A (Consequences of the entropy inequality) *Assume the structure conditions (A1)-(A3):*

(A1)

$$\tilde{e}_{,\eta}(Y) > 0 \quad \text{for all } Y,$$

(A2)

$$\Psi^0 = \lambda^0 q^0,$$

where

$$q^0 := \hat{q}\left(Y^0\right) \quad \text{with} \quad Y^0 := Y|_{\epsilon_t = 0},$$

denotes the stationary energy flux, and λ^0, Ψ^0 are defined similarly;

(A3)

$$q = q^0 - \epsilon_t H \quad \left(q_k = q_k^0 - (\epsilon_t)_{ij} H_{ijk} \right),$$ (2.4)

where $H = (H_{ijk}) = \hat{H}(Y)$ is a third order tensor.

Then the following relations hold:

(i) internal energy

$$\tilde{e} = \hat{\tilde{e}}\left(\epsilon, D\epsilon, \eta\right);$$

(ii) energy multiplier

$$\lambda = \hat{\lambda}\left(\epsilon, D\epsilon, \eta\right) = \frac{1}{\tilde{e}_{,\eta}} > 0;$$

(iii) entropy flux

$$\Psi = \lambda q^0 + \lambda \epsilon_t \left(\tilde{e}_{,D\epsilon} - H\right)$$
$$\left(\Psi_k = \lambda q_k^0 + \lambda(\epsilon_t)_{ij} \left(\tilde{e}_{\epsilon_{ij,k}} - H_{ijk}\right)\right);$$ (2.5)

(iv) stress tensor

$$S = \frac{\delta\tilde{e}}{\delta\epsilon} - \frac{1}{\lambda}\left(\tilde{e}_{,D\epsilon} - H\right)\nabla\lambda + S^{\mathrm{d}}$$
$$\left(S_{ij} = \tilde{e}_{,\epsilon_{ij}} - \partial_k \tilde{e}_{,\epsilon_{ij,k}} - \frac{1}{\lambda}\left(\tilde{e}_{ij,k} - H_{ijk}\right)\lambda_{,k} + S_{ij}^{\mathrm{d}}\right),$$ (2.6)

where

$$\frac{\delta\tilde{e}}{\delta\epsilon} = \tilde{e}_{,\epsilon} - \nabla \cdot \tilde{e}_{,D\epsilon};$$

(v) residual dissipation inequality

$$\epsilon_t \cdot \left(\lambda S^{\mathrm{d}}\right) + \nabla\lambda \cdot q^0 \geq 0$$ (2.7)

for all fields u and η.

We complement the above theorem by the following remarks:

- Assumption (A1) is a nondegeneracy condition for the potential $\tilde{e}$. Under thermal stability postulate the Legendre (duality) transformations show that (A1) means the positivity of temperature.

- The energy and entropy fluxes include extra (unconventional) terms involving an arbitrary third order tensor H. Fluxes Ψ and q are related by the equation
$$\Psi - \lambda q = \lambda \epsilon_t \tilde{e}_{,D\epsilon}.$$ (2.8)

- Entropy inequality does not impose any restrictions on the tensor H.

In the third step of the procedure we presuppose that the energy multiplier λ is an additional independent variable and treat the equation in assertion (ii) of Theorem A as an additional constraint. Then, setting

$$\theta = \frac{1}{\lambda}, \tag{2.9}$$

we have

$$\theta = \bar{e}_{,\eta}(\epsilon, D\epsilon, \eta) > 0. \tag{2.10}$$

With λ as independent variable the residual inequality (2.7) takes on the standard form of the dissipation inequality. Then the application of Edelen's decomposition theorem [10] implies the following thermodynamic relations

$$\frac{S^{\mathrm{d}}}{\theta} = \mathcal{D}_{,\epsilon_t}, \quad q^0 \equiv q^{\mathrm{d}} = \mathcal{D}_{,D(1/\theta)}, \tag{2.11}$$

where $\mathcal{D}$ is a dissipation potential which is convex, nonnegative, and homogeneous of certain degree in ϵ_t and $D(1/\theta)$.

Under thermal stability postulate that the specific heat is strictly positive

$$c_0 = e_{,\theta} = -\theta f_{,\theta\theta} > 0, \tag{2.12}$$

the Legendre transformation is applicable. Then $\tilde{e}$ is defined as the conjugate convex function of free energy f with entropy η as a dual variable:

$$\hat{\tilde{e}}(\epsilon, D\epsilon, \eta) := \sup_{0 < \bar{\theta} < \infty} \left(\bar{\theta}\eta + \hat{f}\left(\epsilon, D\epsilon, \bar{\theta}\right)\right) \leq +\infty. \tag{2.13}$$

At $\bar{\theta} = \theta = \hat{\theta}(\epsilon, D\epsilon, \eta)$ where the supremum is attained,

$$\begin{aligned}\hat{\tilde{e}}(\epsilon, D\epsilon, \eta) - \hat{f}(\epsilon, D\epsilon, \theta) &= \theta\eta, \\ \tilde{e}_{,\eta}(\epsilon, D\epsilon, \eta) &= \theta.\end{aligned} \tag{2.14}$$

Hence, in particular,

$$\tilde{e}_{,\epsilon} = f_{,\epsilon}, \quad \tilde{e}_{,D\epsilon} = f_{,D\epsilon}. \tag{2.15}$$

Due to the convexity, $\tilde{e}_{,\eta\eta} > 0$, the relation between η and θ defines a transformation. Therefore, one can use alternatively (u, η) or (u, θ) as independent variables.

Summing up, for (u, η)-variables the governing potential is the internal energy $\tilde{e} = \tilde{e}(\epsilon, D\epsilon, \eta)$ which is strictly convex in η and such that $\tilde{e}_{,\eta} > 0$. The equations are the balance laws (2.1), where q is given by (2.4), S by (2.6), S^{d} and $q^0 = q^{\mathrm{d}}$ by (2.11), as well as θ is determined by (2.10).

For (u, θ)-variables the potential is the free energy $f(\epsilon, D\epsilon, \theta)$ which is strictly concave with respect to $\theta > 0$, related to $\tilde{e}(\epsilon, D\epsilon, \eta)$ by (2.13). Due to (2.15), equation (2.6) transforms to

$$S = \frac{\delta f}{\delta \epsilon} - \theta\left(f_{,D\epsilon} - H\right)\nabla\frac{1}{\theta} + S^{\mathrm{d}}. \tag{2.16}$$

Thus, in terms of (u, θ)-variables the model equations are the balance laws (2.1) with q given by (2.4), S by (2.16), S^d and $q^0 = q^d$ by (2.11). In accord with the duality relations, internal energy e, entropy η, and free energy f are related by

$$f = e - \theta\eta, \quad \eta = -f_{,\theta}. \tag{2.17}$$

2.2 Physically realistic equations

The presented above constitutive equations involve an extra, unspecified tensor field H. From (2.8) and (2.4), using (2.9) and (2.15), it follows that

$$\Psi = \frac{q^d}{\theta} + \frac{1}{\theta}\epsilon_t \left(f_{,D\epsilon} - H\right) \equiv \frac{q^d}{\theta} + \epsilon_t H^\eta. \tag{2.18}$$

Thus, the extra energy flux, $-\epsilon_t H$, and the extra entropy flux, $\epsilon_t H^\eta$, are linked by the equality

$$\epsilon_t \left(H + \theta H^\eta\right) = \epsilon_t f_{,D\epsilon}. \tag{2.19}$$

In view of the thermodynamic relations (2.17), the equality (2.19) suggests a physically realistic choice of the extra tensors H and H^η:

$$\begin{aligned}
H &= e_{,D\epsilon}\left(\epsilon, D\epsilon, \theta\right) = f_{,D\epsilon}\left(\epsilon, D\epsilon, \theta\right) - \theta f_{,\theta D\epsilon}\left(\epsilon, D\epsilon, \theta\right), \\
H^\eta &= -\eta_{,D\epsilon}\left(\epsilon, D\epsilon, \theta\right) = f_{,\theta D\epsilon}\left(\epsilon, D\epsilon, \theta\right).
\end{aligned} \tag{2.20}$$

For H defined by $(2.20)_1$ the equation (2.16) for S and (2.4) for q become:

$$\begin{aligned}
S &= \frac{\delta f}{\delta \epsilon} - \theta(f_{,D\epsilon} - e_{,D\epsilon})\nabla\left(\frac{1}{\theta}\right) + S^d \\
&= f_{,\epsilon} - \nabla \cdot f_{,D\epsilon} - \eta_{,D\epsilon}\nabla\theta + S^d \\
&= f_{,\epsilon} - \nabla \cdot f_{,D\epsilon} + f_{,\theta D\epsilon}\nabla\theta + S^d \\
&= e_{,\epsilon} - \theta\eta_{,\epsilon} - \nabla \cdot e_{,D\epsilon} + \nabla \cdot (\theta\eta_{,D\epsilon}) - \eta_{,D\epsilon}\nabla\theta + S^d \\
&= \frac{\delta e}{\delta \epsilon} - \theta\frac{\delta\eta}{\delta\epsilon} + S^d
\end{aligned} \tag{2.21}$$

and

$$q = q^d - \epsilon_t e_{,D\epsilon} \tag{2.22}$$

where S^d, q^d are defined by (2.11).

In result, inserting $(2.21)_5$ and (2.22) into (2.1) we arrive at the system

$$\begin{aligned}
&u_{tt} - \nabla \cdot \left(\frac{\delta e}{\delta\epsilon} - \theta\frac{\delta\eta}{\delta\epsilon} + S^d\right) = b, \\
&e_t + \nabla \cdot \left(q^d - \epsilon_t e_{,D\epsilon}\right) - \left(\frac{\delta e}{\delta\epsilon} - \theta\frac{\delta\eta}{\delta\epsilon} + S^d\right) \cdot \epsilon_t = g.
\end{aligned} \tag{2.23}$$

We check now directly that system (2.23) complies with the entropy inequality

$$\eta_t + \nabla \cdot \left(\frac{q^d}{\theta} - \epsilon_t\eta_{,D\epsilon}\right) = \sigma + \frac{g}{\theta}, \tag{2.24}$$

where

$$\sigma = \frac{S^d}{\theta} \cdot \epsilon_t + q^d \cdot \nabla \frac{1}{\theta}$$
$$= \mathcal{D}_{,\epsilon_t} \cdot \epsilon_t + \mathcal{D}_{,D(1/\theta)} \cdot \nabla \frac{1}{\theta} \geq 0$$

is the entropy production, nonnegative by the convexity of $\mathcal{D}$.

This follows by noting that since

$$e_t = (f + \theta\eta)_t$$
$$= f_{,\epsilon} \cdot \epsilon_t + f_{,D\epsilon} \cdot \nabla \epsilon_t + f_{,\theta}\theta_t + \theta_t\eta + \theta\eta_t$$
$$= \theta\eta_t + f_{,\epsilon} \cdot \epsilon_t + f_{,D\epsilon} \cdot \nabla \epsilon_t,$$

the left-hand side of equation $(2.23)_2$ can be rearranged as

$$\theta\eta_t + \nabla \cdot \left(q^d - \epsilon_t e_{,D\epsilon}\right) - \left(e_{,\epsilon} - \nabla \cdot e_{,D\epsilon} - \theta\eta_{,\epsilon} + \theta\nabla \cdot \eta_{,D\epsilon} + S^d\right) \cdot \epsilon_t$$
$$+ f_{,\epsilon} \cdot \epsilon_t + f_{,D\epsilon} \cdot \nabla \epsilon_t$$
$$= \theta\eta_t + \nabla \cdot \left(q^d - \epsilon_t e_{,D\epsilon}\right) + (\nabla \cdot e_{,D\epsilon} - \theta\nabla \cdot \eta_{,D\epsilon}) \cdot \epsilon_t + f_{,D\epsilon} \cdot \nabla \epsilon_t - S^d \cdot \epsilon_t$$
$$= \theta\eta_t + \nabla \cdot q^d - e_{,D\epsilon} \cdot \nabla \epsilon_t - \theta\nabla \cdot (\epsilon_t\eta_{,D\epsilon}) + \theta\eta_{,D\epsilon} \cdot \nabla \epsilon_t + f_{,D\epsilon} \cdot \nabla \epsilon_t - S^d \cdot \epsilon_t$$
$$= \theta\eta_t + \nabla \cdot q^d - \theta\nabla \cdot (\epsilon_t\eta_{,D\epsilon}) - S^d \cdot \epsilon_t.$$

Thus, $(2.23)_2$ turns into

$$\theta\eta_t + \nabla \cdot q^d - \theta\nabla \cdot (\epsilon_t\eta_{,D\epsilon}) - S^d \cdot \epsilon_t = g,$$

which is equivalent to (2.24).

Finally, we derive the temperature form of $(2.23)_2$. Taking into account that

$$e_t = (f - \theta f_{,\theta})_{,\epsilon} \cdot \epsilon_t + (f - \theta f_{,\theta})_{,D\epsilon} \cdot \nabla \epsilon_t + c_0\theta_t,$$

where $c_0 = -\theta f_{,\theta\theta}$ is the specific heat, and using (2.17), $(2.21)_3$, the left-hand side of $(2.23)_2$ can be rearranged as follows:

$$e_t + \nabla \cdot \left(q^d - \epsilon_t f_{,D\epsilon} + \theta\epsilon_t f_{,\theta D\epsilon}\right) - \left(f_{,\epsilon} - \nabla \cdot f_{,D\epsilon} + f_{,\theta D\epsilon}\nabla\theta + S^d\right) \cdot \epsilon_t$$
$$= e_t + \nabla \cdot q^d - f_{,\epsilon} \cdot \epsilon_t - f_{,D\epsilon} \cdot \nabla \epsilon_t + \theta(\nabla \cdot f_{,\theta D\epsilon}) \cdot \epsilon_t + \theta f_{,\theta D\epsilon} \cdot \nabla \epsilon_t - S^d \cdot \epsilon_t$$
$$= c_0\theta_t + \nabla \cdot q^d - \theta f_{,\theta\epsilon} \cdot \epsilon_t + \theta(\nabla \cdot f_{,\theta D\epsilon}) \cdot \epsilon_t - S^d \cdot \epsilon_t.$$

Hence system (2.23), expressed in terms of (u, θ)-variables and free energy f, considered in space-time domain $\Omega \times (0, T)$, takes the form:

$$\begin{cases} u_{tt} - \nabla \cdot \left(f_{,\epsilon} - \nabla \cdot f_{,D\epsilon} + f_{,\theta D\epsilon}\nabla\theta + S^d\right) = b, \\ -\theta f_{,\theta\theta}\theta_t + \nabla \cdot q^d = \theta f_{,\theta\epsilon} \cdot \epsilon_t - \theta(\nabla \cdot f_{,\theta D\epsilon}) \cdot \epsilon_t + S^d \cdot \epsilon_t + g \quad \text{in } \Omega \times (0, T), \end{cases}$$
$$(2.25)$$

with S^d and q^d defined by (2.11).

2.3   "ENERGETIC" AND "ENTROPIC" CASES

We point on two extreme cases of strain-gradient free energy $f = f(\epsilon, D\epsilon, \theta)$, referred to as "energetic" and "entropic" ones. In accord with (2.17) such cases are characterized by the following relations:

(i) Energetic case

$$f_{,D\epsilon} = e_{,D\epsilon} \quad \Leftrightarrow \quad \eta_{,D\epsilon} = -f_{,\theta D\epsilon} = 0. \tag{2.26}$$

This means that the strain gradient, $D\epsilon$, contained in f fully contributes to the internal energy e. Such situation is typical for shape memory models (see e.g. [11, 12, 22, 27]). Then system (2.25) reduces to the form

$$\begin{cases} u_{tt} - \nabla \cdot (f_{,\epsilon} - \nabla \cdot f_{,D\epsilon} + S^d) = b, \\ -\theta f_{,\theta\theta}\theta_t + \nabla \cdot q^d = \theta f_{,\theta\epsilon} \cdot \epsilon_t + S^d \cdot \epsilon_t + g \quad \text{in } \Omega \times (0, T). \end{cases} \tag{2.27}$$

(ii) Entropic case

$$e_{,D\epsilon} = 0 \quad \Leftrightarrow \quad f_{,D\epsilon} = -\theta\eta_{,D\epsilon} = \theta f_{,\theta D\epsilon}$$
$$\Leftrightarrow \quad \left(\frac{f}{\theta}\right)_{,\theta D\epsilon} = 0. \tag{2.28}$$

This means that the strain gradient, $D\epsilon$, contained in f fully contributes to the entropy η. This case is characteristic for polymeric materials (see e.g. [8], and the references in [4]). Then system (2.25) turns into the form (see [22, eq. (61)]):

$$\begin{cases} u_{tt} - \nabla \cdot \left(f_{,\epsilon} - \theta\nabla \cdot \dfrac{f_{,D\epsilon}}{\theta} + S^d\right) = b, \\ -\theta f_{,\theta\theta}\theta_t + \nabla \cdot q^d = \theta f_{,\theta\epsilon} \cdot \epsilon_t - \theta\left(\nabla \cdot \dfrac{f_{,D\epsilon}}{\theta}\right) \cdot \epsilon_t + S^d \cdot \epsilon_t + g \\ \hspace{8cm} \text{in } \Omega \times (0, T). \end{cases} \tag{2.29}$$

2.4   SPECIALIZED EQUATIONS

Let us consider system (2.25) with the free energy f and the dissipation potential $\mathcal{D}$, defined by (1.9) and (1.10), respectively. Then, in accord with (2.11),

$$q^d = \mathcal{D}_{,D(1/\theta)} = k\theta^2 \nabla\frac{1}{\theta} = -k\nabla\theta,$$
$$S^d = \theta\mathcal{D}_{,\epsilon_t} = (\nu_1 + \nu_2\theta)B\epsilon_t.$$

Moreover, in such a case,

$$f_{,\theta\theta} = H_{,\theta\theta}, \quad f_{,\theta\epsilon} = H_{,\theta\epsilon}, \quad f_{,\epsilon} = H_{,\epsilon},$$

and, due to symmetry of tensor A,

$$
\begin{aligned}
Qu &= \nabla \cdot A\epsilon(u) = \nabla\epsilon(u)A, \\
f_{,D\epsilon} &= (\kappa_1 + \kappa_2\theta)A(\nabla\epsilon A) = (\kappa_1 + \kappa_2\theta)AQu, \\
\nabla \cdot f_{,D\epsilon} &= (\kappa_1 + \kappa_2\theta)\nabla \cdot (AQu) + \kappa_2\,(AQu)\,\nabla\theta, \\
&= (\kappa_1 + \kappa_2\theta)A\epsilon(Qu) + \kappa_2\,(AQu)\,\nabla\theta, \\
f_{,\theta D\epsilon} &= \kappa_2 AQu, \\
\nabla \cdot f_{,\theta D\epsilon} &= \kappa_2 A\epsilon(Qu).
\end{aligned} \tag{2.30}
$$

This follows from the following componentwise relations:

$$
\begin{aligned}
(Qu)_i &= (\nabla \cdot A\epsilon(u))_i = (\partial_j A_{ijkl}\epsilon_{kl}(u))_i \\
&= (A_{ijkl}\epsilon_{kl,j}(u))_i = (\epsilon_{kl,j}(u)A_{ijkl})_i \\
&= (\nabla\epsilon(u)A)_i\,,
\end{aligned}
$$

$$
\begin{aligned}
(f_{,D\epsilon})_{pqr} &= \left(\left(\frac{\kappa}{2}\,|Qu|^2\right)_{,D\epsilon}\right)_{pqr} \\
&= \left(\left(\frac{\kappa}{2}\,|\nabla\epsilon(u)A|^2\right)_{,D\epsilon}\right)_{pqr} = \left(\left(\frac{\kappa}{2}\,|\epsilon_{kl,j}A_{klji}|^2\right)_{,\epsilon_{pq,r}}\right)_{pqr} \\
&= \kappa\,((\epsilon_{kl,j}A_{klji})\,A_{pqri})_{pqr} = \kappa\,(A_{pqri}\,(Qu)_i)_{pqr} \\
&= \kappa\,(AQu)_{pqr}\,,
\end{aligned}
$$

$$
\begin{aligned}
(\nabla \cdot f_{,D\epsilon})_{ij} &= (\partial_k\,(\kappa A_{ijkl}\,(Qu)_l))_{ij} \\
&= \kappa\,(\partial_k\,(A_{ijkl}\,(Qu)_l))_{ij} + (A_{ijkl}\,(Qu)_l\,\kappa_{,k})_{ij} \\
&= \kappa\,\left(A_{ijkl}\,(Qu)_{l,k}\right)_{ij} + \kappa_2\,(A_{ijkl}\,(Qu)_l\,\theta_{,k})_{ij} \\
&= \kappa\,(A_{ijkl}\epsilon_{lk}\,(Qu))_{ij} + \kappa_2\,(A_{ijkl}\,(Qu)_l\,\theta_{,k})_{ij} \\
&= \kappa\,(A\epsilon\,(Qu))_{ij} + \kappa_2\,((AQu)\,\nabla\theta)_{ij}\,,
\end{aligned}
$$

where $\kappa = \kappa_1 + \kappa_2\theta$. Then system (2.25) turns into

$$
\begin{cases}
u_{tt} - \nabla \cdot \{H_{,\epsilon} - (\kappa_1 + \kappa_2\theta)\,A\epsilon(Qu) + (\nu_1 + \nu_2\theta)\,B\epsilon(u_t)\} = b, \\
-\theta H_{,\theta\theta}\theta_t - k\triangle\theta = \theta H_{,\theta\epsilon} \cdot \epsilon_t - \kappa_2\theta\,(A\epsilon(Qu)) \cdot \epsilon_t \\
\quad + (\nu_1 + \nu_2\theta)B\epsilon_t \cdot \epsilon_t + g \qquad\qquad\qquad \text{in } \Omega \times (0,T),
\end{cases} \tag{2.31}
$$

which for $b = 0$, $g = 0$ provides equations (1.1). In the energetic case $\kappa_1 > 0$, $\kappa_2 = 0$ whereas in the entropic one, $\kappa_1 = 0$, $\kappa_2 > 0$.

2.5 MASTER STRUCTURE

We point out a master structure of system (2.25) (equivalently (2.23)), comprising (1.1) as a special case. This structure is a direct consequence of the presented derivation procedure. In particular, it provides the total energy conservation, natural boundary conditions and the Lyapunov functional.

Formally, assuming that the functions u and $\theta > 0$ are sufficiently regular and multiplying scalarly equation $(2.1)_1$ by u_t, we obtain the balance equation for the kinetic energy

$$\left(\frac{1}{2}|u_t|^2\right)_t - \nabla \cdot (^t S u_t) + S \cdot \epsilon_t = u_t \cdot b \tag{2.32}$$

with S given by (2.21). Summing up (2.32) and the internal energy balance $(2.23)_2$ gives the total energy balance in the local form

$$\left(e + \frac{1}{2}|u_t|^2\right)_t + \nabla \cdot (-^t S u_t + q^{\mathrm{d}} - \epsilon_t e_{,D\epsilon}) = u_t \cdot b + g. \tag{2.33}$$

Integrating (2.33) over Ω leads to

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(e + \frac{1}{2}|u_t|^2\right) \mathrm{d}x + \int_{\partial\Omega} \left\{-(Sn) \cdot u_t + n \cdot (q^{\mathrm{d}} - \epsilon_t e_{,D\epsilon})\right\} \mathrm{d}S$$
$$= \int_\Omega u_t \cdot b + g \mathrm{d}x,$$

where n denotes the outer unit normal vector on $\partial\Omega$. Hence it follows that if $b = 0$, $g = 0$, and the boundary conditions on $\partial\Omega$ imply that

$$(Sn) \cdot u_t = 0, \quad n \cdot (q^{\mathrm{d}} - \epsilon_t e_{,D\epsilon}) = 0, \tag{2.34}$$

then solutions of system (2.23) satisfy the total energy conservation

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 0 \tag{2.35}$$

with

$$E = \int_\Omega e + \frac{1}{2}|u_t|^2 \, \mathrm{d}x.$$

Further, multiplying the entropy equation (2.24) by a constant $\alpha > 0$ and subtracting from (2.33), we get the so-called availability identity and the corresponding inequality

$$\left(e + \frac{1}{2}|u_t|^2 - \alpha\eta\right)_t + \nabla \cdot \left\{-^t S u_t + (q^{\mathrm{d}} - \epsilon_t e_{,D\epsilon}) - \alpha\left(\frac{q^{\mathrm{d}}}{\theta} - \epsilon_t \eta_{,D\epsilon}\right)\right\}$$
$$= -\alpha\sigma + u_t \cdot b + \left(1 - \frac{\alpha}{\theta}\right) g \le u_t \cdot b + \left(1 - \frac{\alpha}{\theta}\right) g. \tag{2.36}$$

Integrating (2.36) over Ω gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(e + \frac{1}{2}|u_t|^2 - \alpha\eta\right) \mathrm{d}x$$
$$+ \int_{\partial\Omega} \left\{-(Sn) \cdot u_t + n \cdot (q^{\mathrm{d}} - \epsilon_t e_{,D\epsilon}) - \alpha n \cdot \left(\frac{q^{\mathrm{d}}}{\theta} - \epsilon_t \eta_{,D\epsilon}\right)\right\} \mathrm{d}S$$
$$\le \int_\Omega u_t \cdot b + \left(1 - \frac{\alpha}{\theta}\right) g \mathrm{d}x.$$

Hence, it follows that if $b = 0$, $g = 0$, and the boundary conditions on $\partial\Omega$ imply in addition to (2.34) that

$$n \cdot \left(\frac{q^{\mathrm{d}}}{\theta} - \epsilon_t \eta_{,D\epsilon} \right) = 0, \tag{2.37}$$

then solutions of system (2.23) satisfy the Lyapunov inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega e + \frac{1}{2} |u_t|^2 - \alpha \eta \mathrm{d}x \leq 0. \tag{2.38}$$

Setting $\alpha = 1$ we can see with the aid of the total energy conservation (2.35) that

$$F = \int_\Omega (-\eta) \, \mathrm{d}x = \int_\Omega f_{,\theta} \mathrm{d}x \tag{2.39}$$

serves as a particular Lyapunov function satisfying

$$\frac{\mathrm{d}F}{\mathrm{d}t} \leq 0.$$

3   MAIN RESULTS

3.1   THEOREMS ON STABILITIES

The first result of this paper is on the dynamical stability of $\overline{\theta}$-infinitesimally stable stationary solutions of system (1.1)-(1.2).

Theorem 1 *Assume that $\theta \mapsto H(\cdot, \theta)$ is concave. Let $\overline{\theta} > 0$ be a constant and $\overline{u} \in H_0^1 \cap H^2(\Omega)$ an infinitesimally stable critical point of $J_{\overline{\theta}}$ such that $E(\overline{u}, 0, \overline{\theta}) = b = E(u_0, u_1, \theta_0)$. Then $(\overline{u}, \overline{\theta})$ is dynamically stable in the sense that any $\varepsilon > 0$ admits $\delta > 0$ such that*

$$\|\nabla(u_0 - \overline{u})\|_{H^1} < \delta, \quad \left| F(u_0, \theta_0) - F(\overline{u}, \overline{\theta}) \right| < \delta$$

imply $\displaystyle \sup_{t \geq 0} \|\nabla(u(\cdot, t) - \overline{u})\|_{H^1} < \varepsilon, \quad \sup_{t \geq 0} \left| F(u(\cdot, t), \theta(\cdot, t)) - F(\overline{u}, \overline{\theta}) \right| < \varepsilon,$

$$\tag{3.1}$$

where $(u, \theta) = (u(\cdot, t), \theta(\cdot, t))$ is a solution to system (1.1)-(1.2), satisfying $u \in C([0, +\infty), H^2(\Omega))$ and $\theta > 0$.

In the case of system (1.1)-(1.2) with $\kappa_2 = \nu_2 = 0$ this result has been proved in [34].

Assume that $H = H(\epsilon, \theta)$ takes the form (1.4)-(1.5) with functions $W_i(\epsilon)$ satisfying the growth rates

$$|W_{i,\epsilon}(\epsilon)| \leq C|\epsilon|^{K_i - 1} \tag{3.2}$$

for large $|\epsilon|$, where $i = 1, 2$,

$$0 \leq K_i < 6 \quad \text{if } d = 3, \qquad 0 \leq K_i < \infty \quad \text{if } d = 2.$$

Then Theorem 1 implies the following specialized form of the dynamical stability.

Corollary 1 *In addition to the assumption of Theorem 1, suppose that $H = H(\epsilon, \theta)$ takes the form (1.4)-(1.5) with (3.2). Then $(\overline{u}, \overline{\theta})$ is dynamically stable in the sense that any $\varepsilon > 0$ admits $\delta > 0$ such that*

$$\|\nabla(u_0 - \overline{u})\|_{H^1} < \delta, \quad \left| \frac{1}{|\Omega|} \int_{\Omega} f'_*(\theta_0) \mathrm{d}x - f'_*(\overline{\theta}) \right| < \delta$$

imply $\quad \sup_{t \geq 0} \|\nabla(u(\cdot, t) - \overline{u})\|_{H^1} < \varepsilon, \quad \sup_{t \geq 0} \left| \frac{1}{|\Omega|} \int_{\Omega} f'_*(\theta(\cdot, t)) \mathrm{d}x - f'_*(\overline{\theta}) \right| < \varepsilon,$

$$(3.3)$$

where $(u, \theta) = (u(\cdot, t), \theta(\cdot, t))$ is a solution to system (1.1)-(1.2), satisfying $u \in C([0, +\infty), H^2(\Omega))$ and $\theta > 0$.

The second result concerns the dynamical stability of b-infinitesimally stable stationary solutions.

Theorem 2 *Assume that $H = H(\epsilon, \theta)$ takes the form (1.4)-(1.5), $\theta \mapsto H(\cdot, \theta)$ is strictly concave, and (3.2) holds. Let $b = E(u_0, u_1, \theta_0)$, $\overline{u} \in V_b$ be an infinitesimally stable critical point of $\mathcal{J}_b$, and $\overline{\theta} = \Theta(b, \overline{u})$. Then $(\overline{u}, \overline{\theta})$ is dynamically stable in the sense that any $\varepsilon > 0$ admits $\delta > 0$ such that the implication (3.1) holds, where $(u, \theta) = (u(\cdot, t), \theta(\cdot, t))$ is a solution to system (1.1)-(1.2), satisfying $u \in C([0, +\infty), H^2(\Omega))$ and $\theta > 0$.*

Corollary 2 *Under the assumption of Theorem 2, $(\overline{u}, \overline{\theta})$ is dynamically stable in the sense that any $\varepsilon > 0$ admits $\delta > 0$ such that the implication (3.3) holds, where $(u, \theta) = (u(\cdot, t), \theta(\cdot, t))$ is a solution to system (1.1)-(1.2), satisfying $u \in C([0, +\infty), H^2(\Omega))$ and $\theta > 0$.*

Given a stationary solution (b, u), one can conclude the dynamical stabilities both by Theorem 1 and by Theorem 2. These dynamical stabilities are the same, and so there may arise a question concerning the relation between $\overline{\theta}$- and b-stabilities. The following theorem asserts that $\overline{\theta}$-stabilities are stronger than b-stabilities.

Theorem 3 *Assume that $H = H(\epsilon, \theta)$ takes the form (1.4)-(1.5) and $\theta \mapsto H(\cdot, \theta)$ is strictly concave. Let $b = E(u_0, u_1, \theta_0)$, $\theta_0 > 0$, and $\overline{u} \in V_b$ be a critical point of $\mathcal{J}_b$. Then $\overline{u}$ is a critical point of $J_{\overline{\theta}}$ on $H_0^1 \cap H^2(\Omega)$, where $\overline{\theta} = \Theta(b, u)$, and the following facts hold.*

(i) *If $\overline{u}$ is $\overline{\theta}$-infinitesimally stable, then $\overline{u}$ is b-infinitesimally stable.*

(ii) *If $\overline{u}$ is $\overline{\theta}$-locally minimal, then $\overline{u}$ is b-locally minimal.*

(iii) *If $\overline{u}$ is $\overline{\theta}$-linearized stable, then $\overline{u}$ is b-linearized stable.*

Concerning the existence of a stationary solution, we can prove the existence of a global minimizer of the functional $\mathcal{J}_b$. For this purpose, we assume the

growth rates (3.2) and suppose that functions $W_i = W_i(\epsilon)$ are bounded from below, i.e.,

$$W_i(\epsilon) \geq -C_i, \tag{3.4}$$

$i = 1, 2$, which implies that $J_{\overline{\theta}}$ and $\mathcal{J}_b$ are bounded from below and coercive. There exists a global minimizer of $J_{\overline{\theta}}$ provided that $\kappa_1 + \kappa_2 \tilde{\theta} > 0$,

$$H(\epsilon, \overline{\theta}) \geq -C_0, \tag{3.5}$$

and the growth rates

$$\left| H_{,\epsilon}(\epsilon, \overline{\theta}) \right| \leq C|\epsilon|^{K_0 - 1} \tag{3.6}$$

for large $|\epsilon|$, where

$$0 \leq K_0 < 6 \quad \text{if } d = 3, \qquad 0 \leq K_0 < \infty \quad \text{if } d = 2.$$

Theorem 4 *The following facts hold.*

(i) *Assume that* $\kappa_1 + \kappa_2 \overline{\theta} > 0$, *and (3.5), (3.6) hold. Then there exists a global minimizer of* $J_{\overline{\theta}}$ *on* $H_0^1 \cap H^2(\Omega)$.

(ii) *Assume that* $H = H(\epsilon, \theta)$ *takes the form (1.4)-(1.5),* $\theta \mapsto H(\cdot, \theta)$ *is strictly concave, and (3.2), (3.4) hold. Let* $b = E(u_0, u_1, \theta_0)$ *and* $\theta_0 > 0$. *Then there exists a global minimizer of* $\mathcal{J}_b$ *on* V_b.

We note that a global minimizer of $J_{\overline{\theta}}$ does not necessarily correspond to a global minimizer of $\mathcal{J}_b$.

The solution asserted by Theorem 4 (ii) is a global minimizer of $\mathcal{J}_b$, which is a solution to the stationary problem (1.25). Although the infinitesimal stability does not hold in general, we are nevertheless able to show the infinitesimal stability of any local minimizer provided that H is real analytic with respect to ϵ. Here, we assume the growth rates

$$|H_{,\epsilon\epsilon}(\epsilon, \overline{\theta})| \leq C|\epsilon|^{K_0 - 2} \tag{3.7}$$

for large $|\epsilon|$ where

$$0 \leq K_0 < 5 \quad \text{if } d = 3, \qquad 0 \leq K_0 < \infty \quad \text{if } d = 2,$$

which guarantee the elliptic regularity [1] for the stationary problem (1.25).

Theorem 5 *The following facts hold.*

(i) *Let* $\overline{\theta} > 0$ *be a constant. Assume* $H(\cdot, \overline{\theta}) \in C^\omega(\mathrm{Sym}(d, \mathbb{R}), \mathbb{R})$, *and (3.5), (3.7) hold, where* $C^\omega(\mathrm{Sym}(d, \mathbb{R}), \mathbb{R})$ *denotes the set of all real analytic functions on* $\mathrm{Sym}(d, \mathbb{R})$. *Then any local minimizer* $u \in H_0^1 \cap H^2(\Omega)$ *of* $J_{\overline{\theta}}$ *is* $\overline{\theta}$-*infinitesimally stable.*

(ii) *Assume that* $H = H(\epsilon, \theta)$ *takes the form (1.4)-(1.5),* $\theta \mapsto H(\cdot, \theta)$ *is strictly concave, (3.4), (3.7) hold, and* $f_* \in C^\omega(\mathbb{R}_+, \mathbb{R})$, $W_i \in C^\omega(\mathrm{Sym}(d, \mathbb{R}), \mathbb{R})$, $i = 1, 2$. *Let* $b = E(u_0, u_1, \theta_0)$. *Then any local minimizer* $u \in V_b$ *of* $\mathcal{J}_b$ *is* b-*infinitesimally stable.*

The above theorems clarify the interrelations between the stabilities, described in Table 1.

3.2   REMARK ON EXISTENCE RESULTS

So far system (1.1)-(1.2) has been studied in literature only in the case

$$A = B, \quad \kappa_1 = \kappa > 0, \quad \nu_1 = \nu > 0. \tag{3.8}$$

For the one-dimensional situation, $d = 1$, we refer to [3, 5, 16, 17, 31, 32, 42] and the references therein. In the three-dimensional situation the well-posedness of the system has been studied in [27, 28, 23, 24, 25, 41, 43] in the case $f_*(\theta) = -c_v\theta \log \theta$ and in [44, 26] in the case $f_*(\theta) = -c_v\theta^2/2$. The existence of regular solutions on an arbitrary finite time interval has been established. However, the long-time analysis seems still to remain an open problem.

We recall two recent results in the case $f_*(\theta) = -c_v\theta^2/2$.

Theorem B ([44]) *Assume the following conditions with $K_0 = 0$, $0 \leq K_1 < 12/7$, and $0 \leq K_2 < 6$:*

1. *The condition (3.8) holds.*

2. *κ and ν are any positive constants.*

3. *$f_*(\theta) = -c_v\theta^2/2$.*

4. *(N1) W_1, $W_2 \in C^3(\mathrm{Sym}(d, \mathbb{R}), \mathbb{R})$; $W_2 = W_2(\epsilon)$ satisfies the bounds*

$$c_1 |\epsilon|^{K_0} - c_3 \leq W_2(\epsilon) \leq c_2 \left(|\epsilon|^{K_2} + 1 \right) \tag{3.9}$$

   with numbers $0 \leq K_0 \leq K_2$, $0 \leq K_2 < \infty$, c_1, c_2 positive constants and c_3 a real constant.

5. *(N2) $W_1 = W_1(\epsilon)$, $W_2 = W_2(\epsilon)$ satisfy the growth rates*

$$|W_{1,\epsilon}(\epsilon)| \leq c|\epsilon|^{K_1-1}, \quad |W_{1,\epsilon\epsilon}(\epsilon)| \leq c|\epsilon|^{K_1-2}, \quad |W_{1,\epsilon\epsilon\epsilon}(\epsilon)| \leq c|\epsilon|^{K_1-3},$$
$$|W_{2,\epsilon}(\epsilon)| \leq c|\epsilon|^{K_2-1}, \quad |W_{2,\epsilon\epsilon}(\epsilon)| \leq c|\epsilon|^{K_2-2}, \quad |W_{2,\epsilon\epsilon\epsilon}(\epsilon)| \leq c|\epsilon|^{K_2-3} \tag{3.10}$$

   for large $|\epsilon|$, where $0 < K_1 < \infty$.

Let $d = 3$, $T > 0$, and the numbers p, $r \in (1, \infty)$ satisfy the conditions

$$5 < p \leq r < \infty.$$

Then for any $(u_0, u_1, \theta_0) \in B_{p,p}^{4-2/p} \times B_{p,p}^{2-2/p} \times B_{r,r}^{2-2/r}$ with $\underline{\theta} \leq \theta_0 \leq \overline{\theta}$, where $\underline{\theta}$ and $\overline{\theta}$ are positive constants, there exists a unique solution (u, θ) to system (1.1)-(1.2) satisfying

$$(u, \theta) \in W_p^{4,2}(\Omega \times (0, T)) \times W_r^{2,1}(\Omega \times (0, T)),$$
$$0 < \theta_* \leq \theta \leq \theta^* < \infty \quad a.e. \text{ in } \Omega \times (0, T),$$

where θ_* and θ^* depend on T, $\underline{\theta}$, $\overline{\theta}$, and $\|(u_0, u_1, \theta_0)\|_{B_{p,p}^{4-2/p} \times B_{p,p}^{2-2/p} \times B_{r,r}^{2-2/r}}$.

In this theorem, $B_{p,q}^s = B_{p,q}^s(\Omega) = [L^p(\Omega), W_p^j(\Omega)]_{s/j,q}$ is the Besov space, where $W_p^j = W_p^j(\Omega)$ is the standard Sobolev space and $[\cdot,\cdot]_{s/j,q}$ is the real interpolation space. Since the embeddings

$$W_p^{4,2}(\Omega \times I) \hookrightarrow BUC(I, B_{p,p}^{4-2/p}) \hookrightarrow BUC(I, H^2),$$
$$W_q^{2,1}(\Omega \times I) \hookrightarrow BUC(I, B_{q,q}^{2-2/q}) \hookrightarrow BUC(I, L^1)$$

hold for any bounded interval I, see [39], the assumptions of Theorem B guarantee Theorems 1-3 and Corollaries 1-2. If we assume (3.4) in addition to the assumptions of Theorem B, then Theorem 4 also holds true. In such a circumstance, the real analyticity of H with respect to ϵ assures Theorem 5.

Theorem B has been recently generalized in [26] under the so-called viscosity-capillarity relation.

Theorem C ([26]) *Assume the following conditions with $K_0 = 0$, $0 \leq K_1 < 3$, and $0 \leq K_2 < 6$:*

1. *The condition (3.8) holds.*

2. *The viscosity-capillarity relation*

$$0 < 2\sqrt{\kappa} \leq \nu$$

   holds.

3. $f_*(\theta) = -c_\nu \theta^2/2$.

4. *(N1) holds.*

5. *(N2) holds.*

Let $d = 3$, $T > 0$, and the numbers p, q, r, $s \in (1, \infty)$ satisfy the conditions

$$\frac{3}{p} + \frac{2}{q} < 1, \quad \frac{3}{r} + \frac{2}{s} < 1, \quad p \leq r, \quad q \leq s.$$

Then for any $(u_0, u_1, \theta_0) \in B_{p,q}^{4-2/q} \times B_{p,q}^{2-2/q} \times B_{r,s}^{2-2/s}$ with $\underline{\theta} \leq \theta_0 \leq \overline{\theta}$, where $\underline{\theta}$ and $\overline{\theta}$ are positive constants, there exists a unique solution (u, θ) to system (1.1)-(1.2) satisfying

$$(u, \theta) \in W_{p,q}^{4,2}(\Omega \times (0,T)) \times W_{r,s}^{2,1}(\Omega \times (0,T)),$$
$$0 < \theta_* \leq \theta \leq \theta^* < \infty \quad a.e. \text{ in } \Omega \times (0,T),$$

where θ_* and θ^* depend on T, $\underline{\theta}$, $\overline{\theta}$, and $\|(u_0, u_1, \theta_0)\|_{B_{p,q}^{4-2/q} \times B_{p,q}^{2-2/q} \times B_{r,s}^{2-2/s}}$.

In this theorem, $W_{p,q}^{j,j/2} = W_{p,q}^{j,j/2}(\Omega \times (0,T))$ denotes the anisotropic Sobolev space with a mixed norm with respect to space and time variables.

4 Proofs

4.1 Proof of Theorem 1 and Corollary 1

In this subsection we consider the functional $J_{\overline{\theta}}$, and show that the dynamical stability of the stationary solution may be derived from the $\overline{\theta}$-infinitesimal stability.

Proof of Theorem 1 To begin with, inequality (1.36) and

$$\overline{\theta} F(\overline{u}, \overline{\theta}) + b = J_{\overline{\theta}}(\overline{u}).$$

show the relation

$$J_{\overline{\theta}}(\overline{u}) - \overline{\theta} F(\overline{u}, \overline{\theta}) = b \geq J_{\overline{\theta}}(u) - \overline{\theta} F(u, \theta) \qquad (4.1)$$

between the Lyapunov functional F and the variational functional $J_{\overline{\theta}}$. Then it follows from the non-increasing property of F with respect to t that

$$\begin{aligned}
J_{\overline{\theta}}(u(\cdot, t)) - J_{\overline{\theta}}(\overline{u}) &\leq \overline{\theta} \left(F(u(\cdot, t), \theta(\cdot, t)) - F(\overline{u}, \overline{\theta}) \right) \\
&\leq \overline{\theta} \left(F(u_0, \theta_0) - F(\overline{u}, \overline{\theta}) \right).
\end{aligned} \qquad (4.2)$$

Recall that $\overline{u} \in H_0^1 \cap H^2(\Omega)$ is an infinitesimally stable critical point of $J_{\overline{\theta}}$. This means that there is $\varepsilon_0 > 0$ such that any $\varepsilon_1 \in (0, \varepsilon_0/2]$ admits $\delta_0 > 0$ such that $\|\nabla(u - \overline{u})\|_{H^1} < \varepsilon_0$ and $J_{\overline{\theta}}(u) - J_{\overline{\theta}}(\overline{u}) < \delta_0$ imply

$$\|\nabla(u - \overline{u})\|_{H^1} < \varepsilon_1. \qquad (4.3)$$

Therefore, given $\varepsilon > 0$, we can take sufficiently small $\delta \in (0, \varepsilon_0/2]$ such that

$$J_{\overline{\theta}}(u(\cdot, t)) - J_{\overline{\theta}}(\overline{u}) < \min\left(\delta_0, \frac{\varepsilon}{2} \right),$$

by (4.2).

Suppose that $\|\nabla(u(\cdot, t) - \overline{u})\|_{H^1} = \delta \ (\leq \varepsilon_0/2 < \varepsilon_0)$ for some $t > 0$. Then applying (4.3) for $\varepsilon_0 = \delta$, we obtain that $\|\nabla(u(\cdot, t) - \overline{u})\|_{H^1} < \delta$, which is a contradiction. Thus it holds that $\|\nabla(u(\cdot, t) - \overline{u})\|_{H^1} \neq \delta$ for any $t \geq 0$. Since $u = u(\cdot, t) \in C([0, +\infty), H^2(\Omega))$ and $\|\nabla(u_0 - \overline{u})\|_{H^1} < \delta$, we obtain

$$\|\nabla(u(\cdot, t) - \overline{u})\|_{H^1} < \delta, \quad t \geq 0, \qquad (4.4)$$

which completes the proof. $\qquad \qquad \square$

Proof of Corollary 1 In the same way as Theorem 1, we obtain (4.2) and (4.3). Next, it follows that

$$\frac{1}{|\Omega|} |\mathcal{I}_2(u(\cdot, t)) - \mathcal{I}_2(\overline{u})| \leq C_1 \|\nabla(u(\cdot, t) - \overline{u})\|_{H^1} + C_2 \|\nabla(u(\cdot, t) - \overline{u})\|_{H^1}^6, \qquad (4.5)$$

$$\frac{1}{|\Omega|} |\mathcal{I}_2(u_0) - \mathcal{I}_2(\overline{u})| \leq C_1 \|\nabla(u_0 - \overline{u})\|_{H^1} + C_2 \|\nabla(u_0 - \overline{u})\|_{H^1}^6. \qquad (4.6)$$

In fact, it follows from Hölder's inequality that

$$\frac{1}{|\Omega|} \int_\Omega |W_2(\epsilon(u(\cdot,t)) - W_2(\epsilon(\overline{u}))|\, dx$$

$$= \frac{1}{|\Omega|} \int_\Omega |W_{2,\epsilon}\left(\gamma(\cdot,t)\epsilon(u(\cdot,t)) + (1 - \gamma(\cdot,t))\epsilon(\overline{u})\right) \cdot (\epsilon(u(\cdot,t)) - \epsilon(\overline{u}))|\, dx$$

$$\leq \frac{1}{|\Omega|} \|W_{2,\epsilon}\left(\gamma(\cdot,t)\epsilon(u(\cdot,t)) + (1 - \gamma(\cdot,t))\epsilon(\overline{u})\right)\|_{L^{6/5}} \|\nabla(u(\cdot,t) - \overline{u})\|_{L^6},$$

where $\gamma = \gamma(x,t) \in [0,1]$. Using the growth rate (3.2) with $0 \leq K_2 < 6$ and Sobolev's inequality, we have

$$\frac{1}{|\Omega|} \int_\Omega |W_2(\epsilon(u(\cdot,t))) - W_2(\epsilon(\overline{u}))|\, dx$$

$$\leq \tilde{C}_1 \left(\|\gamma(\cdot,t)\epsilon(u(\cdot,t)) + (1 - \gamma(\cdot,t))\epsilon(\overline{u})\|_{L^6}^5 + \tilde{C}_2 \right) \|\nabla(u(\cdot,t) - \overline{u})\|_{H^1}$$

$$\leq \tilde{C}_1 \left\{ (\|\epsilon(\overline{u})\|_{L^6} + \|\gamma(\cdot,t)\left(\epsilon(u(\cdot,t)) - \epsilon(\overline{u})\right)\|_{L^6})^5 + \tilde{C}_2 \right\} \|\nabla(u(\cdot,t) - \overline{u})\|_{H^1}$$

$$\leq \tilde{C}_3 \left(\|\nabla\overline{u}\|_{L^6}^5 + \|\nabla(u(\cdot,t) - \overline{u})\|_{L^6}^5 + \tilde{C}_2 \right) \|\nabla(u(\cdot,t) - \overline{u})\|_{H^1}$$

$$\leq \tilde{C}_3 \left(\tilde{C}_4\|\nabla\overline{u}\|_{H^1}^5 + \tilde{C}_5 \|\nabla(u(\cdot,t) - \overline{u})\|_{H^1}^5 + \tilde{C}_2 \right) \|\nabla(u(\cdot,t) - \overline{u})\|_{H^1}$$

$$= \tilde{C}_6\|\nabla(u(\cdot,t) - \overline{u})\|_{H^1} + C_2\|\nabla(u(\cdot,t) - \overline{u})\|_{H^1}^6,$$

and, therefore, it holds that

$$\frac{1}{|\Omega|} |\mathcal{I}_2(u(\cdot,t)) - \mathcal{I}_2(\overline{u})|$$

$$\leq \frac{\kappa_2}{2|\Omega|}\|Q(u(\cdot,t) - \overline{u})\|_{L^2}^2 + \frac{1}{|\Omega|} \int_\Omega |W_2(\epsilon(u(\cdot,t)) - W_2(\epsilon(\overline{u}))|\, dx$$

$$\leq C_1\|\nabla(u(\cdot,t) - \overline{u})\|_{H^1} + C_2\|\nabla(u(\cdot,t) - \overline{u})\|_{H^1}^6.$$

In the same manner, we also obtain (4.6).

From (4.6), it holds that

$$\frac{1}{|\Omega|} \left\{ F(u_0, \theta_0) - F(\overline{u}, \overline{\theta}) \right\} = \frac{1}{|\Omega|} \int_\Omega H_{,\theta}(\epsilon(u_0), \theta_0) - H_{,\theta}(\epsilon(\overline{u}), \overline{\theta})\, dx$$

$$\leq \left| \frac{1}{|\Omega|} \int_\Omega f_*'(\theta_0)dx - f_*'(\overline{\theta}) \right| + \frac{1}{|\Omega|} |\mathcal{I}_2(u_0) - \mathcal{I}_2(\overline{u})| \qquad (4.7)$$

$$\leq \left| \frac{1}{|\Omega|} \int_\Omega f_*'(\theta_0)dx - f_*'(\overline{\theta}) \right| + C_1\|\nabla(u_0 - \overline{u})\|_{H^1} + C_2\|\nabla(u_0 - \overline{u})\|_{H^1}^6.$$

Therefore, given $\varepsilon > 0$, we can take sufficiently small $\delta \in (0, \varepsilon_0/2]$ such that

$$J_{\overline{\theta}}(u(\cdot,t)) - J_{\overline{\theta}}(\overline{u}) < \min\left(\delta_0, \frac{\varepsilon}{2}\right),$$

by (4.2) and (4.7).

Therefore, as Theorem 1, we obtain (4.4). Moreover, it follows from (4.2), (4.5), (4.7), and (4.4) that

$$\left| \frac{1}{|\Omega|} \int_\Omega f'_*(\theta(\cdot, t)) dx - f'_*(\bar{\theta}) \right|$$
$$\leq \frac{1}{|\Omega|} \left\{ F(u(\cdot, t), \theta(\cdot, t)) - F(\bar{u}, \bar{\theta}) \right\} + \frac{1}{|\Omega|} |\mathcal{I}_2(u(\cdot, t)) - \mathcal{I}_2(\bar{u})|$$
$$\leq \frac{1}{|\Omega|} \left\{ F(u_0, \theta_0) - F(\bar{u}, \bar{\theta}) \right\} + C_1 \|\nabla(u(\cdot, t) - \bar{u})\|_{H^1} + C_2 \|\nabla(u(\cdot, t) - \bar{u})\|_{H^1}^6$$
$$\leq \delta + 2C_1\delta + 2C_2\delta^6,$$
(4.8)

which completes the proof.   □

4.2   PROOF OF THEOREM 2 AND COROLLARY 2

In this subsection we consider the functional $\mathcal{J}_b$, and show that the dynamical stability of the stationary solution may be derived from the b-infinitesimal stability.

Proof of Theorem 2   We first show the semi-unfolding-minimality

$$\frac{1}{|\Omega|} F(u, \bar{\theta}) = \mathcal{J}_b(u) \leq \frac{1}{|\Omega|} F(u, \theta) \qquad (4.9)$$

between the Lyapunov function F and the variational functional $\mathcal{J}_b$. Recall the definition (1.33) of $\mathcal{J}_b$. The monotone decreasing property of $-\Phi$ implies that

$$\mathcal{J}_b(u) = -\Phi\left(\frac{b - \mathcal{I}_1(u)}{|\Omega|}\right) + \frac{1}{|\Omega|}\mathcal{I}_2(u)$$
$$= -\Phi\left(\frac{1}{|\Omega|}\left(\frac{1}{2}\|u_t\|_{L^2}^2 + \int_\Omega e_*(\theta)dx\right)\right) + \frac{1}{|\Omega|}\mathcal{I}_2(u)$$
$$\leq -\Phi\left(\frac{1}{|\Omega|}\int_\Omega e_*(\theta)dx\right) + \frac{1}{|\Omega|}\mathcal{I}_2(u)$$

in the non-stationary state, where $E(u, u_t, \theta) = b = E(u_0, u_1, \theta_0)$. Since $-\Phi$ is convex, by using Jensen's inequality, we obtain

$$\mathcal{J}_b(u) \leq -\Phi\left(\frac{1}{|\Omega|}\int_\Omega e_*(\theta)dx\right) + \frac{1}{|\Omega|}\mathcal{I}_2(u).$$

Here, it follows from the definition (1.32) of Φ that

$$-\Phi(e_*(\theta)) = -\int_{e_*(\tilde{\theta})}^{e_*(\theta)} \frac{d\tau}{e_*^{-1}(\tau)} = \int_{\tilde{\theta}}^{\theta} f''_*(\gamma)d\gamma = f'_*(\theta) - f'_*(\tilde{\theta}),$$

where $\tilde{\theta} \in \mathbb{R}_+$ is a constant. Therefore,

$$\begin{aligned}
\mathcal{J}_b(u) &\leq \frac{1}{|\Omega|} \int_\Omega \left(-\Phi(e_*(\theta))\right) dx + \frac{1}{|\Omega|} \mathcal{I}_2(u) \\
&= \frac{1}{|\Omega|} \int_\Omega f'_*(\theta) dx + \frac{1}{|\Omega|} \mathcal{I}_2(u) - f'_*(\tilde{\theta}) \\
&= \frac{1}{|\Omega|} F(u, \theta) - f'_*(\tilde{\theta}).
\end{aligned}$$

We also obtain

$$\begin{aligned}
\mathcal{J}_b(u) &= -\Phi(e_*(\overline{\theta})) + \frac{1}{|\Omega|} \mathcal{I}_2(u) \\
&= f'_*(\overline{\theta}) + \frac{1}{|\Omega|} \mathcal{I}_2(u) - f'_*(\tilde{\theta}) \\
&= \frac{1}{|\Omega|} F(u, \overline{\theta}) - f'_*(\tilde{\theta}),
\end{aligned}$$

where $\overline{\theta} = \Theta(b, u)$. Consequently, we observe the semi-unfolding-minimality

$$\frac{1}{|\Omega|} F(u, \overline{\theta}) = \mathcal{J}_b(u) + f'_*(\tilde{\theta}) \leq \frac{1}{|\Omega|} F(u, \theta). \tag{4.10}$$

Then it follows from (4.10) and the non-increasing property of F with respect to t that

$$\begin{aligned}
\mathcal{J}_b(u(\cdot, t)) - \mathcal{J}_b(\overline{u}) &\leq \frac{1}{|\Omega|} \left(F(u(\cdot, t), \theta(\cdot, t)) - F(\overline{u}, \overline{\theta})\right) \\
&\leq \frac{1}{|\Omega|} \left(F(u_0, \theta_0) - F(\overline{u}, \overline{\theta})\right).
\end{aligned} \tag{4.11}$$

Recall that $\overline{u} \in V_b$ is an infinitesimally stable critical point of $\mathcal{J}_b$. This means that there is $\varepsilon_0 > 0$ such that any $\varepsilon_1 \in (0, \varepsilon_0/2]$ admits $\delta_0 > 0$ such that $\|\nabla(u - \overline{u})\|_{H^1} < \varepsilon_0$ and $\mathcal{J}_b(u) - \mathcal{J}_b(\overline{u}) < \delta_0$ imply

$$\|\nabla(u - \overline{u})\|_{H^1} < \varepsilon_1. \tag{4.12}$$

Therefore, given $\varepsilon > 0$, we can take sufficiently small $\delta \in (0, \varepsilon_0/2]$ such that

$$\mathcal{J}_b(u(\cdot, t)) - \mathcal{J}_b(\overline{u}) < \min\left(\delta_0, \frac{\varepsilon}{2}\right),$$

by (4.11).

Suppose that $\|\nabla(u(\cdot, t) - \overline{u})\|_{H^1} = \delta$ $(\leq \varepsilon_0/2 < \varepsilon_0)$ for some $t > 0$. Then applying (4.12) for $\varepsilon_0 = \delta$, we obtain $\|\nabla(u(\cdot, t) - \overline{u})\|_{H^1} < \delta$, which is a contradiction. Thus it holds that $\|\nabla(u(\cdot, t) - \overline{u})\|_{H^1} \neq \delta$ for any $t \geq 0$. Like in Theorem 1, we obtain (4.4), which completes the proof. $\quad\square$

Proof of Corollary 2   In the same way as Theorem 2, we obtain (4.10), (4.11), and (4.12). Then in the same fashion as in Corollary 1, with the aid of (4.5), (4.6), and (4.7), we deduce (4.4) and (4.8), which completes the proof. $\quad\square$

4.3   PROOF OF THEOREM 3

(i) To prove the assertion, we show that $\mathcal{J}_b(u) - \mathcal{J}_b(\overline{u}) < \delta_0$ implies

$$\frac{1}{|\Omega|\overline{\theta}}\left(J_{\overline{\theta}}(u) - J_{\overline{\theta}}(\overline{u})\right) < \delta_0. \tag{4.13}$$

From the definition (1.33) of $\mathcal{J}_b$, we have

$$\mathcal{J}_b(u) - \mathcal{J}_b(\overline{u})$$

$$= -\Phi\left(\frac{b - \mathcal{I}_1(u)}{|\Omega|}\right) + \Phi\left(\frac{b - \mathcal{I}_1(\overline{u})}{|\Omega|}\right) + \frac{\mathcal{I}_2(u) - \mathcal{I}_2(\overline{u})}{|\Omega|}$$

$$= \Phi'\left(\frac{b - \gamma_1\mathcal{I}_1(\overline{u}) - (1-\gamma_1)\mathcal{I}_1(u)}{|\Omega|}\right)\frac{\mathcal{I}_1(u) - \mathcal{I}_1(\overline{u})}{|\Omega|} + \frac{\mathcal{I}_2(u) - \mathcal{I}_2(\overline{u})}{|\Omega|}$$

$$= e_*^{-1}\left(\frac{b - \gamma_1\mathcal{I}_1(\overline{u}) - (1-\gamma_1)\mathcal{I}_1(u)}{|\Omega|}\right)^{-1}\frac{\mathcal{I}_1(u) - \mathcal{I}_1(\overline{u})}{|\Omega|} + \frac{\mathcal{I}_2(u) - \mathcal{I}_2(\overline{u})}{|\Omega|},$$

and, therefore, from the monotone increasing property of e_*^{-1}, it follows that

$$\mathcal{J}_b(u) - \mathcal{J}_b(\overline{u})$$

$$= (\gamma_2\Theta(b,\overline{u}) + (1-\gamma_2)\Theta(b,u))^{-1}\frac{\mathcal{I}_1(u) - \mathcal{I}_1(\overline{u})}{|\Omega|} + \frac{\mathcal{I}_2(u) - \mathcal{I}_2(\overline{u})}{|\Omega|},$$

where $\gamma_i \in [0,1]$, $i = 1, 2$. Moreover, $\mathcal{I}_1(u) \geq \mathcal{I}_1(\overline{u})$ and $\mathcal{I}_1(u) \leq \mathcal{I}_1(\overline{u})$ imply that $\Theta(b,u) \leq \Theta(b,\overline{u}) = \overline{\theta}$ and $\Theta(b,u) \geq \Theta(b,\overline{u}) = \overline{\theta}$, respectively. Therefore,

$$\frac{1}{|\Omega|\overline{\theta}}\left(J_{\overline{\theta}}(u) - J_{\overline{\theta}}(\overline{u})\right) \leq \mathcal{J}_b(u) - \mathcal{J}_b(\overline{u}). \tag{4.14}$$

Thus inequality (4.13) holds. Since $\overline{u} \in V_b$ is $\overline{\theta}$-infinitesimally stable, it follows from (4.13) that $\overline{u}$ is b-infinitesimally stable.   □
(ii) The assertion results immediately from inequality (4.14).   □
(iii) It follows from a simple calculation, see (4.19) and (4.22), that

$$Q_{b,\overline{u}}(w, w) = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathcal{J}_b(\overline{u} + sw)\bigg|_{s=0}$$

$$= \frac{1}{\overline{\theta}|\Omega|}\frac{\mathrm{d}^2}{\mathrm{d}s^2}J_{\overline{\theta}}(\overline{u} + sw)\bigg|_{s=0} + R_{b,\overline{u}}(w, w) \tag{4.15}$$

$$= \frac{1}{\overline{\theta}|\Omega|}Q_{\overline{\theta},\overline{u}}(w, w) + R_{b,\overline{u}}(w, w),$$

where $\overline{\theta} = \Theta(b, \overline{u})$ and

$$R_{b,\overline{u}}(w, w)$$

$$= \frac{1}{|\Omega|^2}(e_*^{-1})'\left(\frac{b - \mathcal{I}_1(\overline{u})}{|\Omega|}\right)\left(\kappa_2\left(Q\overline{u}, Qw\right)_{L^2} + \int_\Omega W_{2,\epsilon}(\epsilon(\overline{u}))\cdot\epsilon(w)\mathrm{d}x\right)^2$$

$$\geq 0,$$

because

$$(e_*^{-1})' \left(\frac{b - \mathcal{I}_1(\overline{u})}{|\Omega|} \right) = \frac{1}{e'_*(\overline{\theta})} = \frac{1}{-\overline{\theta} f''_*(\overline{\theta})} > 0.$$

Thus, if $\overline{u} \in V_b$ is $\overline{\theta}$-linearized stable, then $\overline{u}$ is b-linearized stable. ☐

4.4 PROOF OF THEOREM 4

(i) From (3.5), $\kappa_1 + \kappa_2 \overline{\theta} > 0$, and the L^p-estimate $\|u\|_{H^2} \leq C' \|Qu\|_{L^2}$ for elliptic systems [1], it follows that $J_{\overline{\theta}}$ is bounded from below and coercive. Furthermore, the lower semi-continuity of $J_{\overline{\theta}}$ follows from the sub-critical growth rates (3.6) on $H = H(\cdot, \overline{\theta})$ in the Sobolev sense. Then it is easy to show the assertion by a standard variational method. ☐

(ii) Recall that V_b is a nonempty open set in $H_0^1 \cap H^2(\Omega)$, where $b = E(u_0, u_1, \theta_0)$ and $\theta_0 > 0$. First, assumption (3.4) implies that $\mathcal{J}_b$ on V_b is bounded from below. More precisely, it holds that

$$\mathcal{J}_b(u) \geq -\Phi \left(\frac{b}{|\Omega|} + C_1 \right) - C_2.$$

Thus, there is $\{u_j\} \subset V_b$ such that $\mathcal{J}_b(u_j) \to \mathcal{J}_b^*$, where $\mathcal{J}_b^* = \inf_{u \in V_b} \mathcal{J}_b(u)$. Furthermore, $\{u_j\}$ is a bounded sequence in $H_0^1 \cap H^2(\Omega)$. In fact, when V_b is unbounded in $H_0^1 \cap H^2(\Omega)$, $\mathcal{J}_b$ is coercive since we have $s^- = -\infty$, $\lim_{s \downarrow -\infty} \Phi(s) = -\infty$, $\mathcal{I}_1(u) \geq C \|u\|_{H^2} - C$, and $\mathcal{I}_2(u) \geq -C$ for a constant $C > 0$, which can be derived from assumption (3.4) and the L^p-estimate $\|u\|_{H^2} \leq C' \|Qu\|_{L^2}$. Thus $\{u_j\}$ weakly converges to $u^* \in V_b$ after passing to a subsequence.

Hence, it suffices to show the lower semi-continuity of $\mathcal{J}_b$:

$$\mathcal{J}_b(u^*) \leq \liminf_{j \to \infty} \mathcal{J}_b(u_j) = \mathcal{J}_b^*.$$

It holds that

$$
\begin{aligned}
&\mathcal{J}_b(u_j) - \mathcal{J}_b(u^*) \\
&= -\Phi \left(\frac{b - \mathcal{I}_1(u_j)}{|\Omega|} \right) + \Phi \left(\frac{b - \mathcal{I}_1(u^*)}{|\Omega|} \right) + \frac{1}{|\Omega|} \left(\mathcal{I}_2(u_j) - \mathcal{I}_2(u^*) \right) \\
&= \frac{M_j}{|\Omega|} \left(\mathcal{I}_1(u_j) - \mathcal{I}_1(u^*) \right) + \frac{1}{|\Omega|} \left(\mathcal{I}_2(u_j) - \mathcal{I}_2(u^*) \right) \\
&= \frac{M_j}{|\Omega|} \left\{ \frac{\kappa_1}{2} \left(\|Qu_j\|_{L^2}^2 - \|Qu^*\|_{L^2}^2 \right) + \int_\Omega W_1(\epsilon(u_j)) - W_1(\epsilon(u^*)) \mathrm{d}x \right\} \\
&\quad + \frac{\kappa_2}{2|\Omega|} \left(\|Qu_j\|_{L^2}^2 - \|Qu^*\|_{L^2}^2 \right) + \frac{1}{|\Omega|} \left(\int_\Omega W_2(\epsilon(u_j)) - W_2(\epsilon(u^*)) \mathrm{d}x \right),
\end{aligned}
$$

where

$$0 \leq M_j = \Phi' \left(\frac{b - \gamma \mathcal{I}_1(u_*) - (1 - \gamma)\mathcal{I}_1(u_j)}{|\Omega|} \right)$$

$$= e_*^{-1} \left(\frac{b - \gamma \mathcal{I}_1(u_*) - (1 - \gamma)\mathcal{I}_1(u_j)}{|\Omega|} \right)^{-1} < +\infty,$$

and $\gamma \in [0, 1]$. Using the inequality

$$\liminf_{j \to \infty} \|Qu_j\|_{L^2}^2 \geq \|Qu^*\|_{L^2}^2,$$

we have

$$\liminf_{j \to \infty} \mathcal{J}_b(u_j) - \mathcal{J}_b(u^*)$$

$$\geq \frac{M_*}{|\Omega|} \left(\liminf_{j \to \infty} \int_{\Omega} W_1(\epsilon(u_j)) - W_1(\epsilon(u^*)) \mathrm{d}x \right) \tag{4.16}$$

$$+ \frac{1}{|\Omega|} \left(\liminf_{j \to \infty} \int_{\Omega} W_2(\epsilon(u_j)) - W_2(\epsilon(u^*)) \mathrm{d}x \right),$$

where $0 \leq M^* = \liminf_{j \to \infty} M_j < +\infty$. From Hölder's inequality, we deduce

$$\int_{\Omega} |W_i(\epsilon(u_j)) - W_i(\epsilon(u^*))| \, \mathrm{d}x$$

$$= \int_{\Omega} |W_{i,\epsilon} (\gamma_j \epsilon(u_j) + (1 - \gamma_j)\epsilon(u^*)) \cdot (\epsilon(u_j) - \epsilon(u^*))| \, \mathrm{d}x$$

$$\leq \|W_{i,\epsilon} (\gamma_j \epsilon(u_j) + (1 - \gamma_j)\epsilon(u^*))\|_{L^q} \|\nabla(u_j - u^*)\|_{L^p},$$

where $i = 1, 2$, $\gamma_j = \gamma_j(x) \in [0, 1]$, and $1/p + 1/q = 1$. We recall now the growth rate (3.2) with $0 \leq K_i < 6$ and take $q \in (6/5, 6/(K_i - 1))$. Then $p < 6$ and $(K_i - 1)q < 6$ hold, and, therefore, it holds that

$$\int_{\Omega} |W_i(\epsilon(u_j)) - W_i(\epsilon(u^*))| \, \mathrm{d}x$$

$$\leq \tilde{C}_1 \left(\|\gamma_j \epsilon(u_j) + (1 - \gamma_j)\epsilon(u^*)\|_{L^{(K_i - 1)q}}^{K_i - 1} + \tilde{C}_2 \right) \|\nabla(u_j - u^*)\|_{L^p}$$

$$\leq \tilde{C}_1 \left\{ \left(\|\epsilon(u^*)\|_{L^{(K_i-1)q}} + \|\gamma_j(\epsilon(u_j) - \epsilon(u^*))\|_{L^{(K_i-1)q}} \right)^{K_i - 1} + \tilde{C}_2 \right\}$$

$$\times \|\nabla(u_j - u^*)\|_{L^p}$$

$$\leq \tilde{C}_3 \left(\|\nabla u^*\|_{L^{(K_i-1)q}}^{K_i-1} + \|\nabla(u_j - u^*)\|_{L^{(K_i-1)q}}^{K_i-1} + \tilde{C}_2 \right) \|\nabla(u_j - u^*)\|_{L^p}.$$

Since the embeddings $H^1(\Omega) \hookrightarrow L^p(\Omega)$ and $H^1(\Omega) \hookrightarrow L^{(K_i-1)q}(\Omega)$ are compact, we have

$$\int_{\Omega} |W_i(\epsilon(u_j)) - W_i(\epsilon(u^*))| \, \mathrm{d}x \to 0. \tag{4.17}$$

It follows from (4.16) and (4.17) that $\mathcal{J}_b$ is lower semi-continuous, so that $\mathcal{J}_b^* = \mathcal{J}_b(u^*)$. $\qquad \square$

4.5 PROOF OF THEOREM 5

Before stating the proof, we fix some notations concerning the linearized stabili-
ties and the linearized operators. Let $\overline{\theta} > 0$ be a constant and $u \in H_0^1 \cap H^2(\Omega)$ a
critical point of $J_{\overline{\theta}}$. Then the quadratic form in the definition of the $\overline{\theta}$-linearized
stability is

$$
\begin{aligned}
Q_{\overline{\theta},u}(w,w) &= \frac{d^2}{ds^2} J_{\overline{\theta}}(u + sw)\bigg|_{s=0} \\
&= \int_\Omega (\kappa_1 + \kappa_2 \overline{\theta})|Qw|^2 \\
&\quad - [\nabla \cdot \{ (W_{1,\epsilon\epsilon}(\epsilon(u)) + \overline{\theta} W_{2,\epsilon\epsilon}(\epsilon(u))) \cdot \epsilon(w) \}] \cdot w dx,
\end{aligned}
\tag{4.18}
$$

where $w \in H_0^1 \cap H^2(\Omega)$. We define the linearized operator $L_{\overline{\theta},u}$ by

$$
L_{\overline{\theta},u}(w) = (\kappa_1 + \kappa_2 \overline{\theta}) Q^2 w - \nabla \cdot \{ (W_{1,\epsilon\epsilon}(\epsilon(u)) + \overline{\theta} W_{2,\epsilon\epsilon}(\epsilon(u))) \cdot \epsilon(w) \}. \tag{4.19}
$$

We write the eigenvalues of this self-adjoint operator in $L^2(\Omega)$, with the domain

$$
D(L_{\overline{\theta},u}) = \left\{ w \in H^4(\Omega) \ \middle| \ w = Qw = 0 \ \text{ on } \partial\Omega \right\},
$$

as $\mu_1 \leq \mu_2 \leq \cdots$ with counting multiplicities. Then the $\overline{\theta}$-linearized stablity of
$u \in H_0^1 \cap H^2(\Omega)$ means the positivity of the first eigenvalue μ_1 since

$$
Q_{\overline{\theta},u}(w,w) = \left(L_{\overline{\theta},u}(w), w \right)_{L^2}, \tag{4.20}
$$

where $w \in D(L_{\overline{\theta},u})$. Henceforth, let $\phi_1, \ \phi_2, \ \cdots$ be the corresponding L^2-
normalized eigenvectors.

Next, let $b = E(u_0, u_1, \theta_0)$ and $u \in V_b$ be a critical point of $\mathcal{J}_b$. Then the
quadratic form in the definition of the b-linearized stability is

$$
\begin{aligned}
\mathcal{Q}_{b,u}(w,w) &= \frac{d^2}{ds^2} \mathcal{J}_b(u + sw)\bigg|_{s=0} \\
&= \frac{1}{\overline{\theta}|\Omega|} \int_\Omega (\kappa_1 + \kappa_2 \overline{\theta})|Qw|^2 \\
&\quad - [\nabla \cdot \{ (W_{1,\epsilon\epsilon}(\epsilon(u)) + \overline{\theta} W_{2,\epsilon\epsilon}(\epsilon(u))) \cdot \epsilon(w) \}] \cdot w dx \\
&\quad + \frac{1}{|\Omega|^2} (e_\bullet^{-1})' \left(\frac{b - \mathcal{I}_1(u)}{|\Omega|} \right) \left((Qu, Qw)_{L^2} + \int_\Omega \{ \nabla \cdot W_{2,\epsilon}(\epsilon(u)) \} \cdot w dx \right)^2,
\end{aligned}
\tag{4.21}
$$

where $w \in H_0^1 \cap H^2(\Omega)$ and $\overline{\theta} = \Theta(b, u) = e_\bullet^{-1}((b - \mathcal{I}_1(u))/|\Omega|)$. We define the
linearized operator $\mathcal{L}_{b,u}$ by

$$
\begin{aligned}
\mathcal{L}_{b,u}(w) &= (\kappa_1 + \kappa_2 \overline{\theta}) Q^2 w - \nabla \cdot \{ (W_{1,\epsilon\epsilon}(\epsilon(u)) + \overline{\theta} W_{2,\epsilon\epsilon}(\epsilon(u))) \cdot \epsilon(w) \} \\
&\quad + \frac{\overline{\theta}}{|\Omega|} (e_\bullet^{-1})' \left(\frac{b - \mathcal{I}_1(u)}{|\Omega|} \right) \int_\Omega \{ \nabla \cdot W_{2,\epsilon}(\epsilon(u)) \} \cdot w dx \\
&\quad \times (\nabla \cdot W_{2,\epsilon}(\epsilon(u))).
\end{aligned}
\tag{4.22}
$$

We write the eigenvalues of this self-adjoint operator in $L^2(\Omega)$, with the domain

$$D(\mathcal{L}_{b,u}) = \left\{ w \in H^4(\Omega) \mid w = Qw = 0 \quad \text{on } \partial\Omega \right\},$$

as $\bar{\mu}_1 \le \bar{\mu}_2 \le \cdots$ with counting multiplicities. Then the b-linearized stablity of $u \in V_b$ means the positivity of the first eigenvalue $\bar{\mu}_1$ whenever $\bar{\theta} > 0$, because of the equality

$$\mathcal{Q}_{b,u}(w,w) = \frac{1}{\bar{\theta}|\Omega|} \left(\mathcal{L}_{b,u}(w), w\right)_{L^2} \tag{4.23}$$

for $w \in D(\mathcal{L}_{b,u})$. Henceforth, let $\bar{\phi}_1, \bar{\phi}_2, \cdots$ be the corresponding L^2-normalized eigenvectors.

Lemma 1 *Let* $m = 1, 2, \cdots$.

(i) *Under the assumptions of Theorem 5 (i), the function* $\hat{J}$, *defined by*

$$\hat{J}(\tau) := J_{\bar{\theta}}\left(u + \sum_{i=1}^{m} \tau_i \phi_i\right), \quad \tau = (\tau_1, \cdots, \tau_m) \in \mathbb{R}^m,$$

is real analytic on $\mathbb{R}^m$.

(ii) *Under the assumptions of Theorem 5 (ii), the function* $\hat{\mathcal{J}}$, *defined by*

$$\hat{\mathcal{J}}(\tau) := \mathcal{J}_b\left(u + \sum_{i=1}^{m} \tau_i \bar{\phi}_i\right), \quad \tau = (\tau_1, \cdots, \tau_m) \in \mathbb{R}^m,$$

is real analytic at $\tau = 0$.

Proof (i) We take $\tau^* = (\tau_1^*, \cdots, \tau_m^*) \in \mathbb{R}^m$ and $\varepsilon^* > 0$ arbitrarily. We put

$$M = \|\epsilon(u)\|_{L^\infty} + \sum_{i=1}^{m} \left(|\tau_i^*| + \varepsilon^*\right) \|\epsilon(\phi_i)\|_{L^\infty}.$$

Here we note that $u \in H_0^1 \cap H^2(\Omega)$ and $\phi_i \in H_0^1 \cap H^2(\Omega)$, $i = 1, \cdots, m$, are given functions and that their regularity results from the standard elliptic theory with the help of the growth rates (3.7). It follows from the real analyticity of $H(\cdot, \bar{\theta})$ that there exist constants $C_1, C_2 > 0$ such that

$$\left|\partial_\epsilon^{|\alpha|} H(\epsilon, \bar{\theta})\right| \le C_1 C_2^{|\alpha|} |\alpha|! \tag{4.24}$$

for any $|\epsilon| < M$ and $|\alpha| = 1, 2, \cdots$. Then, for any $\tau \in \mathbb{R}^m$ satisfying $|\tau - \tau^*| <$

ε^*, it holds that

$$
\left| \partial_\tau^\alpha \int_\Omega H\left(\epsilon\left(u + \sum_{i=1}^m \tau_i \phi_i\right), \bar\theta\right) dx \right|
$$

$$
= \left| \int_\Omega \partial_\epsilon^{|\alpha|} H\left(\epsilon\left(u + \sum_{i=1}^m \tau_i \phi_i\right), \bar\theta\right) \prod_{i=1}^m \epsilon(\phi_i)^{\alpha_i} dx \right|
$$

$$
\leq \int_\Omega \left| \partial_\epsilon^{|\alpha|} H\left(\epsilon\left(u + \sum_{i=1}^m \tau_i \phi_i\right), \bar\theta\right) \right| \prod_{i=1}^m |\epsilon(\phi_i)|^{\alpha_i} dx,
$$

where $\alpha = (\alpha_1, \cdots, \alpha_m)$ is a multi-index. Since the inequality $|\tau - \tau^*| < \varepsilon^*$ implies that

$$
\left| \epsilon\left(u + \sum_{i=1}^m \tau_i \phi_i\right) \right| \leq \|\epsilon(u)\|_{L^\infty} + \sum_{i=1}^m |\tau_i| \, \|\epsilon(\phi_i)\|_{L^\infty}
$$

$$
\leq \|\epsilon(u)\|_{L^\infty} + \sum_{i=1}^m (|\tau_i^*| + \varepsilon^*) \, \|\epsilon(\phi_i)\|_{L^\infty} = M,
$$

we obtain

$$
\left| \partial_\tau^\alpha \int_\Omega H\left(\epsilon\left(u + \sum_{i=1}^m \tau_i \phi_i\right), \bar\theta\right) dx \right| \leq |\Omega| C_1 C_2^{|\alpha|} |\alpha|! \left(\max_{i=1,\cdots,m} \|\epsilon(\phi_i)\|_{L^\infty} \right)^{|\alpha|}
$$

by using (4.24). On the other hand, it follows that

$$
\partial_{\tau_i} \left(\left\| Q\left(u + \sum_{i=1}^m \tau_i \phi_i\right) \right\|_{L^2}^2 \right) = \int_\Omega \left(Qu + \sum_{i=1}^m \tau_i Q\phi_i \right) \cdot (Q\phi_i) \, dx,
$$

$$
\partial_{\tau_i} \partial_{\tau_{i'}} \left(\left\| Q\left(u + \sum_{i=1}^m \tau_i \phi_i\right) \right\|_{L^2}^2 \right) = \int_\Omega Q\phi_i \cdot Q\phi_{i'} dx,
$$

$$
\partial_\tau^\alpha \left(\left\| Q\left(u + \sum_{i=1}^m \tau_i \phi_i\right) \right\|_{L^2}^2 \right) = 0, \quad |\alpha| \geq 3.
$$

Hence $\hat{J}$ is real analytic on $\mathbb{R}^m$. □

(ii) First, in the same way as (i), we can show that the functions $\hat{\mathcal{I}}_i$, defined by

$$
\hat{\mathcal{I}}_i(\tau) := \mathcal{I}_i\left(u + \sum_{i=1}^m \tau_i \tilde\phi_i\right), \quad \tau = (\tau_1, \cdots, \tau_m) \in \mathbb{R}^m,
$$

are real analytic on $\mathbb{R}^m$. Then $\hat{\mathcal{J}}$ is real analytic at $\tau = 0$ because of $\Phi \in C^\omega((s^-, s^+), \mathbb{R})$, derived from the inverse function theorem. □

Proof of Theorem 5   (i) It holds for $w \in H_0^1 \cap H^2(\Omega)$ that

$$
\begin{aligned}
&J_{\bar{\theta}}(u + w) - J_{\bar{\theta}}(u) \\
&= \frac{1}{2} Q_{\bar{\theta},u}(w, w) + o(\|\nabla w\|_{H^1}^2), \\
&= \frac{1}{2} \sum_{i=1}^{\infty} \mu_i \left| (w, \phi_i)_{L^2} \right|^2 + o(\|\nabla w\|_{H^1}^2), \quad \|\nabla w\|_{H^1} \ll 1.
\end{aligned}
\tag{4.25}
$$

Suppose that the critical point $u \in H_0^1 \cap H^2(\Omega)$ is non-degenerate, that is, the linearized operator $L_{\bar{\theta},u}$ does not have the eigenvalue 0. Then $\mu_1 > 0$ because u is a local minimizer of $J_{\bar{\theta}}$. Thus there is $C > 0$ such that

$$
Q_{\bar{\theta},u}(w, w) \geq C \|\nabla w\|_{H^1}^2. \tag{4.26}
$$

Actually, we have

$$
Q_{\bar{\theta},u}(w, w) \geq \mu_1 \|w\|_{L^2}^2,
$$

and there exist $C_1, C_2 > 0$ such that

$$
Q_{\bar{\theta},u}(w, w) \geq C_1 \|w\|_{H^2}^2 - C_2 \|\nabla w\|_2^2,
$$

because of the L^p-estimate $\|u\|_{H^2} \leq C' \|Qu\|_{L^2}$. Then we obtain (4.26) by using Gagliardo-Nirenberg's inequality.

Therefore, from (4.25) and (4.26), there exists $0 < \varepsilon_0 \ll 1$ such that any $\varepsilon_1 \in (0, \varepsilon_0/4]$ admits $\delta_0 > 0$ such that $\|\nabla w\|_{H^1} < 2\varepsilon_0$ and $J_{\bar{\theta}}(u + w) - J_{\bar{\theta}}(u) < \delta_0$ imply $\|\nabla w\|_{H^1} < \varepsilon_1$, which means that u is $\bar{\theta}$-infinitesimally stable.

Suppose that the critical point $u \in H_0^1 \cap H^2(\Omega)$ is degenerate with the multiplicity $m = 1, 2, \cdots$:

$$
0 = \mu_1 = \cdots = \mu_m < \mu_{m+1} \leq \mu_{m+2} \leq \cdots.
$$

As in the non-degenerate case, it follows from (4.25) that there exists $0 < \varepsilon_0 \ll 1$ such that any $\varepsilon_1 \in (0, \varepsilon_0/4]$ admits $\delta_1 > 0$ such that $\|\nabla w\|_{H^1} < 2\varepsilon_0$ and $J_{\bar{\theta}}(u + w) - J_{\bar{\theta}}(u) < \delta_1$ imply

$$
\|\nabla \overline{w}\|_{H^1} < \frac{\varepsilon_1}{2}. \tag{4.27}
$$

Here and henceforth, we set $s_i = (w, \phi_i)_{L^2}$, $\underline{w} = \sum_{i=1}^{m} s_i \phi_i$, and $\overline{w} = \sum_{i=m+1}^{\infty} s_i \phi_i = w - \underline{w}$.

As $u \in H_0^1 \cap H^2(\Omega)$ is a local minimizer of $J_{\bar{\theta}}$ and $J_{\bar{\theta}}$ is continuous on $H_0^1 \cap H^2(\Omega)$, there is $\bar{\varepsilon}_0 > 0$ such that $0 \leq r_1 \leq r_2 \leq \bar{\varepsilon}_0$ implies

$$
J_{\bar{\theta}}(u + r_1 w) \leq J_{\bar{\theta}}(u + r_2 w) \quad \text{for any } w = \sum_{i=1}^{m} s_i \phi_i, \tag{4.28}
$$

where $s = (s_i)_{i=1,\cdots,m} \in S_1^{m-1}$ and $S_r^{m-1} = \{s \in \mathbb{R}^m \mid |s| = r\}$. We retake sufficiently small $\varepsilon_0 \in (0, \bar{\varepsilon}_0]$. Given $\varepsilon_1 \in (0, \varepsilon_0/4]$,

$$
J_{\bar{\theta}}\left(u + \sum_{i=1}^{m} s_i \phi_i\right), \quad s \in S_{\varepsilon_1/2}^{m-1},
$$

has a global minimizer $s = \varepsilon_1 \hat{s}/2 \in S_{\varepsilon_1/2}^{m-1}$, because $S_{\varepsilon_1/2}^{m-1}$ is compact. Let $\hat{s} = (\hat{s}_i)_{i=1,\cdots,m}$ and $\hat{w} = \sum_{i=1}^{m} \hat{s}_i \phi_i$.

Then the real analyticity of $\hat{J}$ at $\tau = 0$, proven by Lemma 1 (i), implies

$$J_{\overline{g}}(u + r\hat{w}) - J_{\overline{g}}(u) = \hat{J}(r\hat{s}) - \hat{J}(0) = \sum_{j=1}^{\infty} \frac{r}{j!} \left(\sum_{i=1}^{m} \hat{s}_i \partial_{\tau_i} \right)^j \hat{J}(0), \quad |r| \ll 1.$$

Assume that

$$\frac{1}{j!} \left(\sum_{i=1}^{m} \hat{s}_i \partial_{\tau_i} \right)^j \hat{J}(0) = 0 \tag{4.29}$$

for any $j = 1, 2, \cdots$. Then, by the identity theorem for real analytic functions, we have

$$\hat{J}(r\hat{s}) = \hat{J}(0) \tag{4.30}$$

for any $r \in \mathbb{R}$. On the other hand, it holds that

$$\|\nabla(u + r\hat{w})\|_{H^1} \geq |r| \, \|\nabla\hat{w}\|_{H^1} - \|\nabla u\|_{H^1} \to +\infty, \quad |r| \to +\infty.$$

Thus, the coercivity of $J_{\overline{g}}$ implies that

$$\hat{J}(r\hat{s}) \to +\infty, \quad |r| \to +\infty,$$

which contradicts (4.30). Hence, (4.29) does not hold, that is, there is $j = 1, 2, \cdots$ such that

$$\hat{\mu}_j = \frac{1}{j!} \left(\sum_{i=1}^{m} \hat{s}_i \partial_{\tau_i} \right)^j \hat{J}(0) \neq 0.$$

Therefore, there is $\hat{j} = 4, 6, 8, \cdots$ such that

$$\hat{\mu}_{\hat{j}} > 0 \quad \text{and} \quad \hat{\mu}_j = 0 \quad \text{for any } j < \hat{j}, \tag{4.31}$$

because u is a local minimizer. Here, for any $k = 1, 2, \cdots$, it holds that

$$J_{\overline{g}}(u + w) - J_{\overline{g}}(u)$$
$$= \sum_{j=1}^{k} \frac{1}{j!} \mathrm{d}^j J_{\overline{g}}(u)[w, \cdots, w] + o(\|\nabla w\|_{H^1}^k), \quad \|\nabla w\|_{H^1} \ll 1,$$

where

$$\mathrm{d}^j J_{\overline{g}}(u)[w, \cdots, w] = \left. \frac{\mathrm{d}^j}{\mathrm{d}s^j} J_{\overline{g}}(u + sw) \right|_{s=0}, \quad j = 1, 2, \cdots,$$

is a j-linear form. Then, it follows from (4.31) that

$$J_{\overline{g}}(u + r\hat{w}) - J_{\overline{g}}(u) = \frac{1}{\hat{j}!} \mathrm{d}^{\hat{j}} J_{\overline{g}}(u)[r\hat{w}, \cdots, r\hat{w}] + o\left(|r|^{\hat{j}}\right)$$
$$= \hat{\mu}_{\hat{j}} |r|^{\hat{j}} + o\left(|r|^{\hat{j}}\right), \quad |r| \ll 1.$$

More precisely, there exists $0 < r^* < 1$ such that

$$J_{\bar{\theta}}(u + r\hat{w}) - J_{\bar{\theta}}(u) \geq \frac{1}{2}\hat{\mu}_j|r|^{\hat{j}} \qquad (4.32)$$

for any $r \in \mathbb{R}$ satisfying $|r| \leq r^*$. Let $R^* > 0$ be the supremum of r^* satisfying (4.32). We recall that $\varepsilon_0 \in (0, \bar{\varepsilon}_0]$ is sufficiently small and that $\mathbb{R}^m$ is finite-dimensional. Then we can assume $\varepsilon_0 \leq R^*$.

Hence it follows from (4.28) and (4.32) that

$$J_{\bar{\theta}}(u + \underline{w}) - J_{\bar{\theta}}(u) \geq J_{\bar{\theta}}\left(u + \frac{\varepsilon_1}{2}\hat{w}\right) - J_{\bar{\theta}}(u) > \frac{1}{2}\hat{\mu}_j\left(\frac{\varepsilon_1}{2}\right)^{\hat{j}}$$

for any $\underline{w} = \sum_{i=1}^{m} s_i\phi_i$ satisfying $\varepsilon_1/2 \leq |s| < \varepsilon_0$. Therefore, there exists $\delta_2 > 0$ such that $J_{\bar{\theta}}(u + \underline{w}) - J_{\bar{\theta}}(u) < \delta_2$ implies $|s| < \varepsilon_1/2$. Here it holds that

$$\begin{aligned}
&J_{\bar{\theta}}(u + w) - J_{\bar{\theta}}(u + \underline{w}) \\
&= \frac{1}{2}\sum_{i=m+1}^{\infty} \mu_i'|(\overline{w}, \phi_i')_{L^2}|^2 + o(\|\nabla\overline{w}\|_{H^1}^2), \quad \|\nabla\overline{w}\|_{H^1} \ll 1,
\end{aligned}$$

where μ_i' and ϕ_i' denote the eigenvalues and the corresponding L^2-normalized eigenvectors of $L_{\bar{\theta}, u+\underline{w}}$, respectively. From the standard perturbation theory of eigenvalues [18], we have $\mu_i' > 0$ for $i \geq m + 1$, because $\varepsilon_0 > 0$ is sufficiently small. Thus $J_{\bar{\theta}}(u + w) - J_{\bar{\theta}}(u + \underline{w}) > 0$, and therefore, $J_{\bar{\theta}}(u + w) - J_{\bar{\theta}}(u) < \delta_2$ implies

$$|s| < \frac{\varepsilon_1}{2}. \qquad (4.33)$$

As a result, we obtain the $\bar{\theta}$-infinitesimal stability of u by inequalities (4.27) and (4.33). □

(ii) The proof is analogous to that of (i). First, similarly to (4.25), it holds for $w \in H_0^1 \cap H^2(\Omega)$ that

$$\begin{aligned}
&\mathcal{J}_b(u + w) - \mathcal{J}_b(u) \\
&= \frac{1}{2}\mathcal{Q}_{b,u}(w, w) + o(\|\nabla w\|_{H^1}^2), \\
&= \frac{1}{2\bar{\theta}|\Omega|}\sum_{i=1}^{\infty} \tilde{\mu}_i\left|\left(w, \tilde{\phi}_i\right)_{L^2}\right|^2 + o(\|\nabla w\|_{H^1}^2), \quad \|\nabla w\|_{H^1} \ll 1,
\end{aligned}$$

where $\bar{\theta} = \Theta(b, u) > 0$. Then in the same way as (i), recalling Lemma 1 (ii), we can prove the assertion. □

5   EIGENVALUE PROBLEM

5.1   GLOBAL PROPERTIES

When we ignore the positivity of the temperature $\overline{\theta}$ and regard the total energy $b \in \mathbb{R}$ as the eigenvalue, the stationary problem (1.25), expressed as

$$
\begin{cases}
\left(\kappa_1 + \kappa_2 \overline{\theta} \right) Q^2 u = \nabla \cdot H_{,\epsilon}(\epsilon(u), \overline{\theta}) & \text{in } \Omega, \\
u = Qu = 0 & \text{on } \partial\Omega, \\
b = \dfrac{\kappa_1}{2} \|Qu\|_{L^2}^2 + \displaystyle\int_\Omega H(\epsilon(u), \overline{\theta}) - H_{,\theta}(\epsilon(u), \overline{\theta})\overline{\theta}\,\mathrm{d}x,
\end{cases}
\tag{5.1}
$$

is a nonlinear eigenvalue problem with non-local terms. The eigenvalue $b \in \mathbb{R}$ is also called the bifurcation parameter. We consider the total set of (b, u) which solves the stationary problem. On the other hand, for given any stationary temperature $\overline{\theta} > 0$, we can consider the local problem ignoring the total energy conservation $(5.1)_3$. In this case, the total set of $(\overline{\theta}, u)$ which solves the local stationary problem is to be investigated. It is rather easier to analyze the local problem because there is no non-local term. Moreover, the main results formulated in Section 3 and summarized in Table 1 provide a bridge between these two solution sets.

Henceforth, we drop the condition $\overline{\theta} > 0$. Problem (5.1) can be rewritten as

$$
\begin{cases}
\left(\kappa_1 + \kappa_2 \overline{\theta} \right) Q^2 u = \nabla \cdot H_{,\epsilon}(\epsilon(u), \overline{\theta}) & \text{in } \Omega, \\
u = Qu = 0 & \text{on } \partial\Omega, \\
\overline{\theta} = \Theta(b, u),
\end{cases}
\tag{5.2}
$$

where $b \in \mathbb{R}$ is regarded as the bifurcation parameter and $\Theta = \Theta(b, u)$ is defined by (1.29). We denote the solution set of this non-local problem and that of the local problem

$$
\begin{cases}
\left(\kappa_1 + \kappa_2 \overline{\theta} \right) Q^2 u = \nabla \cdot H_{,\epsilon}(\epsilon(u), \overline{\theta}) & \text{in } \Omega, \\
u = Qu = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{5.3}
$$

with the bifurcation parameter $\overline{\theta} \in \mathbb{R}$ as

$$
\mathcal{S} := \{(b, u) \mid \text{classical solutions to (5.2)}\},
$$

$$
\mathcal{S}_{\mathrm{L}} := \{(\overline{\theta}, u) \mid \text{classical solutions to (5.3)}\},
$$

respectively. From the strict convexity of $\theta \mapsto -H(\cdot, \theta)$, the mapping

$$
(b, u) \in \mathcal{S} \quad \mapsto \quad (\overline{\theta}, u) \in \mathcal{S}_{\mathrm{L}},
$$

defined by $\overline{\theta} = \Theta(b, u)$, is a homeomorphism.

Note that $u = 0$ is the *trivial solution* to (5.2). Namely, there is the trivial branch $\{(b, 0) \mid b \in \mathbb{R}\}$ in the (b, u)-space while $\{(\overline{\theta}, 0) \mid \overline{\theta} \in \mathbb{R}\}$ is the trivial branch in the $(\overline{\theta}, u)$-space. Any other solution is said to be a *nontrivial solution*. Concerning the nontrivial solution, we show the upper bound of the temperature $\overline{\theta}$.

Theorem 6 *Assume that $H = H(\epsilon, \theta)$ takes the form (1.4)-(1.5) and*

$$W_{2,\epsilon}(\epsilon) \cdot \epsilon > 0, \quad \epsilon \neq 0. \tag{5.4}$$

Let $(\overline{\theta}, u) \in \mathbb{R} \times H_0^1 \cap H^2(\Omega)$ be a nontrivial solution to (5.3). Furthermore, assume that there is a constant $C_1 > 0$ such that

$$\frac{W_{1,\epsilon}(\epsilon) \cdot \epsilon}{W_{2,\epsilon}(\epsilon) \cdot \epsilon} \geq -C_1$$

for any $\epsilon \neq 0$. Then there exists a constant $\overline{\theta}^* > 0$ such that

$$\overline{\theta} < \overline{\theta}^*.$$

Proof To prove the assertion, we multiply scalarly $(5.3)_1$ by u. Then it holds that

$$\frac{\kappa_1 + \kappa_2\overline{\theta}}{2}\|Qu\|_{L^2}^2 = -\int_\Omega \left\{W_{1,\epsilon}(\epsilon(u)) + \overline{\theta}W_{2,\epsilon}(\epsilon(u))\right\} \cdot \epsilon(u)\mathrm{d}x \geq 0. \tag{5.5}$$

Hence we obtain

$$-C_1 \leq \frac{\int_\Omega W_{1,\epsilon}(\epsilon(u)) \cdot \epsilon(u)\mathrm{d}x}{\int_\Omega W_{2,\epsilon}(\epsilon(u)) \cdot \epsilon(u)\mathrm{d}x} \leq -\overline{\theta},$$

that is, $\overline{\theta} \leq C_1 = \overline{\theta}_*$. $\square$

So far we have described the common properties which can be observed in the energetic and the entropic cases. Concerning the a priori bound of the stationary solution, there is a difference between the energetic case, $\kappa_1 > 0$, $\kappa_2 = 0$, and the non-energetic one, $\kappa_1 \geq 0$, $\kappa_2 > 0$.

Theorem 7 *Assume that $H = H(\epsilon, \theta)$ takes the form (1.4)-(1.5) and (5.4). Let $(\overline{\theta}, u) \in \mathbb{R} \times H_0^1 \cap H^2(\Omega)$ be a nontrivial solution to (5.3) and*

$$\gamma = \begin{cases} -\infty & \text{if } \kappa_1 > 0, \quad \kappa_2 = 0, \\ -\dfrac{\kappa_1}{\kappa_2} & \text{if } \kappa_1 \geq 0, \quad \kappa_2 > 0. \end{cases}$$

Assume that any $\overline{\theta}_* > \gamma$ admits a constant $C_2(\overline{\theta}_*)$ such that

$$\left\{W_{1,\epsilon}(\epsilon) + \overline{\theta}_*W_{2,\epsilon}(\epsilon)\right\} \cdot \epsilon \geq -C_2$$

for any $\epsilon \in \mathrm{Sym}(d, \mathbb{R})$. Then, any $\overline{\theta}_* > \gamma$ admits a constant $C(\overline{\theta}_*) > 0$ such that $\overline{\theta} \geq \overline{\theta}_*$ implies

$$\|\nabla u\|_{H^1} \leq C(\overline{\theta}_*).$$

Proof It follows from the assumption that

$$- \left\{ W_{1,\epsilon}(\epsilon(u)) + \overline{\theta} W_{2,\epsilon}(\epsilon(u)) \right\} \cdot \epsilon(u) \leq - \left\{ W_{1,\epsilon}(\epsilon(u)) + \overline{\theta}_* W_{2,\epsilon}(\epsilon(u)) \right\} \cdot \epsilon(u)$$
$$\leq C_2.$$

By (5.5), we obtain

$$\frac{\kappa_1 + \kappa_2 \overline{\theta}}{2} \|Qu\|_{L^2}^2 \leq C_2 |\Omega|,$$

which completes the proof.   □

If we assume the growth rates (3.7), then the a priori bound of $\|\nabla u\|_{H^1}$ implies that of $\|u\|_{C^{4+\alpha}}$, $\alpha \in (0,1)$, by the elliptic regularity. By Theorems 6 and 7, we have Figure 5.1 which describes the upper bound of the temperature and the a priori estimate of the stationary solution in the $(\overline{\theta}, u)$-space. In Figure 5.1, the horizontal and the vertical axes denote $\overline{\theta}$ and u, e.g. $\|\nabla u\|_{H^1}$, respectively. Namely, the hatched portion is the region where there is no nontrivial solution. In particular, we note that in the entropic case, $\kappa_1 = 0$, $\kappa_2 > 0$, the positivity of the temperature does not signify the existence of an a priori bound of the stationary solution.
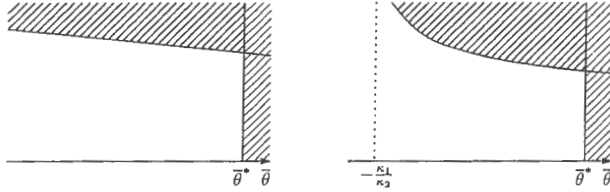
Figure 5.1 Bounds of stationary solutions in the $(\overline{\theta}, u)$-space

5.2   ONE-DIMENSIONAL CASE

Henceforth, we consider the stationary solution in the one-dimensional case, $d = 1$ and $\Omega = (0, l)$ with $l > 0$. Note that the energetic case includes the Falk model on shape memory alloys.

We enumerate assumptions on the elastic energy $H = H(\epsilon, \theta)$. First, we always assume that $H = H(\epsilon, \theta)$ takes the form (1.4)-(1.5) with $f_*, W_1, W_2 \in C^2(\mathbb{R}, \mathbb{R})$ satisfying

$$\theta H_{,\theta\theta} < 0, \tag{5.6}$$

which is expressed equivalently as

$$e_*'(\theta) = -\theta f_*''(\theta) > 0, \quad \theta \in \mathbb{R},$$

where $e_* = e_*(\theta)$ is a C^1-function defined by $e_*(\bar{\theta}) = f_*(\bar{\theta}) - \bar{\theta} f_*'(\bar{\theta})$. We also assume that

$$
\begin{aligned}
&W_1(0) = W_{1,\epsilon}(0) = 0, \\
&W_{2,\epsilon\epsilon}(0) > 0, \qquad W_2(0) = W_{2,\epsilon}(0) = 0, \qquad W_{2,\epsilon}(\epsilon) \neq 0, \quad \epsilon \neq 0.
\end{aligned}
\tag{5.7}
$$

Then $W_2 \geq 0$. We may assume also that

$$
W_1 \geq -C
\tag{5.8}
$$

for a constant $C \geq 0$. Furtheremore, it follows from the isotropic requirement that

$$
W_i(\epsilon) = W_i(-\epsilon), \quad i = 1,\ 2.
\tag{5.9}
$$

The bifurcation problem (5.2) can be written as

$$
\begin{cases}
(\bar{\kappa}_1 + \bar{\kappa}_2\bar{\theta})\partial_x^4 u = \left\{ W_{1,\epsilon}(\partial_x u) + \bar{\theta} W_{2,\epsilon}(\partial_x u) \right\}_x & \text{in } (0, l), \\
u = \partial_x^2 u = 0 & \text{on } \{0, l\}, \\
b = \dfrac{\bar{\kappa}_1}{2}\|\partial_x^2 u\|_{L^2}^2 + l e_*(\bar{\theta}) + \displaystyle\int_0^l W_1(\partial_x u)\mathrm{d}x,
\end{cases}
\tag{5.10}
$$

where $b \in \mathbb{R}$ is regarded as the bifurcation parameter, $e_*(\theta) = f_*(\theta) - \theta f_*'(\theta)$, and $\bar{\kappa}_1 = \kappa_1(\lambda_A + 2\mu_A)^2$, $\bar{\kappa}_2 = \kappa_2(\lambda_A + 2\mu_A)^2$. We denote the solution set of this non-local problem and that of the local problem

$$
\begin{cases}
(\bar{\kappa}_1 + \bar{\kappa}_2\bar{\theta})\partial_x^4 u = \left\{ W_{1,\epsilon}(\partial_x u) + \bar{\theta} W_{2,\epsilon}(\partial_x u) \right\}_x & \text{in } (0, l), \\
u = \partial_x^2 u = 0 & \text{on } \{0, l\}
\end{cases}
\tag{5.11}
$$

with the bifurcation parameter $\bar{\theta} \in \mathbb{R}$ as

$$
S := \left\{ (b, u) \mid \text{classical solutions to (5.10)} \right\},
$$
$$
S_L := \left\{ (\bar{\theta}, u) \mid \text{classical solutions to (5.11)} \right\},
$$

respectively. Then the mapping

$$
(b, u) \in S \quad \mapsto \quad (\bar{\theta}, u) \in S_L,
\tag{5.12}
$$

defined by $\bar{\theta} = \Theta(b, u)$, is a homeomorphism.

By using the abbreviation $\epsilon = \epsilon(u)$, that is, $\epsilon = \epsilon(x) = \partial_x u(x)$, we can see that the problem (5.10) is reduced to

$$
\begin{cases}
-(\bar{\kappa}_1 + \bar{\kappa}_2\bar{\theta})\epsilon_{xx} = -W_{1,\epsilon}(\epsilon) - \bar{\theta} W_{2,\epsilon}(\epsilon) & \text{in } (0, l), \\
\epsilon_x = 0 & \text{on } \{0, l\}, \\
\bar{\theta} = \tilde{\Theta}(b, \epsilon), \quad \displaystyle\int_0^l \epsilon\,\mathrm{d}x = 0.
\end{cases}
\tag{5.13}
$$

Here $\tilde{\Theta} = \tilde{\Theta}(b, \epsilon)$ is given by

$$
\tilde{\Theta}(b, \epsilon) = e_*^{-1}\left(\frac{b - \tilde{I}_1(\epsilon)}{l} \right), \quad \tilde{I}_1(\epsilon) = \frac{\bar{\kappa}_1}{2}\|\epsilon_x\|_{L^2}^2 + \int_0^l W_1(\epsilon)\mathrm{d}x.
$$

If (b, ϵ) solves (5.13), then the corresponding solution $(b, u) \in S$ to (5.10) is given by $u(x) = \int_0^x \epsilon(x')dx'$. Similarly, the problem (5.11) may be reduced to

$$\begin{cases} -(\bar{\kappa}_1 + \bar{\kappa}_2\bar{\theta})\epsilon_{xx} = -W_{1,\epsilon}(\epsilon) - \bar{\theta}W_{2,\epsilon}(\epsilon) & \text{in } (0, l), \\ \epsilon_x = 0 & \text{on } \{0, l\}, \\ \int_0^l \epsilon dx = 0. \end{cases} \quad (5.14)$$

If $(\bar{\theta}, \epsilon)$ solves (5.14), then the corresponding solution $(\bar{\theta}, u) \in S_L$ to (5.11) is given by $u(x) = \int_0^x \epsilon(x')dx'$.

The quadratic form in the definition of the $\bar{\theta}$-linearized stability, see (4.18), is written as

$$\begin{aligned} Q_{\bar{\theta},u}(w, w) &= \left.\frac{d^2}{ds^2} J_{\bar{\theta}}(u + sw)\right|_{s=0} \\ &= \int_0^l (\bar{\kappa}_1 + \bar{\kappa}_2\bar{\theta})|\partial_x\varphi|^2 - \left(W_{1,\epsilon\epsilon}(\epsilon) + \bar{\theta}W_{2,\epsilon\epsilon}(\epsilon)\right)\varphi^2 dx, \end{aligned} \quad (5.15)$$

where $\epsilon = \partial_x u$, $\varphi = \partial_x w \in H^1(0, l)$. We define the linearized operator $\hat{L}_{\bar{\theta},\epsilon}$ by

$$\hat{L}_{\bar{\theta},\epsilon}(\varphi) = (\bar{\kappa}_1 + \bar{\kappa}_2\bar{\theta})\partial_x^2\varphi - \left(W_{1,\epsilon\epsilon}(\epsilon) + \bar{\theta}W_{2,\epsilon\epsilon}(\epsilon)\right)\varphi. \quad (5.16)$$

Then the $\bar{\theta}$-linearized stability means the positivity of the first eigenvalue of this self-adjoint operator in $L^2(0, l)$, with the domain

$$D(\hat{L}_{\bar{\theta},\epsilon}) = \left\{\varphi \in H^2(0, l) \ \middle| \ \varphi_x = 0 \ \text{on } \{0, l\}, \ \int_0^l \varphi dx = 0\right\},$$

because of

$$Q_{\bar{\theta},u}(w, w) = \left(\hat{L}_{\bar{\theta},\epsilon}(\varphi), \varphi\right)_{L^2}. \quad (5.17)$$

Similarly, the quadratic form in the definition of the b-linearized stability, see (4.21), is written as

$$\begin{aligned} Q_{b,u}(w, w) &= \left.\frac{d^2}{ds^2} J_b(u + sw)\right|_{s=0} \\ &= \frac{1}{\bar{\theta}l} \int_0^l (\bar{\kappa}_1 + \bar{\kappa}_2\bar{\theta})|\partial_x\varphi|^2 - \left(W_{1,\epsilon\epsilon}(\epsilon) + \bar{\theta}W_{2,\epsilon\epsilon}(\epsilon)\right)\varphi^2 dx \\ &\quad + \frac{1}{l^2}(e_*^{-1})'\left(\frac{b - \tilde{I}_1(\epsilon)}{l}\right)\left(\bar{\kappa}_2\left(\partial_x\epsilon, \partial_x\varphi\right)_{L^2} + \int_0^l W_{2,\epsilon}(\epsilon)\varphi dx\right)^2, \end{aligned} \quad (5.18)$$

where $\bar{\theta} = \Theta(b, u)$. We define the linearized operator $\hat{L}_{b,\epsilon}$ by

$$\begin{aligned} \hat{L}_{b,\epsilon}(\varphi) &= (\bar{\kappa}_1 + \bar{\kappa}_2\bar{\theta})\partial_x^2\varphi - \left(W_{1,\epsilon\epsilon}(\epsilon) + \bar{\theta}W_{2,\epsilon\epsilon}(\epsilon)\right)\varphi \\ &\quad + \frac{\bar{\theta}}{l}(e_*^{-1})'\left(\frac{b - \tilde{I}_1(\epsilon)}{l}\right)\left(\bar{\kappa}_2\left(\partial_x\epsilon, \partial_x\varphi\right)_{L^2} + \int_0^l W_{2,\epsilon}(\epsilon)\varphi dx\right) \\ &\quad \times \left(-\bar{\kappa}_2\partial_x^2\epsilon + W_{2,\epsilon}(\epsilon)\right). \end{aligned} \quad (5.19)$$

Then the b-linearized stability means the positivity of the first eigenvalue of this self-adjoint operator in $L^2(0, l)$, with the domain

$$D(\hat{\mathcal{L}}_{b,\epsilon}) = \left\{ \varphi \in H^2(0, l) \,\middle|\, \varphi_x = 0 \quad \text{on } \{0, l\}, \quad \int_0^l \varphi \mathrm{d}x = 0 \right\},$$

because of

$$Q_{b,u}(w, w) = \frac{1}{\bar{\theta}l} \left(\hat{\mathcal{L}}_{b,\epsilon}(\varphi), \varphi \right)_{L^2}. \tag{5.20}$$

As in (4.15), we observe that

$$\begin{aligned} Q_{b,u}(w, w) &= \frac{\mathrm{d}^2}{\mathrm{d}s^2} \mathcal{J}_b(u + sw) \bigg|_{s=0} \\ &= \frac{1}{\bar{\theta}l} \frac{\mathrm{d}^2}{\mathrm{d}s^2} J_{\bar{\theta}}(u + sw) \bigg|_{s=0} + \hat{R}_{b,\epsilon}(\varphi, \varphi) \\ &= \frac{1}{\bar{\theta}l} Q_{\bar{\theta},u}(w, w) + \hat{R}_{b,\epsilon}(\varphi, \varphi), \end{aligned} \tag{5.21}$$

where

$$\hat{R}_{b,\epsilon}(\varphi, \varphi) = \frac{1}{l^2} (e_*^{-1})' \left(\frac{b - \tilde{\mathcal{I}}_1(\epsilon)}{l} \right) \left(\bar{\kappa}_2 \left(\partial_x \epsilon, \partial_x \varphi \right)_{L^2} + \int_0^l W_{2,\epsilon}(\epsilon) \varphi \mathrm{d}x \right)^2 \geq 0$$

because

$$(e_*^{-1})' \left(\frac{b - \tilde{\mathcal{I}}_1(\epsilon)}{l} \right) = \frac{1}{e_*'(\bar{\theta})} > 0.$$

We formulate now several results concerning stationary one-dimensional solutions. The proofs of these results are presented in Subsection 5.3.

Theorem 8 *Assume (5.7) and (5.9). Let $(b_*, \epsilon_*) \in \mathbb{R} \times C^2[0, l]$ be any nontrivial solution to (5.13). Then in any sufficiently small neighborhood of (b_*, ϵ_*), solutions to (5.13) generate a unique branch (one-dimensional manifold) in $\mathbb{R} \times C^2[0, l]$.*

Lemma 2 *Assume (5.7) and (5.9). Then the following facts hold.*

(i) *Let $(b_*, \epsilon_*) \in \mathbb{R} \times C^2[0, l]$ be any nontrivial solution to (5.13). If the linearized operator $\hat{\mathcal{L}}_{b_*, \epsilon_*}$, defined by (5.19), has the eigenvalue 0, then the eigenfunction $\tilde{\psi} = \tilde{\psi}(x)$ of $\hat{\mathcal{L}}_{b_*, \epsilon_*}$ associated with the eigenvalue 0 satisfies*

$$\bar{\kappa}_2 \left(\partial_x \epsilon_*, \partial_x \tilde{\psi} \right)_{L^2} + \int_0^l W_{2,\epsilon}(\epsilon_*) \tilde{\psi} \mathrm{d}x \neq 0. \tag{5.22}$$

(ii) Let $(\overline{\theta}_*, \epsilon_*) \in \mathbb{R} \times C^2[0, l]$ be any nontrivial solution to (5.14). If the linearized operator $\hat{L}_{\overline{\theta}_*, \epsilon_*}$, defined by (5.16), has the eigenvalue 0, then the eigenfunction $\psi = \psi(x)$ of $\hat{L}_{\overline{\theta}_*, \epsilon_*}$ associated with the eigenvalue 0 satisfies

$$\bar{\kappa}_2 \left(\partial_x \epsilon_*, \partial_x \psi \right)_{L^2} + \int_0^l W_{2,\epsilon}(\epsilon_*) \psi \mathrm{d}x \neq 0. \tag{5.23}$$

Lemma 3 *Assume (5.7) and (5.9). Then the following facts hold.*

(i) *Let* $(b_*, \epsilon_*) \in \mathbb{R} \times C^2[0, l]$ *be any solution to (5.13). If the linearized operator* $\hat{\mathcal{L}}_{b_*, \epsilon_*}$, *defined by (5.19), has the eigenvalue 0, then it is simple.*

(ii) *Let* $(\overline{\theta}_*, \epsilon_*) \in \mathbb{R} \times C^2[0, l]$ *be any solution to (5.14). Then any eigenvaule of the linearized operator* $\hat{L}_{\overline{\theta}_*, \epsilon_*}$, *defined by (5.16), is simple.*

Theorem 9 *Assume (5.7). The bifurcation points on the branch* $\{(b, 0) \mid b \in \mathbb{R}\}$ *consisting of the trivial solutions to (5.13) are* $b = b_j$, *where*

$$\begin{aligned} b_j &= l \left(e_*(\overline{\theta}_j) - W_1(0) \right), \\ \overline{\theta}_j &= -\frac{l^2 W_{1,\epsilon\epsilon}(0) + j^2 \pi^2 \bar{\kappa}_1}{l^2 W_{2,\epsilon\epsilon}(0) + j^2 \pi^2 \bar{\kappa}_2}, \end{aligned} \tag{5.24}$$

$j = 1, 2, \cdots$. *In a neighborhood of the bifurcation point* $(b_j, 0)$, *the bifurcated branch consisting of solutions to (5.13) can be described as follows:*

$$\mathcal{C}_j = \left\{ (b(s), \epsilon(s)) \in \mathbb{R} \times C^2[0, l] \mid s \in \mathcal{I} \right\}, \quad \epsilon(s) = \tilde{\psi}_j + z(s),$$

where $\mathcal{I}$ *is an open interval containing 0,* $b(0) = b_j$, $z(0) = \dot{z}(0) = 0$,

$$b : \mathcal{I} \to \mathbb{R}, \quad z : \mathcal{I} \to Z, \quad \dot{} = \frac{\partial}{\partial s}, \quad \tilde{\psi}_j(x) = \sqrt{\frac{2}{l}} \cos \frac{j\pi x}{l},$$

and Z *is a complement of* $\operatorname{span}\{\tilde{\psi}_j\}$ *in* $C^2[0, l]$. *Moreover,* $\dot{b}(0) = 0$ *and*

$$\begin{aligned} \ddot{b}(0) = W_{1,\epsilon\epsilon}(0) + \frac{j^2 \pi^2}{l^2} \Bigg\{ \bar{\kappa}_1 &- \frac{e_*'(\overline{\theta}_j)}{2} \left(\partial_\epsilon^4 W_1(0) + \overline{\theta}_j \partial_\epsilon^4 W_2(0) \right) \\ &\left(W_{2,\epsilon\epsilon}(0) + \frac{j^2 \pi^2 \bar{\kappa}_2}{l^2} \right)^{-1} \Bigg\}. \end{aligned} \tag{5.25}$$

if $e_* \in C^3(\mathbb{R})$, $W_1, W_2 \in C^4(\mathbb{R})$.

Theorem 10 *Assume (5.7). Let* $u \in V_b$ *be a critical point of* $\mathcal{J}_b$ *satisfying* $\mathcal{Q}_{b,u}(w, w) \geq 0$ *for any* $w \in H_0^1 \cap H^2(0, l)$. *Then* $u = 0$ *or* u *has a definite sign in* $(0, l)$.

Aanalysis similar to that in the previous subsection allows to conclude the following facts.

Theorem 11 *Assume (5.7) and (5.8). Let $(\overline{\theta}, \epsilon)$ be a nontrivial solution to (5.14). Then there exists a constant $\overline{\theta}^* > 0$ such that*

$$\overline{\theta} < \overline{\theta}^*.$$

Theorem 12 *Assume (5.7) and (5.8). Let $(\overline{\theta}, \epsilon)$ be a nontrivial solution to (5.14) and*

$$\gamma = \begin{cases} -\infty & \text{if } \kappa_1 > 0, \quad \kappa_2 = 0, \\ -\dfrac{\kappa_1}{\kappa_2} & \text{if } \kappa_1 \geq 0, \quad \kappa_2 > 0, \end{cases}$$

Assume, furthermore, that any $\overline{\theta}_* > \gamma$ admits a constant $C_1 > 0$ such that

$$\frac{W_1(\epsilon)}{W_2(\epsilon)} \geq -C_1, \qquad \frac{W_1(\epsilon)}{W_2(\epsilon)} \to +\infty, \quad |\epsilon| \to \infty. \tag{5.26}$$

Then any $\overline{\theta}_* > \gamma$ admits a constant $C(\overline{\theta}_*)$ such that $\overline{\theta} \geq \overline{\theta}_*$ implies

$$\|\epsilon\|_{L^\infty} \leq C(\overline{\theta}_*).$$

In the case $\tilde{\kappa}_2 = 0$, these results has been proved in [34]. Concerning the stationary solutions, see also [35] for the boundary condition $u_x = \partial_x^2 u_x = 0$ on $\{0, l\}$, and [15] for the Falk model

$$W_1(\epsilon) = \alpha_3 \epsilon^6 - \alpha_2 \epsilon^4 - \alpha_1 \theta_c \epsilon^2, \quad W_2(\epsilon) = \alpha_1 \epsilon^2,$$

where α_1, α_2, α_3, and the critical temperature θ_c are positive physical constants.

We are now able to draw the bifurcation structure by means of the above presented theorems together with the main results formulated in Section 3. First, the bifurcation points from the trivial branch $\{(b, 0) \mid b \in \mathbf{R}\}$ to (5.10) are $b = b_j$, $j = 1, 2, \cdots$, given by (5.24). In the $(\overline{\theta}, u)$-space, the associated bifurcation points are $\overline{\theta} = \overline{\theta}_j$, $j = 1, 2, \cdots$, also given by (5.24). Here,

$$\overline{\theta}_j \downarrow \begin{cases} -\infty & \text{if } \kappa_2 = 0, \\ -\frac{\kappa_1}{\kappa_2} & \text{if } \kappa_2 > 0, \end{cases}$$

as $j \to \infty$. Considering the typical case $f_*(\theta) = -c_v \theta \log \theta$, we have $e_*(\theta) = f_*(\theta) - \theta f_*'(\theta) = c_v \theta$. If we define $e_*(\theta) = c_v \theta$ as a function on $\mathbf{R}$ and assume $W_1(0) = 0$, then we also have

$$b_j \downarrow \begin{cases} -\infty & \text{if } \kappa_2 = 0, \\ -\frac{c_v \kappa_1 l}{\kappa_2} & \text{if } \kappa_2 > 0, \end{cases}$$

see Figures 5.2 and 5.3.

Next, from Theorem 8, the total set of the stationary solutions with the bifurcation parameter $b \in \mathbf{R}$ is composed of the trivial branch and the nontrivial branches which may intersect only the trivial branch. Such a structure observed
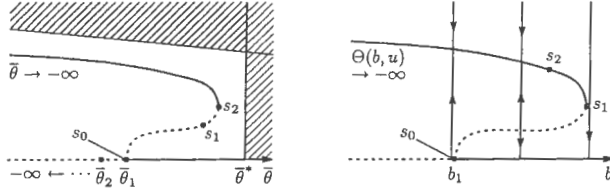
Figure 5.2 Bifurcation diagrams in the energetic case
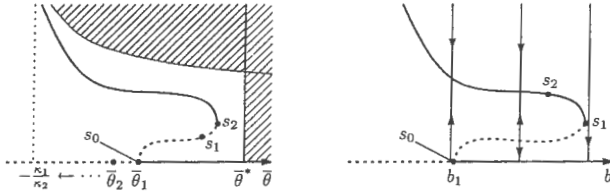
Figure 5.3 Bifurcation diagrams in the non-energetic case

in the (b, u)-space is also the case in the $(\bar{\theta}, u)$-space by virtue of the homeomorphism (5.12). It can be directly proven as in Theorem 8. Furthermore, any nontrivial branch consists of nontrivial solutions which have $j - 1$ nodal zeros, $j = 1, 2, \cdots$, whose number are invariant along the branch by a standard argument concerning the number of nodal zeros. More precisely, we have the following.

Corollary 3 *Assume (5.7) and (5.9). Then the following facts hold.*

(i) *The solution set S to (5.10) is composed of the trivial branch $\{(b, 0) \mid b \in \mathbb{R}\}$ and the nontrivial branches which may intersect only the trivial branch at $b = b_j$, $j = 1, 2, \cdots$, given by (5.24).*

(ii) *The solution set S_L to (5.11) is composed of the trivial branch $\{(\bar{\theta}, 0) \mid \bar{\theta} \in \mathbb{R}\}$ and the nontrivial branches which may intersect only the trivial branch at $\bar{\theta} = \bar{\theta}_j$, $j = 1, 2, \cdots$, given by (5.24).*

On the other hand, Theorem 12 (or Theorems 6 and 7) ensures the upper bound $\bar{\theta}^* < +\infty$ of the temperature $\bar{\theta}$ for the existence of the nontrivial solution

and the a priori estimate $\|\epsilon(u)\|_{L^\infty} \leq C(\overline{\theta}_*)$ for the solution $(\overline{\theta}, u)$ satisfying $\overline{\theta} \geq \overline{\theta}_*$. Consequently, we can assert the following, see [30].

Corollary 4 *Under the assumptions of Theorem 9 and Corollary 3, any non-trivial branch belonging to the solution set S_L to (5.11) is unbounded in the sense that $\overline{\theta} \to -\infty$ if $\kappa_2 = 0$ (energetic case) and $\overline{\theta} \to -\kappa_1/\kappa_2$ if $\kappa_2 > 0$ (non-energetic case). In other words, any nontrivial branch belonging to the solution set S to (5.10) is unbounded in the sense that $\Theta(b, u) \to -\infty$ if $\kappa_2 = 0$ (energetic case) and $\Theta(b, u) \to -\kappa_1/\kappa_2$ if $\kappa_2 > 0$ (non-energetic case).*

Let us take into consideration the stability of the stationary solutions and the positivity of the corresponding temperature, $\overline{\theta} = \Theta(b, u) > 0$. In the case $b < b_1$ the trivial solution $u = 0$ is not linearized stable so that there is a nontrivial global minimizer of $\mathcal{J}_b$. In view of (5.25) in Theorem 9, we can see that there arise both cases of super- and sub-critical directions of the bifurcated branch from $(b_1, 0)$ which produce stable and unstable solutions. Note that any other bifurcated object consists of nodal solutions, which are therefore unstable by Theorem 11. We note also that any local minimizer has a definite sign by Theorem 11.

Some physical parameters comply with the sub-critical condition, $\ddot{b}(0) > 0$ for (5.25) (see e.g. [11, 5]). Moreover, the temperature $\overline{\theta}_1 = \Theta(b_1, 0)$ at the first bifurcation point is often positive. In this case, the temperature $\overline{\theta} = \Theta(b, u)$ is positive at least near the first bifurcation point $(b_1, 0)$.

Henceforth we focus on the first branch bifurcated from $(b_1, 0)$ and suppose $\ddot{b}(0) > 0$ for (5.25) and $\overline{\theta}_1 = \Theta(b_1, 0) > 0$. Then we can obtain Figures 5.2 and 5.3 as possible bifurcation diagrams.

From Corollary 4, the bifurcated branch has at least one turning point in the $(\overline{\theta}, u)$-space and the temperature $\overline{\theta}$ finally goes to $-\infty$ in the energetic case, $\kappa_2 = 0$, and to $-\kappa_1/\kappa_2$ in the non-energetic case, $\kappa_2 > 0$. Hence by the comparison of the bifurcation structures to (5.10) and (5.11) along with Theorem 3, which implies the b-stability of any $\overline{\theta}$-stable critical point, we can say that the bifurcated branch has at least one turning point also in the (b, u)-space.

More precisely, we can parametrize the branch emerged from $(b_1, 0)$ by $s \in \mathbb{R}$ such that $(b(s), u(s)) = (b(s_0), u(s_0))$, $(b(s), u(s)) = (b(s_1), u(s_1))$, and $(b(s), u(s)) = (b(s_2), u(s_2))$ denote the bifurcation point, the first turning point in the (b, u)-space, and the first turning point in the $(\overline{\theta}, u)$-space, respectively. Here $s_0 = 0$, $s_1 > 0$, $s_2 > 0$, and, in particular, $(b(0), u(0)) = (b_1, 0)$. The temperature $\overline{\theta} = \Theta(b(s), u(s))$ is non-decreasing in $(0, s_2)$, and so there is $\delta > 0$ such that $\Theta(b(s), u(s)) > 0$ for $s \in (s_1, s_2 + \delta)$ since $\overline{\theta}_1 = \Theta(b_1, 0) > 0$. Then we obtain $s_1 \leq s_2$ by Theorem 3.

We can show that the inverse assertion of Theorem 3 does not hold in general by presenting a counterexample. It follows from Lemmas 2 and 3 that $\hat{R}_{b(s_1), \epsilon(u(s_1))} > 0$ and $\hat{R}_{b(s_2), \epsilon(u(s_2))} > 0$ in (5.21). Hence the turning point $(b(s_2), u(s_2))$ is not $\overline{\theta}$-linearized stable (degenerate) but b-linearized stable, which is a counterexample to the inverse assertion of Theorem 3. Thus we

have $s_1 < s_2$, see Figures 5.2 and 5.3. Furthermore, $\{(b(s), u(s)) \mid s \in (s_1, s_2)\}$ consists of not $\bar{\theta}$-linearized stable but b-linearized stable (or degenerate) solutions.

Thus, from $\bar{\theta}_1 = \Theta(b_1, 0) > 0$, there is $\delta > 0$ such that $(b(s), u(s))$ satisfies $\Theta(b(s), u(s)) > 0$ and is b-linearized stable (or degenerate) for any $s \in (s_1, s_2 + \delta)$. By virtue of Theorem 2 or Corollary 2, there is a dynamically stable nontrivial solution to (1.25) which coexists with the stable trivial solution $u = 0$. Consequently, there arises a hysteresis related to the change of stable stationary states as b decreases and increases. The existence of the hetero-clinic orbits and the hysteretic cycle may be suggested by the bifurcation diagram.

5.3   PROOFS

Proof of Lemma 2   (i) Let $\epsilon_* = \epsilon_*(x)$ be j-times symmetric, $j = 1, 2, \cdots$. This means that $\psi_1 = \partial_x \epsilon_* = \partial_x \epsilon_*(x)$ has exactly $j - 1$ nodal zeros $x = l/j$, $2l/j, \cdots, (j-1)l/j$ in $(0, l)$. Then the isotropic assumption (5.9) implies that $\epsilon_* = \epsilon_*(x)$ has exactly j nodal zeros $x = l/(2j), 3l/(2j), \cdots, (2j-1)l/(2j)$ in $(0, l)$.

Suppose

$$\bar{\kappa}_2 \left(\partial_x \epsilon_*, \partial_x \bar{\psi} \right)_{L^2} + \int_0^l W_{2,\epsilon}(\epsilon_*) \bar{\psi} \, dx = 0.$$

In view of (5.16) and (5.19), we have

$$\hat{\mathcal{L}}_{b_*, \epsilon_*}(\bar{\psi}) = \hat{\mathcal{L}}_{\bar{\theta}_*, \epsilon_*}(\bar{\psi}) = 0,$$

that is,

$$\begin{cases} -(\bar{\kappa}_1 + \bar{\kappa}_2 \bar{\theta}) \partial_x^2 \bar{\psi} + \left(W_{1,\epsilon\epsilon}(\epsilon_*) + \bar{\theta}_* W_{2,\epsilon\epsilon}(\epsilon_*) \right) \bar{\psi} = 0 & \text{in } (0, l), \\ \partial_x \bar{\psi} = 0 & \text{on } \{0, l\}, \end{cases}$$

where $\bar{\theta}_* = \bar{\Theta}(b_*, \epsilon_*)$. On the other hand, differentiating (5.13) with respect to x, we have

$$\begin{cases} -(\bar{\kappa}_1 + \bar{\kappa}_2 \bar{\theta}) \partial_x^2 \psi_1 + \left(W_{1/\epsilon\epsilon}(\epsilon_*) + \bar{\theta}_* W_{2/\epsilon\epsilon}(\epsilon_*) \right) \psi_1 = 0 & \text{in } (0, l), \\ \psi_1 = 0 & \text{on } \{0, l\} \end{cases}$$

for $\psi_1 = \partial_x \epsilon_*$. As $\psi_1 = \psi_1(x)$ has a definite sign in $(0, l/j)$, it follows from (5.9) and Sturm's comparison theorem (see e.g. [6]) that $\bar{\psi} = \bar{\psi}(x)$ has exactly one nodal zero $x = l/(2j)$ in $(0, l/j)$. Without loss of generality, we assume that $\bar{\psi} > 0$ and $\epsilon_* > 0$ in $(0, l/(2j))$ and that $\bar{\psi} < 0$ and $\epsilon_* < 0$ in $(l/(2j), l/j)$. Then it holds that $\partial_x \epsilon_* < 0$ and $\partial_x \bar{\psi} < 0$ in $(0, l/j)$. Moreover, $W_{2,\epsilon}(\epsilon_*) \bar{\psi} > 0$ in $(0, l/j) \setminus \{l/(2j)\}$ from (5.7). Hence by a reflection argument, we have

$$\bar{\kappa}_2 \left(\partial_x \epsilon_*, \partial_x \bar{\psi} \right)_{L^2(0, l)} + \int_0^l W_{2,\epsilon}(\epsilon_*) \bar{\psi} \, dx$$

$$= j \bar{\kappa}_2 \left(\partial_x \epsilon_*, \partial_x \bar{\psi} \right)_{L^2(0, l/j)} + j \int_0^{l/j} W_{2,\epsilon}(\epsilon_*) \bar{\psi} \, dx > 0,$$

which is a contradiction. This completes the proof.   □
(ii) In the same way as (i), we may prove the assertion by a contradiction.   □

Proof of Lemma 3   (i) If $\epsilon_* = 0$, then $\int_0^l W_{2,\epsilon}(\epsilon_*)\tilde{\phi}_x \mathrm{d}x = 0$, and, therefore, the simplicity of any eigenvalue follows from the Sturm-Liouville theory. Thus we assume that (b_*, ϵ_*) is a nontrivial solution. Let $\tilde{\psi}_1$ and $\tilde{\psi}_2$ be eigenfunctions of $\hat{\mathcal{L}}_{b_*,\epsilon_*}$ associated with the eigenvalue 0: $\hat{\mathcal{L}}_{b_*,\epsilon_*}(\tilde{\psi}_1) = \hat{\mathcal{L}}_{b_*,\epsilon_*}(\tilde{\psi}_2) = 0$. It follows from Lemma 2 that

$$\bar{\kappa}_2\left(\partial_x\epsilon_*, \partial_x\tilde{\psi}_1\right)_{L^2} + \int_0^l W_{2,\epsilon}(\epsilon_*)\tilde{\psi}_1\mathrm{d}x \neq 0,$$

$$\bar{\kappa}_2\left(\partial_x\epsilon_*, \partial_x\tilde{\psi}_2\right)_{L^2} + \int_0^l W_{2,\epsilon}(\epsilon_*)\tilde{\psi}_2\mathrm{d}x \neq 0.$$

Then there exist nonzero constants c_1 and c_2 such that $\tilde{\psi}_3 = c_1\tilde{\psi}_1 + c_2\tilde{\psi}_2$ satisfies

$$\bar{\kappa}_2\left(\partial_x\epsilon_*, \partial_x\tilde{\psi}_3\right)_{L^2} + \int_0^l W_{2,\epsilon}(\epsilon_*)\tilde{\psi}_3\mathrm{d}x = 0, \quad \hat{\mathcal{L}}_{b_*,\epsilon_*}(\tilde{\psi}_3) = 0.$$

Hence it holds that $\tilde{\psi}_3 \equiv 0$ by Lemma 2, that is, $\tilde{\psi}_1$ and $\tilde{\psi}_2$ are linearly dependent.   □
(ii) The assertion is obvious thanks to the Sturm-Liouville theory.   □

Proof of Theorem 8   If $\hat{\mathcal{L}}_{b_*,\epsilon_*}$ does not have the eigenvalue 0, then the assertion follows from the implicit function theorem. Thus suppose that $\hat{\mathcal{L}}_{b_*,\epsilon_*}$ has the eigenvalue 0. From Lemma 3, it is simple. Let $\tilde{\psi}$ be the eigenfunction associated with the eigenvalue 0: $\hat{\mathcal{L}}_{b_*,\epsilon_*}(\tilde{\psi}) = 0$. Then it holds that

$$\mathrm{Ker}(\hat{\mathcal{L}}_{b_*,\epsilon_*}) = \left\{ \alpha\tilde{\psi} \ \middle|\ \alpha \in \mathbb{R} \right\}.$$

Define the operator $\Psi : \mathbb{R} \times \mathbb{R} \times \hat{X} \to C[0, l]$ by

$$\Psi(s, \gamma, v)$$
$$= -(\bar{\kappa}_1 + \bar{\kappa}_2\bar{\theta})\partial_x^2(\epsilon_* + s\tilde{\psi} + v) - W_{1,\epsilon}(\epsilon_* + s\tilde{\psi} + v)$$
$$\quad - \tilde{\Theta}(b_* + \gamma, \epsilon_* + s\tilde{\psi} + v)W_{2,\epsilon}(\epsilon_* + s\tilde{\psi} + v),$$

where

$$\hat{X} = \left\{ v \in C^2[0, l] \ \middle|\ \int_0^l v\tilde{\psi}\mathrm{d}x = 0, \quad v_x = 0 \ \text{ on } \{0, l\} \right\}.$$

Zeros of Ψ have one-to-one correspondence with solutions to (5.13) and it holds that $\Psi(0, 0, 0) = 0$. Then the linearized operator $\Psi_{(\gamma,v)}(0, 0, 0) : \mathbb{R} \times \hat{X} \to C[0, l]$, given by

$$\Psi_{(\gamma,v)}(0, 0, 0) = \left(\frac{1}{e_*'(\bar{\theta}_*)l}\left(-\bar{\kappa}_2\partial_x^2\epsilon_* + W_{2,\epsilon}(\epsilon_*)\right) \quad \hat{\mathcal{L}}_{b_*,\epsilon_*} \right).$$

is a homeomorphism by the aid of Lemma 2, where $\overline{\theta}_* = \tilde{\Theta}(b_*, \epsilon_*)$ and $e'_*(\overline{\theta}_*) > 0$. Therefore, it follows from the implicit function theorem that for $|s| \ll 1$, there exists a unique C^1 mapping

$$s \mapsto (\gamma(s), v(s)) \in \mathbb{R} \times \hat{X}$$

such that

$$\gamma(0) = 0, \quad v(0) = 0, \quad \Psi(s, \gamma(s), v(s)) = 0.$$

Consequently,

$$\mathcal{C}^* = \{ (b(s), \epsilon(s)) \in \mathbb{R} \times X \mid |s| \ll 1 \}$$

is the unique branch in the assertion, where $b(s) = b_* + \gamma(s)$, $\epsilon(s) = \epsilon_* + s\tilde{\psi} + v(s)$. $\square$

Proof of Theorem 9   The bifurcation points from the trivial branch $\{(b,0) \mid b \in \mathbb{R}\}$ are obtained by the standard bifurcation theory from simple eigenvalues, see Crandall-Rabinowitz [7]. In fact, we have

$$\mathrm{Ker}\left(\hat{\mathcal{L}}_j\right) = \left\{\alpha\tilde{\psi}_j \mid \alpha \in \mathbb{R}\right\},$$

$$\mathrm{Ran}\left(\hat{\mathcal{L}}_j\right) = \left\{v \in C[0,l] \;\middle|\; \int_0^l v\tilde{\psi}_j \mathrm{d}x = 0\right\},$$

$$\overline{H}_{,b\epsilon}(b_j, 0)\tilde{\psi}_j = \frac{W_{2,\epsilon\epsilon}(0)}{e'_*(\overline{\theta}_j)l}\tilde{\psi}_j \notin \mathrm{Ran}\left(\hat{\mathcal{L}}_j\right),$$

$$\tilde{\psi}_j(x) = \sqrt{\frac{2}{l}}\cos\frac{j\pi x}{l},$$

where $\hat{\mathcal{L}}_j = \hat{\mathcal{L}}_{b_j,0}$ and

$$\overline{H}(b, \epsilon) = -(\tilde{\kappa}_1 + \tilde{\kappa}_2\tilde{\Theta}(b,\epsilon))\partial_x^2\epsilon + W_{1,\epsilon}(\epsilon) + \tilde{\Theta}(b,\epsilon)W_{2,\epsilon}(\epsilon).$$

Next, we consider the branch $\mathcal{C}_j$ emerged from $(b_j, 0)$. For $(b(s), \epsilon(s)) \in \mathcal{C}_j$, it holds that

$$\overline{H}(b(s), \epsilon(s)) = -(\tilde{\kappa}_1 + \tilde{\kappa}_2\overline{\theta}(s))\partial_x^2\epsilon(s) + W_{1,\epsilon}(\epsilon(s)) + \overline{\theta}(s)W_{2,\epsilon}(\epsilon(s)) = 0, \quad (5.27)$$

where

$$\overline{\theta}(s) = \tilde{\Theta}(b(s), \epsilon(s)) = e_*^{-1}(\beta(s)),$$

$$\beta(s) = \frac{b(s) - \tilde{\mathcal{I}}_1(\epsilon(s))}{l}.$$

We note that

$$\overline{\theta}(0) = \tilde{\Theta}(b_j, 0) = e_*^{-1}\left(\frac{b_j}{l}\right) = \overline{\theta}_j, \quad \beta(0) = \frac{b_j}{l}.$$

Differentiating (5.27) with respect to s, we have

$$
\begin{aligned}
0 = &-(\tilde{\kappa}_1 + \tilde{\kappa}_2\overline{\theta}(s))\partial_x^2(\tilde{\psi}_j + \dot{z}(s)) - \tilde{\kappa}_2\dot{\overline{\theta}}(s)\partial_x^2\epsilon(s) + + W_{1,\epsilon\epsilon}(\tilde{\psi}_j + \dot{z}(s)) \\
&+ \dot{\overline{\theta}}(s)W_{2,\epsilon}(\epsilon(s)) + \overline{\theta}(s)W_{2,\epsilon\epsilon}(\epsilon(s))(\tilde{\psi}_j + \dot{z}(s)),
\end{aligned}
\tag{5.28}
$$

where

$$
\dot{\overline{\theta}}(s) = \frac{\dot{\beta}(s)}{e'_*(\overline{\theta}(s))},
$$

$$
\dot{\beta}(s) = \frac{1}{l}\left\{\dot{b}(s) - \tilde{\kappa}_1\int_0^l \partial_x\epsilon(s)\partial_x(\tilde{\psi}_j + \dot{z}(s))\mathrm{d}x - \int_0^l W_{1,\epsilon}(\epsilon(s))(\tilde{\psi}_j + \dot{z}(s))\mathrm{d}x\right\}.
$$

Note that

$$
\dot{\overline{\theta}}(0) = \frac{\dot{b}(0)}{e'_*(\overline{\theta}_j)l}, \quad \dot{\beta}(0) = \frac{\dot{b}(0)}{l}.
$$

Differentiating (5.28) once more, we have

$$
\begin{aligned}
0 = &-(\tilde{\kappa}_1 + \tilde{\kappa}_2\overline{\theta}(s))\partial_x^2\ddot{z}(s) - 2\tilde{\kappa}_2\dot{\overline{\theta}}(s)\partial_x^2(\tilde{\psi}_j + \dot{z}(s)) - \tilde{\kappa}_2\ddot{\overline{\theta}}(s)\partial_x^2\epsilon(s) \\
&+ \partial_\epsilon^3 W_1(\epsilon(s))(\tilde{\psi}_j + \dot{z}(s))^2 + W_{1,\epsilon\epsilon}(\epsilon(s))\ddot{z}(s) \\
&+ \ddot{\overline{\theta}}(s)W_{2,\epsilon}(\epsilon(s)) + 2\dot{\overline{\theta}}(s)W_{2,\epsilon\epsilon}(\epsilon(s))(\tilde{\psi}_j + \dot{z}(s)) \\
&+ \overline{\theta}(s)\left\{\partial_\epsilon^3 W_2(\epsilon(s))(\tilde{\psi}_j + \dot{z}(s))^2 + W_{2,\epsilon\epsilon}(\epsilon(s))\ddot{z}(s)\right\},
\end{aligned}
\tag{5.29}
$$

where

$$
\ddot{\overline{\theta}}(s) = \frac{\ddot{\beta}(s)}{e'_*(\overline{\theta}(s))} - \frac{e''_*(\overline{\theta}(s))\dot{\beta}(s)^2}{e'_*(\overline{\theta}(s))^3},
$$

$$
\ddot{\beta}(s) = \frac{1}{l}\left\{\ddot{b}(s) - \tilde{\kappa}_1\int_0^l |\partial_x(\tilde{\psi}_j + \dot{z}(s))|^2 + \partial_x\epsilon(s)\partial_x\ddot{z}(s)\mathrm{d}x \right.
$$
$$
\left. - \int_0^l W_{1,\epsilon\epsilon}(\epsilon(s))(\tilde{\psi}_j + \dot{z}(s))^2 + W_{1,\epsilon}(\epsilon(s))\ddot{z}(s)\mathrm{d}x\right\}.
$$

We note that

$$
\ddot{\overline{\theta}}(0) = \frac{\ddot{\beta}(0)}{e'_*(\overline{\theta}_j)} - \frac{e''_*(\overline{\theta}_j)\dot{\beta}(0)^2}{e'_*(\overline{\theta}_j)^3},
$$

$$
\ddot{\beta}(0) = \frac{1}{l}\left\{\ddot{b}(0) - \tilde{\kappa}_1\int_0^l |\partial_x\tilde{\psi}_j|^2\mathrm{d}x - W_{1,\epsilon\epsilon}(0)\int_0^l \tilde{\psi}_j^2\mathrm{d}x\right\}
$$
$$
= \frac{1}{l}\left\{\ddot{b}(0) - \frac{j^2\pi^2\tilde{\kappa}_1}{l^2} - W_{1,\epsilon\epsilon}(0)\right\},
$$

because $\|\tilde{\psi}_j\|_2^2 = 1$ and $\|\partial_x\tilde{\psi}_j\|_2^2 = j^2\pi^2/l^2$. Hence, taking $s = 0$ in (5.29), we obtain

$$
\hat{\mathcal{L}}_j(\ddot{z}(0)) + \frac{2\dot{b}(0)}{e'_*(\overline{\theta}_j)l}\left(W_{2,\epsilon\epsilon}(0)\tilde{\psi}_j - \tilde{\kappa}_2\partial_x^2\tilde{\psi}_j\right) = 0,
$$

and, therefore,
$$\dot{b}(0) = \ddot{z}(0) = \dot{\beta}(0) = \dot{\overline{\theta}}(0) = 0.$$
By differentiating (5.29) once more, we have

$$
\begin{aligned}
0 = &-(\tilde{\kappa}_1 + \tilde{\kappa}_2\overline{\theta}(s))\partial_x^2\partial_s^3 z(s) - 3\tilde{\kappa}_2\dot{\overline{\theta}}(s)\partial_x^2\ddot{z}(s) - 3\tilde{\kappa}_2\ddot{\overline{\theta}}(s)\partial_x^2(\tilde{\psi}_j + \dot{z}(s)) \\
&+ \partial_\epsilon^4 W_1(\epsilon(s))(\tilde{\psi}_j + \dot{z}(s))^3 + 3\partial_\epsilon^3 W_1(\epsilon(s))(\tilde{\psi}_j + \dot{z}(s))\ddot{z}(s) \\
&+ W_{1,\epsilon\epsilon}(\epsilon(s))\partial_s^3 z(s) \\
&- \tilde{\kappa}_2\partial_s^3\overline{\theta}(s)\partial_x^2\epsilon(s) + \partial_s^3\overline{\theta}(s)W_{2,\epsilon}(\epsilon(s)) + 3\ddot{\overline{\theta}}(s)W_{2,\epsilon\epsilon}(\epsilon(s))(\tilde{\psi}_j + \dot{z}(s)) \\
&+ 3\dot{\overline{\theta}}(s)\left\{\partial_\epsilon^3 W_2(\epsilon(s))(\tilde{\psi}_j + \dot{z}(s))^2 + W_{2,\epsilon\epsilon}(\epsilon(s))\ddot{z}(s)\right\} \\
&+ \overline{\theta}(s)\left\{\partial_\epsilon^4 W_2(\epsilon(s))(\tilde{\psi}_j + \dot{z}(s))^3 + 3\partial_\epsilon^3 W_2(\epsilon(s))(\tilde{\psi}_j + \dot{z}(s))\ddot{z}(s) \right. \\
&\left. + W_{2,\epsilon\epsilon}(\epsilon(s))\partial_s^3 z(s)\right\},
\end{aligned}
\tag{5.30}
$$

where

$$
\partial_s^3\overline{\theta}(s) = \frac{\partial_s^3\beta(s)}{e_\ast'(\overline{\theta}(s))} - \frac{3e_\ast''(\overline{\theta}(s))}{e_\ast'(\overline{\theta}(s))^3}\dot{\beta}(0)\ddot{\beta}(0) + \left(\frac{3e_\ast''(\overline{\theta}(s))^2}{e_\ast'(\overline{\theta}(s))^5} - \frac{e_\ast^{(3)}(\overline{\theta}(s))}{e_\ast'(\overline{\theta}(s))^4}\right)\dot{\beta}(s)^3,
$$

$$
\begin{aligned}
\partial_s^3\beta(s) = \frac{1}{l}\Bigg\{ &\partial_s^3 b(s) - \tilde{\kappa}_1\int_0^l 3\partial_x(\tilde{\psi}_j + \dot{z}(s))\partial_x\ddot{z}(s) + \partial_x\epsilon(s)\partial_x\partial_s^3 z(s)\mathrm{d}x \\
&- \int_0^l \partial_\epsilon^3 W_1(\epsilon(s))(\tilde{\psi}_j + \dot{z}(s))^3 + 3W_{1,\epsilon\epsilon}(\epsilon(s))(\tilde{\psi}_j + \dot{z}(s))\ddot{z}(s) \\
&+ W_{1,\epsilon}(\epsilon(s))\partial_s^3 z(s)\mathrm{d}x\Bigg\}.
\end{aligned}
$$

Hence, taking $s = 0$ in (5.30), we obtain

$$
\begin{aligned}
0 = &\hat{\mathcal{L}}_j(\partial_s^3 z(0)) + 3\ddot{\overline{\theta}}(0)\left(-\tilde{\kappa}_2\partial_x^2 + W_{2,\epsilon\epsilon}(0)\right)\tilde{\psi}_j \\
&+ \left(\partial_\epsilon^4 W_1(0) + \overline{\theta}(0)\partial_\epsilon^4 W_2(0)\right)\tilde{\psi}_j^3 \\
= &\hat{\mathcal{L}}_j(\partial_s^3 z(0)) + \frac{3}{e_\ast'(\overline{\theta}_j)l}\left\{\ddot{b}(0) - \frac{j^2\pi^2\tilde{\kappa}_1}{l^2} - W_{1,\epsilon\epsilon}(0)\right\}\left(-\tilde{\kappa}_2\partial_x^2 + W_{2,\epsilon\epsilon}(0)\right)\tilde{\psi}_j \\
&+ \left(\partial_\epsilon^4 W_1(0) + \overline{\theta}_j\partial_\epsilon^4 W_2(0)\right)\tilde{\psi}_j^3,
\end{aligned}
$$

and, therefore,

$$
\begin{aligned}
\ddot{b}(0) = &W_{1,\epsilon\epsilon}(0) + \frac{j^2\pi^2\tilde{\kappa}_1}{l^2} - \frac{e_\ast'(\overline{\theta}_j)}{2}\left(\partial_\epsilon^4 W_1(0) + \overline{\theta}_j\partial_\epsilon^4 W_2(0)\right) \\
&\|\tilde{\psi}_j\|_4^4\left(\tilde{\kappa}_2\|\partial_x\tilde{\psi}_j\|_2^2 W_{2,\epsilon\epsilon}(0) + \|\tilde{\psi}_j\|_2^2\right)^{-1} \\
= &W_{1,\epsilon\epsilon}(0) + \frac{j^2\pi^2}{l^2}\left\{\tilde{\kappa}_1 - \frac{e_\ast'(\overline{\theta}_j)}{2}\left(\partial_\epsilon^4 W_1(0) + \overline{\theta}_j\partial_\epsilon^4 W_2(0)\right)\right. \\
&\left.\left(W_{2,\epsilon\epsilon}(0) + \frac{j^2\pi^2\tilde{\kappa}_2}{l^2}\right)^{-1}\right\}.
\end{aligned}
$$

This completes the proof. □

Proof of Theorem 10 Suppose that $u = u(x)$ has $j - 1$ nodal zeros in $(0, l)$, $j = 2, 3, \cdots$. Differentiating (5.13) with respect to x, we have

$$\begin{cases} -(\bar{\kappa}_1 + \bar{\kappa}_2\bar{\theta})\partial_x^2\psi_1 + \left(W_{1,\epsilon\epsilon}(\epsilon) + \bar{\theta}W_{2,\epsilon\epsilon}(\epsilon)\right)\psi_1 = 0 & \text{in } (0, l), \\ \psi_1 = 0 & \text{on } \{0, l\}, \end{cases}$$

where $\psi_1 = \partial_x\epsilon$, $\epsilon = \partial_x u$, and $\bar{\theta} = \Theta(b, u)$. Here $\psi_1 = \psi_1(x)$ has exactly $j - 1$ nodal zeros $x = l/j, 2l/j, \cdots, (j-1)l/j$ in $(0, l)$. Define the function $\hat{\psi}_1 \in H^1(0, l)$ by

$$\hat{\psi}_1 = \hat{\psi}_1(x) = \begin{cases} \psi_1(x) & \left(x \in \left[0, \dfrac{2l}{j}\right]\right) \\ 0 & \left(x \in \left[\dfrac{2l}{j}, l\right]\right) \end{cases}.$$

Then $\int_0^l \hat{\psi}_1 dx = 0$. Moreover, $\hat{\psi}_1 = \hat{\psi}_1(x)$ and $\partial_x\epsilon = \partial_x\epsilon(x)$ are antisymmetric while $\epsilon = \epsilon(x)$ and $\partial_x\hat{\psi}_1 = \partial_x\hat{\psi}_1(x)$ are symmetric with respect to $x = l/j$ in $[0, 2l/j]$. Hence we obtain

$$\bar{\kappa}_2\left(\partial_x\epsilon, \partial_x\hat{\psi}_1\right)_{L^2} + \int_0^l W_{2,\epsilon}(\epsilon)\hat{\psi}_1 dx$$

$$= \bar{\kappa}_2\left(\partial_x\epsilon, \partial_x\hat{\psi}_1\right)_{L^2(0,2l/j)} + \int_0^{2l/j} W_{2,\epsilon}(\epsilon)\hat{\psi}_1 dx = 0.$$

Consequently, it holds that $Q_{b,u}(\hat{w}, \hat{w}) = 0$ for $\hat{w} = \hat{w}(x) = \int_0^x \hat{\psi}_1(x')dx'$. Recall that $Q_{b,u}(w, w) \geq 0$ for any $w \in H^2 \cap H_0^1(0, l)$. Then $w = \hat{w} \in H^2 \cap H_0^1(0, l)$ is a global minimizer of $Q_{b,u} = Q_{b,u}(w, w)$ and it satisfies

$$\begin{cases} -(\bar{\kappa}_1 + \bar{\kappa}_2\bar{\theta})\partial_x^2\hat{\psi}_1 + \left(W_{1,\epsilon\epsilon}(\epsilon) + \bar{\theta}W_{2,\epsilon\epsilon}(\epsilon)\right)\hat{\psi}_1 = 0 & \text{in } (0, l), \\ \hat{\psi}_1(0) = \partial_x\hat{\psi}_1(0) = 0. \end{cases}$$

This implies $\hat{\psi}_1 \equiv 0$ because of the uniqueness of solutions to the initial value problems for ODEs, which is a contradiction. □

Proof of Theorem 11 Let $x = x_* \in [0, l]$ be any critical point of $\epsilon = \epsilon(x)$. It holds that

$$\frac{(\bar{\kappa}_1 + \bar{\kappa}_2\bar{\theta})}{2}\epsilon_x^2 = W_1(\epsilon) + \bar{\theta}W_2(\epsilon) - W_1(\epsilon(x_*)) - \bar{\theta}W_2(\epsilon(x_*)) \quad \text{in } [0, l].$$

Any nontrivial solution $\epsilon = \epsilon(x)$ has a simple nodal zero $x = x_1 \in (0, l)$, that is, $\epsilon(x_1) = 0$ and $\epsilon_x(x_1) \neq 0$, because $\int_0^l \epsilon dx = 0$. Hence we have

$$\frac{(\bar{\kappa}_1 + \bar{\kappa}_2\bar{\theta})}{2}\epsilon_x(x_1)^2 = -W_1(\epsilon(x_*)) - \bar{\theta}W_2(\epsilon(x_*)) > 0. \tag{5.31}$$

Next, it holds that

$$
\begin{aligned}
W_1(v) &= \frac{1}{2}W_{1,\epsilon\epsilon}(0)v^2 + o(v^2), \\
W_2(v) &= \frac{1}{2}W_{2,\epsilon\epsilon}(0)v^2 + o(v^2)
\end{aligned}
\tag{5.32}
$$

for $|v| \ll 1$. Suppose the assertion does not hold. Let $\bar{\theta} \geq \bar{\theta}_1$ be a constant, where $\bar{\theta}_1 > 0$ is a constant satisfying $\bar{\theta}_1 > |W_{1,\epsilon\epsilon}(0)/W_{2,\epsilon\epsilon}(0)|$. It follows from (5.32) that there exists a constant $\delta_1 > 0$ such that

$$
W_2(v) > \frac{1}{2}W_{2,\epsilon\epsilon}(0)\delta_1^2 \qquad \text{for } |v| \geq \delta_1, \tag{5.33}
$$

$$
-W_1(v) - \bar{\theta}W_2(v) \leq 0 \qquad \text{for } 0 \leq |v| \leq \delta_1, \tag{5.34}
$$

by (5.7) and (5.32). Moreover, if $\bar{\theta} \geq 0$, then it follows from (5.8) and (5.33) that

$$
-W_1(v) - \bar{\theta}W_2(v) \leq -\bar{\theta}W_2(v) + C \leq -\frac{1}{2}W_{2,\epsilon\epsilon}(0)\delta_1^2\bar{\theta} + C \qquad \text{for } |v| \geq \delta_1. \tag{5.35}
$$

Here, we take a constant $\bar{\theta}_2 \geq \bar{\theta}_1$ satisfying $\bar{\theta}_2 > 2C/(W_{2,\epsilon\epsilon}(0)\delta_1^2)$ and $\bar{\theta}_2 > |W_{1,\epsilon\epsilon}(0)/W_{2,\epsilon\epsilon}(0)|$, and retake $\bar{\theta} \geq \bar{\theta}_2$. Then inequalities (5.34) and (5.35) still hold true because of $W_2 \geq 0$. Hence it holds that

$$
-W_1(v) - \bar{\theta}W_2(v) \leq 0 \qquad \text{for any } v \in \mathbb{R}
$$

if $\bar{\theta} \geq \bar{\theta}_2$. Thus the assertion follows from (5.31). $\qquad\square$

Proof of Theorem 12 Let $x = x_0 \in [0, l]$ be a maximal point of $|\epsilon| = |\epsilon(x)|$: $\|\epsilon\|_{L^\infty} = |\epsilon(x_0)|$. It holds that

$$
\frac{(\bar{\kappa}_1 + \bar{\kappa}_2\bar{\theta})}{2}\epsilon_x^2 = W_1(\epsilon) + \bar{\theta}W_2(\epsilon) - W_1(\epsilon(x_0)) - \bar{\theta}W_2(\epsilon(x_0)) \quad \text{in } [0, l].
$$

Any nontrivial solution $\epsilon = \epsilon(x)$ has a simple nodal zero $x = x_1 \in (0, l)$, that is, $\epsilon(x_1) = 0$ and $\epsilon_x(x_1) \neq 0$, because $\int_0^l \epsilon dx = 0$. Hence it follows that:

$$
\frac{\bar{\kappa}_1}{2}\epsilon_x(x_1)^2 = -W_1(\epsilon(x_0)) - \bar{\theta}W_2(\epsilon(x_0)) > 0 \tag{5.36}
$$

in the energetic case, $\bar{\kappa}_2 = 0$, and that

$$
\left(\bar{\theta} + \frac{\bar{\kappa}_1}{\bar{\kappa}_2}\right) = -\frac{2}{\bar{\kappa}_2\epsilon_x(x_1)^2}\left(W_1(\epsilon(x_0)) + \bar{\theta}W_2(\epsilon(x_0))\right) > 0 \tag{5.37}
$$

in the non-energetic case, $\bar{\kappa}_2 > 0$. Here it holds that $W_2(\epsilon) > 0$ for any $\epsilon \neq 0$ from (5.7). Therefore, it follows from (5.36) that

$$
\frac{W_1(\epsilon(x_0))}{W_2(\epsilon(x_0))} < -\bar{\theta} \leq -\bar{\theta}_*
$$

if $\bar{\theta} \geq \bar{\theta}_*$. Thus assumption (5.26) implies $\|\epsilon\|_{L^\infty} \leq C(\bar{\theta}_*)$. $\qquad\square$

REFERENCES

[1] S. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II*, Comm. Pure Appl. Math., 17 (1964), 35–92.

[2] E.C. Aifantis, *The physics of plastic deformation*, Int. J. Plasticity, 3 (1987), 211–247.

[3] T. Aiki, *Weak solutions for Falk's model of shape memory alloys*, Math. Methods Appl. Sci., 23 (2000), 299–319.

[4] H.W. Alt and I. Pawlow, *On the entropy principle of phase transition models with a conserved order parameter*, Adv. Math. Sci. Appl., 6 (1996), 291–376.

[5] M. Brokate and J. Sprekels, "Hysteresis and Phase Transitions," Springer, New York, 1996.

[6] E.A. Coddington and N. Levinson, "Theory of Ordinary Differential Equations," McGraw-Hill, New York, 1955.

[7] M.G. Crandall and P.H. Rabinowitz, *Bifurcation from simple eigenvalues*, J. Funct. Anal., 8 (1971), 321–340.

[8] P.G. de Gennes, "Scaling Concepts in Polymer Physics," Cornell Univ. Press, Ithaca, 1985.

[9] J.E. Dunn and J. Serrin, *On the thermomechanics of interstitial working*, Arch. Ration. Mech. Anal., 88 (1985), 95–133.

[10] D.G.B. Edelen, *On the existence of symmetry relations and dissipation potentials*, Arch. Ration. Mech. Anal., 51 (1973), 218–227.

[11] F. Falk, *Ginzburg-Landau theory of static domain walls in shape-memory alloys*, Z. Phys. B Condensed Matter, 51 (1983), 177–185.

[12] F. Falk, *Ginzburg-Landau theory and solitary waves in shape-memory alloys*, Z. Phys. B Condensed Matter, 54 (1984), 159–167.

[13] F. Falk, *Elastic phase transitions and nonconvex energy functions*, In: "Free Boundary Problems: Theory and Applications, vol. I,", Eds. K. Hoffmann and J. Sprekels, Longman, London, 1990, pp, 45–59.

[14] F. Falk and P. Konopka, *Three-dimensional Landau theory describing the martensitic phase transformation of shape memory alloys*, J. Phys.: Condensed Matter, 2 (1990), 61–77.

[15] A. Friedman and J. Sprekels, *Steady states of austenitic-martensitic domains in the Ginzburg-Landau theory of shape memory alloys*, Contin. Mech. Thermodyn., 2 (1990), 199–213.

[16] T. Herdman, P. Morin, and R.D. Spies, *Convergent spectral approximations for the thermomechanical processes in shape memory alloys*, Nonlinear Anal., 39 (2000), 11–32.

[17] K.H. Hoffmann and A. Żochowski, *Existence of solutions to some nonlinear thermoelastic systems with viscosity*, Math. Methods Appl. Sci., 15 (1992), 187–204.

[18] T. Kato, "Perturbation Theory for Linear Operators," Springer, Berlin, 1966.

[19] I.S. Liu, *Method of Lagrange multipliers for exploitation of the entropy principle*, Arch. Ration. Mech. Anal., 46 (1972), 131–148.

[20] I. Müller, "Thermodynamics," Pitman, London, 1985.

[21] C. Papenfuss and S. Forest, *Thermodynamical frameworks for higher grade material theories with internal variables or additional degrees of freedom*, J. Non-Equilib. Thermodyn., 31 (2006), 319–353.

[22] I. Pawłow, *Three-dimensional model of the thermomechanical equation of shape memory materials*, Control Cybernet., 29 (2000), 341–365.

[23] I. Pawłow and W.M. Zajączkowski, *Global existence to a three-dimensional nonlinear thermoelasticity system arising in shape memory materials*, Math. Methods Appl. Sci., 28 (2005), 407–442.

[24] I. Pawłow and W.M. Zajączkowski, *Unique global solvability in two-dimensional nonlinear thermoelasticity system arising in shape memory materials*, Math. Methods Appl. Sci., 28 (2005), 551–592.

[25] I. Pawłow and W.M. Zajączkowski, *New existence result for a 3-D shape memory model*, In: "Dissipative Phase Transitions," Eds. P. Colli, N. Kenmochi and J. Sprekels, World Scientific Publishing Co. Pte. Ltd., Singapore, 2006, pp. 201–224.

[26] I. Pawłow and W.M. Zajączkowski, *Unique solvability of a nonlinear thermoviscoelasticity system in Sobolev space with a mixed norm*, Discrete Contin. Dyn. Syst. Ser. S, 4 (2011), 441–466.

[27] I. Pawłow and A. Żochowski, *Nonlinear thermoelastic system with viscosity and nonlocality*, In: "Free Boundary Problems: Theory and Applications, vol. I," Eds. N. Kenmochi, Gakkotosho, Tokyo, 2000, pp. 251–265.

[28] I. Pawłow and A. Żochowski, *Existence and uniqueness of solutions for a three-dimensional thermoelastic system*, Dissertationes Math., 406 (2002), 1–46.

[29] R. Peyroux and U. Stefanelli, *Analysis of a 1-D thermoviscoelastic model with temperature-dependent viscosity*, In: "Dissipative Phase Transitions", Eds. P. Colli, N. Kenmochi, and J. Sprekels, World Scientific Publishing Co. Pte. Ltd., Singapore, 2006, pp. 225–246.

[30] P.H. Rabinowitz, *Some global results for nonlinear eigenvalue problems*, J. Funct. Anal., 7 (1971), 487–513.

[31] J. Sprekels, *Global existence for thermomechanical processes with nonconvex free energies of Ginzburg-Landau form*, J. Math. Anal. Appl., 141 (1989), 333–348.

[32] J. Sprekels and S. Zheng, *Global solutions to the equations of a Ginzburg-Landau theory for structural phase transitions in shape memory alloys*, Physica D, 39 (1989), 59–76.

[33] T. Suzuki, "Mean Field Theories and Dual Variation," Atlantis Press, Paris, 2008.

[34] T. Suzuki and S. Tasaki, *Stationary solutions to a thermoelastic system on shape memory materials*, Nonlinearity, 23 (2010), 2623–2656.

[35] T. Suzuki and S. Tasaki, *Stationary solutions to the Falk system on shape memory alloys*, Math. Methods Appl. Sci., 33 (2010), 994–1011.

[36] T. Suzuki and S. Yoshikawa, *Stability of the steady state for the Falk model system of shape memory alloys*, Math. Methods Appl. Sci., 30 (2007), 2233–2245.

[37] T. Suzuki and S. Yoshikawa, *Stability of the steady state for multi-dimensional thermoelastic systems of shape memory alloys*, Discrete Contin. Dyn. Syst. Ser. S, 5 (2012), 209–217.

[38] S. Tasaki, "Stationary states and their stability in mathematical models of non-equilibrium thermodynamics", Doctoral Thesis, Osaka University, Osaka, Japan, 2010.

[39] H. Triebel, "Interpolation Theory, Function Spaces, Differential Operators," North-Holland Publishing Co., Amsterdam-New York, 1978.

[40] P. Ván, *Weakly nonlocal non-equilibrium thermodynamic - variational principles and second law*, Applied Wave Mathematics, 3 (2009), 153–186.

[41] S. Yoshikawa, *Unique global existence for a three-dimensional thermoelastic system of shape memory alloys*, Adv. Math. Sci. Appl., 15 (2005), 603–627.

[42] S. Yoshikawa, *Weak solutions for the Falk model system of shape memory alloys in energy class*, Math. Methods Appl. Sci., 28 (2005), 1423–1443.

[43] S. Yoshikawa, I. Pawłow, and W.M. Zajączkowski, *Quasilinear thermoelasticity system arising in shape memory materials*, SIAM J. Math. Anal., 38 (2007), 1733–1759.

[44] S. Yoshikawa, I. Pawłow, and W.M. Zajączkowski, *A quasilinear thermoelasticity system for shape memory alloys with temperature dependent specific heat*, Commun. Pure Appl. Anal., 8 (2009), 1093–1115.