2/2011

Raport Badawczy

Research Report

RB/11/2011

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I. Pawłow, W.M. Zajączkowski

Instytut Badań Systemowych
Polska Akademia Nauk

Systems Research Institute
Polish Academy of Sciences

POLSKA AKADEMIA NAUK

Instytut Badań Systemowych

ul. Newelska 6

01-447 Warszawa

tel.: (+48) (22) 3810100

fax: (+48) (22) 3810105

Kierownik Zakładu zgłaszający pracę:
Prof. nadzw. dr hab. inż. Antoni Żochowski

Warszawa 2011

GLOBAL REGULAR SOLUTIONS TO A KELVIN-VOIGT TYPE THERMOVISCOELASTIC SYSTEM*

IRENA PAWŁOW† AND WOJCIECH M. ZAJĄCZKOWSKI‡

Abstract. A classical 3-D thermoviscoelastic system of Kelvin-Voigt type is considered. The existence and uniqueness of a global regular solution is proved without small data assumption. The existence proof is based on the successive approximation method. The crucial part constitute a priori estimates on an arbitrary finite time interval, which are derived with the help of the theory of anisotropic Sobolev spaces with a mixed norm.

Key words. thermoviscoelastic system, Kelvin-Voigt type materials, Sobolev spaces with a mixed norm, global existence, a priori estimates

AMS subject classifications. Primary 74B20, 35K50; Secondary: 35Q72, 74F05

1. Introduction.

1.1. Motivation and goal. This article is concerned with the existence and uniqueness of global regular solutions to a classical 3-D thermoviscoelastic system at small strains. The system describes materials which have the properties both of elasticity and viscosity. Such materials are usually referred to as Kelvin-Voigt type.

As noted in the recent paper on this subject by Roubíček [21] – and according to our best knowledge as well – the existence of global solutions to a thermoviscoelastic system with constant both specific heat and heat conductivity is, in spite of great effort through many decades, still open in dimensions $n \geq 2$. In dimension $n = 1$ it was established in the pioneering papers by Slemrod [22], Dafermos [5] and Defermos-Hsiao [6].

The local in time existence and global uniqueness of a weak solution to 3-D thermoviscoelastic system with constant specific heat and heat conductivity has been proved by Bonetti-Bonfanti [3]. Other known results on multidimensional thermoviscoelasticity deal with a modified energy equation. Modifications involve either nonconstant specific heat or nonconstant heat conductivity. A thermoviscoelastic system with temperature-dependent specific heat has been addressed by Blanchard-Guibé [2] where the existence of global, weak-renormalized solutions has been proved, and recently in [21] where the existence of a very weak solution has been established. We mention also that the framework of renormalized solutions has been applied in [26] for 3-D thermoviscoelastic system arising in structural phase transitions.

In a more general setting allowing for large strains a 3-D thermoviscoelastic system has been studied under small data assumption by Shibata [23] and recently by Gawinecki-Zajączkowski [11].

For thermoviscoelastic problems with a modified heat conductivity we refer to Eck-Jarušek-Krbec [8] and the references therein.

*Partially supported by Polish Grant NN 201 396 937

†Systems Research Institute, Polish Academy of Sciences, Newelska 6, 01-447 Warsaw, Poland and
Institute of Mathematics and Cryptology, Cybernetics Faculty, Military University of Technology,S. Kaliskiego 2, 00-908 Warsaw, Poland E-mail: pawlow@ibspan.waw.
‡Institute of Mathematics, Polish Academy of Sciences, Śniadeckich 8, 00-950 Warsaw, Poland and
Institute of Mathematics and Cryptology, Cybernetics Faculty, Military University of Technology,S. Kaliskiego 2, 00-908 Warsaw, Poland E-mail: E-mail: wz@impan.gov.pl

In the present paper we consider a thermoviscoelastic system with specific heat linearly increasing with temperature and with constant heat conductivity. Such setting is a particular case of systems addressed in [2] and [21].

The novelty of the existence result presented in this paper concerns the regularity of a 3-D global solution corresponding to sufficiently smooth but arbitrary in size initial data. The proof of the existence theorem is based on the successive approximation method. The key regularity estimates are derived with the help of the parabolic theory in anisotropic Sobolev spaces $W^{2,1}_{p,p_0}(\Omega^T)$, $\Omega^T = \Omega \times (0,T)$, $p, p_0 \in (1,\infty)$, with a mixed norm with respect to space and time variables. Such framework has been previously applied by the authors [19] to the thermoviscoelastic system arising in shape memory alloys. It allowed to generalize the former results on this subject in [27].

As known, in deriving a priori estimates for a solution of a system of balance laws it is common to begin with estimates arising from the conservation of a total energy. Such estimates provide L_∞-time regularity for the conserved quantities. To take advantage of such time regularity in deriving subsequent regularity estimates it is desirable to work in Sobolev spaces with a mixed norm, for example $W^{2,1}_{p,p_0}(\Omega^T)$, where the space exponent p is determined by the energy structure and the time exponent p_0 may be arbitrarily large. This is the idea behind using the framework of Sobolev spaces with a mixed norm to the thermoviscoelastic system under considerations. The theory of IBVP's in Sobolev spaces with a mixed norm is the subject of recent theoretical studies. We apply the general results due to Krylov [13] and Denk-Hieber-Prüss [7].

1.2. Thermoviscoelastic system. The system under consideration has the following form:

$$u_{tt} - \nabla \cdot [A_1\varepsilon_t + A_2(\varepsilon - \theta\alpha)] = b, \tag{1.1}$$

$$c_v\theta\theta_t - k\Delta\theta = -\theta(A_2\alpha)\cdot\varepsilon_t + (A_1\varepsilon_t)\cdot\varepsilon_t + g \quad \text{in } \Omega^T = \Omega \times (0,T), \tag{1.2}$$

where

$$\varepsilon \equiv \varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T), \quad \varepsilon_t \equiv \varepsilon(u_t) = \frac{1}{2}(\nabla u_t + (\nabla u_t)^T).$$

Here $\Omega \subset \mathbb{R}^3$ is a bounded domain occupied by a body in a fixed reference configuration, and $(0,T)$ is the time interval. The system is completed by appropriate boundary and initial conditions. Here we assume

$$u = 0, \quad n\cdot\nabla\theta = 0 \quad \text{on } S^T = S \times (0,T), \tag{1.3}$$

$$u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1, \quad \theta|_{t=0} = \theta_0 \quad \text{in } \Omega, \tag{1.4}$$

where S is the boundary of Ω and n is the unit outward normal to S.

The field $u: \Omega^T \to \mathbb{R}^3$ is the displacement, $\theta: \Omega^T \to \mathbb{R}_+ = (0,\infty)$ is the absolute temperature, the second order tensors $\varepsilon = (\varepsilon_{ij})_{i,j=1,2,3}$ and $\varepsilon_t = ((\varepsilon_t)_{ij})_{i,j=1,2,3}$ denote respectively the linearized strain and the strain rate.

Equation (1.1) is the linear momentum balance with the stress tensor given by a linear thermoviscoelastic law of the Kelvin-Voigt type (cf. [8], Chap. 5.4)

$$S = A_1\varepsilon_t + A_2(\varepsilon - \theta\alpha).$$

The fourth order tensors $\boldsymbol{A}_1 = ((A_1)_{ijkl})_{i,j,k,l=1,2,3}$ and $\boldsymbol{A}_2 = ((A_2)_{ijkl})_{i,j,k,l=1,2,3}$ are respectively the linear viscosity and the elasticity tensors, defined by

$$\varepsilon \mapsto \boldsymbol{A}_p\varepsilon = \lambda_p tr\varepsilon \boldsymbol{I} + 2\mu_p\varepsilon, \quad p = 1,2, \tag{1.5}$$

where λ_1, μ_1 are the viscosity constants, and λ_2, μ_2 are the Lamé constants, both λ_1, μ_1 and λ_2, μ_2 with the values within the elasticity range

$$\mu_p > 0, \quad 3\lambda_p + 2\mu_p > 0, \quad p = 1,2; \tag{1.6}$$

$\boldsymbol{I} = (\delta_{ij})_{i,j=1,2,3}$ is the identity tensor.

The second order symmetric tensor $\boldsymbol{\alpha} = (\alpha_{ij})_{i,j=1,2}$ with constant α_{ij}, represents the thermal expansion. The vector field $\boldsymbol{b}: \Omega^T \to \mathbb{R}^3$ is the external body force.

Above and hereafer the summation convention over the repeated indices is used, vectors and tensors are denoted by bold letters, and the dot denotes the inner product of tensors, e.g.

$$(\boldsymbol{A}\varepsilon) \cdot \varepsilon = A_{ijkl}\varepsilon_{kl}\varepsilon_{ij}.$$

Moreover,

$$\boldsymbol{A}\varepsilon = (A_{ijkl}\varepsilon_{kl})_{i,j=1,2,3} \quad \text{and} \quad \nabla \cdot (\boldsymbol{A}\varepsilon) = \left(\frac{\partial}{\partial x_j}(A_{ijkl}\varepsilon_{kl})\right)_{i=1,2,3}.$$

Equation (1.2) is the energy balance in which the linear Fourier law for the heat flux, $q = -k\nabla\theta$ with constant heat conductivity $k > 0$, and temperature-dependent specific heat, $c_v\theta$ with $c_v > 0$, have been adopted. The first two terms on the right-hand side of (1.2) represent heat sources created by the deformation of the material and by the viscosity. The field $g: \Omega^T \to \mathbb{R}$ is the external heat source.

The boundary conditions in (1.3) mean that the body is fixed at the boundary S and thermally isolated. The initial conditions (1.4) prescribe displacement, velocity and temperature at $t = 0$.

The system (1.1)–(1.2) can be derived by various arguments of thermodynamics, see e.g. [9, 3]. In Section 2 we summarize its thermodynamic basis.

1.3. Linear elasticity and viscosity operators. Assumptions. For further analysis we formulate problem (1.1)–(1.4) in terms of the linear viscosity and elasticity operators, $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$, defined by

$$\boldsymbol{u} \mapsto \boldsymbol{Q}_p\boldsymbol{u} = \nabla \cdot (\boldsymbol{A}_p\varepsilon(\boldsymbol{u})) = \mu_p\Delta\boldsymbol{u} + (\lambda_p + \mu_p)\nabla(\nabla \cdot \boldsymbol{u}), \quad p = 1,2, \tag{1.7}$$

with domains $D(\boldsymbol{Q}_p) = \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}_0^1(\Omega)$.

Then system (1.1), (1.2) takes the form

$$\begin{aligned} &\boldsymbol{u}_{tt} - \boldsymbol{Q}_1\boldsymbol{u}_t = \boldsymbol{Q}_2\boldsymbol{u} - \nabla \cdot (\theta\boldsymbol{A}_2\boldsymbol{\alpha}) + \boldsymbol{b}, \\ &c_v\theta\theta_t - k\Delta\theta = -\theta(\boldsymbol{A}_2\boldsymbol{\alpha}) \cdot \varepsilon_t + (\boldsymbol{A}_1\varepsilon_t) \cdot \varepsilon_t + g \quad \text{in } \Omega^T, \end{aligned} \tag{1.8}$$

with boundary and initial conditions (1.3), (1.4).

Throughout we shall assume that

(A1) $\Omega \subset \mathbb{R}^3$ is a bounded domain with the boundary S of class at least C^2; $T > 0$ is an arbitrary finite number;

(A2) $\alpha = (\alpha_{ij})_{i,j=1,2,3}$ is a second order symmetric tensor with constant α_{ij};

(A3) The fourth order tensors A_1 and A_2 are defined by (1.5) with the coefficients μ_p, λ_p, $p = 1, 2$, satisfying (1.6).

We list the implications of assumption **(A3)** which are used in further analysis. The condition (1.6) ensures the symmetry of tensors A_p:

$$(A_p)_{ijkl} = (A_p)_{jikl} = (A_p)_{klij}, \quad p = 1, 2, \tag{1.9}$$

and their coercivity and boundedness

$$a_{p*}|\varepsilon|^2 \le (A_p\varepsilon)\cdot\varepsilon \le a_p^*|\varepsilon|^2, \quad p = 1, 2, \tag{1.10}$$

where

$$a_{p*} = \min\{3\lambda_p + 2\mu_p, 2\mu_p\}, \quad a_p^* = \max\{3\lambda_p + 2\mu_p, 2\mu_p\}.$$

Moreover, (1.6) ensures the following properties of operators Q_p, $p = 1, 2$:

— Q_p are strongly elliptic (property holding true under weaker assumption $\mu_p > 0$, $\lambda_p + 2\mu_p > 0$, (see [20], Sect. 7)) and satisfy the estimate [17], Lemma 3.2:

$$c_p\|u\|_{H^2(\Omega)} \le \|Q_p u\|_{L_2(\Omega)} \quad \text{for} \ \ u \in D(Q_p), \ \ p = 1, 2, \tag{1.11}$$

with positive constants c_p depending on Ω. Since clearly,

$$\|Q_p u\|_{L_2(\Omega)} \le \bar{c}_p\|u\|_{H^2(\Omega)},$$

it follows that the norms $\|Q_p u\|_{L_2(\Omega)}$ and $\|u\|_{H^2(\Omega)}$ are equivalent on $D(Q_p)$;

— the operators Q_p are self-adjoint on $D(Q_p)$:

$$\begin{aligned}(Q_p u, v)_{L_2(\Omega)} &= -\mu_p(\nabla u, \nabla v)_{L_2(\Omega)} - (\lambda_p + \mu_p)(\nabla\cdot u, \nabla\cdot v)_{L_2(\Omega)} \\ &= (u, Q_p v)_{L_2(\Omega)} \quad \text{for} \ \ u, v \in D(Q_p);\end{aligned} \tag{1.12}$$

— the operators $-Q_p$ are positive on $D(Q_p)$:

$$\begin{aligned}(-Q_p u, u) = \mu_p\|\nabla u\|^2_{L_2(\Omega)} + (\lambda_p + \mu_p)\|\nabla\cdot u\|^2_{L_2(\Omega)} \ge 0 \\ \text{for} \ \ u \in D(Q_p).\end{aligned} \tag{1.13}$$

Hence, there exist fractional powers $Q_p^{1/2}$ with the domains $D(Q_p^{1/2}) = H_0^1(\Omega)$, satisfying

$$\begin{aligned}(Q_p^{1/2}u, Q_p^{1/2}v)_{L_2(\Omega)} = (-Q_p u, v)_{L_2(\Omega)} = (u, -Q_p v)_{L_2(\Omega)} \\ \text{for} \ \ u, v \in D(Q_p).\end{aligned} \tag{1.14}$$

Let us also notice that by (1.10) and the Korn inequality

$$d^{1/2}\|u\|_{H^1(\Omega)} \le \|\varepsilon(u)\|_{L_2(\Omega)} \quad \text{for} \quad u \in H_0^1(\Omega), \ \ d > 0, \tag{1.15}$$

it follows that

$$\begin{aligned}\|Q_p^{1/2}u\|^2_{L_2(\Omega)} &= \mu_p\|\nabla u\|^2_{L_2(\Omega)} + (\lambda_p + \mu_p)\|\nabla\cdot u\|^2_{L_2(\Omega)} \\ &= (A_p\varepsilon(u), \varepsilon(u))_{L_2(\Omega)} \ge a_{p*}\|\varepsilon(u)\|^2_{L_2(\Omega)} \ge a_{p*}d\|u\|^2_{H^1(\Omega)}.\end{aligned} \tag{1.16}$$

Thus, the norms $\|Q_p^{1/2}u\|_{L_2(\Omega)}$ and $\|u\|_{H^1(\Omega)}$ are equivalent on $D(Q_p^{1/2})$.

1.4. Main result.

THEOREM A. *(Existence) Let the assumptions (A1)–(A3) hold, $S \in C^2$, $T > 0$ finite and*

$$u_0 \in W^2_{12}(\Omega), \quad u_1 \in B^{11/6}_{12,12}(\Omega), \quad \theta_0 \in B^{5/3}_{6,6}(\Omega),$$
$$g \in L_{\infty,12}(\Omega^T), \quad b \in L_{12}(\Omega^T), \qquad g \geq 0, \quad \theta_0 \geq \underline{\theta} > 0,$$

where $\underline{\theta}$ is a constant. Then there exists a solution to problem (1.1)–(1.4) such that $u \in C([0,T]; W^2_{12}(\Omega))$, $u_t \in W^{2,1}_{12}(\Omega^T)$, $\theta \in W^{2,1}_6(\Omega^T)$ and $\theta(t) \geq \underline{\theta}\exp(-c_0 T) \equiv \theta_* > 0$, where the positive constant c_0 depends on a_{1*}, a_2^*, $|\alpha|$, c_v.
Moreover, the following estimates are satisfied

$$\|u\|_{C([0,T];W^2_{12}(\Omega))} \leq c\|u_t\|_{W^{2,1}_{12}(\Omega^T)},$$
$$\|u_t\|_{W^{2,1}_{12}(\Omega^T)} + \|\theta\|_{W^{2,1}_6(\Omega^T)} \leq \varphi(T, \|u_0\|_{W^2_{12}(\Omega)} + \|u_1\|_{B^{11/6}_{12,12}(\Omega)}$$
$$+ \|\theta_0\|_{B^{5/3}_{6,6}(\Omega)} + \|b\|_{L_{12}(\Omega)} + \|g\|_{L_{\infty,12}(\Omega^T)}),$$

where φ is an increasing positive function of its arguments.

THEOREM B. *(Uniqueness) Let us assume that tensors A_p, $p = 1,2$, satisfy (1.10). Then any solution (u, θ) to problem (1.1)–(1.4) satisfying*

$$\begin{aligned} &\varepsilon_t \in L_2(0,T; L_3(\Omega)), \\ &\theta \in L_2(0,T; L_\infty(\Omega)), \quad \theta_t \in L_2(0,T; L_3(\Omega)), \\ &0 < \theta_* < \theta, \end{aligned} \tag{1.17}$$

is uniquely defined.

COROLLARY 1.1. *The regular solution in Theorem A is uniquely defined.*

1.5. Relation to other results. We comment on the connections of our result to the two other global existence results in three space dimensions. Firstly, we mention the result by Roubíček [21] who proved the existence of a very weak solution to the thermoviscoelasticity system (1.1)–(1.2) involving monotone viscosity of a p-Laplacian type, $(A_1\varepsilon_t) \cdot \varepsilon_t \sim |\varepsilon_t|^p$, and the specific heat having $(\omega - 1)$-polynomial growth, $c_*(\theta) \sim c_v\theta^{\omega-1}$. This result, based on the Galerkin method, was obtained for L^1-data under the conditions $p \geq 2$, $\omega \geq 1$ and $p > 1 + \frac{3}{2\omega}$ (in 3-D). In the case of linear viscosity, $p = 2$, the latter condition implies that $\omega > 3/2$, that is the growth of the specific heat is greater than 1/2.

Our result concerns the case $p = 2$ and $\omega = 2$. We have to restrict ourselves to the linear viscosity, $p = 2$, because the proof relies on the results by Krylov [13] and Solonnikov [24] on the solvability of the linear problem

$$u_{tt} - \nabla \cdot A_1\varepsilon(u_t) = f$$

with the boundary and initial conditions (1.3), (1.4) (see Lemma 3.4).
Concerning the specific heat growth exponent, $\omega - 1$, it seems that after some additional technical effort it would be possible to admit $\omega < 2$. However, in the case of a constant specific heat, i.e., $\omega = 1$, we have been faced with a serious mathematical obstacle.

As already mentioned in Subsection 1.1, the local existence result in such a case was obtained by Bonetti and Bonfanti [3].

Secondly, we recall the multidimensional result by Blanchard and Guibé [2] who addressed problem (1.1)–(1.2) equally in the prototype case $p = 2$ and $\omega = 2$, and in the more general setting involving linear viscosity, $p = 2$, specific heat with $(\omega - 1)$-polynomial growth and a nonlinear thermoelastic coupling; more precisely, the term $\nabla\theta$ in (1.1) was replaced by $\nabla f(\theta)$ with f hawing an α-polynomial growth. The existence of solutions in the weak-renormalized sense was proved there by the Schauder fixed point theorem. It is worth to remark that in the case of the linear thermoelastic coupling, $\alpha = 1$, the result in [2] requires the specific heat to have growth of the order greater than $1/2$, as in the result by Roubíček [21].

1.6. Outline. In Section 2 we present a thermodynamic basis of system (1.1), (1.2). Section 3 recalls basic results on the Sobolev spaces with a mixed norm and on the solvability of boundary-value problems for linear parabolic equations in such spaces. In Section 4 we derive a priori estimates for problem (1.1)–(1.4). The procedure consists in a recursive improvement of the basic energy estimates. The main tool in this procedure are the results on the solvability of linear parabolic problems in Sobolev spaces with a mixed norm. Section 5 presents the proof of Theorem A, which is based on the successive approximation method. The proof of the uniqueness, stated in Theorem B, is given in Section 6.

Since a priori estimates in Section 4 are crucial for the proof of the global existence we advertise here the main steps of the procedure of deriving such estimates. First we prove the energy type estimate (see Lemma 4.2)

$$\|u_t\|_{L_{2,\infty}(\Omega^T)} + \|\varepsilon\|_{L_{2,\infty}(\Omega^T)} + \|\theta\|_{L_{2,\infty}(\Omega^T)} \le \text{data}. \tag{1.18}$$

In Lemma 4.6 we show the estimates

$$\|u_t\|_{L_{2,\infty}(\Omega^T)} + \|u\|_{L_\infty(0,T;H^1(\Omega))} + \|u_t\|_{L_2(0,T;H^1(\Omega))} \le c\|\theta\|_{L_2(\Omega^T)} + \text{data},$$

and

$$\|u_t\|_{L_\infty(0,T;H^1(\Omega))} + \|u\|_{L_\infty(0,T;H^2(\Omega))} + \|u_t\|_{L_2(0,T;H^2(\Omega))} \le c\|\nabla\theta\|_{L_2(\Omega^T)} + \text{data}.$$

The norms of θ will be later removed by some interpolation inequalities based on estimate (1.18).
In Lemma 4.7 we obtain the estimate

$$\|\theta\|_{L_\infty(0,T;L_3(\Omega))} + \|\theta\|_{L_2(0,T;H^1(\Omega))} + \|\varepsilon_t\|_{V_2(\Omega^T)} \le \text{data}, \tag{1.19}$$

and next in Lemma 4.8,

$$\|\theta_t\|_{L_2(\Omega^T)} + \|\nabla\theta\|_{L_\infty(0,T;L_2(\Omega))} \le \text{data}.$$

To deduce the boundedness of θ we first prove in Lemma 4.10 the estimate

$$\|\theta\|_{L_{r,\infty}(\Omega^T)} \le \text{data}, \quad r < \infty, \tag{1.20}$$

and in Lemma 4.17,

$$\|\theta\|_{L_\infty(\Omega^T)} \le \text{data}. \tag{1.21}$$

To get (1.21) we make use of the important inequality (see Lemma 4.9)

$$\|\varepsilon_t\|_{L_{p,\sigma}(\Omega^T)} \le c(t)(\|\theta\|_{L_{p,\sigma}(\Omega^T)} + \text{data}), \quad p,\sigma \in (1,\infty),$$

and the fact that the coefficient near θ_t in (1.2) is proportional to θ.
To establish the continuity of θ (proved in Lemma 4.10) we need (1.21), estimates $\|\varepsilon_t\|_{L_2(0,T;L_\infty(\Omega))} \le$ data (see Corollary 4.16), and

$$\|\theta\|_{W_2^{2,1}(\Omega^T)} \le \text{data} \quad \text{(see Corollary 4.18)}.$$

Having the previous estimates for ε_t and the continuity of θ we finally prove in Lemma 4.23 that

$$\theta \in W_6^{2,1}(\Omega^T), \quad u_t \in W_{12}^{2,1}(\Omega^T).$$

2. Thermodynamic basis. System (1.1), (1.2) represents balance laws for the linear momentum and energy in a referential description, with the referential mass density assumed constant, normalized to unity, $\rho_0 = 1$:

$$\begin{aligned} u_{tt} - \nabla\cdot S &= b, \\ e_t + \nabla\cdot q - S\cdot\varepsilon_t &= g, \end{aligned} \tag{2.1}$$

where S is the stress tensor, q – the referential heat flux, and e – the specific internal energy.

The system is governed by two thermodynamic potentials: the free energy $f = \hat{f}(\varepsilon,\theta)$, which by a thermodynamic requirement is strictly concave with respect to θ, and the dissipation potential (called pseudopotential of dissipation in [10], [3]) $\mathcal{D} = \hat{\mathcal{D}}(\varepsilon_t, \nabla\theta; \varepsilon, \theta)$, which by a thermodynamic requirement is nonnegative, convex in $(\varepsilon_t, \nabla\theta)$ and such that $\hat{\mathcal{D}}(0,0;\varepsilon,\theta) = 0$.

In the case of (1.1), (1.2) the free energy is specified by

$$f(\varepsilon,\theta) = f_*(\theta) + W(\varepsilon,\theta), \tag{2.2}$$

where

$$f_*(\theta) = -\frac{1}{2}c_v\theta^2, \quad c_v = \text{const} > 0, \tag{2.3}$$

is the caloric energy, and

$$\begin{aligned} W(\varepsilon,\theta) &= \frac{1}{2}(\varepsilon - \theta\alpha)\cdot A_2(\varepsilon-\theta\alpha) - \frac{\theta^2}{2}\alpha\cdot(A_2\alpha) \\ &= \frac{1}{2}\varepsilon\cdot(A_2\varepsilon) - \theta\varepsilon\cdot(A_2\alpha) \end{aligned} \tag{2.4}$$

is the elastic energy; we recall that A_2 stands for the fourth order elasticity tensor and α for the second order thermal expansion tensor.

The caloric energy (2.3) is associated with temperature-dependent caloric specific heat

$$c_*(\theta) = -\theta f_*''(\theta) = c_v\theta, \tag{2.5}$$

which gives rise to the term $c_v\theta\theta_t$ in energy equation (1.2).

We remark that in the case f_* is given by the standard formula

$$f_*(\theta) = -c_v\theta \log\frac{\theta}{\theta_1} + c_v\theta + \tilde{c}, \tag{2.6}$$

where c_v, θ_1, $\tilde{c}$ are positive constants, the caloric specific heat is constant

$$c_* = -\theta f_*''(\theta) = c_v. \tag{2.7}$$

This gives rise to the usual parabolic term $c_v\theta_t$ in (1.2) in place of $c_v\theta\theta_t$. As mentioned in Section 1, in the case of a constant caloric heat there are serious mathematical obstacles in the proof of the global existence.

A thermoviscoelastic system with the specific heat $c_*(\theta)$ given by (2.5) has been considered in [9], [2] and [21], where also more general forms of $c_*(\theta)$ have been analysed. Moreover, we mention that a fourth order thermoviscoelastic systems with temperature-dependent specific heat, arising in shape memory materials, have been studied in [27] and [19].

The dissipation potential corresponding to system (1.1), (1.2) is given by

$$\mathcal{D} = \frac{1}{2\theta}\varepsilon_t \cdot (A_1\varepsilon_1) + \frac{k}{2}\theta^2\left|\nabla\frac{1}{\theta}\right|^2 = \frac{1}{2\theta}\varepsilon_t \cdot (A_1\varepsilon_t) + \frac{k}{2}|\nabla \log\theta|^2, \tag{2.8}$$

where A_1 is the viscosity tensor and $k > 0$ the constant heat conductivity.

In accord with the basic thermodynamic relations the internal energy e and the entropy η are related to the free energy f by the equations

$$e = f + \theta\eta, \quad \eta = -f_{,\theta}. \tag{2.9}$$

For the free energy f defined by (2.2)–(2.4) this gives

$$e = \frac{1}{2}c_v\theta^2 + \frac{1}{2}\varepsilon \cdot (A_2\varepsilon), \quad \eta = c_v\theta + (A_2\alpha)\cdot\varepsilon. \tag{2.10}$$

As a consequence of the second law of thermodynamics expressed by the Clausius-Duhem inequality, the stress tensor S and the heat flux q satisfy the following relations:

$$S = \frac{\partial f}{\partial\varepsilon} + \theta\frac{\partial\mathcal{D}}{\partial\varepsilon_t}, \quad q = \frac{\partial\mathcal{D}}{\partial\nabla\frac{1}{\theta}}. \tag{2.11}$$

For f given by (2.2)–(2.4) and $\mathcal{D}$ by (2.8) the formulas (2.11) yield the standard forms of the stress tensor and the heat flux

$$S = A_2(\varepsilon - \theta\alpha) + A_1\varepsilon_t, \quad q = k\theta^2\nabla\frac{1}{\theta} = -k\nabla\theta. \tag{2.12}$$

Thus, S consists of two terms: the nondissipative equilibrium term determined by f, and the dissipative one determined by $\mathcal{D}$. The dissipative heat flux q is entirely determined by $\mathcal{D}$.

Inserting the relations $(2.10)_1$ and (2.12) into balance laws (2.1) one arrives at the system (1.1)–(1.2).

For further purposes (see Lemma 4.2) it is of interest to notice that on account of the identity

$$e_t = (f + \theta\eta)_t = f_t + \theta_t\eta + \theta\eta_t = \theta\eta_t + \frac{\partial f}{\partial\varepsilon}\cdot\varepsilon_t,$$

along with the relation $(2.11)_1$, the energy balance $(2.1)_2$ admits the form

$$\theta\eta_t + \nabla\cdot q = \theta\frac{\partial\mathcal{D}}{\partial\varepsilon_t}\cdot\varepsilon_t + g. \tag{2.13}$$

For $\mathcal{D}$ given by (2.8) this leads to the following equivalent form of equation (1.2):

$$\theta\eta_t - k\Delta\theta = (A_1\varepsilon_t)\cdot\varepsilon_t + g. \tag{2.14}$$

Let us also notice that assuming $\theta > 0$ and using $(2.11)_2$, the equation (2.13) may be expressed as

$$\eta_t + \nabla\cdot\frac{q}{\theta} = \sigma + \frac{g}{\theta}, \tag{2.15}$$

where

$$\sigma = \frac{\partial\mathcal{D}}{\partial\nabla\frac{1}{\theta}}\cdot\nabla\frac{1}{\theta} + \frac{\partial\mathcal{D}}{\partial\varepsilon_t}\cdot\varepsilon_t = k\theta^2\left|\nabla\frac{1}{\theta}\right|^2 + \frac{1}{\theta}(A_1\varepsilon_t)\cdot\varepsilon_t \geq 0$$

is the specific entropy production. From (2.15) it follows that system (1.1), (1.2) complies with the Clausius-Duhem inequality

$$\eta_t + \nabla\cdot\frac{q}{\theta} \geq \frac{g}{\theta}. \tag{2.16}$$

3. Notation and auxiliary results.

3.1. Notation. Let $\Omega\subset\mathbb{R}^n$ be an open bounded subset of $\mathbb{R}^n$, $n\geq 1$, with a smooth boundary S, and $\Omega^T = \Omega\times(0,T)$, $S^T = S\times(0,T)$, $T>0$ finite.
We introduce the following spaces: $W_p^k(\Omega)$, $k\in\mathbb{N}\cup\{0\}$, $p\in[1,\infty)$ – the Sobolev space on Ω with the finite norm

$$\|u\|_{W_p^k(\Omega)} = \bigg(\sum_{|\alpha|\leq k}\int_\Omega |D_x^\alpha u|^p dx\bigg)^{1/p},$$

where $\alpha = (\alpha_1,\ldots,\alpha_n)$ is a multiindex, $\alpha_i\in\mathbb{N}\cup\{0\}$, $|\alpha| = \alpha_1+\alpha_2+\ldots+\alpha_n$, $D_x^\alpha = \partial_{x_1}^{\alpha_1}\ldots\partial_{x_n}^{\alpha_n}$; $H^k(\Omega) = W_2^k(\Omega)$; $L_{p,p_0}(\Omega^T) = L_{p_0}(0,T;L_p(\Omega))$, $p,p_0\in[1,\infty)$ – the space of functions $u:\ (0,T)\to L_p(\Omega)$ with the finite norm

$$\|u\|_{L_{p,p_0}(\Omega^T)} = \bigg(\int_0^T \|u(t)\|_{L_p(\Omega)}^{p_0}dt\bigg)^{1/p_0};$$

$V_2(\Omega^T) = L_\infty(0,T;L_2(\Omega))\cap L_2(0,T;H^1(\Omega))$ – the space of functions $u:(0,T)\to H^1(\Omega)$ with the finite norm

$$\|u\|_{V_2(\Omega^T)} = \operatorname*{ess\,sup}_{t\in[0,T]}\|u(t)\|_{L_2(\Omega)} + \|\nabla u\|_{L_2(\Omega^T)};$$

$V_2^{1,0}(\Omega^T) = V_2(\Omega^T)\cap C([0,T];L_2(\Omega))$ – the space with the finite norm

$$\|u\|_{V_2^{1,0}(\Omega^T)} = \max_{t\in[0,T]}\|u(t)\|_{L_2(\Omega)} + \|\nabla u\|_{L_2(\Omega^T)};$$

$W^{k,k/2}_{p,p_0}(\Omega^T)$, $k, k/2 \in \mathbb{N} \cup \{0\}$, $p, p_0 \in [1,\infty)$ – the Sobolev space with a mixed norm, which is a completion of $C^\infty(\Omega^T)$-functions under the finite norm

$$\|u\|_{W^{k,k/2}_{p,p_0}(\Omega^T)} = \bigg(\int_0^T \bigg(\sum_{|\alpha|+2a\le k} \int_\Omega |D^\alpha_x \partial^a_t u|^p dx\bigg)^{p_0/p} dt\bigg)^{1/p_0};$$

$W^{s,s/2}_{p,p_0}(\Omega^T)$, $s \in \mathbb{R}_+$, $p, p_0 \in [1,\infty)$ – the Sobolev-Slobodecki space with the finite norm

$$\begin{aligned}\|u\|_{W^{s,s/2}_{p,p_0}(\Omega^T)} &= \sum_{|\alpha|+2a\le [s]} \|D^\alpha_x \partial^a_t u\|_{L_{p,p_0}(\Omega^T)} \\ &+ \bigg[\int_0^T \bigg(\int_\Omega\int_\Omega \sum_{|\alpha|+2a=[s]} \frac{|D^\alpha_x \partial^a_t u(x,t) - D^\alpha_{x'}\partial^a_t u(x',t)|^p}{|x-x'|^{n+p(s-[s])}} dx dx'\bigg)^{p_0/p} dt\bigg]^{1/p} \\ &+ \bigg[\int_0^T\int_0^T \bigg(\int_\Omega \sum_{|\alpha|+2a=[s]} \frac{|D^\alpha_x \partial^a_t u(x,t) - D^\alpha_x \partial^a_{t'} u(x,t')|^p}{|t-t'|^{1+p(\frac{s}{2}-[\frac{s}{2}])}} dx\bigg)^{p_0/p} dt dt'\bigg]^{1/p_0},\end{aligned}$$

where $a \in \mathbb{N} \cup \{0\}$ and $[s]$ is the integer part of s. For s odd the last term in the above norm vanishes whereas for s even the two last terms vanish.
$B^l_{p,p_0}(\Omega)$, $l \in \mathbb{R}_+$, $p, p_0 \in [1,\infty)$ – the Besov space with the finite norm

$$\|u\|_{B^l_{p,p_0}(\Omega)} = \|u\|_{L_p(\Omega)} + \bigg(\sum_{i=1}^n \int_0^\infty \frac{\|\Delta^m_i(h,\Omega)\partial^k_{x_i} u\|^{p_0}_{L_p(\Omega)}}{h^{1+(l-k)p_0}} dh\bigg)^{1/p_0},$$

where:

$k \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$, $m > l - k > 0$,
$\Delta^j_i(h,\Omega)u$, $j \in \mathbb{N}$, $h \in \mathbb{R}_+$, is the finite difference of the order j of the function $u(x)$ with respect to x_i, with

$$\begin{aligned}\Delta^1_i(h,\Omega)u &= \Delta_i(h,\Omega)u \\ &= u(x_1,\dots,x_{i-1},x_i+h,x_{i+1},\dots,x_n) - u(x_1,\dots,x_n),\end{aligned}$$

$$\Delta^j_i(h,\Omega)u = \Delta_i(h,\Omega)\Delta^{j-1}_i(h,\Omega)u,$$

and

$$\Delta^j_i(h,\Omega)u = 0 \quad \text{for} \quad x + jh \notin \Omega.$$

In [12] it has been proved that the norms of the Besov space $B^l_{p,p_0}(\Omega)$ are equivalent for different m and k satisfying the condition $m > l - k > 0$.

By c we denote a generic positive constant which changes its value from formula to formula and depends at most on the imbedding constants, constants of the considered problem and the regularity of the boundary.

By $\varphi = \varphi(\sigma_1,\dots,\sigma_k)$, $k \in \mathbb{N}$, we denote a generic function which is a positive increasing function of its arguments $\sigma_1,\dots,\sigma_k$, and may change its form from formula to formula.

3.2. Auxiliary results. We need the following interpolation lemma

LEMMA 3.1. *[1, Chap. 4, Sect. 18] Let* $u \in W^{s,s/2}_{p,p_0}(\Omega^T)$, $s \in \mathbb{R}_+$, $p, p_0 \in [1,\infty]$, $\Omega \subset \mathbb{R}^3$. *Let* $\sigma \in \mathbb{R}_+ \cup \{0\}$, *and*

$$\kappa = \frac{3}{p} + \frac{2}{p_0} - \frac{3}{q} - \frac{2}{q_0} + |\alpha| + 2a + \sigma < s.$$

Then $D^\alpha_x \partial^a_t u \in W^{\sigma,\sigma/2}_{q,q_0}(\Omega^T)$, $q \geq p$, $q_0 \geq p_0$, *and there exists* $\varepsilon \in (0,1)$ *such that*

$$\|D^\alpha_x \partial^a_t u\|_{W^{\sigma,\sigma/2}_{q,q_0}(\Omega^T)} \leq \varepsilon^{s-\kappa}\|u\|_{W^{s,s/2}_{p,p_0}(\Omega^T)} + c\varepsilon^{-\kappa}\|u\|_{L_{p,p_0}(\Omega^T)}.$$

We recall from [4] the trace and the inverse trace theorems for Sobolev spaces with a mixed norm.

LEMMA 3.2. *(Traces in* $W^{s,s/2}_{p,p_0}(\Omega^T)$*)*

(i) *Let* $u \in W^{s,s/2}_{p,p_0}(\Omega^T)$, $s \in \mathbb{R}_+$, $p, p_0 \in (1,\infty)$.
Then $u(x,t_0) \equiv u(x,t)|_{t=t_0}$ *for* $t_0 \in [0,T]$, *belongs to* $B^{s-2/p_0}_{p,p_0}(\Omega)$, *and*

$$\|u(\cdot,t_0)\|_{B^{s-2/p_0}_{p,p_0}(\Omega)} \leq c\|u\|_{W^{s,s/2}_{p,p_0}(\Omega^T)},$$

where constant c does not depend on u.

(ii) *For a given* $\tilde{u} \in B^{s-2/p_0}_{p,p_0}(\Omega)$, $s \in \mathbb{R}_+$, $s > 2/p_0$, $p, p_0 \in (1,\infty)$, *there exists a function* $u \in W^{s,s/2}_{p,p_0}(\Omega^T)$ *such that* $u|_{t=t_0} = \tilde{u}$ *for* $t_0 \in [0,T]$, *and*

$$\|u\|_{W^{s,s/2}_{p,p_0}(\Omega^T)} \leq c\|\tilde{u}\|_{B^{s-2/p_0}_{p,p_0}(\Omega)},$$

where constant c does not depend on $\tilde{u}$.

We recall also (see [1]) that if $l > 1/p$ then every function from $B^l_{p,p_0}(\Omega)$ has a trace on the boundary S belonging to $B^{l-1/p}_{p,p_0}(S)$, and

$$\|u\|_{B^{l-1/p}_{p,p_0}(S)} \leq c\|u\|_{B^l_{p,p_0}(\Omega)}.$$

We apply the following imbeddings between Besov spaces.

LEMMA 3.3. *[25, Theorem 4.6.1] Let* $\Omega \subset \mathbb{R}^n$ *be an arbitrary domain.*
(a) Let $s \in \mathbb{R}_+$, $\varepsilon > 0$, $p \in (1,\infty)$ *and* $1 \leq q_1 \leq q_2 \leq \infty$. *Then*

$$B^{s+\varepsilon}_{p,\infty}(\Omega) \subset B^s_{p,1}(\Omega) \subset B^s_{p,q_1}(\Omega) \subset B^s_{p,q_2}(\Omega) \subset B^s_{p,\infty}(\Omega) \subset B^{s-\varepsilon}_{p,1}(\Omega).$$

(b) Let $\infty > q \geq p > 1$, $1 \leq r \leq \infty$, $0 \leq t \leq s < \infty$, *and*

$$t + \frac{n}{p} - \frac{n}{q} \leq s.$$

Then

$$B^s_{p,r}(\Omega) \subset B^t_{q,r}(\Omega).$$

We recall now from [19] a result on the solvability of a linear parabolic system with elasticity operator $\boldsymbol{Q}$ in Sobolev space with a mixed norm. This result will be repeatedly used in Section 4 in deriving a priori estimates for viscoelasticity system

(1.1). It generalizes the result by Krylov [13] from the single parabolic equation to the following parabolic system

$$
\begin{aligned}
&u_t - Qu = f && \text{in } \Omega^T = \Omega \times (0,T),\\
&u = 0 && \text{on } S^T = S \times (0,T),\\
&u|_{t=0} = u_0 && \text{in } \Omega,
\end{aligned}
\tag{3.1}
$$

where $\Omega \subset \mathbb{R}^3$, $S = \partial\Omega$, $f = (f_i)$, and

$$Qu = \mu\Delta u + \nu\nabla(\nabla\cdot u)$$

with $\mu > 0$, $\nu > 0$. Let us notice that letting

$$Q \equiv Q_1, \quad \mu \equiv \mu_1, \quad \nu \equiv \lambda_1 + \mu_1,$$

the assumption (1.6) implies that $\mu > 0$ and $\nu > 0$.

LEMMA 3.4. *(Parabolic system in $W^{2,1}_{p,p_0}(\Omega^T)$ [13, 19, 24])*

(i) *Assume that $f \in L_{p,p_0}(\Omega^T)$, $u_0 \in B^{2-2/p_0}_{p,p_0}(\Omega)$, $p, p_0 \in (1,\infty)$, $S \in C^2$. If $2 - 2/p_0 - 1/p > 0$ the compatibility condition $u_0|_S = 0$ is assumed. Then there exists a unique solution to problem (3.1) such that $u \in W^{2,1}_{p,p_0}(\Omega^T)$ and*

$$\|u\|_{W^{2,1}_{p,p_0}(\Omega^T)} \le c(\|f\|_{L_{p,p_0}(\Omega^T)} + \|u_0\|_{B^{2-2/p_0}_{p,p_0}(\Omega)}) \tag{3.2}$$

with a constant c depending on Ω, S, p, p_0.

(ii) *Assume that $f = \nabla\cdot g + b$, $g = (g_{ij})$, $b = (b_i)$, $g, b \in L_{p,p_0}(\Omega^T)$, $u_0 \in B^{2-2/p_0}_{p,p_0}(\Omega)$. Assume the compatibility condition*

$$u_0|_S = 0 \quad \textit{if} \quad 1 - 2/p_0 - 1/p > 0.$$

Then there exists a unique solution to (3.1) such that $u \in W^{1,1/2}_{p,p_0}(\Omega^T)$ and

$$
\begin{aligned}
\|u\|_{W^{1,1/2}_{p,p_0}(\Omega^T)} &\le c(\|g\|_{L_{p,p_0}(\Omega^T)} + \|b\|_{L_{p,p_0}(\Omega^T)}\\
&\quad + \|u_0\|_{B^{2-2/p_0}_{p,p_0}(\Omega)})
\end{aligned}
\tag{3.3}
$$

with a constant c depending on Ω, S, p, p_0.

In the proof of Theorem A we shall apply also the following regularity result for a linear parabolic equation. This result is the special case of a more general theorem due to Denk-Hieber-Prüss [7, Theorem 2.3].

LEMMA 3.5. *(Parabolic equation in $W^{2,1}_{p,p_0}(\Omega^T)$) Let us consider the problem*

$$
\begin{aligned}
&\theta_t - \varrho\Delta\theta = g && \textit{in } \Omega^T,\\
&n\cdot\nabla\theta = 0 && \text{on } S^T,\\
&\theta|_{t=0} = \theta_0 && \textit{in } \Omega,
\end{aligned}
\tag{3.4}
$$

where $\varrho(x,t)$ is a continuous function on Ω^T such that $\inf_\Omega \varrho > 0$. Assume that $g \in L_{p,p_0}(\Omega^T)$, $\theta_0 \in B^{2-2/p_0}_{p,p_0}(\Omega)$, $p, p_0 \in (1,\infty)$, $S \in C^2$, and the corresponding compatibility condition is satisfied. Then there exists a unique solution to problem (3.4) such that $\theta \in W^{2,1}_{p,p_0}(\Omega^T)$ and

$$\|\theta\|_{W^{2,1}_{p,p_0}(\Omega^T)} \le c(\|g\|_{L_{p,p_0}(\Omega^T)} + \|\theta_0\|_{B^{2-2/p_0}_{p,p_0}(\Omega)}) \tag{3.5}$$

with a constant c depending on Ω, T, S, $\inf_{\Omega^T}\varrho$ and $\sup_{\Omega^T}\varrho$.

REMARK 3.6. *The constants c in Lemmas 3.4 and 3.5 do not depend on T. For T small the proof of this fact is evident whereas for T large it can be deduced by applying the same arguments as in the proof of Theorem 3.1.1 in [26, Ch. 3].*

4. A priori estimates. In this section we derive a priori estimates for solutions of problem (1.1)–(1.4) on an arbitrary finite time interval $(0,T)$. The estimates are essential for the existence proof by the successive approximation method, presented in Section 5.

The procedure of deriving a priori estimates consists in a recursive improvement of the basic energy estimates. The main tool used in this procedure is Lemma 3.4 which provides the solvability of the viscoelasticity system $(1.8)_1$ in the Sobolev space $W^{2,1}_{p,p_0}(\Omega^T)$. The applied procedure is aimed to establish the continuity of temperature θ and finally to apply Lemma 3.5 on the solvability of parabolic equation in Sobolev space $W^{2,1}_{q,q_0}(\Omega^T)$.

Throughout this section we assume that assumptions (A1)–(A3) (see Sect. 1.3) hold, and

$$\theta_0 \geq \underline{\theta} > 0 \quad \text{in } \Omega, \quad g \geq 0 \quad \text{in } \Omega^T, \tag{4.1}$$

where $\underline{\theta}$ is a positive constant.

First we prove the lower bound on θ by using similar arguments as in [16, Lemma 3.7], [27, Lemma 3.3].

LEMMA 4.1. *(Lower bound on θ) Let us assume that (4.1) holds. Then there exists a positive constant c depending only on parameters a_{1*}, a_2^* (from (1.10), $|\alpha|$ (see A2), c_v (see 1.2)), such that*

$$\theta(t) \geq \underline{\theta}\exp(-cT) \equiv \theta_* > 0 \quad \textit{for} \;\; t \in [0,T]. \tag{4.2}$$

Proof. For $m \in \mathbb{R}_+$ let us define the truncation

$$\theta_m = \max\left\{\theta, \frac{1}{m}\right\}$$

and

$$\Omega_m(t) = \left\{x \in \Omega : \; \theta(x,t) > \frac{1}{m}\right\}.$$

Multiplying (1.2) by $-\theta_m^{-\varrho}$ with $\varrho > 2$ (admissible test function) and integrating over $\Omega_m(t)$ gives

$$\begin{aligned} &- c_v \int\limits_{\Omega_m(t)} \theta\theta_t\theta_m^{-\varrho}dx + k \int\limits_{\Omega_m(t)} \theta_m^{-\varrho}\Delta\theta dx + \int\limits_{\Omega_m(t)} \frac{(A_1\varepsilon_t)\cdot\varepsilon_t}{\theta_m^{\varrho}}dx \\ &+ \int\limits_{\Omega_m(t)} \frac{g}{\theta_m^{\varrho}}dx = \int\limits_{\Omega_m(t)} \frac{\theta}{\theta_m^{\varrho}}(A_2\alpha)\cdot\varepsilon_t dx. \end{aligned} \tag{4.3}$$

The first term on the left-hand side of (4.3) is equal to

$$\begin{aligned} -c_v \int\limits_{\Omega_m(t)} \theta_m\theta_{m,t}\theta_m^{-\varrho}dx &= \frac{c_v}{\varrho-2} \int\limits_{\Omega_m(t)} \partial_t\theta_m^{2-\varrho}dx \\ &= \frac{c_v}{\varrho-2}\int\limits_{\Omega} \partial_t\theta_m^{2-\varrho}dx = \frac{c_v}{\varrho-2}\frac{d}{dt}\int\limits_{\Omega} \theta_m^{2-\varrho}dx, \end{aligned} \tag{4.4}$$

because $\partial_t \theta_m^{2-\varrho} = 0$ for $x \in \Omega \setminus \Omega_m(t) = \{x \in \Omega : \theta_m(t) = \frac{1}{m}\}$.
The second term on the left-hand side of (4.3) equals

$$(4.5) \qquad k \int\limits_{\Omega_m(t)} \theta_m^{-\varrho} \Delta\theta_m dx = k \int\limits_{\Omega} \theta_m^{-\varrho} \Delta\theta_m dx = \frac{4k\varrho}{(\varrho-1)^2} \int\limits_{\Omega} \left| \nabla\left(\frac{1}{\theta_m^{\frac{\varrho-1}{2}}} \right) \right|^2 dx,$$

because $\nabla\theta_m = \nabla\theta$ for $x \in \Omega_m(t)$ and $\nabla\theta_m = 0$ for $x \in \Omega \setminus \Omega_m(t)$. On account of (1.10) the third term on the left-hand side of (4.3) is bounded from below by

$$(4.6) \qquad \int\limits_{\Omega_m(t)} \frac{(A_1\varepsilon_t)\cdot\varepsilon_t}{\theta_m^\varrho} dx \geq a_{1*} \int\limits_{\Omega_m(t)} \frac{|\varepsilon_t|^2}{\theta_m^\varrho} dx,$$

and the fourth one by

$$(4.7) \qquad \int\limits_{\Omega_m(t)} \frac{g}{\theta_m^\varrho} dx \geq 0.$$

In view of the boundedness of A_2 and α, the integral on the right-hand side of (4.3) is estimated as follows:

$$(4.8) \qquad \begin{aligned} \int\limits_{\Omega_m(t)} \frac{\theta}{\theta_m^\varrho}(A_2\alpha)\cdot\varepsilon_t dx &= \int\limits_{\Omega_m(t)} \frac{\theta_m}{\theta_m^{\varrho/2}}(A_2\alpha)\cdot\frac{\varepsilon_t}{\theta_m^{\varrho/2}} dx \\ &\leq \frac{\delta}{2} \int\limits_{\Omega_m(t)} \frac{|\varepsilon_t|^2}{\theta_m^\varrho} dx + \frac{c}{2\delta} \int\limits_{\Omega_m(t)} \frac{\theta_m^2}{\theta_m^\varrho} dx, \quad \delta > 0. \end{aligned}$$

Now, setting $\delta/2 = a_{1*}$ and incorporating (4.4)–(4.8) into (4.3) we arrive at

$$\begin{aligned} \frac{c_v}{\varrho-2}\frac{d}{dt}\int\limits_{\Omega} \theta_m^{2-\varrho} dx &+ \frac{4k\varrho}{(\varrho-1)^2} \int\limits_{\Omega} \left| \nabla\left(\frac{1}{\theta_m^{\frac{\varrho-1}{2}}} \right) \right|^2 dx \\ &\leq c \int\limits_{\Omega_m(t)} \theta_m^{2-\varrho} dx \leq c \int\limits_{\Omega} \theta_m^{2-\varrho} dx, \end{aligned}$$

where in the last inequality we used the fact that $\theta_m > 0$ in Ω. Hence, by the Gronwall inequality, it follows that

$$\int\limits_{\Omega} \theta_m^{2-\varrho}(t) dx \leq \int\limits_{\Omega} \theta_m^{2-\varrho}(0) dx \exp\left[\frac{c(\varrho-2)}{c_v} t \right] \quad \text{for} \quad t \in [0,T],$$

that is,

$$(4.9) \qquad \|\theta_m^{-1}(t)\|_{L_{\varrho-2}(\Omega)} \leq \|\theta_m^{-1}(0)\|_{L_{\varrho-2}(\Omega)} \exp(cT)$$

with a constant c independent of ϱ and m. Letting $\varrho \to \infty$, (4.9) yields the bound

$$\theta_m(t) \geq \theta_m(0)\exp(-cT).$$

Further, letting $m \to \infty$ and noting that for sufficiently large m, $\theta_m(0) = \max\{\theta_0, \frac{1}{m}\} \geq \underline{\theta}$, we conclude the bound (4.2). □

LEMMA 4.2. *(Energy estimates) Let us assume that (4.1) holds, $\theta > 0$, and*

$$u_0 \in H^1(\Omega), \quad u_1 \in L_2(\Omega), \quad \theta_0 \in L_2(\Omega), \quad b \in L_{2,1}(\Omega^T), \quad g \in L_1(\Omega^T).$$

Then a sufficiently smooth solution (u,θ) to (1.1)–(1.4) satisfies the estimate

$$\begin{aligned} &\|u_{t'}\|_{L_{2,\infty}(\Omega^t)} + \|\varepsilon\|_{L_{2,\infty}(\Omega^t)} + \|\theta\|_{L_{2,\infty}(\Omega^t)} \\ &\quad + \|\theta^{-1}\nabla\theta\|_{L_2(\Omega^t)} + \|\theta^{-1/2}\varepsilon_{t'}\|_{L_2(\Omega^t)} \\ &\le c(\|\varepsilon(u_0)\|_{L_2(\Omega)} + \|u_1\|_{L_2(\Omega)} + \|\theta_0\|_{L_2(\Omega)} \\ &\quad + \|b\|_{L_{2,1}(\Omega^t)} + \|g\|_{L_1(\Omega^t)} + 1) \equiv cA_0, \end{aligned} \tag{4.10}$$

where $t \le T$.

Proof. Note that the positivity of θ is ensured by Lemma 4.1. Multiplying (1.1) by u_t, integrating over Ω and integrating by parts using boundary condition $(1.3)_1$, gives

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega |u_t|^2dx + \int_\Omega (A_1\varepsilon_t)\cdot\varepsilon_t dx + \int_\Omega A_2(\varepsilon - \theta\alpha)\cdot\varepsilon_t dx \\ &= \int_\Omega b\cdot u_t dx. \end{aligned} \tag{4.11}$$

Further, integrating (1.2) over Ω and by parts using $(1.3)_2$ yields

$$\frac{c_v}{2}\frac{d}{dt}\int_\Omega \theta^2 dx + \int_\Omega \theta(A_2\alpha)\cdot\varepsilon_t dx - \int_\Omega (A_1\varepsilon_t)\cdot\varepsilon_t dx = \int_\Omega g dx. \tag{4.12}$$

Next, multiplying (1.2) by $1/\theta$, integrating over Ω and integrating by parts leads to

$$\frac{d}{dt}\int_\Omega [c_v\theta + (A_2\alpha)\cdot\varepsilon]dx - k\int_\Omega \frac{|\nabla\theta|^2}{\theta^2}dx - \int_\Omega \frac{(A_1\varepsilon_t)\cdot\varepsilon_t}{\theta}dx = \int_\Omega \frac{g}{\theta}dx. \tag{4.13}$$

Adding by sides (4.11) and (4.12) gives

$$\frac{d}{dt}\int_\Omega \left[\frac{1}{2}c_v\theta^2 + \frac{1}{2}(A_2\varepsilon)\cdot\varepsilon + \frac{1}{2}|u_t|^2\right]dx = \int_\Omega (b\cdot u_t + g)dx. \tag{4.14}$$

Now, multiplying (4.13) by a positive constant β, and subtracting by sides from (4.14), we get

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \left[\frac{1}{2}c_v\theta^2 + \frac{1}{2}(A_2\varepsilon)\cdot\varepsilon + \frac{1}{2}|u_t|^2 - \beta c_v\theta - \beta(A_2\alpha)\cdot\varepsilon\right]dx \\ &\quad + k\beta\int_\Omega \frac{|\nabla\theta|^2}{\theta^2}dx + \beta\int_\Omega \frac{(A_1\varepsilon_t)\cdot\varepsilon_t}{\theta}dx = \int_\Omega \left[b\cdot u_t + \left(1 - \frac{\beta}{\theta}\right)g\right]dx. \end{aligned} \tag{4.15}$$

Integrating (4.15) with respect to time, noting that by the boundedness of A_2 and α,

$$\begin{aligned} &\frac{1}{2}c_v\theta^2 + \frac{1}{2}(A_2\varepsilon)\cdot\varepsilon + \frac{1}{2}|u_t|^2 - \beta c_v\theta - \beta(A_2\alpha)\cdot\varepsilon \\ &\qquad \ge \frac{1}{4}c_v\theta^2 + \frac{1}{4}(A_2\varepsilon)\cdot\varepsilon + \frac{1}{2}|u_t|^2 - c, \end{aligned}$$

and using the coercivity of A_1 (see (1.10)), we get

$$\begin{aligned}&\|\theta(t)\|^2_{L_2(\Omega)}+\|\varepsilon(t)\|^2_{L_2(\Omega)}+\|u_t(t)\|^2_{L_2(\Omega)}+\|\theta^{-1}\nabla\theta\|^2_{L_2(\Omega^t)}\\&\quad+\|\theta^{-1/2}\varepsilon_{t'}\|^2_{L_2(\Omega^t)}\le c(\|\theta_0\|^2_{L_2(\Omega)}+\|\varepsilon(u_0)\|^2_{L_2(\Omega)}+\|u_1\|^2_{L_2(\Omega)})\\&\quad+\|u_{t'}\|_{L_{2,\infty}(\Omega^t)}\|b\|_{L_{2,1}(\Omega^t)}+\|g\|_{L_1(\Omega^t)}+c\end{aligned}$$

for $t\le T$, with a constant c depending only on parameters.
Now, applying the Young inequality to the second term on the right-hand side of the above inequality yields (4.10). □

REMARK 4.3. *By integrating the identity (4.14) with respect to time one can immediately conclude that*

$$\|u_{t'}\|_{L_{2,\infty}(\Omega^t)}+\|\varepsilon\|_{L_{2,\infty}(\Omega^t)}+\|\theta\|_{L_{2,\infty}(\Omega^t)}\le cA_0, \tag{4.16}$$

which is a part of inequality (4.10).
On the contrary to (4.10) this inequality does not require the assumption $\theta>0$. We point out that in deriving further estimates we shall use just the bounds in (4.16) loosing the information contained in the two dissipative terms of (4.10). This information may be of importance in the analysis of the long time behaviour of solutions.

REMARK 4.4. *We complement Lemma 4.2 by some physical interpretations. In view of* $(2.10)_1$, *(2.12) and boundary conditions (1.3), identity (4.14) represents the balance equation for the total energy*

$$\frac{d}{dt}\int_\Omega\left(e+\frac12|u_t|^2\right)dx+\int_\Omega[-(Sn)\cdot u_t+n\cdot q]dS=\int_\Omega(b\cdot u_t+g)dx.$$

On the other hand, in view of $(2.10)_2$, $(2.12)_2$ *and the boundary condition* $(1.3)_2$, *identity (4.13) represents the balance equation for the entropy*

$$\frac{d}{dt}\int_\Omega\eta dx+\int_S n\cdot\frac{q}{\theta}dS=\int_\Omega\sigma dx+\int_\Omega\frac{g}{\theta}dx$$

with the entropy production

$$\sigma=\frac{k}{\theta^2}|\nabla\theta|^2+\frac1\theta(A_1\varepsilon_t)\cdot\varepsilon_t\ge0.$$

Equation (4.15) represents the so-called availability identity

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\left(e+\frac12|u_t|^2-\beta\eta\right)dx+\int_S\left[-(Sn)\cdot u_t+\left(1-\frac\beta\theta\right)n\cdot q\right]dS\\&\quad+\beta\int_\Omega\Sigma dx=\int_\Omega\left[b\cdot u_t+\left(1-\frac\beta\theta\right)g\right]dx,\end{aligned}\tag{4.17}$$

where $\beta=\text{const}>0$. *Hence, since $\sigma\ge0$, it follows that if the external sources vanish*

$$b=0,\quad g=0,$$

and if the boundary conditions on S imply that

$$(Sn)\cdot u_t=0,\quad\left(1-\frac\beta\theta\right)n\cdot q=0,\tag{4.18}$$

then

$$\frac{d}{dt}\int_\Omega \left(e+\frac{1}{2}|u_t|^2-\beta\eta\right)dx \le 0.$$

This provides the Lyapunov functional $\int_\Omega \left(e+\frac{1}{2}|u_t|^2-\beta\eta\right)dx$, *which is nonincreasing on solutions paths. Let us notice that the boundary conditions (1.3) ensure (4.18). The identity (4.17) has been used in deriving energy estimates in Lemma 4.2.*

Our goal now is to derive further regularity properties from the energy estimates (4.10). To this purpose we use the regularity results for parabolic systems in Sobolev space with a mixed norm, stated in Lemmas 3.4 and 3.5.

Let us consider the viscoelasticity system (1.1) with boundary and initial conditions $(1.3)_1$, $(1.4)_1$, expresses in the form:

$$\begin{aligned} &u_{tt}-Qu_t=\nabla\cdot[A_2(\varepsilon-\theta\alpha)]+b && \text{in } \Omega^T,\\ &u=0 && \text{on } S^T,\\ &u|_{t=0}=u_0,\quad u_t|_{t=0}=u_1 && \text{in } \Omega, \end{aligned} \tag{4.19}$$

where $Q=Q_1$ is the viscosity operator (1.7).

Applying Lemma 3.4 (i), (ii) to system (4.19) we deduce, in view of the boundedness of A_2 and α, the following

COROLLARY 4.5. *Let us assume that*

$$u_1\in B^{2-2/\sigma}_{p,\sigma}(\Omega),\quad b\in L_{p,\sigma}(\Omega^T),\quad p,\sigma\in(1,\infty),$$

and if $2-2/\sigma-1/p>0$ *then the compatibility condition* $u_0|_S=0$ *holds.*

(i) *If* $\varepsilon\in L_{p,\sigma}(\Omega^T)$ *and* $\theta\in L_{p,\sigma}(\Omega^T)$, $p,\sigma\in(1,\infty)$, *then the solution* u *to problem (4.19) satisfies*

$$\begin{aligned} \|\varepsilon_{t'}\|_{L_{p,\sigma}(\Omega^t)} &\le c\|u_{t'}\|_{W^{1,1/2}_{p,\sigma}(\Omega^T)} \le c(\|\varepsilon\|_{L_{p,\sigma}(\Omega^t)}+\|\theta\|_{L_{p,\sigma}(\Omega^t)}\\ &\quad+\|b\|_{L_{p,\sigma}(\Omega^t)}+\|u_1\|_{B^{2-2/\sigma}_{p,\sigma}(\Omega)}) \end{aligned} \tag{4.20}$$

for $t\in(0,T]$, *with a constant* c *depending on* Ω, S, T, p *and* σ.

(ii) *If* $\nabla\varepsilon\in L_{p,\sigma}(\Omega^T)$ *and* $\nabla\theta\in L_{p,\sigma}(\Omega^T)$, $p,\sigma\in(1,\infty)$, *then the solution* u *to (4.19) satisfies*

$$\begin{aligned} \|\varepsilon_{t'}\|_{W^{1,1/2}_{p,\sigma}(\Omega^t)} &\le c\|u_{t'}\|_{W^{2,1}_{p,\sigma}(\Omega^t)} \le c(\|\nabla\varepsilon\|_{L_{p,\sigma}(\Omega^t)}+\|\nabla\theta\|_{L_{p,\sigma}(\Omega^t)}\\ &\quad+\|b\|_{L_{p,\sigma}(\Omega^t)}+\|u_1\|_{B^{2-2/\sigma}_{p,\sigma}(\Omega)}) \end{aligned} \tag{4.21}$$

for $t\in(0,T]$, *with a constant* c *depending on* Ω, S, T, p *and* σ.

Using (4.10) in (4.20) for $p=2$ and σ arbitrary finite we have

$$\begin{aligned} \|\varepsilon_{t'}\|_{L_{2,\sigma}(\Omega^t)} &\le c(A_0+\|u_1\|_{B^{2-2/\sigma}_{2,\sigma}(\Omega)}+\|b\|_{L_{2,\sigma}(\Omega^t)})\\ &\equiv cA_1(\sigma),\quad \sigma\in(1,\infty),\quad t\le T, \end{aligned} \tag{4.22}$$

where A_0 is defined in (4.10).

For further purposes (see the proof of Lemma 4.7) we prepare now some inequalities between the norms of u and θ.

Let us consider viscoelasticity system (4.19) rewritten in the form

$$
(4.23)\qquad
\begin{aligned}
&u_{tt} - Qu_t - Qu = \nabla\cdot[(A_2 - A_1)\varepsilon - \theta A_2\alpha] + b && \text{in } \Omega^T,\\
&u = 0 && \text{on } S^T,\\
&u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1 && \text{in } \Omega,
\end{aligned}
$$

where $Q \equiv Q_1$.

LEMMA 4.6.

(i) *Let* $u_0 \in H_0^1(\Omega)$, $u_1 \in L_2(\Omega)$, $b \in L_2(\Omega^T)$ *and* $\theta \in L_2(\Omega^T)$. *Then a solution* u *to system (4.23) satisfies*

$$
(4.24)\qquad
\begin{aligned}
&\|u_{t'}\|_{L_{2,\infty}(\Omega^t)} + \|Q^{1/2}u\|_{L_{2,\infty}(\Omega^t)} + \|Q^{1/2}u_{t'}\|_{L_2(\Omega^t)}\\
&\le c(t)(\|\theta\|_{L_2(\Omega^t)} + \|Q^{1/2}u_0\|_{L_2(\Omega)} + \|u_1\|_{L_2(\Omega)}\\
&\quad + \|b\|_{L_2(\Omega^t)}) \qquad \text{for } t \le T,
\end{aligned}
$$

with a constant $c(t)$ *exponentially depending on* t.

(ii) *Let* $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$, $b \in L_2(\Omega^T)$ *and* $\nabla\theta \in L_2(\Omega)$. *Then a solution* u *to system (4.23) satisfies*

$$
(4.25)\qquad
\begin{aligned}
&\|Q^{1/2}u_{t'}\|_{L_{2,\infty}(\Omega^t)} + \|Qu\|_{L_{2,\infty}(\Omega^t)} + \|Qu_{t'}\|_{L_2(\Omega^t)}\\
&\le c(t)(\|\nabla\theta\|_{L_2(\Omega^t)} + \|Q^{1/2}u_1\|_{L_2(\Omega)} + \|Qu_0\|_{L_2(\Omega)}\\
&\quad + \|b\|_{L_2(\Omega^t)}) \qquad \text{for } t \le T,
\end{aligned}
$$

with a constant $c(t)$ *exponentially depending on* t.

Proof. (i) Multiplying $(4.23)_1$ by u_t, integrating over Ω and using the boundary condition $(4.23)_2$ gives

$$
(4.26)\qquad
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega (|u_t|^2 + |Q^{1/2}u|^2)dx + \int_\Omega |Q^{1/2}u_t|^2 dx\\
&= \int_\Omega [(A_1 - A_2)\varepsilon + \theta A_2\alpha]\cdot\varepsilon_t dx + \int_\Omega b\cdot u_t dx.
\end{aligned}
$$

Using the estimate

$$
\int_\Omega [(A_1 - A_2)\varepsilon + \theta A_2\alpha]\cdot\varepsilon_t dx \le \delta_1\int_\Omega |Q^{1/2}u_t|^2 dx + c(1/\delta_1)\int_\Omega (|\varepsilon|^2 + \theta^2)dx,
$$

which results on account of the Young inequality and (1.16), we conclude that

$$
(4.27)\qquad
\begin{aligned}
&\frac{d}{dt}\int_\Omega (|u_t|^2 + |Q^{1/2}u|^2)dx + \int_\Omega |Q^{1/2}u_t|^2 dx\\
&\le c\int_\Omega (\theta^2 + |b|^2)dx + c_1\int_\Omega (|u_t|^2 + |Q^{1/2}u|^2)dx,
\end{aligned}
$$

where we distinguished the constant c_1.
Hence, omitting the last integral on the left-hand side, it follows that

$$
\frac{d}{dt}\left[\int_\Omega (|u_t|^2 + |Q^{1/2}u|^2)dx e^{-c_1 t}\right] \le ce^{-c_1 t}\int_\Omega (\theta^2 + |b|^2)dx,
$$

which after integrating with respect to $t' \in (0,t)$ gives

$$\begin{aligned}\int\limits_{\Omega}(|u_t|^2 + |Q^{1/2}u|^2)dx &\le ce^{c_1 t}\int\limits_{\Omega^t}(\theta^2 + |b|^2)dxdt' \\ &+ e^{c_1 t}\int\limits_{\Omega}(|u_1|^2 + |Q^{1/2}u_0|^2)dx.\end{aligned} \tag{4.28}$$

Now, using (4.28) in (4.27) and again integrating the result with respect to $t' \in (0,t)$ leads to

$$\begin{aligned}&\int\limits_{\Omega}(|u_t|^2 + |Q^{1/2}u|^2)dx + \int\limits_{\Omega^t}|Q^{1/2}u_t|^2dxdt' \\ &\le c(te^{c_1 t}+1)\int\limits_{\Omega^t}(\theta^2 + |b|^2)dxdt' + (te^{c_1 t}+1)\int\limits_{\Omega}(|u_1|^2 + |Q^{1/2}u_0|^2)dx,\end{aligned}$$

which proves (4.24).

(ii) Multiplying $(4.23)_1$ by Qu_t, integrating over Ω and integrating by parts yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega}(|Q^{1/2}u_t|^2 + |Qu|^2)dx + \int\limits_{\Omega}|Qu_t|^2dx \\ &= \int\limits_{\Omega}(\nabla\cdot[(A_1 - A_2)\varepsilon + \theta A_2\alpha])\cdot Qu_t dx - \int\limits_{\Omega} b\cdot Qu_t dx \equiv \mathcal{R}.\end{aligned} \tag{4.29}$$

In view of the boundedness of A_1, A_2 and α,

$$\mathcal{R} \le \delta_2\int\limits_{\Omega}|Qu_t|^2dx + c(1/\delta_2)\int\limits_{\Omega}(|\nabla\varepsilon|^2 + |\nabla\theta|^2 + |b|^2)dx.$$

Hence, choosing δ_2 suffciently small and recalling the ellipticity of the operator Q, we get

$$\begin{aligned}&\frac{d}{dt}\int\limits_{\Omega}(|Q^{1/2}u_t|^2 + |Qu|^2)dx + \int\limits_{\Omega}|Qu_t|^2dx \\ &\le c\int\limits_{\Omega}(|\nabla\theta|^2 + |b|^2)dx + c_2\int\limits_{\Omega}(|Q^{1/2}u_t|^2 + |Qu|^2)dx,\end{aligned} \tag{4.30}$$

where we distinguished the constant c_2. Omitting the last integral on the left-hand side the latter inequality leads to

$$\frac{d}{dt}\left[\int\limits_{\Omega}(|Q^{1/2}u_t|^2 + |Qu|^2)dxe^{-c_2 t}\right] \le ce^{-c_2 t}\int\limits_{\Omega}(|\nabla\theta|^2 + |b|^2)dx,$$

which after integrating with respect to $t' \in (0,t)$, leads to

$$\begin{aligned}\int\limits_{\Omega}(|Q^{1/2}u_t|^2 + |Qu|^2)dx &\le ce^{c_2 t}\int\limits_{\Omega^t}(|\nabla\theta|^2 + |b|^2)dxdt' \\ &+ e^{c_2 t}\int\limits_{\Omega}(|Q^{1/2}u_1|^2 + |Qu_0|^2)dx.\end{aligned} \tag{4.31}$$

Inserting (4.31) into (4.30) and again integrating the result with respect to $t' \in (0,t)$ gives

$$\int_{\Omega}(|Q^{1/2}u_t|^2 + |Qu|^2)dx + \int_{\Omega^t}|Qu_{t'}|^2dxdt'$$
$$\le c(te^{c_2t}+1)\int_{\Omega^t}(|\nabla\theta|^2 + |b|^2)dxdt' + (te^{c_2t}+1)\int_{\Omega}(|Q^{1/2}u_1|^2 + |Qu_0|^2)dx,$$

which proves (4.25). □

From (4.24) we conclude that

$$\begin{aligned}(4.32)\qquad &\|u_{t'}\|_{L_{2,\infty}(\Omega^t)} + \|u\|_{L_\infty(0,t;H^1(\Omega))} + \|u_{t'}\|_{L_2(0,t;H^1(\Omega))}\\ &\le c(t)(\|\theta\|_{L_2(\Omega^t)} + \|u_0\|_{H^1(\Omega)} + \|u_1\|_{L_2(\Omega)} + \|b\|_{L_2(\Omega^t)}), \quad t\le T.\end{aligned}$$

Similarly, from (4.25) it follows that

$$\begin{aligned}(4.33)\qquad &\|u_{t'}\|_{L_\infty(0,t;H^1(\Omega))} + \|u_{t'}\|_{L_2(0,t;H^2(\Omega))} + \|u\|_{L_\infty(0,t;H^2(\Omega))}\\ &\le c(t)(\|\nabla\theta\|_{L_2(\Omega^t)} + \|u_0\|_{H^2(\Omega)} + \|u_1\|_{H^1(\Omega)} + \|b\|_{L_2(\Omega^t)}), \quad t\le T.\end{aligned}$$

Hence, by the definition of ε,

$$\begin{aligned}(4.34)\qquad &\|\varepsilon_{t'}\|_{V_2(\Omega^t)} \equiv \|\varepsilon_{t'}\|_{L_{2,\infty}(\Omega^t)} + \|\varepsilon_{t'}\|_{L_2(0,t;H^1(\Omega))}\\ &\le c(t)(\|\nabla\theta\|_{L_2(\Omega^t)} + \|u_0\|_{H^2(\Omega)} + \|u_1\|_{H^1(\Omega)} + \|b\|_{L_2(\Omega^t)}),\end{aligned}$$

where $t \le T$. With the help of this inequality we prove

LEMMA 4.7. *Assume that* $u_0 \in H^2(\Omega)$, $u_1 \in B^{2-1/5}_{2,10}(\Omega)$, $\theta_0 \in L_3(\Omega)$, $b \in L_{2,10}(\Omega^t)$, $g \in L_{2,1}(\Omega^t)$, $t \le T$.
Then

$$(4.35)\qquad \|\theta(t)\|_{L_3(\Omega)} + \|\theta\|_{L_2(0,t;H^1(\Omega))} + \|\varepsilon_{t'}\|_{V_2(\Omega^t)} \le cA_2, \quad t \le T,$$

where

$$\begin{aligned}(4.36)\qquad A_2 &= \varphi(A_0, A_1(10), \|u_0\|_{H^2(\Omega)}, \|u_1\|_{H^1(\Omega)}, \|\theta_0\|_{L_3(\Omega)}, \|b\|_{L_2(\Omega^t)}, \|g\|_{L_{2,1}(\Omega^t)})\\ &\le \varphi(\|u_0\|_{H^2(\Omega)}, \|u_1\|_{B^{2-1/5}_{2,10}(\Omega)}, \|\theta_0\|_{L_3(\Omega)}, \|b\|_{L_{2,10}(\Omega^t)}, \|g\|_{L_{2,1}(\Omega^t)}) \equiv A_3,\end{aligned}$$

with $A_1(\cdot)$ *defined in (4.22) and* A_0 *in (4.10).*

Proof. Multiplying (1.2) by θ and integrating over Ω^t we get

$$\begin{aligned}(4.37)\qquad &\|\theta(t)\|^3_{L_3(\Omega)} + \int_0^t \|\nabla\theta(t')\|^2_{L_2(\Omega)}dt'\\ &\le c\int_0^t\int_{\Omega}\theta^2|\varepsilon_{t'}|dxdt' + c\int_0^t\int_{\Omega}\theta|\varepsilon_{t'}|^2dxdt' + c\int_0^t\int_{\Omega}\theta|g|dxdt' + \|\theta_0\|^3_{L_3(\Omega)}.\end{aligned}$$

With the use of the Hölder inequality the first term on the right-hand side of (4.37) is estimated by

$$\begin{aligned}
&\int_0^t \|\theta\|_{L_3(\Omega)}\|\theta\|_{L_6(\Omega)}\|\varepsilon_{t'}\|_{L_2(\Omega)}dt' \\
&\le \sup_t \|\theta(t)\|_{L_3(\Omega)}\int_0^t \|\theta(t')\|_{L_6(\Omega)}\|\varepsilon_{t'}(t')\|_{L_2(\Omega)}dt' \\
&\le \sup_t \|\theta(t)\|_{L_3(\Omega)}\|\theta\|_{L_{6,2}(\Omega^t)}\|\varepsilon_{t'}\|_{L_2(\Omega^t)} \\
&\le \delta_1 \sup_t \|\theta(t)\|^3_{L_3(\Omega)} + c(1/\delta_1)\|\theta\|^{3/2}_{L_{6,2}(\Omega^t)}\|\varepsilon_{t'}\|^{3/2}_{L_2(\Omega^t)} \equiv I_1.
\end{aligned}$$

Using (4.22) for $\sigma = 2$ yields

$$I_1 \le \delta_1 \sup_t \|\theta(t)\|^3_{L_3(\Omega)} + c(1/\delta_1)\|\theta\|^{3/2}_{L_2(0,t;H^1(\Omega))}A_1^{3/2}(2).$$

The second term on the right-hand side of (4.37) is estimated by

$$\begin{aligned}
&\int_0^t \|\theta\|_{L_6(\Omega)}\|\varepsilon_{t'}\|_{L_3(\Omega)}\|\varepsilon_{t'}\|_{L_2(\Omega)}dt' \\
&\le \|\theta\|_{L_{6,2}(\Omega^t)}\|\varepsilon_{t'}\|_{L_{3,s_1}(\Omega^t)}\|\varepsilon_{t'}\|_{L_{2,\sigma_1}(\Omega^t)} \\
&\le \delta_2\|\theta\|^2_{L_2(0,t;H^1(\Omega))} + c(1/\delta_2)\|\varepsilon_{t'}\|^2_{L_{3,s_1}(\Omega^t)}A_1^2(\sigma_1),
\end{aligned}$$

where (4.22)was used with $\sigma = \sigma_1$, $\frac{1}{s_1} + \frac{1}{\sigma_1} = \frac{1}{2}$, $s_1 > 2$ but close to 2, because σ_1 can be an arbitrary positive finite number.
Now we examine the integral

$$\begin{aligned}
&\left(\int_0^t \left|\int_\Omega |\varepsilon_{t'}|^3 dx\right|^{s_1/3} dt'\right)^{1/s_1} \le \left(\int_0^t \|\varepsilon_{t'}(t')\|^{2s_1/3}_{L_4(\Omega)}\|\varepsilon_{t'}(t')\|^{s_1/3}_{L_2(\Omega)}dt'\right)^{1/s_1} \\
&\le \left(\int_0^t \|\varepsilon_{t'}(t')\|^{2s_1\lambda_1/3}_{L_4(\Omega)}dt'\right)^{1/s_1\lambda_1}\left(\int_0^t \|\varepsilon_{t'}(t')\|^{s_1\lambda_2/3}_{L_2(\Omega)}dt'\right)^{1/s_1\lambda_2} \equiv I_2,
\end{aligned}$$

where $1/\lambda_1 + 1/\lambda_2 = 1$. Setting $s_1\lambda_1 = 3$, $s_1\lambda_2 = \frac{3s_1}{3-s_1}$, we obtain

$$I_2 = \left(\int_0^t \|\varepsilon_{t'}(t')\|^2_{L_4(\Omega)}dt'\right)^{1/3}\left[\left(\int_0^t \|\varepsilon_{t'}(t')\|^{\frac{s_1}{3-s_1}}_{L_2(\Omega)}dt'\right)^{\frac{3-s_1}{s_1}}\right]^{1/3}.$$

Now, using (4.22) with $\sigma = \frac{s_1}{3-s_1}$ gives

$$I_2 \le c\left(\int_0^t \|\varepsilon_{t'}(t')\|^2_{H^1(\Omega)}dt'\right)^{1/3} A_1^{1/3}\left(\frac{s_1}{3-s_1}\right)$$

for any $s_1 \in (2,3)$.

Finally, the third term on the right-hand side of (4.37) is estimated by

$$c\|\theta\|_{L_{2,\infty}(\Omega^t)}\|g\|_{L_{2,1}(\Omega^t)} \le cA_0\|g\|_{L_{2,1}(\Omega^t)}.$$

Inserting the above estimates into (4.37) and assuming that δ_1, δ_2 are sufficiently small, we arrive at

$$\begin{aligned}(4.38)\qquad &\|\theta(t)\|^3_{L_3(\Omega)} + \|\theta\|^2_{L_2(0,t;H^1(\Omega))} \le \|\theta\|^2_{L_2(\Omega^t)} + cA_1^{3/2}(2)\|\theta\|^{3/2}_{L_2(0,t;H^1(\Omega))}\\ &+ cA_1^2(\sigma_1)A_1^{2/3}\left(\frac{s_1}{3-s_1}\right)\|\varepsilon_{t'}\|^{4/3}_{L_2(0,t;H^1(\Omega))} + cA_0\|g\|_{L_{2,1}(\Omega^t)} + \|\theta_0\|^3_{L_3(\Omega)},\end{aligned}$$

where $\frac{1}{s_1}+\frac{1}{\sigma_1}=\frac{1}{2}$, $2<s_1<3$.
Let us choose $s_1=\frac{5}{2}$. Then $\frac{s_1}{3-s_1}=5$ and $\sigma_1=10$. Since $A_1(\sigma)$ is an increasing function of σ it follows from (4.38) that

$$\begin{aligned}(4.39)\qquad &\|\theta(t)\|^3_{L_3(\Omega)} + \|\theta\|^2_{L_2(0,t;H^1(\Omega))} \le \|\theta\|^2_{L_2(\Omega^t)}\\ &+ c(A_1^6(10) + A_1^{8/3}(10)\|\varepsilon_{t'}\|^{4/3}_{L_2(0,t;H^1(\Omega))} + A_0\|g\|_{L_{2,1}(\Omega^t)}) + \|\theta_0\|^3_{L_3(\Omega)},\end{aligned}$$

where we used the Young inequality in the second term on the right-hand side of (4.38).

By virtue of (4.34) and (4.10) we obtain from (4.39) the inequality

$$\begin{aligned}(4.40)\qquad &\|\varepsilon_{t'}\|_{V_2(\Omega^t)} \le c(t)(A_0 + A_1^3(10) + A_1^4(10) + A_0^{1/2}\|g\|^{1/2}_{L_{2,1}(\Omega^t)}\\ &+ \|\theta_0\|^{3/2}_{L_3(\Omega)} + \|u_0\|_{H^2(\Omega)} + \|u_1\|_{H^1(\Omega)} + \|b\|_{L_2(\Omega^t)}).\end{aligned}$$

Employing (4.40) in (4.39) yields (4.35). This completes the proof. □

Using (4.35) in (4.33) implies

$$\begin{aligned}(4.41)\qquad &\|u_{t'}\|_{L_\infty(0,t;H^1(\Omega))} + \|u_{t'}\|_{L_2(0,t;H^2(\Omega))} + \|u\|_{L_\infty(0,t;H^2(\Omega))}\\ &\le c(t)(A_2 + \|u_0\|_{H^2(\Omega)} + \|u_1\|_{H^1(\Omega)} + \|b\|_{L_2(\Omega^t)}) \le c(t)A_3,\end{aligned}$$

with A_3 defined in (4.36).

From what it has already been proved we deduce

LEMMA 4.8. *Assume that* $u_0 \in H^2(\Omega)$, $u_1 \in B^{2-1/5}_{2,10}(\Omega) \cap B^{2-2/\sigma}_{2,\sigma}(\Omega)$, $\theta_0 \in H^1(\Omega)$, $b \in L_{2,10}(\Omega^t) \cap L_{2,\sigma}(\Omega^t)$, $g \in L_2(\Omega^t)$, $\sigma > 4$.
Then

$$(4.42)\qquad \|\theta_t\|_{L_2(\Omega^t)} + \|\nabla\theta\|_{L_\infty(0,t;L_2(\Omega))} \le \varphi(A_4(\sigma)), \quad \sigma > 4,$$

where

$$\begin{aligned}(4.43)\qquad A_4(\sigma) = &\|u_0\|_{H^2(\Omega)} + \|u_1\|_{B^{2-1/5}_{2,10}(\Omega)} + \|u_1\|_{B^{2-2/\sigma}_{2,\sigma}(\Omega)}\\ &+ \|\theta_0\|_{H^1(\Omega)} + \|b\|_{L_{2,10}(\Omega^t)} + \|b\|_{L_{2,\sigma}(\Omega^t)} + \|g\|_{L_2(\Omega^t)}.\end{aligned}$$

Proof. Multiplying (1.2) by θ_t and integrating over Ω gives

$$\begin{aligned}(4.44)\qquad &c_v\int_\Omega \theta\theta_t^2dx + \frac{k}{2}\frac{d}{dt}\int_\Omega |\nabla\theta|^2dx\\ &\le c\int_\Omega \theta|\varepsilon_t|\,|\theta_t|dx + c\int_\Omega |\varepsilon_t|^2|\theta_t|dx + \int_\Omega |g|\,|\theta_t|dx.\end{aligned}$$

Applying the Hölder and the Young inequalities the first term on the right-hand side of (4.44) is estimated by

$$\Big(\int_\Omega \theta\theta_t^2 dx\Big)^{1/2}\Big(\int_\Omega \theta|\varepsilon_t|^2 dx\Big)^{1/2} \le \delta_1 \int_\Omega \theta\theta_t^2 dx + c(1/\delta_1)\int_\Omega \theta|\varepsilon_t|^2 dx,$$

where on account of (4.35) the second integral is bounded by

$$c\Big(\int_\Omega \theta^3 dx\Big)^{1/3}\Big(\int_\Omega |\varepsilon_t|^3 dx\Big)^{2/3} \le cA_2\|\varepsilon_t\|^2_{L_3(\Omega)}.$$

The second term on the right-hand side of (4.44) can be estimated by

$$c\|\theta_t\|_{L_2(\Omega)}\|\varepsilon_t\|_{L_6(\Omega)}\|\varepsilon_t\|_{L_3(\Omega)} \le \delta_2\|\theta_t\|^2_{L_2(\Omega)} + c(1/\delta_2)\|\varepsilon_t\|^2_{L_6(\Omega)}\|\varepsilon_t\|^2_{L_3(\Omega)}.$$

Finally, the third term on the right-hand side of (4.44) is bounded by

$$\delta_3\|\theta_t\|^2_{L_2(\Omega)} + c(1/\delta_3)\|g\|^2_{L_2(\Omega)}.$$

Employing the above estimates in (4.44), assuming δ_i, $i = 1, 2, 3$, sufficiently small, recalling that $\theta \ge \theta_* > 0$, and integrating the result with respect to time, we conclude on account of (4.41) that

$$\begin{aligned} &\|\theta_t\|^2_{L_2(\Omega^t)} + \|\nabla\theta(t)\|^2_{L_2(\Omega)} \le cA_2A_3^2 \\ &\quad + cA_3^2\|\varepsilon_{t'}\|^2_{L_\infty(0,t;L_3(\Omega))} + c\|g\|^2_{L_2(\Omega^t)} + c\|\nabla\theta_0\|^2_{L_2(\Omega)}. \end{aligned} \tag{4.45}$$

In view (4.41) and (4.45), applying Corollary 4.5 to problem (4.19) we conclude that

$$\begin{aligned} \|u_{t'}\|_{W^{2,1}_{2,\sigma}(\Omega^t)} &\le c(A_3 + A_2^{1/2}A_3 + A_3\|\varepsilon_{t'}\|_{L_\infty(0,t;L_3(\Omega))} \\ &\quad + \|g\|_{L_2(\Omega^t)} + \|b\|_{L_{2,\sigma}(\Omega^t)} + \|\theta_0\|_{H^1(\Omega)} + \|u_1\|_{B^{2-2/\sigma}_{2,\sigma}(\Omega)}) \\ &\le cA_3\|\varepsilon_{t'}\|_{L_\infty(0,t;L_3(\Omega))} + c\varphi(A_4(\sigma)), \end{aligned} \tag{4.46}$$

where σ is an arbitrary finite number.
By the definition of the tensor ε, inequality (4.46) implies

$$\|\varepsilon_{t'}\|_{W^{1,1/2}_{2,\sigma}(\Omega^t)} \le cA_3\|\varepsilon_{t'}\|_{L_\infty(0,t;L_3(\Omega))} + cA_4(\sigma). \tag{4.47}$$

In view of the interpolation inequality

$$\|\varepsilon_{t'}\|_{L_\infty(0,t;L_3(\Omega))} \le \delta\|\varepsilon_{t'}\|_{W^{1,1/2}_{2,\sigma}(\Omega^t)} + c(1/\delta)\|\varepsilon_{t'}\|_{L_{2,\sigma}(\Omega^t)}, \tag{4.48}$$

which holds for $\sigma > 4$, it follows from (4.47) and (4.35) that

$$\|\varepsilon_{t'}\|_{W^{1,1/2}_{2,\sigma}(\Omega^t)} \le \varphi(A_3, A_4(\sigma)), \quad \sigma > 4. \tag{4.49}$$

Hence,

$$\|\varepsilon_{t'}\|_{L_\infty(0,t;L_3(\Omega))} \le \varphi(A_3, A_4(\sigma)), \quad \sigma > 4. \tag{4.50}$$

Applying (4.50) in (4.45) gives (4.42). This completes the proof. □

Let us note that since t is finite, estimate (4.49) in conjunction with the Hölder inequality implies that

$$\|\varepsilon_{t'}\|_{W^{1,1/2}_{2,\sigma_0}(\Omega^t)} \le \varphi(t, A_3, A_4(\sigma)), \tag{4.51}$$

where $\sigma > 4$ and $\sigma_0 \ge 1$.
Similarly, by (4.46) and (4.50), it follows that

$$\|u_{t'}\|_{W^{2,1}_{2,\sigma_0}(\Omega^t)} \le \varphi(t, A_3, A_4(\sigma)), \tag{4.52}$$

where $\sigma > 4$ and $\sigma_0 \ge 1$.

LEMMA 4.9. *Assume that* $\theta \in L_{p,\sigma}(\Omega^t)$, $b \in L_{p,\sigma}(\Omega^t)$, $u_0 \in W^1_p(\Omega)$, $u_1 \in B^{2-2/\sigma}_{p,\sigma}(\Omega)$, $p \in (1,\infty)$, $\sigma \in (1,\infty)$.
Then

$$\|u_{t'}\|_{W^{1,1/2}_{p,\sigma}(\Omega^t)} \le c(t)[\|\theta\|_{L_{p,\sigma}(\Omega^t)} + A_5(p,\sigma)], \tag{4.53}$$

where

$$A_5(p,\sigma) = \|u_0\|_{W^1_p(\Omega)} + \|u_1\|_{B^{2-2/\sigma}_{p,\sigma}(\Omega)} + \|b\|_{L_{p,\sigma}(\Omega^t)},$$

and the constant $c(t)$ depends exponentially on t.

Proof. Let us consider system (4.19) and apply the inequality (4.20). Representing ε by

$$\varepsilon(t) = \int_0^t \varepsilon_{t'}(t')dt' + \varepsilon(0), \tag{4.54}$$

and using the generalized Minkowski inequality, we obtain

$$\begin{aligned}\|\varepsilon\|^\sigma_{L_{p,\sigma}(\Omega^t)} &\le c\int_0^t \left\|\int_0^{t'} \varepsilon_{t''}(t'')dt''\right\|^\sigma_{L_p(\Omega)} dt' + c\int_0^t \|\varepsilon(u_0)\|^\sigma_{L_p(\Omega)}dt' \\ &\le c\int_0^t \left(\int_0^{t'} \|\varepsilon_{t''}(t'')\|_{L_p(\Omega)}dt''\right)^\sigma dt' + ct\|\varepsilon(u_0)\|^\sigma_{L_p(\Omega)} \\ &\le c\int_0^t \left(\int_0^{t'} \|\varepsilon_{t''}(t'')\|^\sigma_{L_p(\Omega)}dt''\right)(t')^{\sigma-1}dt' + ct\|\varepsilon(u_0)\|^\sigma_{L_p(\Omega)}.\end{aligned}$$

Consequently, denoting

$$a(t) = \|\varepsilon_{t'}\|^\sigma_{L_{p,\sigma}(\Omega^t)} = \int_0^t \|\varepsilon_{t'}(t')\|^\sigma_{L_p(\Omega)}dt',$$

we deduce from (4.20) the inequality

$$a(t) \le \int_0^t \alpha(t')a(t')dt' + A(t),$$

where

$$\alpha(t) = ct^{\sigma-1},$$
$$A(t) = c(\|\theta\|^\sigma_{L_{p,\sigma}(\Omega^t)} + \|b\|^\sigma_{L_{p,\sigma}(\Omega^t)} + t\|\varepsilon(u_0)\|^\sigma_{L_p(\Omega)} + \|u_1\|^\sigma_{B^{2-2/\sigma}_{p,\sigma}(\Omega)}).$$

Hence, by the Gronwall inequality, it follows that

$$\begin{aligned} a(t) &\le A(t) + \int_0^t \alpha(t')A(t')e^{\int_{t'}^t \alpha(t'')dt''}dt' \\ &\le A(t)(1+\alpha_1(t)e^{\alpha_1(t)}), \quad \alpha_1(t) = t\alpha(t). \end{aligned}$$

Thus,

$$\begin{aligned} \|\varepsilon_{t'}\|^\sigma_{L_{p,\sigma}(\Omega^t)} &\le c(t)(\|\theta\|^\sigma_{L_{p,\sigma}(\Omega^t)} + t\|\varepsilon(u_0)\|^\sigma_{L_p(\Omega)} \\ &\quad + \|u_1\|^\sigma_{B^{2-2/\sigma}_{p,\sigma}(\Omega)} + \|b\|^\sigma_{L_{p,\sigma}(\Omega^t)}) \\ &\equiv c(t)(\|\theta\|^\sigma_{L_{p,\sigma}(\Omega^t)} + D^\sigma(t)) \end{aligned} \tag{4.55}$$

with $c(t) = c(1+\alpha_1(t)e^{\alpha_1(t)})$.
Using (4.55) in (4.54) yields the analogous bound on $\|\varepsilon\|^\sigma_{L_{p,\sigma}(\Omega^t)}$. Consequently, on account of (4.20) the corresponding bound on $\|u_{t'}\|_{W^{1,1/2}_{p,\sigma}(\Omega^t)}$ follows as well. The proof is completed. □

On the basis of Lemma 4.9 we prove now

LEMMA 4.10. *Assume that* $u_0 \in W^1_r(\Omega)$, $u_1 \in B^{2-2/r}_{r,r}(\Omega)$, $\theta_0 \in L_r(\Omega)$, $b \in L_r(\Omega^T)$, $g \in L_r(\Omega^T)$, $r \in (1,\infty)$.
Then

$$\|\theta\|_{L_{r,\infty}(\Omega^t)} \le A_6(r,r,t), \quad r < \infty, \tag{4.56}$$

where

$$A_6(r,r,t) = c(t)(\|\theta_0\|_{L_r(\Omega)} + \sqrt[r]{r}A_5(r,r) + \|g\|_{L_r(\Omega^t)} + 1), \tag{4.57}$$

and $A_5(p,\sigma)$ *is defined in (4.53).*

Proof. Multiplying (1.2) by θ^r, $r > 1$, and integrating over Ω gives

$$\begin{aligned} &\frac{c_v}{r+2}\frac{d}{dt}\|\theta\|^{r+2}_{L_{r+2}(\Omega)} + \frac{4kr}{(r+1)^2}\int_\Omega \left|\nabla\theta^{\frac{r+2}{2}}\right|^2 dx \\ &= -\int_\Omega \theta^{r+1}(A_2\alpha)\cdot\varepsilon_t dx + \int_\Omega \theta^r(A_1\varepsilon_t)\cdot\varepsilon_t dx + \int_\Omega \theta^r g dx. \end{aligned} \tag{4.58}$$

Hence, after integrating with respect to time,

$$\begin{aligned} &\frac{c_v}{r+2}\|\theta(t)\|^{r+2}_{L_{r+2}(\Omega)} + \frac{4kr}{(r+1)^2}\int_{\Omega^t}\left|\nabla\theta^{\frac{r+2}{2}}\right|^2 dxdt' \\ &\le c\int_{\Omega^t}\theta^{r+1}|\varepsilon_{t'}|dxdt' + c\int_{\Omega^t}\theta^r|\varepsilon_{t'}|^2dxdt' + \int_{\Omega^t}\theta^r g dxdt' + \frac{c_v}{r+2}\|\theta_0\|^{r+2}_{L_{r+2}(\Omega)}. \end{aligned} \tag{4.59}$$

Here let us recall Lemma 4.9 which provides the inequality

$$\|\varepsilon_{t'}\|_{L_r(\Omega^t)} \le c(t)(\|\theta\|_{L_r(\Omega^t)} + A_5(r,r)) \tag{4.60}$$

for $r \in (1,\infty)$.

On account of (4.60) the second integral on the right-hand side of (4.59) is estimated as follows

$$\begin{aligned}\int_{\Omega^t} \theta^r |\varepsilon_{t'}|^2 dx dt' &\le \|\theta\|^r_{L_{r+2}(\Omega^t)} \|\varepsilon_{t'}\|^2_{L_{r+2}(\Omega^t)} \\ &\le c(t)\|\theta\|^r_{L_{r+2}(\Omega^t)}(\|\theta\|^2_{L_{r+2}(\Omega^t)} + A_5^2(r+2,r+2)) \\ &\le c(t)(\|\theta\|^{r+2}_{L_{r+2}(\Omega^t)} + A_5^{r+2}(r+2,r+2)).\end{aligned} \tag{4.61}$$

Now, using (4.61) we estimate the first integral on the right-hand side of (4.59) by

$$\begin{aligned}&\int_{\Omega^t} \theta^{r+1}|\varepsilon_{t'}| dx dt' = \int_{\Omega^t} \theta^{\frac{r}{2}+1}\theta^{\frac{r}{2}}|\varepsilon_{t'}| dx dt' \\ &\le \left\|\theta^{\frac{r}{2}+1}\right\|_{L_2(\Omega^t)} \left\|\theta^{\frac{r}{2}}|\varepsilon_{t'}|\right\|_{L_2(\Omega^t)} = \|\theta\|^{\frac{r+2}{2}}_{L_{r+2}(\Omega^t)} \|\theta^r|\varepsilon_{t'}|^2\|^{1/2}_{L_1(\Omega^t)} \\ &\le c(t)\|\theta\|^{\frac{r+2}{2}}_{L_{r+2}(\Omega^t)}(\|\theta\|^{\frac{r+2}{2}}_{L_{r+2}(\Omega^t)} + A_5^{\frac{r+2}{2}}(r+2,r+2)) \\ &\le c(t)(\|\theta\|^{r+2}_{L_{r+2}(\Omega^t)} + A_5^{r+2}(r+2,r+2)).\end{aligned}$$

Finally, with the use of Hölder's and Young's inequalities the third integral on the right-hand side of (4.59) is estimated by

$$\int_{\Omega^t} \theta^r g dx dt' \le \|\theta\|^r_{L_{r+2}(\Omega^t)} \|g\|_{L_{(r+2)/2}(\Omega^t)} \le c\|\theta\|^{r+2}_{L_{r+2}(\Omega^t)} + c(\|g\|^{r+2}_{L_{r+2}(\Omega^t)} + 1).$$

Inserting the above estimates into (4.59) leads to

$$\begin{aligned}\|\theta(t)\|^{r+2}_{L_{r+2}(\Omega)} \le \frac{r+2}{c_v} c(t)\bigg(\int_0^t \|\theta(t')\|^{r+2}_{L_{r+2}(\Omega)} dt' + A_5^{r+2}(r+2,r+2) \\ + \|g\|^{r+2}_{L_{r+2}(\Omega^t)} + 1\bigg) + \|\theta_0\|^{r+2}_{L_{r+2}(\Omega)}.\end{aligned}$$

Hence, by the Gronwall inequality, we conclude that

$$\begin{aligned}\|\theta(t)\|^{r+2}_{L_{r+2}(\Omega)} &\le [\|\theta_0\|^{r+2}_{L_{r+2}(\Omega)} + (r+2)c(t)(A_5^{r+2}(r+2,r+2) \\ &\quad + \|g\|^{r+2}_{L_{r+2}(\Omega^t)} + 1)] \exp(c(t)(r+2)) \\ &\le A_6^{r+2}(r+2,r+2,t)\end{aligned}$$

for $t \in (0,T)$, with A_6 defined by (4.57). This gives (4.56). The proof is completed. □

COROLLARY 4.11. *Taking into account that $\sqrt[r]{r}$ is bounded let us define*

$$\begin{aligned}A_7(r,r,t) = c(t)(\|u_0\|_{W_r^1(\Omega)} + \|u_1\|_{B^{2-2/r}_{r,r}(\Omega)} + \|\theta_0\|_{L_r(\Omega)} \\ + \|b\|_{L_{r,r}(\Omega^t)} + \|g\|_{L_r(\Omega^t)}).\end{aligned} \tag{4.62}$$

Then, according to (4.56) and the definition of $A_5(r,r)$ in Lemma 4.9,

$$\|\theta\|_{L_{r,\infty}(\Omega^t)} \le A_7(r,r,t), \quad r\in(1,\infty), \quad t\le T. \tag{4.63}$$

Moreover, by (4.53) and (4.63),

$$\begin{aligned}\|\varepsilon_{t'}\|_{L_{r,\sigma}(\Omega^t)} &\le c\|u_{t'}\|_{W^{1,1/2}_{r,\sigma}(\Omega^t)}\\ &\le c(t)A_7(r,r,t), \quad (r,\sigma)\in(1,\infty).\end{aligned} \tag{4.64}$$

Let us consider now the elliptic problem resulting from (1.2) and $(1.3)_2$:

$$\begin{aligned} &-k\Delta\theta = -c_v\theta\theta_t - \theta(A_2\alpha)\cdot\varepsilon_t + (A_1\varepsilon_t)\cdot\varepsilon_t + g && \text{in } \Omega,\\ &n\cdot\nabla\theta = 0 && \text{on } S.\end{aligned} \tag{4.65}$$

We have

LEMMA 4.12. *Asume that $u_0\in H^2(\Omega)\cap W^1_{\frac{2s}{2-s}}(\Omega)\cap B^{2-2/\sigma}_{2,\sigma}(\Omega)$, $u_1\in B^{9/5}_{2,10}(\Omega)\cap B^{\frac{3s-2}{s}}_{\frac{2s}{2-s},\frac{2s}{2-s}}(\Omega)$, $\theta_0\in H^1(\Omega)\cap L_{\frac{2s}{2-s}}(\Omega)$, $b\in L_{2,10}(\Omega^T)\cap L_{2,\sigma}(\Omega^T)\cap L_{\frac{2s}{2-s}}(\Omega^T)$, $g\in L_2(\Omega^T)\cap L_{\frac{2s}{2-s}}(\Omega^T)$. Then*

$$\|\theta\|_{W^{2,1}_s(\Omega^t)} \le \varphi(A_8(s,\sigma,t)), \quad t\le T, \tag{4.66}$$

where

$$\begin{aligned} A_8(s,\sigma,t) &= c(t)(\|u_0\|_{H^2(\Omega)} + \|u_0\|_{W^1_{\frac{2s}{2-s}}(\Omega)}\\ &\quad + \|u_1\|_{B^{9/5}_{2,10}(\Omega)} + \|u_1\|_{B^{2-2/\sigma}_{2,\sigma}(\Omega)} + \|u_1\|_{B^{\frac{3s-2}{s}}_{\frac{2s}{2-s},\frac{2s}{2-s}}(\Omega)}\\ &\quad + \|\theta_0\|_{H^1(\Omega)} + \|\theta_0\|_{L_{\frac{2s}{2-s}}(\Omega)} + \|b\|_{L_{2,10}(\Omega^t)} + \|b\|_{L_{2,\sigma}(\Omega^t)}\\ &\quad + \|b\|_{L_{\frac{2s}{2-s}}(\Omega^t)} + \|g\|_{L_2(\Omega^t)} + \|g\|_{L_{\frac{2s}{2-s}}(\Omega^t)}),\end{aligned} \tag{4.67}$$

and $1<s<2$, $\sigma>4$.

Proof. On account of (4.42), (4.63) and (4.64) it follows from $(4.65)_1$ that

$$\begin{aligned}\|\Delta\theta\|_{L_s(\Omega^t)} &\le c(\|\theta\|_{L_{\frac{2s}{2-s}}(\Omega^t)}\|\theta_t\|_{L_2(\Omega^t)}\\ &\quad + \|\theta\|_{L_{2s}(\Omega^t)}\|\varepsilon_{t'}\|_{L_{2s}(\Omega^t)} + \|\varepsilon_{t'}\|^2_{L_{2s}(\Omega^t)} + \|g\|_{L_s(\Omega^t)})\\ &\le c(t)\bigg(A_7\Big(\frac{2s}{2-s},\frac{2s}{2-s},t\Big)\varphi(A_4(\sigma))\\ &\quad + A_7^2(2s,2s,t) + \|g\|_{L_s(\Omega^t)}\bigg),\end{aligned} \tag{4.68}$$

where $s<2$ is close to 2, $\sigma>4$ and $t\le T$.
In view of the estimate

$$\|\theta\|_{W^2_s(\Omega)} \le c(\|\Delta\theta\|_{L_s(\Omega)} + \|\theta\|_{L_s(\Omega)}), \tag{4.69}$$

which holds for the Neumann problem (4.65), we deduce from (4.68) that

$$\begin{aligned}\|\theta\|_{L_s(0,t;W^2_s(\Omega))} &\le c\|\theta\|_{L_s(\Omega^t)} + c(t)\bigg(A_7\Big(\frac{2s}{2-s},\frac{2s}{2-s},t\Big)\varphi(A_4(\sigma))\\ &\quad + A_7^2(2s,2s,t) + \|g\|_{L_s(\Omega^t)}\bigg),\end{aligned} \tag{4.70}$$

where $\sigma > 4$, $s < 2$ close to 2, $t \le T$.
Now, combining (4.70) with (4.42) and using (4.10) we conclude that

$$\begin{aligned}\|\theta\|_{W_s^{2,1}(\Omega^t)} &\le c(t)(A_0 + \varphi(A_4(\sigma)))\\ &+ c(t)\biggl(A_7\Bigl(\frac{2s}{2-s}, \frac{2s}{2-s}, t\Bigr)\varphi(A_4(\sigma))\\ &+ A_7^2(2s, 2s, t) + \|g\|_{L_s(\Omega^t)}\biggr) \le \varphi(A_8(s, \sigma, t)),\end{aligned} \tag{4.71}$$

where the latter inequality follows from the definitions of the quantities A_0 (see (4.10)), $A_4(\sigma)$ (see (4.43)), $A_7(r, r, t)$ (see (4.62)) and $A_8(s, \sigma, t)$ (see (4.67)). This proves the lemma. □

COROLLARY 4.13. *In view of the imbedding*

$$\nabla W_s^{2,1}(\Omega^T) \subset L_{p',s}(\Omega^T), \quad s \in (1,2), \tag{4.72}$$

where $s < p' \le \frac{3}{3/s - 1}$, *and the Hölder inequality with respect to the integral over* Ω *in the case* $1 < p' < s$, *we have*

$$\|\nabla v\|_{L_{p',s}(\Omega^T)} \le c\|v\|_{W_s^{2,1}(\Omega^T)} \tag{4.73}$$

for any $v \in W_s^{2,1}(\Omega^T)$, $1 < p' \le \frac{3}{3/s - 1}$.
Applying the imbedding inequality (4.73) to (4.66) we deduce that

$$\|\nabla\theta\|_{L_{p',s}(\Omega^t)} \le c(t)\varphi(A_8(s, \sigma, t)) \tag{4.74}$$

where $1 < p' \le \frac{3}{3/s - 1}$, $s \in (1,2)$, $\sigma > 4$, *and* $A_8(s, \sigma, t)$ *is defined in (4.67).*

We shall use (4.74) to get more regularity estimates on ε. To this purpose we return to the viscoelasticity system (4.19) and prove the following result analogous to Lemma 4.9.

LEMMA 4.14. *Assume that* $\nabla\theta \in L_{p,\sigma}(\Omega^T)$, $b \in L_{p,\sigma}(\Omega^T)$, $u_1 \in B_{p,\sigma}^{2-2/\sigma}(\Omega)$, $p, \sigma \in (1, \infty)$.
Then

$$\begin{aligned}\|\varepsilon_{t'}\|_{W_{p,\sigma}^{1,1/2}(\Omega^t)} &\le c\|u_{t'}\|_{W_{p,\sigma}^{2,1}(\Omega^t)}\\ &\le c(t)(\|\nabla\theta\|_{L_{p,\sigma}(\Omega^t)} + \|b\|_{L_{p,\sigma}(\Omega^t)} + \|u_1\|_{B_{p,\sigma}^{2-2/\sigma}(\Omega)})\\ &\equiv c(t)(\|\nabla\theta\|_{L_{p,\sigma}(\Omega^t)} + A_9(p, \sigma)),\end{aligned} \tag{4.75}$$

where $t \le T$, $p, \sigma \in (1, \infty)$, *and*

$$A_9(p, \sigma) = \|u_1\|_{B_{p,\sigma}^{2-2/\sigma}(\Omega)} + \|b\|_{L_{p,\sigma}(\Omega^t)}.$$

Proof. Let us consider system (4.19). By inequality (4.21) we have

$$\begin{aligned}\|\nabla\varepsilon_{t'}\|_{L_{p,\sigma}(\Omega^t)} &\le \|u_{t'}\|_{W_{p,\sigma}^{2,1}(\Omega^t)} \le c(\|\nabla\varepsilon\|_{L_{p,\sigma}(\Omega^t)} + \|\nabla\theta\|_{L_{p,\sigma}(\Omega^t)}\\ &+ \|u_1\|_{B_{p,\sigma}^{2-2/\sigma}(\Omega)} + \|b\|_{L_{p,\sigma}(\Omega^t)})\end{aligned} \tag{4.76}$$

for $t \le T$. We use now the formula

$$\nabla\varepsilon(t) = \int_0^t \nabla\varepsilon_{t'}(t')dt' + \nabla\varepsilon(0), \quad \nabla\varepsilon(0) = \nabla\varepsilon(u_0), \tag{4.77}$$

and repeat the proof of Lemma 4.9 with $\nabla\varepsilon$, $\nabla\theta$ in place of ε, θ. In result we conclude that

$$\|\nabla\varepsilon_{t'}\|_{L_{p,\sigma}(\Omega^t)} \le c(t)(\|\nabla\theta\|_{L_{p,\sigma}(\Omega^t)} + \|u_1\|_{B^{2-2/\sigma}_{p,\sigma}(\Omega)} + \|b\|_{L_{p,\sigma}(\Omega^t)}). \tag{4.78}$$

Using (4.78) in (4.77) implies the analogous bound on $\|\nabla\varepsilon\|_{L_{p,\sigma}(\Omega^t)}$ and then, by (4.76), on $\|u_{t'}\|_{W^{2,1}_{p,\sigma}(\Omega^t)}$ as well. □

COROLLARY 4.15. *Applying estimate (4.74) in (4.75) yields*

$$\|\varepsilon_{t'}\|_{W^{1,1/2}_{p',s}(\Omega^t)} \le c(t)(\varphi(A_8(s,\sigma,t)) + A_9(p',s)) \tag{4.79}$$

for $1 < p' \le \frac{3}{3/s-1}$, $s \in (1,2)$, $\sigma > 4$, $t \le T$.

COROLLARY 4.16. *Let us consider the imbedding*

$$W^{1,1/2}_{p',s}(\Omega^T) \subset L_{\infty,2}(\Omega^T), \quad s \in (1,2), \tag{4.80}$$

which holds true provided $p' > \frac{3}{2-\frac{2}{s}}$. *This condition together with* $p' \le \frac{3}{\frac{3}{s}-1}$ *implies that*

$$\frac{5}{3} < s < 2.$$

Consequently, it follows from (4.79) that

$$\|\varepsilon_{t'}\|_{L_{\infty,2}(\Omega^t)} \le c(t)(\varphi(A_8(s,\sigma,t)) + A_9(p',s)) \equiv A_{10}(s,\sigma,p',t) \tag{4.81}$$

for $\frac{3}{2-\frac{2}{s}} < p' \le \frac{3}{\frac{3}{s}-1}$, $s \in \left(\frac{5}{3}, 2\right)$, $\sigma > 4$.

Estimate (4.81) plays the key role in getting $L_\infty(\Omega^T)$-norm bound for θ.

LEMMA 4.17. *Let the assumptions of Lemma 4.12 be satisfied. Moreover, let* $u_1 \in B^{2-2/s}_{p',s}(\Omega)$, $\theta_0 \in L_\infty(\Omega)$, $b \in L_{p',s}(\Omega^T)$, $g \in L_{\infty,1}(\Omega^T)$, *where*

$$s \in \left(\frac{5}{3}, 2\right), \quad \frac{3}{2-\frac{2}{s}} < p' \le \frac{3}{\frac{3}{s}-1}, \quad \sigma > 4.$$

Then

$$\begin{aligned} \|\theta\|_{L_\infty(\Omega^t)} \le c(t)[&A_{10}(s,\sigma,p',t) + A^2_{10}(s,\sigma,p',t) \\ &+ \|\theta_0\|_{L_\infty(\Omega)} + \|g\|_{L_{\infty,1}(\Omega^t)}] \equiv c(t)A_{11}(s,\sigma,p',t), \end{aligned} \tag{4.82}$$

where $A_{10}(s,\sigma,p',t)$ *is defined in (4.81).*

Proof. Let us consider once more the identity (4.58). Regarding (4.81), we have

$$\begin{aligned} \frac{1}{r+2}\frac{d}{dt}\|\theta\|^{r+2}_{L_{r+2}(\Omega)} \le\ & c\|\theta\|^{r+1}_{L_{r+1}(\Omega)}\|\varepsilon_t\|_{L_\infty(\Omega)} \\ &+ c\|\theta\|^r_{L_r(\Omega)}\|\varepsilon_t\|^2_{L_\infty(\Omega)} + \|\theta\|^r_{L_r(\Omega)}\|g\|_{L_\infty(\Omega)}. \end{aligned} \tag{4.83}$$

Taking into account that $\theta \ge \theta_* > 0$, and using the inequality

$$\|\theta\|^{r+1}_{L_{r+1}(\Omega)} \le |\Omega|^{\frac{1}{r+2}}\|\theta\|^{r+1}_{L_{r+2}(\Omega)} \le c\|\theta\|^{r+1}_{L_{r+2}(\Omega)},$$

and

$$\|\theta\|^r_{L_r(\Omega)} \le |\Omega|^{\frac{2}{r+2}} \|\theta\|^r_{L_{r+2}(\Omega)} \le |\Omega|^{\frac{2}{r+2}} \frac{1}{\|\theta_*\|_{L_{r+2}(\Omega)}} \|\theta\|^{r+1}_{L_{r+2}(\Omega)}$$
$$\le \frac{1}{\theta_*} |\Omega|^{\frac{3}{r+2}} \|\theta\|^{r+1}_{L_{r+2}(\Omega)} \le c\|\theta\|^{r+1}_{L_{r+2}(\Omega)},$$

where constant c is independent of r, we deduce from (4.83) that

$$\|\theta\|^{r+1}_{L_{r+2}(\Omega)} \frac{d}{dt} \|\theta\|_{L_{r+2}(\Omega)} \le c\|\theta\|^{r+1}_{L_{r+2}(\Omega)} (\|\varepsilon_t\|_{L_\infty(\Omega)} + \|\varepsilon_t\|^2_{L_\infty(\Omega)} + \|g\|_{L_\infty(\Omega)}).$$

Hence,

$$\frac{d}{dt} \|\theta\|_{L_{r+2}(\Omega)} \le c(\|\varepsilon_t\|_{L_\infty(\Omega)} + \|\varepsilon_t\|^2_{L_\infty(\Omega)} + \|g\|_{L_\infty(\Omega)}),$$

which in conjunction with (4.81) leads to

$$\begin{aligned} \|\theta(t)\|_{L_{r+2}(\Omega)} &\le \|\theta_0\|_{L_{r+2}(\Omega)} + c(\|\varepsilon_{t'}\|_{L_{\infty,1}(\Omega^t)} + \|\varepsilon_{t'}\|^2_{L_{\infty,2}(\Omega^t)} \\ &+ \|g\|_{L_{\infty,1}(\Omega^t)}) \le c(t)(A_{10}(s,\sigma,p',t) + A^2_{10}(s,\sigma,p',t) \\ &+ \|\theta_0\|_{L_{r+2}(\Omega)} + \|g\|_{L_{\infty,1}(\Omega^t)}). \end{aligned} \tag{4.84}$$

Now, letting $r \to \infty$ in (4.84) we get the assertion. □

COROLLARY 4.18. *Repeating the proof of Lemma 4.12 with the use of the upper bound (4.82) allows us to deduce that*

$$\|\theta\|_{W^{2,1}_2(\Omega^t)} \le \varphi(t, A_{11}(s,\sigma,p',t)), \tag{4.85}$$

where $t \le T$, $s \in (\frac{5}{3}, 2)$, $\frac{3}{2-2/s} < p' \le \frac{3}{3/s-1}$, $\sigma > 4$.

COROLLARY 4.19. *Applying (4.82) in (4.53) yields*

$$\|\varepsilon_{t'}\|_{L_{p,\sigma'}(\Omega^t)} \le c(t)[A_{11}(s,\sigma,p',t) + A_5(p,\sigma')], \tag{4.86}$$

where $t \le T$, $s \in (\frac{5}{3}, 2)$, $\frac{3}{2-2/s} < p' \le \frac{3}{3/s-1}$, $\sigma > 4$, $p, \sigma' \in (1,\infty)$, $A_5(p,\sigma)$ *is defined by (4.53) and* $A_{11}(s,\sigma,p',t)$ *by (4.82).*

The next step consists in obtaining a "better" estimate for θ by means of the parabolic regularity result stated in Lemma 3.5. To apply this result to the quasilinear equation (1.2) we need to prove first that θ is a continuous function on Ω^T. We are able to prove more, namely the Hölder continuity by means of the parabolic De Giorgi method in the same way as in [18, Lemma 6.1].

Since the above reference concerns much more general situation, we present here for reader's convenience a direct, simpler proof.

Following [14, Chap. II. 7] we record the definition of the space $\mathcal{B}_2(\Omega^T, M, \gamma, r, \delta, \kappa)$, where $\Omega^T = \Omega \times (0,T)$, $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, and $M, \gamma, r, \delta, \kappa$ are positive numbers.

The function $u \in \mathcal{B}_2(\Omega^T, M, \gamma, r, \delta, \kappa)$ if:

(i) $u \in V^{1,0}_2(\Omega^T) := C(0,T; L_2(\Omega)) \cap L_2(0,T; W^1_2(\Omega))$,
(ii) $\operatorname{ess\,sup}_{\Omega^T} |u| \le M$,
(iii) the function $w(x,t) = \pm u(x,t)$ satisfies the inequalities

$$\max_{t_0 \le t \le t_0+\tau} \|(w-k)_+\|^2_{L_2(B_{\varrho-\sigma_1\varrho}(x_0))} \le \|(w-k)_+(\cdot,t_0)\|^2_{L_2(B_\varrho(x_0))}$$
$$+ \gamma[(\sigma_1\varrho)^{-2} \|(w-k)_+\|^2_{L_2(Q(\varrho,\tau))} + \mu^{\frac{2}{r}(1+\kappa)}(k,\varrho,\tau)]$$

and

$$\begin{aligned}&\|(w-k)_+\|^2_{V_2(Q(\varrho-\sigma_1\varrho,\tau-\sigma_2\tau))}\\&\qquad\le \gamma\{[(\sigma_1\varrho)^{-2}+(\sigma_2\tau)^{-1}]\|(w-k)_+\|^2_{L_2(Q(\varrho,\tau))}+\mu^{\frac{2}{r}(1+\kappa)}(k,\varrho,\tau)\}.\end{aligned}$$

Here the following notation is used:

$$\begin{aligned}&(w-k)_+=\max\{w-k,0\}\ -\ \text{the truncation of } w,\\&B_\varrho(x_0)=\{x\in\Omega:\ |x-x_0|<\varrho\}\ -\ \text{a ball in }\Omega,\\&Q(\varrho,\tau)=B_\varrho(x_0)\times(t_0,t_0+\tau)=\{(x,t)\in\Omega^T:\ |x-x_0|<\varrho,\\&\qquad\qquad t_0<t<t_0+\tau\}\ -\ \text{a cylinder in }\Omega^T,\end{aligned}$$

where ϱ, τ are arbitrary positive numbers, σ_1, σ_2 are arbitrary numbers from the interval $(0,1)$, and k is an arbitrary number such that

$$\operatorname*{ess\,sup}_{Q(\varrho,\tau)} w(x,t)-k<\delta.$$

Moreover,

$$\begin{aligned}&V_2(\Omega^T)=L_\infty(0,T;L_2(\Omega))\cap L_2(0,T;W^1_2(\Omega)),\\&\mu(k,\varrho,\tau)=\int\limits_{t_0}^{t_0+\tau}\text{meas}^{r/q}A_{k,\varrho}(t)dt,\end{aligned}$$

where

$$A_{k,\varrho}(t)=\{x\in B_\varrho(x_0):\ w(x,t)>k\},$$

and positive numbers q, r are linked by the relation

$$\frac{1}{r}+\frac{n}{2q}=\frac{n}{4},$$

with the admissible ranges

$$\begin{aligned}&q\in\left(2,\frac{2n}{n-2}\right], && r\in[2,\infty) && \text{for } n\ge 3,\\&q\in(2,\infty), && r\in(2,\infty) && \text{for } n=2,\\&q\in(2,\infty], && r\in[4,\infty) && \text{for } n=1.\end{aligned}$$

LEMMA 4.20. *Let the assumptions of Lemma 4.17 be satisfied, and there exist constants θ_* and M such that*

$$\begin{aligned}&\theta\ge\theta_*>0,\\&M\equiv\|\theta\|_{L_\infty(\Omega^T)}\le c(T)A_{11}(s,\sigma,p',T),\\&\|\theta\|_{W^{2,1}_2(\Omega^T)}\le\varphi(T,A_{11}(s,\sigma,p',T)),\\&\|\varepsilon_t\|_{L_p(\Omega^T)}\le c(T)[A_{11}(s,\sigma,p',T)+A_5(p,p)],\end{aligned}\tag{4.87}$$

where $p \in (1,\infty)$, $A_{11}(s,\sigma,p',T)$ *is defined by (4.82) and* $A_5(p,p)$ *by (4.53). Besides, let* k *be a positive number such that*

$$\sup_{\Omega} \theta_0(x) < k \tag{4.88}$$

and

$$M - k < \delta \quad \text{with some} \quad \delta > 0.$$

Then

$$\theta \in B_2(\Omega^T, M, \gamma, r, \delta, \kappa) \tag{4.89}$$

with

$$\begin{aligned} r = q &= \frac{10}{3}, \quad \kappa \in \left(0, \frac{2}{3}\right), \\ \gamma &= c(t)\bigg(1 + A_{11}^2(s,\sigma,p',T) + A_5^2\left(\frac{5}{2-3\kappa}, \frac{5}{2-3\kappa}\right) \\ &\quad + A_5^2\left(\frac{10}{2-3\kappa}, \frac{10}{2-3\kappa}\right) + \|g\|_{L_{\frac{5}{2-3\kappa}}(\Omega^T)}\bigg), \end{aligned}$$

$s \in \left(\frac{5}{3}, 2\right)$, $\frac{3}{2-2/s} < p' \le \frac{3}{3/s-1}$, $\sigma > 4$.

Proof. Note that the bound $(4.87)_1$ is ensured by Lemma 4.1, the bound $(4.87)_2$ by Lemma 4.17 and $(4.87)_{3,4}$ by Corollaries 4.18 and 4.19.
We check that θ satisfies the conditions (i)–(iii) in the definition of $B_2(\Omega^T, M, \gamma, r, \delta, \kappa)$.

From [1, Chap. 3, Sec. 10] we have the imbedding $W_2^{2,1}(\Omega^T) \subset V_2^{1,0}(\Omega^T)$. Since $\theta \in W_2^{2,1}(\Omega^T)$ the condition (i) is satisfied.

Condition (ii) is automatically satisfied on account of $(4.87)_2$.

Let us check that θ satisfies the second inequality in condition (iii). By virtue of $(4.87)_1$ it sufficies to consider (iii) with $w(x,t) = \theta(x,t)$.

Let $Q(\varrho,\tau) = B_\varrho(x_0) \times (t_0, t_0+\tau)$ be an arbitrary cylinder in Ω^T, and $\zeta(x,t)$ be a smooth function such that $\operatorname{supp}\zeta(x,t) \subset Q(\varrho,\tau)$ and $\zeta(x,t) = 1$ for $(x,t) \in Q(\varrho - \sigma_1\varrho, \tau - \sigma_2\tau)$, where $\sigma_1, \sigma_2 \in (0,1)$. Moreover, let

$$A_{k,\varrho}(t) = \{x \in B_\varrho(x_0) : \ \theta(x,t) > k\}.$$

Multiplying equation (1.2) by $\zeta^2(\theta-k)_+$ and integrating over Ω gives

$$\begin{aligned} &\frac{c_v}{2}\int_\Omega \theta\zeta^2\partial_t(\theta-k)_+^2 dx + k_0\int_\Omega |\nabla(\theta-k)_+|^2\zeta^2 dx \\ &+ 2k_0\int_\Omega \zeta(\theta-k)_+\nabla(\theta-k)_+\cdot\nabla\zeta dx = \int_\Omega G\zeta^2(\theta-k)_+ dx, \end{aligned} \tag{4.90}$$

where, for simplicity, the right-hand side of (1.2) is denoted by

$$G \equiv -\theta(A_2\alpha)\cdot\varepsilon_t + (A_1\varepsilon_t)\cdot\varepsilon_t + g,$$

and to avoid the notational collision the letter k for heat conductivity is replaced by k_0 (for this proof only).

Let us rearrange the first integral on the left-hand side of (4.90) to the form

$$\begin{aligned}\frac{c_v}{2}\int_\Omega \theta\zeta^2\partial_t(\theta-k)_+^2dx &= \frac{c_v}{2}\frac{d}{dt}\int_\Omega \theta(\theta-k)_+^2\zeta^2dx\\ &-\frac{c_v}{2}\int_\Omega \theta_t(\theta-k)_+^2\zeta^2dx - c_v\int_\Omega \theta(\theta-k)_+^2\zeta\zeta_tdx.\end{aligned}\tag{4.91}$$

Further, the term with θ_t in (4.91) is rearranged as follows

$$\begin{aligned}-\frac{c_v}{2}\int_\Omega \theta_t(\theta-k)_+^2\zeta^2dx &= -\frac{c_v}{2}\int_\Omega (\theta-k)_+^2\partial_t(\theta-k)_+\zeta^2dx\\ &= -\frac{c_v}{6}\int_\Omega \partial_t(\theta-k)_+^3\zeta^2dx\\ &= -\frac{c_v}{6}\frac{d}{dt}\int_\Omega (\theta-k)_+^3\zeta^2dx + \frac{c_v}{3}\int_\Omega (\theta-k)_+^3\zeta\zeta_tdx.\end{aligned}\tag{4.92}$$

Inserting (4.91) and (4.92) into (4.90) we obtain the identity

$$\begin{aligned}&\frac{c_v}{2}\frac{d}{dt}\int_\Omega \theta(\theta-k)_+^2\zeta^2dx + k_0\int_\Omega |\nabla(\theta-k)_+|^2\zeta^2dx\\ &= \frac{c_v}{6}\frac{d}{dt}\int_\Omega (\theta-k)_+^3\zeta^2dx - \frac{c_v}{3}\int_\Omega (\theta-k)_+^3\zeta\zeta_tdx\\ &+ c_v\int_\Omega \theta(\theta-k)_+^2\zeta\zeta_tdx - 2k_0\int_\Omega \zeta(\theta-k)_+\nabla(\theta-k)_+\cdot\nabla\zeta dx\\ &+\int_\Omega G\zeta^2(\theta-k)_+dx.\end{aligned}\tag{4.93}$$

Integrating (4.93) with respect to time, taking into account $(4.87)_1$ and the fact that by (4.88), $(\theta_0-k)_+=0$, we conclude that

$$\begin{aligned}&\frac{c_v\theta_*}{2}\int_\Omega (\theta-k)_+^2\zeta^2dx + k_0\int_{\Omega^t} |\nabla(\theta-k)_+|^2\zeta^2dxdt'\\ &\le c\int_\Omega (\theta-k)_+^3\zeta^2dx + c\int_{\Omega^t}(\theta-k)_+^3|\zeta|\,|\zeta_{t'}|dxdt'\\ &+c\int_{\Omega^t}\theta(\theta-k)_+^2|\zeta|\,|\zeta_{t'}|dxdt' + c\int_{\Omega^t}(\theta-k)_+|\nabla(\theta-k)_+|\,|\zeta|\,|\nabla\zeta|dxdt'\\ &+c\int_{\Omega^t}|G|(\theta-k)_+\zeta^2dxdt' \equiv \sum_{i=1}^5 I_i.\end{aligned}\tag{4.94}$$

Since, by the assumption $M-k<\delta$, it holds

$$(\theta-k)_+^3 \le \delta(\theta-k)_+^2 \quad \text{with arbitrary } \delta>0,\tag{4.95}$$

the integral I_1 can be absorbed by the left-hand side of (4.94). Further, using $(4.87)_2$, (4.95) and the bound $|\zeta| \le 1$,

$$I_2 + I_3 \le M \int\limits_{\Omega^t} (\theta - k)_+^2 |\zeta_{t'}| dx dt'.$$

Next, by Young's inequality

$$I_4 \le \frac{k_0}{2} \int\limits_{\Omega^t} |\nabla(\theta - k)_+|^2 \zeta^2 dx dt' + \frac{c}{2k_0} \int\limits_{\Omega^t} (\theta - k)_+^2 |\nabla \zeta|^2 dx dt',$$

so the first integral on the right-hand side of the latter inequality is absorbed by the left-hand side of (4.94). In result, incorporating the above estimates into (4.94), we arrive at

$$\begin{aligned} &\int\limits_{\Omega} (\theta - k)_+^2 \zeta^2 dx + \int\limits_{\Omega^t} |\nabla(\theta - k)_+|^2 \zeta^2 dx dt' \\ &\le c(M+1) \int\limits_{\Omega^t} (\theta - k)_+^2 (|\nabla \zeta|^2 + |\zeta_{t'}|) dx dt' + I_5, \end{aligned} \tag{4.96}$$

where

$$I_5 \equiv c \int\limits_{\Omega^t} |G| (\theta - k)_+ \zeta^2 dx dt'.$$

By the definition of ζ, it holds

$$\begin{aligned} &M \int\limits_{\Omega^t} (\theta - k)_+^2 (|\nabla \zeta|^2 + |\zeta_{t'}|^2) dx dt' \\ &\le M[(\sigma_1 \varrho)^{-2} + (\sigma_2 \tau)^{-1}] \int\limits_{Q(\varrho,\tau)} (\theta - k)_+^2 dx dt'. \end{aligned} \tag{4.97}$$

It remains to estimate the integral I_5. Recalling $(4.87)_2$ again and applying Hölder's inequality yields

$$\begin{aligned} I_5 &= c \int\limits_{t_0}^{t_0+\tau} \int\limits_{A_{k,\varrho}(t')} |G| (\theta - k)_+ \zeta^2 dx dt' \\ &\le M \Big(\int\limits_{t_0}^{t_0+\tau} \int\limits_{A_{k,\varrho}(t')} |G|^{\lambda_1} dx dt' \Big)^{1/\lambda_1} \Big(\int\limits_{t_0}^{t_0+\tau} \operatorname{meas} A_{k,\varrho}(t') dt' \Big)^{1/\lambda_2}, \end{aligned}$$

where $1/\lambda_1 + 1/\lambda_2 = 1$. To satisfy the conditions in the definition of the space $\mathcal{B}_2(\Omega^T, M, \gamma, r, \delta, \kappa)$ we set

$$\frac{1}{\lambda_2} = \frac{2}{r}(1 + \kappa) \quad \text{and} \quad r = q \quad \text{with} \quad \frac{1}{r} + \frac{3}{2r} = \frac{3}{4}.$$

Then

$$\lambda_2 = \frac{5}{3(1+\kappa)}, \quad \lambda_1 = \frac{5}{2-3\kappa} \in \left(\frac{5}{2}, \infty\right) \quad \text{for} \quad \kappa \in \left(0, \frac{2}{3}\right).$$

Consequently,

$$I_5 \le M\|G\|_{L_{\lambda_1}(\Omega^T)}\mu^{1/\lambda_2}(k,\varrho,\tau).$$

By virtue of $(4.87)_2$ and $(4.87)_4$, we obtain

$$\begin{aligned}
&\|G\|_{L_{\frac{5}{2-3\kappa}}(\Omega^T)} \le c(\|\theta\|_{L_\infty(\Omega^T)}\|\varepsilon_t\|_{L_{\frac{5}{2-3\kappa}}(\Omega^T)} + \|\varepsilon_t\|^2_{L_{\frac{10}{2-3\kappa}}(\Omega^T)}\\
&+ \|g\|_{L_{\frac{5}{2-3\kappa}}(\Omega^T)}) \le c(T)\bigg(A_{11}^2(s,\sigma,p',T) + A_5^2\Big(\frac{5}{2-3\kappa}, \frac{5}{2-3\kappa}\Big)\\
&+ A_5^2\Big(\frac{10}{2-3\kappa}, \frac{10}{2-3\kappa}\Big) + \|g\|_{L_{\frac{5}{2-3\kappa}}(\Omega^T)}\bigg) \equiv \gamma_1
\end{aligned}$$

Hence,

$$I_5 \le M\gamma_1\mu^{\frac{3}{5}(1+\kappa)}(k,\varrho,\tau). \tag{4.98}$$

In result, applying estimates (4.97) and (4.98) in (4.96) leads to

$$\begin{aligned}
&\|(\theta-k)_+\|^2_{V_2(Q(\varrho-\sigma_1\varrho,\tau-\sigma_2\tau))} \equiv \operatorname*{ess\,sup}_{t\in[0,T]}\int_\Omega(\theta-k)_+^2\zeta^2dx + \int_{\Omega^T}|\nabla(\theta-k)_+|^2\zeta^2dxdt\\
&\le \gamma\{[(\sigma_1\varrho)^{-2} + (\sigma_2\tau)^{-1}]\|(\theta-k)_+\|^2_{L_2(Q(\varrho,\tau))} + \mu^{\frac{3}{5}(1+\kappa)}(k,\varrho,\tau)\}.
\end{aligned}$$

This proves the second inequality in condition (iii) with the positive number

$$\gamma = c(T)(A_{11}^2(s,\sigma,p',T) + 1 + \gamma_1). \tag{4.99}$$

The first inequality in (iii) follows by multiplying (1.2) by $\zeta_0^2(\theta-k)_+$, where $\zeta_0(x)$ is a smooth function such that $\operatorname{supp}\zeta_0(x) \subset B_\varrho(x_0)$, $\zeta_0(x) = 1$ for $x \in B_{\varrho-\sigma_1\varrho}(x_0)$, $\sigma_1 \in (0,1)$, and next integrating over $\Omega\times(t_0,t_0+\tau)$. Then, by repeating the presented above estimates we arrive at the following inequality in place of (4.96):

$$\begin{aligned}
&\int_{B_\varrho(x_0)}(\theta-k)_+^2\zeta_0^2dx + \int_{Q(\varrho,\tau)}|\nabla(\theta-k)_+|^2\zeta_0dxdt\\
&\le c(M+1)\bigg[\int_{B_\varrho(x_0)}(\theta(t_0)-k)_+^2\zeta_0^2dx + \int_{Q(\varrho,\tau)}(\theta-k)_+^2|\nabla\zeta_0|^2dxdt\bigg]\\
&+ c\int_{Q(\varrho,\tau)}|G|(\theta-k)_+\zeta_0^2dxdt.
\end{aligned} \tag{4.100}$$

The last two integrals on the right-hand side of (4.100) are estimated respectively by (4.97) with $(\sigma_2\tau)^{-1} = 0$, and by (4.98). In result, (4.100) provides the first inequality in the condition (iii) with γ defined in (4.99). The proof is complete. □

COROLLARY 4.21. *By virtue of the imbedding (cf. [14, Thm II. 7.1])*

$$B_2(\Omega^T, M, \gamma, r, \delta, \kappa) \subset C^{\alpha,\alpha/2}(\Omega^T), \quad \alpha \in (0,1),$$

it follows from (4.89) that

$$\theta \in C^{\alpha,\alpha/2}(\Omega^T) \tag{4.101}$$

with Hölder's exponent $\alpha \in (0,1)$ depending on M, γ, r, δ and κ, where

$$\begin{aligned}
&M = \sup_{\Omega^T} \theta \le c(T) A_{11}(s,\sigma,p',T),\\
&\gamma(s,\sigma,p',\kappa,T) = c(T)\bigg(1 + A_{11}^2(s,\sigma,p',T) + A_5^2\Big(\frac{5}{2-3\kappa},\frac{5}{2-3\kappa}\Big)\\
&\quad + A_5^2\Big(\frac{10}{2-3\kappa},\frac{10}{2-3\kappa}\Big) + \|g\|_{L_{\frac{5}{2-3\kappa}}(\Omega^T)}\bigg),\\
&s \in \Big(\frac{5}{3},2\Big),\ \frac{3}{2-2/s} < p' < \frac{3}{3/s-1}, \quad \sigma > 4, \quad \kappa \in \Big(0,\frac{2}{3}\Big).
\end{aligned}$$

In view of Hölder's continuity of θ we can apply Lemma 3.5 and deduce the final estimates on θ and u.

PROPOSITION 4.22. *Let the quantity*

$$\begin{aligned}
A(t) = {}& \|u_0\|_{W^2_{12/5}(\Omega)\cap W^1_{p'}(\Omega)} + \|u_1\|_{B^{11/6}_{12,12}(\Omega)\cap B^{2-2/\sigma'}_{p',\sigma'}(\Omega)}\\
& + \|\theta_0\|_{W^1_{p_*}(\Omega)} + \|g\|_{L_{\infty,12}(\Omega^t)} + \|b\|_{L_{12}(\Omega^t)\cap L_{p',\sigma'}(\Omega^t)},\\
& p_* > 3, \quad p',\sigma' \in (1,\infty),
\end{aligned} \tag{4.102}$$

be finite.
Then the following a priori estimate

$$\|\theta\|_{L_\infty(\Omega^t)} + \|\theta\|_{C^{\alpha,\alpha/2}(\Omega^t)} + \|\varepsilon_{t'}\|_{L_{p',\sigma'}(\Omega^t)} \le \varphi(A(t)), \quad t \le T, \tag{4.103}$$

is valid.

Proof. From (4.82), (4.86) and (4.101) it follows that the estimates on $\|\theta\|_{L_\infty(\Omega^t)}$, $\|\varepsilon_t\|_{L_{p,\sigma'}(\Omega^t)}$ and $\|\theta\|_{C^{\alpha,\alpha/2}(\Omega^t)}$ involve the parameters $s \in \left(\frac{5}{3},2\right)$, $p' \in \left(\frac{3}{2-2/s},\frac{3}{3/5-1}\right]$, $\sigma > 4$, $(p,\sigma') \in (1,\infty)$ and $\kappa \in \left(0,\frac{2}{3}\right)$. We select these parameters in such a way to express the above mentioned estimates in an explicit, sufficiently simple way.

Let us recall that the estimates depend on the quantities:

$$
\begin{aligned}
&A_8(s,\sigma,t) = c(t)[\|u_0\|_{H^2(\Omega)} + \|u_0\|_{W^1_{\frac{2s}{2-s}}(\Omega)} + \|u_1\|_{B^{9/5}_{2,10}(\Omega)}\\
&\quad + \|u_1\|_{B^{2-2/\sigma}_{2,\sigma}(\Omega)} + \|u_1\|_{B^{\frac{3s-2}{s}}_{\frac{2s}{2-s},\frac{2s}{2-s}}(\Omega)} + \|\theta_0\|_{H^1(\Omega)}\\
&\quad + \|\theta_0\|_{L_{\frac{2s}{2-s}}(\Omega)} + \|b\|_{L_{2,10}(\Omega^t)} + \|b\|_{L_{2,\sigma}(\Omega^t)} + \|b\|_{L_{\frac{2s}{2-s}}(\Omega^t)}\\
&\quad + \|g\|_{L_2(\Omega^t)} + \|g\|_{L_{\frac{2s}{2-s}}(\Omega^t)}],\\
&A_9(p',s) = \|u_1\|_{B^{2-2/s}_{p',s}(\Omega)} + \|b\|_{L_{p',s}(\Omega^t)},\\
&A_{10}(s,\sigma,p',t) = c(t)(\varphi(A_8(s,\sigma,t)) + A_9(p',s)),\\
&A_{11}(s,\sigma,p',t) = c(t)(A_{10}(s,\sigma,p',t) + A_{10}^2(s,\sigma,p',t)\\
&\quad + \|\theta_0\|_{L_\infty(\Omega)} + \|g\|_{L_{\infty,1}(\Omega^t)}),
\end{aligned}
$$

$$
\begin{aligned}
&\gamma(s,\sigma,p',\kappa,t) = c\bigg(1 + A_{11}^2(s,\sigma,p',t) + A_5^2\Big(\frac{5}{2-3\kappa},\frac{5}{2-3\kappa}\Big)\\
&\quad + A_5^2\Big(\frac{10}{2-3\kappa},\frac{10}{2-3\kappa}\Big) + \|g\|_{L_{\frac{5}{2-3\kappa}}(\Omega^t)}\bigg),\\
&A_5\Big(\frac{5}{2-3\kappa},\frac{5}{2-3\kappa}\Big) = \|u_0\|_{W^1_{\frac{5}{2-3\kappa}}(\Omega)} + \|u_1\|_{B^{\frac{6(1+\kappa)}{5}}_{\frac{5}{2-3\kappa},\frac{5}{2-3\kappa}}(\Omega)}\\
&\quad + \|b\|_{L_{\frac{5}{2-3\kappa}}(\Omega^t)},\\
&A_5\Big(\frac{10}{2-3\kappa},\frac{10}{2-3\kappa}\Big) = \|u_0\|_{W^1_{\frac{10}{2-3\kappa}}(\Omega)} + \|u_1\|_{B^{\frac{8+3\kappa}{5}}_{\frac{10}{2-3\kappa},\frac{10}{2-3\kappa}}(\Omega)}\\
&\quad + \|b\|_{L_{\frac{10}{2-3\kappa}}(\Omega)},\\
&A_5(p,\sigma') = \|u_0\|_{W^1_p(\Omega)} + \|u_1\|_{B^{2-2/\sigma'}_{p,\sigma'}(\Omega)} + \|b\|_{L_{p,\sigma'}(\Omega^t)},
\end{aligned}
\tag{4.104}
$$

where $s \in (\frac{5}{3},2)$, $p' \in \Big(\frac{3}{2-\frac{3}{s}},\frac{3}{\frac{3}{s}-1}\Big]$, $\sigma > 4$, $\kappa \in (0,\frac{2}{3})$.

To have the above quantities finite we need

$$
\begin{aligned}
&u_0 \in H^2(\Omega) \cap W^1_{\frac{2s}{2-s}}(\Omega) \cap W^1_{\frac{10}{2-3\kappa}}(\Omega) \cap W^1_p(\Omega),\\
&u_1 \in B^{9/5}_{2,10}(\Omega) \cap B^{2-2/\sigma}_{2,\sigma}(\Omega) \cap B^{\frac{3s-2}{s}}_{\frac{2s}{2-s},\frac{2s}{2-s}}(\Omega) \cap B^{2-2/s}_{p',s}(\Omega)\\
&\quad \cap B^{\frac{6(1+\kappa)}{5}}_{\frac{5}{2-3\kappa},\frac{5}{2-3\kappa}}(\Omega) \cap B^{\frac{8+3\kappa}{5}}_{\frac{10}{2-3\kappa},\frac{10}{2-3\kappa}}(\Omega) \cap B^{2-2/r'}_{p,\sigma'}(\Omega),\\
&\theta_0 \in L_\infty(\Omega) \cap H^1(\Omega),\\
&b \in L_{2,10}(\Omega^t) \cap L_{2,\sigma}(\Omega^t) \cap L_{\frac{2s}{2-s}}(\Omega^t) \cap L_{p',s}(\Omega^t) \cap L_{\frac{10}{2-3\kappa}}(\Omega^t)\\
&\quad \cap L_{p,\sigma'}(\Omega^t),\\
&g \in L_2(\Omega^t) \cap L_{\frac{2s}{2-s}}(\Omega^t) \cap L_{\infty,1}(\Omega^t) \cap L_{\frac{5}{2-3\kappa}}(\Omega^t),
\end{aligned}
\tag{4.105}
$$

where $s \in (\frac{5}{3},2)$, $p' \in \Big(\frac{3}{2-\frac{3}{s}},\frac{3}{\frac{3}{s}-1}\Big]$, $\sigma > 4$, $\kappa \in (0,\frac{2}{3})$.

Let us set

$$
s = \frac{12}{7}, \quad p' = 4, \quad \kappa = \frac{1}{9}, \quad \sigma = 10.
$$

Then $\frac{2s}{2-s} = 12$, $\frac{5}{2-3\kappa} = 3$, $\frac{10}{2-3\kappa} = 6$, and (4.105) takes the form

$$
\begin{aligned}
&u_0 \in H^2(\Omega) \cap W^1_{12}(\Omega) \cap W^1_p(\Omega),\\
&u_1 \in B^{9/5}_{2,10}(\Omega) \cap B^{11/6}_{12,12}(\Omega) \cap B^{5/6}_{6,6}(\Omega) \cap B^{5/6}_{4,12/7}(\Omega) \cap B^{2-2/\sigma'}_{p,\sigma'}(\Omega),\\
&\theta_0 \in L_\infty(\Omega) \cap H^1(\Omega),\\
&b \in L_{12}(\Omega^t) \cap L_{p,\sigma'}(\Omega^t),\\
&g \in L_{\infty,1}(\Omega^t) \cap L_{12}(\Omega^t) = L_{\infty,12}(\Omega^t).
\end{aligned}
\tag{4.106}
$$

Using the imbeddings

$$
\begin{aligned}
&W^2_{12/5}(\Omega) \subset H^2(\Omega) \cap W^1_{12}(\Omega),\\
&B^{11/6}_{12,12}(\Omega) \subset B^{9/5}_{2,10}(\Omega) \cap B^{5/6}_{4,12/7}(\Omega),\\
&W^1_p(\Omega) \subset L_\infty(\Omega) \cap H^1(\Omega) \quad \text{for } p > 3,
\end{aligned}
$$

we replace (4.106) by

$$
\begin{aligned}
&u_0 \in W^2_{12/5}(\Omega) \cap W^1_p(\Omega), \quad u_1 \in B^{11/6}_{12,12}(\Omega) \cap B^{2-2/\sigma'}_{p,\sigma'}(\Omega),\\
&\theta_0 \in W^1_{p_*}(\Omega), \quad p_* > 3,\\
&b \in L_{12}(\Omega^t) \cap L_{p,\sigma'}(\Omega^t), \qquad g \in L_{\infty,12}(\Omega^t).
\end{aligned}
\tag{4.107}
$$

The assumptions (4.107) ensure that the quantity $A(t)$ in (4.102) is finite. Then estimates (4.82), (4.86) and (4.101) imply (4.103). Let us note that p is replaced by p'. This completes the proof. □

LEMMA 4.23. *Assume that the data satisfy (4.107), $S \in C^2$, and moreover,*

$$
\begin{aligned}
&u_0 \in W^1_{2q}(\Omega), \qquad u_1 \in B^{2-1/q_0}_{2q,2q_0}(\Omega) \cap B^{2-2/p_0}_{p,p_0}(\Omega),\\
&\theta_0 \in B^{2-2/q_0}_{q,q_0}(\Omega), \quad b \in L_{p,p_0}(\Omega^t) \cap L_{2q,2q_0}(\Omega^t),\\
&g \in L_{q,q_0}(\Omega^t),
\end{aligned}
\tag{4.108}
$$

with any numbers $p, p_0, q, q_0 \in (1, \infty)$ such that

$$\frac{3}{q} + \frac{2}{q_0} - \frac{3}{p} - \frac{2}{p_0} \le 1.$$

Then

$$
\begin{aligned}
\|\theta\|_{W^{2,1}_{q,q_0}(\Omega^t)} \le \varphi(A(t), t) + c(&\|b\|^2_{L_{2q,2q_0}(\Omega^t)}\\
&+ \|u_0\|^2_{W^1_{2q}(\Omega)} + \|u_1\|_{B^{2-1/q_0}_{2q,2q_0}(\Omega)} + \|g\|_{L_{q,q_0}(\Omega^t)}\\
&+ \|\theta_0\|_{B^{2-2/q_0}_{q,q_0}(\Omega)}) \equiv B_1(q, q_0, t)
\end{aligned}
\tag{4.109}
$$

and

$$\|u_t\|_{W^{2,1}_{p,p_0}(\Omega^t)} \le c(t) B_1(q, q_0, t) + c(\|b\|_{L_{p,p_0}(\Omega^t)} + \|u_1\|_{B^{2-2/p_0}_{p,p_0}(\Omega)}), \tag{4.110}$$

where $A(t)$ is defined by (4.102), and $t \le T$.

We remark that, by Lemmas 3.4 and 3.5, the constant c in (4.109) and (4.110) depends on p, p_0, q, q_0 and C^2-norm of the boundary S.

Proof. In view of the Hölder continuity of θ along with its lower and upper boundedness we deduce from Lemma 3.5 and Proposition 4.22 that

$$
\begin{aligned}
\|\theta\|_{W^{2,1}_{q,q_0}(\Omega^t)} &\le c(\|\theta(A_2\alpha)\cdot\varepsilon_{t'}\|_{L_{q,q_0}(\Omega^t)} + \|(A_1\varepsilon_{t'})\cdot\varepsilon_{t'}\|_{L_{q,q_0}(\Omega^t)} \\
&\quad + \|g\|_{L_{q,q_0}(\Omega^t)} + \|\theta_0\|_{B^{2-2/q_0}_{q,q_0}(\Omega)}) \\
&\le \varphi(A(t),t)\|\varepsilon_{t'}\|_{L_{q,q_0}(\Omega^t)} + c(\|\varepsilon_{t'}\|^2_{L_{2q,2q_0}(\Omega^t)} \\
&\quad + \|g\|_{L_{q,q_0}(\Omega^t)} + \|\theta_0\|_{B^{2-2/q_0}_{q,q_0}(\Omega)}), \quad t\le T.
\end{aligned}
\tag{4.111}
$$

From (4.103) with the parameters $(r,r_0)=(p,\sigma')$ selected in Proposition 4.22, we have

$$
\|\varepsilon_{t'}\|_{L_{r,r_0}(\Omega^t)} \le \varphi(A(t),t),
\tag{4.112}
$$

where (r,r_0) is equal $(2q,2q_0)$.
Using (4.112) in (4.111) we conclude (4.109).

Let us consider now the imbedding

$$
\nabla W^{2,1}_{q,q_0}(\Omega^T)\subset L_{q',q_0'}(\Omega^T),
$$

which holds under the condition

$$
\frac{3}{q}+\frac{2}{q_0}-\frac{3}{q'}-\frac{2}{q_0'}\le 1,\quad q'\ge q,\quad q_0'\ge q_0.
\tag{4.113}
$$

Then it follows from (4.75) and (4.112) that

$$
\begin{aligned}
&\|u_{t'}\|_{W^{2,1}_{p,p_0}(\Omega^t)} \le c(t)(\|\nabla\theta\|_{L_{p,p_0}(\Omega^t)} + \|b\|_{L_{p,p_0}(\Omega^t)} + \|u_1\|_{B^{2-2/p_0}_{p,p_0}(\Omega)}) \\
&\le c(t)(\|\nabla\theta\|_{L_{q',q_0'}(\Omega^t)} + \|b\|_{L_{p,p_0}(\Omega^t)} + \|u_1\|_{B^{2-2/p_0}_{p,p_0}(\Omega)}) \\
&\le c(t)(\|\theta\|_{W^{2,1}_{q,q_0}(\Omega^t)} + \|b\|_{L_{p,p_0}(\Omega^t)} + \|u_1\|_{B^{2-2/p_0}_{p,p_0}(\Omega)}),
\end{aligned}
\tag{4.114}
$$

where $p\le q'$, $p_0\le q_0'$, and q',q_0' satisfy (4.113). Finally, we set $p=q'$, $p_0=q_0'$. This establishes (4.110) and thereby completes the proof. □

REMARK 4.24. *Since* $u=\int_0^t u_{t'}dt'+u_0$, $u_t\in W^{2,1}_{p,p_0}(\Omega^T)$, $p,p_0\in(1,\infty)$, *and* $u_0\in B^{2-2/\sigma_0}_{p,\sigma_0}(\Omega)$, *it follows that* $u\in W^{2,1}_{p,\sigma_0}(\Omega^T)$ *where* $\sigma_0\in[1,\infty)$.

COROLLARY 4.25. *Comparing assumptions (4.107) with* $p=p'$ *and (4.108) we deduce that*

$$
\begin{aligned}
&u_0\in W^2_{12/5}(\Omega)\cap W^1_{p'}(\Omega)\cap W^1_{2q}(\Omega),\\
&u_1\in B^{11/6}_{12,12}(\Omega)\cap B^{2-2/\sigma'}_{p',\sigma'}(\Omega)\cap B^{2-1/q_0}_{2q,2q_0}(\Omega)\cap B^{2-2/p_0}_{p,p_0}(\Omega).
\end{aligned}
\tag{4.115}
$$

Setting $p'=2q$, $\sigma'=2q_0$, $q=q_0=6$, $p=p_0=12$, *we get*

$$
u_0\in W^2_{12/5}(\Omega),\quad u_1\in B^{11/6}_{12,12}(\Omega),
$$

and

$$
\begin{aligned}
&\theta_0\in W^1_{p_*}(\Omega)\cap B^{5/3}_{6,6}(\Omega)=B^{5/3}_{6,6}(\Omega)\quad\text{for}\ \ 3<p_*<6,\\
&b\in L_{12}(\Omega^t)\cap L_{p,p_0}(\Omega^t)\cap L_{2q,2q_0}(\Omega^t)=L_{12}(\Omega^t),\\
&g\in L_{\infty,12}(\Omega^t)\cap L_{q,q_0}(\Omega^t)=L_{\infty,12}(\Omega^t).
\end{aligned}
$$

Introducing the quantity

$$B(t) = \|u_0\|_{W^2_{12/5}(\Omega)\cap W^1_{12}(\Omega)} + \|u_1\|_{B^{11/6}_{12,12}(\Omega)}$$
$$+ \|b\|_{L_{12}(\Omega^t)} + \|\theta_0\|_{B^{5/3}_{6,6}(\Omega)} + \|g\|_{L_{\infty,12}(\Omega^t)},$$

we conclude in place of (4.109) and (4.110) the estimate

$$\|u_t\|_{W^{2,1}_{12}(\Omega^t)} + \|\theta\|_{W^{2,1}_{6}(\Omega^t)} \le \varphi(B(t)). \tag{4.116}$$

5. Existence (proof of Theorem A). To prove the existence of solutions to problem (1.1)–(1.4) we use the following method of successive approximations:

$$\begin{aligned}
&u^{n+1}_{tt} - \nabla\cdot(A_1\varepsilon(u^{n+1}_t)) = \nabla\cdot[A_2\varepsilon(u^n) - (A_2\alpha)\theta^n] + b && \text{in } \Omega^T,\\
&c_v\theta^n\theta^{n+1}_t - k\Delta\theta^{n+1} = \\
&\qquad - \theta^n(A_2\alpha)\cdot\varepsilon(u^n_t) + A_1\varepsilon(u^n_t)\cdot\varepsilon(u^n_t) + g && \text{in } \Omega^T,\\
&u^{n+1} = 0 && \text{on } S^T,\\
&n\cdot\nabla\theta^{n+1} = 0 && \text{on } S^T,\\
&u^{n+1}|_{t=0} = u_0, \quad u^{n+1}_t|_{t=0} = u_1 && \text{in } \Omega,\\
&\theta^{n+1}|_{t=0} = \theta_0 && \text{in } \Omega,
\end{aligned} \tag{5.1}$$

where u^n, θ^n, $n \in \mathbb{N}\cup\{0\}$, are treated as given.
Moreover, the approximation (u^0, θ^0) is constructed by an extension of the initial data in such a way that

$$u^0|_{t=0} = u_0, \quad u^0_t|_{t=0} = u_1, \quad \theta^0|_{t=0} = \theta_0 \quad \text{in } \Omega, \tag{5.2}$$

and

$$u^0 = 0, \quad n\cdot\nabla\theta^0 = 0 \quad \text{on } S^T. \tag{5.3}$$

First we show that the sequence $\{u^n, \theta^n\}$ is uniformly bounded.

LEMMA 5.1. *(Boundedness of the approximation) Assume that*

$$D = D(p, p_0, q, q_0) \equiv \|u_0\|_{W^2_p(\Omega)} + \|u_1\|_{B^{2-2/p_0}_{p,p_0}(\Omega)} + \|\theta_0\|_{B^{2-2/q_0}_{q,q_0}(\Omega)}$$
$$+ \|b\|_{L_{p,p_0}(\Omega^\tau)} + \|g\|_{L_{q,q_0}(\Omega^\tau)} < \infty,$$

where $p, p_0, q, q_0 \in (1,\infty)$ satisfy the conditions

$$\frac{3}{q} + \frac{2}{q_0} - \frac{3}{p} - \frac{2}{p_0} < 1,$$
$$\frac{3}{p} + \frac{2}{p_0} - \frac{3}{2q} - \frac{2}{2q_0} < 1.$$

Moreover, assume that $\theta_0 \ge \underline{\theta} > 0$, $\tau > 0$ is sufficiently small, and $n \in \mathbb{N}\cup\{0\}$. Then there exists a constant A independent of n but depending on D, p, p_0, q, q_0 such that solutions to problem (5.1) satisfy the estimates

$$\begin{aligned}
&\|u^n_t\|_{W^{2,1}_{p,p_0}(\Omega^\tau)} + \|\theta^n\|_{W^{2,1}_{q,q_0}(\Omega^\tau)} \le A,\\
&\frac{1}{2}\underline{\theta} \le \sup_{\Omega^\tau}\theta^n \le A
\end{aligned} \tag{5.4}$$

for $n \in \mathbb{N}\cup\{0\}$.

Proof. Let $u^n \in W^{2,1}_{p,p_0}(\Omega^\tau)$, $\theta^n \in W^{2,1}_{q,q_0}(\Omega^\tau)$. Then Lemma 3.4 ensures the existence of solutions to problem $(5.1)_{1,3,5}$ and the estimate

$$
\begin{aligned}
\|u_t^{n+1}\|_{W^{2,1}_{p,p_0}(\Omega^\tau)} &\le c(\|\nabla^2 u^n\|_{L_{p,p_0}(\Omega^\tau)} + \|\nabla\theta^n\|_{L_{p,p_0}(\Omega^\tau)} \\
&\quad + \|b\|_{L_{p,p_0}(\Omega^\tau)} + \|u_1\|_{B^{2-2/p_0}_{p,p_0}(\Omega)}),
\end{aligned}
\tag{5.5}
$$

where constant c does not depend on τ.
With the use of the formula

$$
u^n(x,t) = \int_0^t u^n_{t'}(x,t')dt' + u_0(x),
$$

the first term on the right-hand side of (5.5) is estimated by

$$
\begin{aligned}
&\|\nabla^2 u^n\|_{L_{p,p_0}(\Omega^\tau)} \le \left\| \int_0^t \nabla^2 u^n_{t'} dt' \right\|_{L_{p,p_0}(\Omega^\tau)} + \tau^{1/p_0}\|\nabla^2 u_0\|_{L_p(\Omega)} \\
&\le \tau\|\nabla^2 u^n_t\|_{L_{p,p_0}(\Omega^\tau)} + \tau^{1/p_0}\|\nabla^2 u_0\|_{L_p(\Omega)},
\end{aligned}
\tag{5.6}
$$

where in the last line the Hölder inequality was used.
The second term on the right-hand side of (5.5) is estimated with the help of the interpolation inequality:

$$
\begin{aligned}
&\|\nabla\theta^n\|_{L_{p,p_0}(\Omega^\tau)} \le \delta\|\theta^n\|_{W^{2,1}_{q,q_0}(\Omega^\tau)} + c(1/\delta)\|\theta^n\|_{L_{q,q_0}(\Omega^\tau)} \\
&\le \delta\|\theta^n\|_{W^{2,1}_{q,q_0}(\Omega^\tau)} + c(1/\delta)\left\| \int_0^t \theta^n_{t'}dt' + \theta_0 \right\|_{L_{q,q_0}(\Omega^\tau)} \\
&\le \delta\|\theta^n\|_{W^{2,1}_{q,q_0}(\Omega^\tau)} + c(1/\delta)(\tau\|\theta^n\|_{W^{2,1}_{q,q_0}(\Omega^\tau)} + \tau^{1/q_0}\|\theta_0\|_{L_q(\Omega)}),
\end{aligned}
\tag{5.7}
$$

where $c(1/\delta) \sim \delta^{-a}$, $a > 0$. The interpolation inequality in the first line holds under the restriction

$$
\frac{3}{q} + \frac{2}{q_0} - \frac{3}{p} - \frac{2}{p_0} < 1, \quad p \ge q, \quad p_0 \ge q_0,
\tag{5.8}
$$

where the last two restrictions can be relaxed on account of the Hölder inequality

$$
\|v\|_{L_{r,r_0}(\Omega^\tau)} \le |\Omega|^{1/r-1/\sigma}\tau^{1/r_0-1/\sigma_0}\|v\|_{L_{\sigma,\sigma_0}(\Omega^\tau)}
$$

with $r \le \sigma$, $r_0 \le \sigma_0$.
Applying estimates (5.6) and (5.7) in (5.5) leads to

$$
\begin{aligned}
&\|u_t^{n+1}\|_{W^{2,1}_{p,p_0}(\Omega^\tau)} \le c[\tau\|u^n_t\|_{W^{2,1}_{p,p_0}(\Omega^\tau)} \\
&\quad + (\delta + c(1/\delta)\tau)\|\theta^n\|_{W^{2,1}_{q,q_0}(\Omega^\tau)} + \tau^{1/p_0}\|\nabla^2 u_0\|_{L_p(\Omega)} \\
&\quad + c(1/\delta)\tau^{1/q_0}\|\theta_0\|_{L_q(\Omega)} + \|b\|_{L_{p,p_0}(\Omega^\tau)} + \|u_1\|_{B^{2-2/p_0}_{p,p_0}(\Omega)}] \\
&\le c[\tau^a(\|u^n_t\|_{W^{2,1}_{p,p_0}(\Omega^\tau)} + \|\theta^n\|_{W^{2,1}_{q,q_0}(\Omega^\tau)}) \\
&\quad + \|u_0\|_{W^2_p(\Omega)} + \|u_1\|_{B^{2-2/p_0}_{p,p_0}(\Omega)} + \|\theta_0\|_{L_q(\Omega)} + \|b\|_{L_{p,p_0}(\Omega^\tau)}],
\end{aligned}
\tag{5.9}
$$

where p, p_0, q, q_0 satisfy (5.8) with $p, q, p_0, q_0 \in (1,\infty)$, τ is sufficiently small, and $a > 0$.

By virtue of Lemma 3.3 there exists a solution to problem $(5.1)_{2,4,6}$, and the following estimate holds

$$
\begin{aligned}
\|\theta^{n+1}\|_{W^{2,1}_{q,q_0}(\Omega^\tau)} &\le \varphi\Big(\sup_{\Omega^\tau}\frac{1}{\theta^n}, \sup_{\Omega^\tau}\theta^n\Big)[\|\theta^n\varepsilon(u^n_t)\|_{L_{q,q_0}(\Omega^\tau)} \\
&\quad + \|\,|\varepsilon(u_t)|^2\|_{L_{q,q_0}(\Omega^\tau)} + \|g\|_{L_{q,q_0}(\Omega^\tau)} + \|\theta_0\|_{B^{2-2/q_0}_{q,q_0}(\Omega)}],
\end{aligned}
\tag{5.10}
$$

where φ is an increasing positive function of its arguments and τ is sufficiently small. By the Hölder inequality it follows from (5.10) that

$$
\begin{aligned}
\|\theta^{n+1}\|_{W^{2,1}_{q,q_0}(\Omega^\tau)} &\le \varphi\Big(\sup_{\Omega^\tau}\frac{1}{\theta^n}, \sup_{\Omega^\tau}\theta^n\Big)[\|\theta^n\|_{L_{2q,2q_0}(\Omega^\tau)}\|\nabla u^n_t\|_{L_{2q,2q_0}(\Omega^\tau)} \\
&\quad + \|\nabla u^n_t\|^2_{L_{2q,2q_0}(\Omega^\tau)} + \|g\|_{L_{q,q_0}(\Omega^\tau)} + \|\theta_0\|_{B^{2-2/q_0}_{q,q_0}(\Omega)}].
\end{aligned}
\tag{5.11}
$$

We proceed now to estimate the norms on the right-hand side of (5.11). First we examine

$$
\begin{aligned}
\sup_{t\le\tau}\|\theta^n\|_{L_\infty(\Omega)} &\le \sup_{t\le\tau}(\delta\|\theta^n\|_{B^{2-2/q_0}_{q,q_0}(\Omega)} + c(1/\delta)\|\theta^n\|_{L_q(\Omega)}) \\
&\le \sup_{t\le\tau}\bigg[\delta\|\theta^n\|_{B^{2-2/q_0}_{q,q_0}(\Omega)} + c(1/\delta)\bigg\|\int_0^t \theta^n_{t'}dt' + \theta_0\bigg\|_{L_q(\Omega)}\bigg] \\
&\le \sup_{t\le\tau}\bigg[\delta\|\theta^n\|_{B^{2-2/q_0}_{q,q_0}(\Omega)} + c(1/\delta)t^{1/q_0'}\bigg(\int_0^t \|\theta^n_{t'}\|^{q_0}_{L_q(\Omega)}dt'\bigg)^{1/q_0} \\
&\quad + c(1/\delta)\|\theta_0\|_{L_q(\Omega)}\bigg] \\
&\le (\delta + c(1/\delta)\tau^{1/q_0'})\|\theta^n\|_{W^{2,1}_{q,q_0}(\Omega^\tau)} + c(1/\delta)\|\theta_0\|_{B^{2-2/q_0}_{q,q_0}(\Omega)},
\end{aligned}
\tag{5.12}
$$

where $c(1/\delta) \sim \delta^{-a}$, $a > 0$, $1/q_0 + 1/q_0' = 1$. The first inequality in (5.12) expresses the interpolation inequality which holds for

$$3/q < 2 - 2/q_0. \tag{5.13}$$

The last inequality in (5.12) follows from the imbedding

$$\sup_{t\le\tau}\|u(t)\|_{B^{2-2/q_0}_{q,q_0}(\Omega)} \le c(\|u\|_{W^{2,1}_{q,q_0}(\Omega^\tau)} + \|u(0)\|_{B^{2-2/q_0}_{q,q_0}(\Omega)}) \tag{5.14}$$

with a constant c independent of τ, which holds true for any solution of a parabolic equation with vanishing boundary conditions and sufficiently smooth coefficients.

To estimate θ^n from below by a positive constant we use the assumption $\theta_0 \ge \underline{\theta} > 0$. Since by (5.13), $\theta^n \in W^{2,1}_{q,q_0}(\Omega^\tau) \subset C^\alpha(\Omega^\tau)$ with some $\alpha \in (0,1)$, it follows that

$$\theta^n = \theta^n - \theta_0 + \theta_0 \ge \underline{\theta} - |\theta^n - \theta_0| \ge \underline{\theta} - \sup_{x\in\Omega}\|\theta^n\|_{C^{\alpha/2}(0,\tau)}\tau^{\alpha/2}.$$

Hence for τ so small that

$$\|\theta^n\|_{L_\infty(\Omega;C^{\alpha/2}(0,\tau))}\tau^{\alpha/2} \le c\|\theta^n\|_{W^{2,1}_{q,q_0}(\Omega^\tau)}\tau^{\alpha/2} \le \frac{1}{2}\underline{\theta}, \tag{5.15}$$

we have

$$\theta^n \ge \frac{1}{2}\underline{\theta}. \tag{5.16}$$

To estimate the first two terms in the square bracket on the right-hand side of (5.11) we consider the factors $\|\theta^n\|_{L_{2q,2q_0}(\Omega^T)}$ and $\|\nabla u_t^n\|_{L_{2q,2q_0}(\Omega^\tau)}$. We have

$$\begin{aligned}\|\theta^n\|_{L_{2q,2q_0}(\Omega^\tau)} &\le \delta\|\theta^n\|_{W^{2,1}_{q,q_0}(\Omega^\tau)} + c(1/\delta)\|\theta^n\|_{L_{q,q_0}(\Omega^\tau)}\\ &\le \delta\|\theta^n\|_{W^{2,1}_{q,q_0}(\Omega^\tau)} + c(1/\delta)\bigg\|\int_0^t \theta^n_{t'}dt' + \theta_0\bigg\|_{L_{q,q_0}(\Omega^\tau)}\\ &\le (\delta + c(1/\delta)\tau)\|\theta^n\|_{W^{2,1}_{q,q_0}(\Omega^\tau)} + \tau^{1/q_0}\|\theta_0\|_{L_q(\Omega)},\end{aligned} \tag{5.17}$$

where the first inequality is the interpolation inequality which holds under the condition

$$\frac{3}{q} + \frac{2}{q_0} - \frac{3}{2q} - \frac{2}{2q_0} < 2,$$

that is

$$\frac{3}{q} + \frac{2}{q_0} < 4. \tag{5.18}$$

Next,

$$\begin{aligned}\|\nabla u_t^n\|_{L_{2q,2q_0}(\Omega^\tau)} &\le \delta\|u_t^n\|_{W^{2,1}_{p,p_0}(\Omega^\tau)} + c(1/\delta)\|u_t^n\|_{L_{p,p_0}(\Omega^\tau)}\\ &\le \delta\|u_t^n\|_{W^{2,1}_{p,p_0}(\Omega^\tau)} + c(1/\delta)\bigg\|\int_0^t u^n_{tt'}dt' + u_t(0)\bigg\|_{L_{p,p_0}(\Omega^\tau)}\\ &\le (\delta + c(1/\delta)\tau)\|u_t^n\|_{W^{2,1}_{p,p_0}(\Omega^\tau)} + c(1/\delta)\tau^{1/p_0}\|u_1\|_{L_p(\Omega)},\end{aligned} \tag{5.19}$$

where the first inequality holds under the condition

$$\frac{3}{p} + \frac{2}{p_0} - \frac{3}{2q} - \frac{2}{2q_0} < 1, \quad p, p_0, q, q_0 \in (1,\infty). \tag{5.20}$$

Using (5.12), (5.16), (5.17), (5.19) and choosing appropriately δ so that to get the factors of $\|\theta^n\|_{W^{2,1}_{q,q_0}(\Omega^T)}$ and $\|u_t^n\|_{W^{2,1}_{p,p_0}(\Omega^T)}$ proportional to τ^a with $a > 0$, we infer from (5.11) that

$$\begin{aligned}\|\theta^{n+1}\|_{W^{2,1}_{q,q_0}(\Omega^\tau)} &\le \varphi(\tau^a\|\theta^n\|_{W^{2,1}_{q,q_0}(\Omega^\tau)} + c\|\theta_0\|_{B^{2-2/q_0}_{q,q_0}(\Omega)}, \underline{\theta})\cdot[\tau^a\|\theta^n\|^2_{W^{2,1}_{q,q_0}(\Omega^\tau)}\\ &+ \tau^a\|u_t^n\|^2_{W^{2,1}_{p,p_0}(\Omega^\tau)} + \|\theta_0\|^2_{L_q(\Omega)} + \|u_1\|^2_{L_p(\Omega)} + \|g\|_{L_{q,q_0}(\Omega^\tau)}\\ &+ \|\theta_0\|_{B^{2-2/q_0}_{q,q_0}(\Omega)}]\end{aligned} \tag{5.21}$$

for $q, q_0, p, p_0 \in (1,\infty)$, restricted by (5.13) and (5.20) ((5.18) is inactive).
Let us denote

$$X_n(\tau) = \|u_t^n\|_{W^{2,1}_{p,p_0}(\Omega^\tau)} + \|\theta^n\|_{W^{2,1}_{q,q_0}(\Omega^\tau)} \tag{5.22}$$

and

$$D = \|u_0\|_{W_p^2(\Omega)} + \|u_1\|_{B_{p,p_0}^{2-2/p_0}(\Omega)} + \|\theta_0\|_{B_{q,q_0}^{2-2/q_0}(\Omega)} + \|b\|_{L_{p,p_0}(\Omega^\tau)} + \|g\|_{L_{q,q_0}(\Omega^\tau)}. \tag{5.23}$$

Then inequalities (5.9) and (5.21) give

$$\|u_t^{n+1}\|_{W_{p,p_0}^{2,1}(\Omega^\tau)} \le c[\tau^a X_n(\tau) + D], \tag{5.24}$$

and

$$\|\theta^{n+1}\|_{W_{q,q_0}^{2,1}(\Omega^\tau)} \le \varphi(\tau^a X_n(\tau), D)[(\tau^a X_n(\tau))^2 + D + D^2], \tag{5.25}$$

where $p, p_0, q, q_0 \in (1,\infty)$ satisfy (5.8), (5.13) and (5.20), $a > 0$, τ is sufficiently small, and c does not depend on τ.

It follows from (5.24) and (5.25) that

$$\begin{aligned} &X_{n+1}(\tau) \le \varphi(\tau^a X_n(\tau), D)[\tau^a X_n(\tau) + (\tau^a X_n(\tau))^2 + D + D^2] \\ &\text{for } \ n \in \mathbb{N} \cup \{0\}, \end{aligned} \tag{5.26}$$

where

$$X_0(\tau) = \|u_t^0\|_{W_{p,p_0}^{2,1}(\Omega^\tau)} + \|\theta^0\|_{W_{q,q_0}^{2,1}(\Omega^\tau)},$$

and $p, p_0, q, q_0 \in (1,\infty)$ satisfy (5.8), (5.13) and (5.20).
Regarding the fact that φ stands for a generic, positive, increasing function of its arguments, (5.26) can be expressed in the following simpler form

$$X_{n+1}(\tau) \le \varphi(\tau^a X_n(\tau), D) \quad \text{for } \ n \in \mathbb{N} \cup \{0\}. \tag{5.27}$$

For τ sufficiently small there exists a positive constant A such that

$$X_0(\tau) \le A \quad \text{and} \quad \varphi(\tau^a A, D) \le A. \tag{5.28}$$

Then (5.27) implies that

$$X_n(\tau) \le A \quad \text{for any } \ n \in \mathbb{N}.$$

This establishes $(5.4)_1$. Moreover, in view of (5.15) and the imbedding $W_{q,q_0}^{2,1}(\Omega^\tau) \subset C^\alpha(\Omega^\tau)$, the bounds $(5.4)_2$ hold. Thereby the proof is completed. □

To show the convergence of the sequence $\{u^n, \theta^n\}$ we introduce the differences

$$U^n(t) = u^n(t) - u^{n-1}(t), \quad \vartheta^n(t) = \theta^n(t) - \theta^{n-1}(t), \quad n \in \mathbb{N} \cup \{0\}, \tag{5.29}$$

which are solutions to the following problems:

$$\begin{aligned} &U_{tt}^{n+1} - \nabla \cdot (A_1 \varepsilon(U_t^{n+1})) = \nabla \cdot (A_2 \varepsilon(U^n)) - \nabla \cdot (A_2 \alpha \vartheta^n) && \text{in } \ \Omega^T, \\ &U^{n+1} = 0 && \text{on } \ S^T, \\ &U^{n+1}|_{t=0} = 0, \quad U_t^{n+1}|_{t=0} = 0 && \text{in } \ \Omega, \end{aligned} \tag{5.30}$$

and

$$\begin{aligned} &c_v \theta^n \vartheta_t^{n+1} - k\Delta \vartheta^{n+1} = -c_v \vartheta^n \theta_t^n - \theta^n (A_2 \alpha) \cdot \varepsilon(U_t^n) \\ &\quad - \vartheta^n (A_2 \alpha) \cdot \varepsilon(u_t^{n-1}) + A_1 \varepsilon(U_t^n) \cdot \varepsilon(u_t^n) + A_1 \varepsilon(u_t^{n-1}) \cdot \varepsilon(U_t^n) && \text{in } \ \Omega^T, \\ &\bar{n} \cdot \nabla \vartheta^n = 0 && \text{on } \ S^T, \\ &\vartheta^{n+1}|_{t=0} = 0 && \text{in } \ \Omega, \end{aligned} \tag{5.31}$$

where

$$\varepsilon(u^{-1}) = 0, \quad \varepsilon(u_t^{-1}) = 0, \quad \theta^{-1} = 0,$$
$$\varepsilon(U^0) = \varepsilon(u_0), \quad \varepsilon(U_t^0) = \varepsilon(u_1), \quad \vartheta^0 = \theta_0.$$

Let us introduce the quantity

$$Y^n(\tau) = \|U_t^n\|_{W^{2,1}_{\bar{p},\bar{p}_0}(\Omega^\tau)} + \|\vartheta^n\|_{W^{2,1}_{\bar{q},\bar{q}_0}(\Omega^\tau)}, \tag{5.32}$$

where $\bar{p}, \bar{p}_0, \bar{q}, \bar{q}_0 \in (1,\infty)$.

LEMMA 5.2. *(Convergence of the approximation) Let the assumptions of Lemma 5.1 hold. Moreover, let the numbers $\bar{p}, \bar{p}_0, \bar{q}, \bar{q}_0 \in (1,\infty)$ satisfy the following conditions:*

$$q = 2\bar{q}, \quad q_0 = 2\bar{q}_0,$$
$$\frac{3}{\bar{q}} + \frac{2}{\bar{q}_0} - \frac{3}{\bar{p}} - \frac{2}{\bar{p}_0} < 1,$$
$$\frac{3}{\bar{p}} + \frac{2}{\bar{p}_0} - \frac{3}{2\bar{q}} - \frac{2}{2\bar{q}_0} < 1,$$
$$\frac{3}{p} + \frac{2}{p_0} - \frac{3}{2\bar{q}} - \frac{2}{2\bar{q}_0} < 1.$$

Then there exists a positive constant d which depends on A, D, $\underline{\theta}$, $p, p_0, q, q_0, \bar{p}, \bar{p}_0$ such that

$$Y^{n+1}(\tau) \le d\tau^a Y^n(\tau), \tag{5.33}$$

where $n \in \mathbb{N} \cup \{0\}$, $a > 0$, and

$$Y^0 = \|u^0\|_{W^{2,1}_{\bar{p},\bar{p}_0}(\Omega^\tau)} + \|\theta^0\|_{W^{2,1}_{\bar{q},\bar{q}_0}(\Omega^\tau)}.$$

Proof. By Lemma 3.4 the solutions to problem (5.30) satisfy the estimate

$$\|U_t^{n+1}\|_{W^{2,1}_{\bar{p},\bar{p}_0}(\Omega^\tau)} \le c_1(\|\nabla^2 U^n\|_{L_{\bar{p},\bar{p}_0}(\Omega^\tau)} + \|\nabla\vartheta^n\|_{L_{\bar{p},\bar{p}_0}(\Omega^\tau)}), \tag{5.34}$$

where c_1 does not depend on τ.
The first term on the right-hand side of (5.34) is estimated by

$$\|\nabla^2 U^n\|_{L_{\bar{p},\bar{p}_0}(\Omega^\tau)} = \left\| \int_0^t \nabla^2 U_{t'}^n dt' \right\|_{L_{\bar{p},\bar{p}_0}(\Omega^\tau)} \le c\tau \|U_t^n\|_{W^{2,1}_{\bar{p},\bar{p}_0}(\Omega^\tau)},$$

and the second one by

$$\|\nabla\vartheta^n\|_{L_{\bar{p},\bar{p}_0}(\Omega^\tau)} \le \delta\|\vartheta^n\|_{W^{2,1}_{\bar{q},\bar{q}_0}(\Omega^\tau)} + c(1/\delta)\|\vartheta^n\|_{L_{\bar{q},\bar{q}_0}(\Omega^\tau)}$$
$$\le (\delta + c(1/\delta)\tau)\|\vartheta^n\|_{W^{2,1}_{\bar{q},\bar{q}_0}(\Omega^\tau)},$$

where $c(1/\delta) = \delta^{-a}$, $a > 0$, and the interpolation inequality in the first line holds under the restriction

$$\frac{3}{\bar{q}} + \frac{2}{\bar{q}_0} - \frac{3}{\bar{p}} - \frac{2}{\bar{p}_0} < 1, \quad \bar{p}, \bar{p}_0, \bar{q}, \bar{q}_0 \in (1,\infty). \tag{5.35}$$

Hence,

$$\begin{aligned}(5.36)\qquad \|U_t^{n+1}\|_{W^{2,1}_{\bar p,\bar p_0}(\Omega^\tau)} &\le c\tau^a(\|U_t^n\|_{W^{2,1}_{\bar p,\bar p_0}(\Omega^\tau)} + \|\vartheta^n\|_{W^{2,1}_{\bar q,\bar q_0}(\Omega^\tau)})\\ &\le c\tau^a Y_n(\tau).\end{aligned}$$

For the solutions to problem (5.31) we have

$$\begin{aligned}(5.37)\qquad \|\vartheta^{n+1}\|_{W^{2,1}_{\bar q,\bar q_0}(\Omega^\tau)} &\le \varphi\Big(\sup_{\Omega^\tau}\theta^n, \sup_{\Omega^\tau}\frac{1}{\theta^n}\Big)\Big[\|\vartheta^n\theta_t^n\|_{L_{\bar q,\bar q_0}(\Omega^\tau)}\\ &+ \|\theta^n|\nabla U_t^n|\|_{L_{\bar q,\bar q_0}(\Omega^\tau)}\|\vartheta^n|\nabla u_t^{n-1}|\|_{L_{\bar q,\bar q_0}(\Omega^\tau)}\\ &+ \|\,|\nabla U_t^n|\,|\nabla u_t^n|\,\|_{L_{\bar q,\bar q_0}(\Omega^\tau)} + \|\,|\nabla U_t^n|\,|\nabla u_t^{n-1}|\,\|_{L_{\bar q,\bar q_0}(\Omega^\tau)}\Big].\end{aligned}$$

By virtue of Lemma 5.1 the arguments of the function φ are estimated by A and $\underline{\theta}/2$. The successive terms under the square bracket on the right-hand side of (5.37) are estimated as follows.

The first term by

$$\|\vartheta^n\|_{L_{2\bar q,2\bar q_0}(\Omega^\tau)}\|\theta_t^n\|_{L_{2\bar q,2\bar q_0}(\Omega^\tau)} \le \varphi(A)\tau^a\|\vartheta^n\|_{W^{2,1}_{\bar q,\bar q_0}(\Omega^\tau)}.$$

This estimate follows on account of $(5.4)_1$ and similar arguments as in (5.7) under the restrictions

$$\begin{aligned}(5.38)\qquad & q = 2\bar q, \quad q_0 = 2\bar q_0, \quad \text{and}\\ & \frac{3}{\bar q} + \frac{2}{\bar q_0} - \frac{3}{2\bar q} - \frac{2}{2\bar q_0} < 2, \quad \text{so} \quad \frac{3}{2\bar q} + \frac{2}{2\bar q_0} < 2.\end{aligned}$$

The second term is estimated by

$$\begin{aligned}\|\theta^n\|_{L_{2\bar q,2\bar q_0}(\Omega^\tau)}\|\nabla U_t^n\|_{L_{2\bar q,2\bar q_0}(\Omega^\tau)} &\le \|\theta^n\|_{L_{q,q_0}(\Omega^\tau)}\cdot\tau^a\|U_t^n\|_{W^{2,1}_{\bar p,\bar p_0}(\Omega^\tau)}\\ &\le \varphi(A)\tau^a\|U_t^n\|_{W^{2,1}_{\bar p,\bar p_0}(\Omega^\tau)},\end{aligned}$$

where we assumed that

$$(5.39)\qquad \frac{3}{\bar p} + \frac{2}{\bar p_0} - \frac{3}{2\bar q} - \frac{2}{2\bar q_0} < 1.$$

The third term is estimated by

$$\|\vartheta^n\|_{L_{2\bar q,2\bar q_0}(\Omega^\tau)}\|\nabla u_t^{n-1}\|_{L_{2\bar q,2\bar q_0}(\Omega^\tau)} \le \varphi(A)\tau^a\|\vartheta^n\|_{W^{2,1}_{\bar q,\bar q_0}(\Omega^\tau)},$$

where we assumed (5.38) and applied (5.4) under the condition

$$(5.40)\qquad \frac{3}{p} + \frac{2}{p_0} - \frac{3}{2\bar q} - \frac{2}{2\bar q_0} < 1.$$

In view of (5.39) and (5.40) the fourth term is estimated by

$$\|\nabla U_t^n\|_{L_{2\bar q,2\bar q_0}(\Omega^\tau)}\|\nabla u_t^n\|_{L_{2\bar q,2\bar q_0}(\Omega^\tau)} \le \varphi(A)\tau^a\|U_t^n\|_{W^{2,1}_{\bar p,\bar p_0}(\Omega^\tau)}.$$

A similar estimate holds for the last fifth term.

Using the above estimates in (5.37) yields

$$\|\vartheta^{n+1}\|_{W^{2,1}_{q,q_0}(\Omega^\tau)} \le \varphi(A, D, \underline{\theta})\tau^a Y^n(\tau). \tag{5.41}$$

From (5.36) and (5.41) we conclude (5.33). The proof is complete. □

Lemmas 5.1 and 5.2 imply the existence of a local solution.

LEMMA 5.3. *(Local existence)*
Assume that $u_0 \in W^2_p(\Omega)$, $u_1 \in B^{2-2/p_0}_{p,p_0}(\Omega)$, $\theta_0 \in B^{2-2/q_0}_{q,q_0}(\Omega)$, $b \in L_{p,p_0}(\Omega^\tau)$, $g \in L_{q,q_0}(\Omega^\tau)$, $\theta_0 \ge \underline{\theta} > \theta_* > 0$, *where* θ_* *is given by (4.2), and* $p, p_0, q, q_0 \in (1,\infty)$ *satisfy*

$$\begin{aligned} &\frac{3}{q} + \frac{2}{q_0} - \frac{3}{p} - \frac{2}{p_0} < 1, \quad &&\frac{3}{p} + \frac{2}{p_0} < 2,\\ &\frac{3}{p} + \frac{2}{p_0} - \frac{3}{2q} - \frac{2}{2q_0} < 1, \quad &&\frac{3}{q} + \frac{2}{q_0} < 2. \end{aligned} \tag{5.42}$$

Then for τ *sufficiently small there exists a solution to problem (1.1)–(1.4) such that*

$$u_t \in W^{2,1}_{p,p_0}(\Omega^\tau), \quad \theta \in W^{2,1}_{q,q_0}(\Omega^\tau), \quad u \in C([0,\tau]; W^2_p(\Omega)),$$

and

$$\begin{aligned} &\|u_t\|_{W^{2,1}_{p,p_0}(\Omega^\tau)} + \|\theta\|_{W^{2,1}_{q,q_0}(\Omega^\tau)} + \|u\|_{C([0,\tau];W^2_p(\Omega))}\\ &\le \varphi(D, \theta_*, p, p_0, q, q_0), \end{aligned} \tag{5.43}$$

where

$$\begin{aligned} D = &D(p, p_0, q, q_0, \tau) \equiv \|u_0\|_{W^2_p(\Omega)} + \|u_1\|_{B^{2-2/p_0}_{p,p_0}(\Omega)}\\ &+ \|\theta_0\|_{B^{2-2/q_0}_{q,q_0}(\Omega)} + \|b\|_{L_{p,p_0}(\Omega^\tau)} + \|g\|_{L_{q,q_0}(\Omega^\tau)}. \end{aligned} \tag{5.44}$$

Proof. To satisfy the assumptions of Lemmas 5.1 and 5.2 we choose $q = 2\bar{q}$, $q_0 = 2\bar{q}_0$ and $\frac{3}{\bar{p}} + \frac{2}{\bar{p}_0} \ge \frac{3}{p} + \frac{2}{p_0}$. Moreover, the assumptions of Lemma 5.1 imply that $\frac{3}{q} + \frac{2}{q_0} < 4$, $\frac{3}{p} + \frac{2}{p_0} < 2$, and Lemma 5.2 gives $\frac{3}{q} + \frac{2}{q_0} < 2$, $\frac{3}{\bar{p}} + \frac{2}{\bar{p}_0} < 1$. Hence, we choose $\frac{3}{p} + \frac{2}{p_0} < 2$, $\frac{3}{\bar{p}} + \frac{2}{\bar{p}_0} < 1$, $\frac{3}{q} + \frac{2}{q_0} < 2$. Then the sequence (u^n_t, θ^n), $n \in \mathbb{N}$, of solutions to the approximate problem (5.1) is for sufficiently small τ uniformly bounded in the space $W^{2,1}_{p,p_0}(\Omega^\tau) \times W^{2,1}_{q,q_0}(\Omega^\tau)$ and convergent in $W^{2,1}_{\bar{p},\bar{p}_0}(\Omega^\tau) \times W^{2,1}_{\bar{q},\bar{q}_0}(\Omega^\tau)$.
Hence, by virtue of the well known result (see the proof of Lemma 2.1 from [15, Ch. 2]) we conclude that the limit

$$(u_t = \lim_{n\to\infty} u^n_t, \theta = \lim_{n\to\infty} \theta^n) \in W^{2,1}_{p,p_0}(\Omega^\tau) \times W^{2,1}_{q,q_0}(\Omega^\tau)$$

and satisfies problem (1.1)–(1.4). □

REMARK 5.4. *Under the assumptions*

$$\begin{aligned} &u \in C([0,T]; W^2_p(\Omega)), \qquad &&u_t \in C([0,T]; B^{2-2/p_0}_{p,p_0}(\Omega)),\\ &\theta \in C([0,T]; B^{2-2/q_0}_{q,q_0}(\Omega)), \qquad &&b \in L_{p,p_0}(\Omega \times (t, t+\tau)),\\ &g \in L_{q,q_0}(\Omega \times (t, t+\tau)), \qquad &&\theta(t) \ge \theta_* > 0, \quad t \in [0,T], \end{aligned}$$

with p, p_0, q, q_0 *satisfying (5.42), the assertion of Lemma 5.3 holds true for the interval* $[t, t+\tau]$, *where* τ *is a sufficiently small number depending on the data.*

LEMMA 5.5. *(Global existence) Assume that*

$$u_0 \in W_r^2(\Omega), \quad u_1 \in B_{r,r_0}^{2-2/r_0}(\Omega), \quad \theta_0 \in B_{\sigma,\sigma_0}^{2-2/\sigma_0}(\Omega),$$
$$g \in L_{\sigma,\sigma_0}(\Omega^T) \cap L_{\infty,r_0}(\Omega^T), \quad b \in L_{r,r_0}(\Omega^T),$$

with $r \ge \max\{12, p\}$, $r_0 \ge \max\{12, p_0\}$, $\sigma \ge \max\{6, q\}$, $\sigma_0 \ge \max\{6, q_0\}$, *where* (p, p_0) *and* (q, q_0) *satisfy (5.42). Then there exists a solution to problem (1.1)-(1.4) such that*

$$u_t \in W_{r,r_0}^{2,1}(\Omega^T), \quad \theta \in W_{\sigma,\sigma_0}^{2,1}(\Omega^T), \quad u \in C([0,T]; W_r^2(\Omega)),$$

satisfying the estimate

$$\begin{aligned} &\|u_t\|_{W_{r,r_0}^{2,1}(\Omega^T)} + \|u\|_{C([0,T];W_r^2(\Omega))} + \|\theta\|_{W_{\sigma,\sigma_0}^{2,1}(\Omega^T)} \\ &\le \varphi(D(r, r_0, \sigma, \sigma_0, T), \theta_*, r, r_0, \sigma, \sigma_0, T), \end{aligned} \tag{5.45}$$

where $D(r, r_0, \sigma, \sigma_0, T)$ *is defined by (5.44) and* φ *is a generic function.*

Proof. To extend the local solution from Lemma 5.3 step by step we choose parameters p, p_0, q, q_0 in such a way to satisfy simultaneously the restriction (4.115) required by a priori estimate (4.116) and the assumptions of Lemma 5.3. Then, in accord with (4.115) we assume that

$$\begin{aligned} &\|u_0\|_{W_{12/5}^2(\Omega)\cap W_p^2(\Omega)} \le c\|u_0\|_{W_r^2(\Omega)} \\ &\text{for } r \ge \max\left\{\frac{12}{5}, p\right\}, \\ &\|u_1\|_{B_{p,p_0}^{2-2/p_0}(\Omega)\cap B_{12,12}^{11/6}(\Omega)} \le c\|u_1\|_{B_{r,r_0}^{2-2/r_0}(\Omega)} \\ &\text{for } r \ge \max\{12, p\}, \quad r_0 \ge \max\{12, p_0\}, \\ &\|\theta_0\|_{B_{q,q_0}^{2-2/q_0}(\Omega)\cap B_{6,6}^{5/3}(\Omega)} \le c\|\theta_0\|_{B_{\sigma,\sigma_0}^{2-2/\sigma_0}(\Omega)} \\ &\text{for } \sigma \ge \max\{6, q\}, \quad \sigma_0 \ge \max\{6, q_0\}. \end{aligned} \tag{5.46}$$

For r and σ specified in (5.46), the conditions (5.42) are satisfied, because $\frac{3}{r} + \frac{2}{r_0} \le \frac{5}{12}$ and $\frac{3}{\sigma} + \frac{2}{\sigma_0} \le \frac{5}{6}$.

Let $t \in [0, T]$. By (4.116) in Corollary 4.25 and by the direct trace theorem we have

$$\begin{aligned} &\|u_t(t)\|_{B_{r,r_0}^{2-2/r_0}(\Omega)} + \|\theta(t)\|_{B_{\sigma,\sigma_0}^{2-2/\sigma_0}(\Omega)} \\ &\le \varphi(D, \theta_*, r, r_0, \sigma, \sigma_0, T). \end{aligned} \tag{5.47}$$

Moreover,

$$\begin{aligned} \|u(t)\|_{W_r^2(\Omega)} &\le \int_0^t \|u_{t'}(t')\|_{W_r^2(\Omega)} dt' + \|u_0\|_{W_r^2(\Omega)} \\ &\le T^{1-1/r_0} \left(\int_0^T \|u_{t'}(t')\|_{W_r^2(\Omega)} dt' \right)^{1/r_0} + \|u_0\|_{W_r^2(\Omega)} \\ &\le T^{1-1/r_0} \|u_t\|_{W_{r,r_0}^{2,1}(\Omega^T)} + \|u_0\|_{W_r^2(\Omega)} \\ &\le \varphi(D_0, \theta_*, r, r_0, \sigma, \sigma_0, T) \end{aligned} \tag{5.48}$$

for any $t \in [0,T]$. Let $N \in \mathbb{N}$ be given so large that $\frac{T}{N+1} \le \tau$, where τ is determined by Remark 5.4 and Lemma 5.3. Using (5.47), (5.48) for $t = k\tau$ and taking into account that

$$\sup_{0\le k\le N} \|b\|_{L_{r,r_0}(\Omega\times(k\tau,(k+1)\tau))} \le \|b\|_{L_{r,r_0}(\Omega^T)},$$

$$\sup_{0\le k\le N} \|g\|_{L_{\sigma,\sigma_0}(\Omega\times(k\tau,(k+1)\tau))\cap L_{\infty,r_0}(\Omega\times(k\tau,(k+1)\tau))}$$

$$\le \|g\|_{L_{\sigma,\sigma_0}(\Omega^t)\cap L_{\infty,r_0}(\Omega^T)},$$

it follows from Remark 5.4 that there exists a solution (u,θ) on the interval $[k\tau,(k+1)\tau]$, $0 \le k \le N$, where τ does not depend on k.
This implies the existence of the solution of problem (1.1)–(1.4) on the whole interval $[0,T]$, which ends the proof. □

Choosing $r = r_0 = 12$, $\sigma = \sigma_0 = 6$ in Lemma 5.5 and recalling Lemma 4.1 we complete the proof of Theorem A.

6. Uniqueness (Proof of Theorem B). Assume that we have two solutions (u_1,θ_1), (u_2,θ_2) to problem (1.1)–(1.4). Introducing the differences

$$U = u_1 - u_2, \quad \vartheta = \theta_1 - \theta_2, \quad E = \varepsilon_1 - \varepsilon_2, \tag{6.1}$$

we see that they satisfy the problem

$$U_{tt} - \nabla\cdot[A_1E_t + A_2(E-\vartheta\alpha)] = 0, \tag{6.2}$$

$$\begin{aligned} c_v(\theta_2\vartheta_t + \vartheta\theta_{1t}) - k\Delta\vartheta &= -[\vartheta(A_2\alpha)\cdot\varepsilon_{1t} + \theta_2(A_2\alpha)\cdot E_t] \\ &\quad + A_1E_t\cdot\varepsilon_{1t} + A_1\varepsilon_{2t}\cdot E_t \quad \text{in } \Omega^T, \end{aligned} \tag{6.3}$$

$$U|_{S^T} = 0, \quad n\cdot\nabla\vartheta|_{S^T} = 0 \quad \text{on } S^T, \tag{6.4}$$

$$U|_{t=0} = 0, \quad U_t|_{t=0} = 0, \quad \vartheta|_{t=0} = 0 \quad \text{in } \Omega. \tag{6.5}$$

Multiplying (6.2) by U_t, integrating over Ω and integrating by parts we obtain

$$\frac{1}{2}\frac{d}{dt}\bigg(\int_\Omega U_t^2dx + \int_\Omega (A_2E)\cdot E dx\bigg) + \int_\Omega A_1E_t\cdot E_t dx = \int_\Omega (A_2\vartheta\alpha)\cdot E_t dx.$$

In view of (1.10) this implies

$$\frac{1}{2}\frac{d}{dt}\bigg(\int_\Omega U_t^2dx + \int_\Omega (A_2E)\cdot E dx\bigg) + a_{1*}\int_\Omega |E_t|^2dx \le \int_\Omega (A_2\vartheta\alpha)\cdot E_t dx$$

which, after applying the Hölder and the Young inequalities to the right hand side, yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (U_t^2 + (A_2E)\cdot E)dx + \frac{1}{2}a_{1*}\int_\Omega |E_t|^2dx \le c_1\int_\Omega \vartheta^2dx. \tag{6.6}$$

Further, multiplying (6.3) by ϑ, integrating over Ω, integrating by parts and using the boundary conditions implies

$$\frac{1}{2}\int_\Omega c_v\Big(\theta_2\frac{\partial}{\partial t}\vartheta^2+\theta_{1t}\vartheta^2\Big)dx+k\int_\Omega|\nabla\vartheta|^2dx \le c_2\int_\Omega(\vartheta^2|\varepsilon_{1t}|+\theta_2|E_t|\,|\vartheta|)dx.$$

Continuing, we have

$$\tag{6.7} \begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega c_v\theta_2\vartheta^2dx+k\int_\Omega|\nabla\vartheta|^2dx &\le c_3\int_\Omega(|\theta_{1t}|+|\theta_{2t}|)\vartheta^2dx\\ &+c_2\int_\Omega(\vartheta^2|\varepsilon_{1t}|+\theta_2|E_t|\,|\vartheta|)dx.\end{aligned}$$

From (6.7) we get

$$\tag{6.8} \begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega c_v\theta_2\vartheta^2dx+k\|\vartheta\|^2_{H^1(\Omega)} &\le \varepsilon_1\|\vartheta\|^2_{L_6(\Omega)}\\ &+c(1/\varepsilon_1)(\|\theta_{1t}\|^2_{L_3(\Omega)}+\|\theta_{2t}\|^2_{L_3(\Omega)})\|\vartheta\|^2_{L_2(\Omega)}\\ &+\varepsilon_2\|\vartheta\|^2_{L_6(\Omega)}+c(1/\varepsilon_2)\|\varepsilon_{1t}\|^2_{L_3(\Omega)}\|\vartheta\|^2_{L_2(\Omega)}\\ &+\varepsilon_3\|E_t\|^2_{L_2(\Omega)}+c(1/\varepsilon_3)\|\theta_2\|^2_{L_\infty(\Omega)}\|\vartheta\|^2_{L_2(\Omega)}+k\|\vartheta\|^2_{L_2(\Omega)}.\end{aligned}$$

Adding (6.6) and (6.8), assuming that ε_1, ε_2, ε_3 are sufficiently small we obtain

$$\tag{6.9} \begin{aligned}&\frac{d}{dt}\int_\Omega(U_t^2+(A_2E)\cdot E+c_v\theta_2\vartheta^2)dx+\frac{1}{2}a_{1*}\int_\Omega|E_t|^2dx\\ &+k\|\vartheta\|^2_{H^1(\Omega)}\le c(c_1+\|\theta_{1t}\|^2_{L_3(\Omega)}+\|\theta_{2t}\|^2_{L_3(\Omega)}+\|\varepsilon_{1t}\|^2_{L_3(\Omega)}\\ &+\|\theta_2\|^2_{L_\infty(\Omega)}+k)\|\vartheta\|^2_{L_2(\Omega)}.\end{aligned}$$

By virtue of (4.2) it holds

$$\theta_2\ge\theta_*>0.$$

Thus, introducing

$$\begin{aligned}X(t)&=\int_\Omega(U_t^2+(A_2E)\cdot E+c_v\theta_2\vartheta^2)dx,\\ A(t)&=c_1+k+\|\theta_{1t}\|^2_{L_3(\Omega)}+\|\theta_{2t}\|^2_{L_3(\Omega)}+\|\varepsilon_{1t}\|^2_{L_3(\Omega)}+\|\theta_2\|^2_{L_\infty(\Omega)},\end{aligned}$$

we conclude from (6.9) the inequality

$$\tag{6.10} \frac{d}{dt}X\le AX\quad\text{for}\quad t\in(0,T).$$

Hence,

$$\tag{6.11} X(t)\le X(0)\exp\Big(\int_0^t A(t')dt'\Big).$$

Since $X(0) = 0$ and, by the assumption (1.17), $\int_0^t A(t')dt' < \infty$, it follows that $X(t) = 0$ for $t \in (0,T)$. This proves the uniquenes of the solution.

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