1/2011

Raport Badawczy
Research Report

RB/10/2011

On a class of sixth order viscous Cahn- Hilliard type equations

I. Pawłow, W.M. Zajączkowski

Instytut Badań Systemowych
Polska Akademia Nauk

Systems Research Institute
Polish Academy of Sciences

POLSKA AKADEMIA NAUK

Instytut Badań Systemowych

ul. Newelska 6

01-447 Warszawa

tel.: (+48) (22) 3810100

fax: (+48) (22) 3810105

Kierownik Zakładu zgłaszający pracę:
Prof. nadzw. dr hab. inż. Antoni Żochowski

Warszawa 2011

Manuscript submitted to
AIMS' Journals
Volume X, Number 0X, XX 200X

Website: http://AIMsciences.org

pp. X–XX

ON A CLASS OF SIXTH ORDER VISCOUS CAHN-HILLIARD TYPE EQUATIONS

Dedicated to Professor Michel Frémond on the occasion of his 70th birthday

IRENA PAWLOW

Systems Research Institute, Polish Academy of Sciences,
Newelska 6, 01-447 Warsaw, Poland
Institute of Mathematics and Cryptology, Cybernetics Faculty,
Military University of Technology, S. Kaliskiego 2,
00-908 Warsaw, Poland

WOJCIECH M. ZAJĄCZKOWSKI

Institute of Mathematics, Polish Academy of Sciences,
Śniadeckich 8, 00-950 Warsaw, Poland
Institute of Mathematics and Cryptology, Cybernetics Faculty,
Military University of Technology, S. Kaliskiego 2,
00-908 Warsaw, Poland

ABSTRACT. An initial-boundary-value problem for a class of sixth order viscous Cahn-Hilliard type equations with a nonlinear diffusion is considered. The study is motivated by phase-field modelling of various spatial structures, for example arising in oil-water-surfactant mixtures and in modelling of crystal growth on atomic length, known as phase field crystal model. For such problem we prove the existence and uniqueness of a global in time regular solution. First the finite-time existence is proved by means of the Leray-Schauder fixed point theorem. Then, due to suitable estimates, the finite-time solution is extended step by step on the infinite time interval.

1. Introduction.

1.1. Motivation and aim. In recent literature one can observe a remarkable interest in higher order phase field models of the Cahn-Hilliard and Landau-Ginzburg (Allen-Cahn) types for microstructure evolution, see e.g. [8, 21] for overviews and up-to-date references.

In this article we are concerned with an initial-boundary-value problem for a class of sixth order Cahn-Hilliard type equations with a nonlinear diffusion and viscous effects. For such problem we prove the existence and uniqueness of a global in time regular solution.

The study is motivated by two physical problems described by second order free energies of the Landau-Ginzburg type: the model of microstructure evolution in oil-water-surfactant mixtures and the so-called phase field crystal (PFC) atomistic model of crystal growth, proposed by Elder et al. [6, 7, 1, 2].

2000 *Mathematics Subject Classification.* Primary: 35K60; Secondary: 35Q72, 35L205 .

Key words and phrases. Sixth order viscous Cahn-Hilliard type equation,existence of global regular solutions, oil-water-surfactant mixtures.

Partially supported by Polish Grant NN 201 396 937.

The second order Landau-Ginzburg free energy for oil-water-surfactant mixtures has been proposed in a series of papers by Gompper et al. [9, 10, 11]. A corresponding sixth order equation, extending the classical fourth order Cahn-Hilliard equation, has been recently studied in [18]. The existence and uniqueness of a regular solution on an arbitrary finite time interval have been proved there, provided given sufficiently smooth initial datum. Here we incorporate viscous effects associated with the rates of the order parameter and its spatial gradients into the sixth order model. Such effects – typical for soft-matter systems – may be relevant for the description of oil-water-surfactant mixtures. Thanks to the viscous structure of the model we are able to prove the global in time existence of regular solutions under a weaker assumption on the initial datum than that postulated in [18]. The key tool in the proof is an absorbing type estimate. Such estimate allows not only to extend the finite-time solution step by step on the inifinite time interval but also to conclude the existence of an absorbing set. The long-time analysis of solutions is postponed for a future work.

We mention that a fourth order Cahn-Hilliard system (governed by a first order gradient free energy) with a similar viscous structure and additional cross-coupling terms has been derived and studied in [5] (for details see Section 2).

1.2. **Problem statement.** Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary S, and $T > 0$ be a final time. We consider the following system of Cahn-Hilliard type for the order parameter χ and the chemical potential μ:

$$\chi_t - M\Delta\mu = 0 \quad \text{in} \quad \Omega^T := \Omega \times (0,T), \tag{1.1}$$

$$\mu = f_{0,\chi} - \frac{1}{2}\varkappa_{1,\chi}|\nabla\chi|^2 - \varkappa_1\Delta\chi + \varkappa_2\Delta^2\chi + \beta\chi_t - \gamma\Delta\chi_t \quad \text{in} \quad \Omega^T, \tag{1.2}$$

with the initial condition

$$\chi|_{t=0} = \chi_0 \quad \text{in} \quad \Omega, \tag{1.3}$$

and the boundary conditions

$$\boldsymbol{n}\cdot\nabla\chi = 0 \quad \text{on} \quad S^T := S \times (0,T), \tag{1.4}$$

$$\boldsymbol{n}\cdot\nabla\Delta\chi = 0 \quad \text{on} \quad S^T, \tag{1.5}$$

$$\boldsymbol{n}\cdot\nabla\mu = 0 \quad \text{on} \quad S^T, \tag{1.6}$$

where M, $\varkappa_2$, β, γ are positive constants, $f_0 = f_0(\chi)$, $\varkappa_1 = \varkappa_1(\chi)$ are given functions specified below, $\boldsymbol{n}$ is the unit outward vector normal to S, $\chi_t = \partial\chi/\partial t$, $f_{,\chi} = df/d\chi$, the dot, $\cdot$, means the scalar product, $\nabla\cdot$ stands for the spatial divergence, vectors and tensors are denoted by bold letters and the summation convention is used.

The constants M, $\varkappa_2$, β, γ denote the mobility, the second gradient energy coefficient, and the two viscosity coefficients respectively. Later on, for simplicity, we set $M \equiv 1$.

The function $f_0(\chi)$ denotes the multiwell volumetric free energy density and $\varkappa_1(\chi)$ is the first gradient energy coefficient which may be of arbitrary sign. We shall assume the polynomial forms of f_0 and $\varkappa_1$ which comprise the oil-water-surfactant model and the PFC model as the particular cases (see Sect. 1.5):

$$f_0(\chi) = \sum_{i=0}^{2k} a_i\chi^i \quad \text{with} \quad a_i \in \mathbb{R}, \quad a_{2k} > 0, \quad k > 1, \tag{1.7}$$

and

$$\varkappa_1(\chi) = \sum_{i=0}^{2l} b_i \chi^i \quad \text{with} \quad b_i \in \mathbb{R}, \quad b_{2l} > 0, \quad l > 1. \tag{1.8}$$

The assumption that the leading coefficients a_{2k} and b_{2l} are positive is motivated by physical examples.

As in the standard fourth order Cahn-Hilliard problem the boundary conditions (1.4) and (1.5) arise in a natural way from the free energy potential (see (1.18) below) whereas (1.6) represents the mass isolation at the boundary S. Other types of boundary conditions are also possible. In particular, it is relevant (and mathematically simpler) to consider periodic boundary conditions.

System (1.1)–(1.6) with $M = 1$ can be equivalently expressed as the following initial-boundary-value problem for the sixth order viscous Cahn-Hilliard type equation:

$$\begin{aligned} &\chi_t - \beta\Delta\chi_t + \gamma\Delta^2\chi_t - \varkappa_2\Delta^3\chi \\ &= \Delta\left(f_{0,\chi} - \frac{1}{2}\varkappa_{1,\chi}|\nabla\chi|^2 - \varkappa_1\Delta\chi\right) \quad \text{in} \ \Omega^T, \end{aligned} \tag{1.9}$$

$$\chi|_{t=0} = \chi_0 \quad \text{in} \ \Omega, \tag{1.10}$$

$$\boldsymbol{n}\cdot\nabla\chi = 0 \quad \text{on} \ S^T, \tag{1.11}$$

$$\boldsymbol{n}\cdot\nabla\Delta\chi = 0 \quad \text{on} \ S^T, \tag{1.12}$$

$$\varkappa_2\boldsymbol{n}\cdot\nabla\Delta^2\chi = \boldsymbol{n}\cdot\nabla\left(\frac{1}{2}\varkappa_{1,\chi}|\nabla\chi|^2\right) \quad \text{on} \ S^T. \tag{1.13}$$

We note that the coefficient $\varkappa_1(\chi)$ gives rise to the nonlinear boundary condition (1.13).

1.3. Main result.

Theorem 1.1. *(Global existence and uniqueness) Let us assume that $\Omega \subset \mathbb{R}^3$ is a bounded domain with boundary S of class C^6, $T > 0$ is a given number, functions f_0 and $\varkappa_1$ are defined by (1.7), (1.8), and the initial datum χ_0 is such that $\chi_0 \in H^5(\Omega)$ with the spatial mean*

$$\fint_\Omega \chi_0 dx = \frac{1}{|\Omega|}\int_\Omega \chi_0 dx =: \chi_m, \quad |\Omega| = \text{meas}\ \Omega,$$

and satisfies the compatibility conditions

$$\boldsymbol{n}\cdot\nabla\chi_0 = 0, \quad \boldsymbol{n}\cdot\nabla\Delta\chi_0 = 0 \quad \text{on} \ S. \tag{1.14}$$

Then problem (1.9)–(1.13) (equivalent to (1.1)–(1.6)) has a unique global in time solution such that

$$\begin{aligned} &\chi \in L_2(\mathbb{R}_+; H^6(\Omega)) \cap H^1(\mathbb{R}_+; H^4(\Omega)), \\ &\chi|_{t=0} = \chi_0, \quad \text{and} \quad \fint_\Omega \chi(t)dx = \chi_m \quad \text{for all} \ \ t \in \mathbb{R}_+ := (0,\infty), \end{aligned} \tag{1.15}$$

satisfying the energy estimate

$$\|\chi\|_{L_\infty(R_+;H^2(\Omega))} + \|\chi_t\|_{L_2(\mathbb{R}_+;H^1(\Omega))} + \|\nabla\mu\|_{L_2(\mathbb{R}_+;L_2(\Omega))} \le c_1 \tag{1.16}$$

with a constant $c_1 = \varphi(\|\chi_0\|_{H^2(\Omega)}, |\chi_m|)$.

Moreover, there exists a positive constant A_* depending on the data and T but independent of $k \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$, such that

$$\|\chi\|_{L_2(kT,(k+1)T;H^6(\Omega))} + \|\chi\|_{H^1(kT,(k+1)T;H^1(\Omega))} \le \varphi(A_*, c_1, T), \tag{1.17}$$

where φ is a positive, increasing function of its arguments.

Remark 1. Our main goal behind this result was to investigate the influence of viscous effects on the regularity of solutions. In the case of absent viscous terms ($\beta = \gamma = 0$) the regularity question has been previously addressed in [18].

Of course, it is not necessary to restrict the analysis to regular solutions. The weak solvability of systems similar to (1.1)–(1.6) was studied in [5] and [20]. It seems to be possible to adapt the Galerkin approximation approach of [5] to system (1.1)–(1.6) and its more general version (2.7) (see Section 2) to prove the existence of a weak solution for the initial datum in the energy class. This question is left for a future study.

1.4. **Variational structure.** System (1.1)–(1.2) is associated with the following two gradient type potentials: the free energy

$$f = f(\chi, \nabla\chi, \nabla^2\chi) = f_0(\chi) + \frac{1}{2}\varkappa_1(\chi)|\nabla\chi|^2 + \frac{1}{2}\varkappa_2|\Delta\chi|^2, \tag{1.18}$$

and the dissipation potential

$$\mathcal{D} = \mathcal{D}(\chi_t, \nabla\chi_t, \nabla\mu) = \frac{1}{2}\beta\chi_t^2 + \frac{1}{2}\gamma|\nabla\chi_t|^2 + \frac{1}{2}M|\nabla\mu|^2. \tag{1.19}$$

In terms of these potentials (1.1) and (1.2) read as

$$\begin{aligned} &\chi_t - \nabla\cdot\frac{\partial\mathcal{D}}{\partial\nabla\mu} = 0 \quad \text{in } \Omega^T,\\ &\mu = \frac{\delta f}{\delta\chi} + \frac{\delta\mathcal{D}}{\delta\chi_t} \qquad \text{in } \Omega^T, \end{aligned} \tag{1.20}$$

where $\delta f/\delta\chi$ (resp. $\delta\mathcal{D}/\delta\chi_t$) denotes the first variation defined by the condition that

$$\frac{d}{d\lambda}\int_\Omega f(\chi+\lambda\zeta, \nabla\chi+\lambda\nabla\zeta, \nabla^2\chi+\lambda\nabla^2\zeta)dx\big|_{\lambda=0} =: \int_\Omega \frac{\delta f}{\delta\chi}\zeta dx$$

must hold for all test functions $\zeta \in C_0^\infty(\Omega)$.
In fact, for f and $\mathcal{D}$ defined by (1.18), (1.19) we have

$$\begin{aligned} &\frac{\delta f}{\delta\chi} = f_{0,\chi} + \frac{1}{2}\varkappa_{1,\chi}|\nabla\chi|^2 - \nabla\cdot(\varkappa_1\nabla\chi) + \varkappa_2\Delta^2\chi,\\ &\frac{\delta\mathcal{D}}{\delta\chi_t} = \beta\chi_t - \gamma\Delta\chi_t, \quad \frac{\partial\mathcal{D}}{\partial\nabla\mu} = M\nabla\mu, \end{aligned}$$

which lead to (1.1), (1.2).

From (1.20) one can immediately deduce the energy equality. Formally, testing $(1.20)_1$ by μ, $(1.20)_2$ by $-\chi_t$, adding the obtained relations, integrating over Ω and integrating by parts using the no-flux boundary conditions (1.4)–(1.6) one arrives at

$$\frac{d}{dt}\int_\Omega f(\chi, \nabla\chi, \nabla^2\chi)dx + \int_\Omega \sigma dx = 0, \tag{1.21}$$

where the quantity

$$\sigma = \frac{\partial \mathcal{D}}{\partial \chi_t}\chi_t + \frac{\partial \mathcal{D}}{\partial \nabla \chi_t} \cdot \nabla \chi_t + \frac{\partial \mathcal{D}}{\partial \nabla \mu} \cdot \nabla \mu = \beta \chi_t^2 + \gamma |\nabla \chi_t|^2 + M|\nabla \mu|^2 \geq 0$$

denotes the dissipation density.

1.5. **Model examples.** Gompper et al. [9, 10, 11] have proposed a phenomenological Landau-Ginzburg theory for oil-water-surfactant mixtures. This theory is based on the free energy (1.18) with constant $\varkappa_2 > 0$ and functions f_0, $\varkappa_1$ approximated, respectively, by a sixth and a second order polynomials:

$$f_0(\chi) = (\chi+1)^2(\chi^2+h_0)(\chi-1)^2, \quad \varkappa_1(\chi) = g_0 + g_2\chi^2, \tag{1.22}$$

where h_0, g_0, g_2 are constants, $g_2 > 0$ and h_0, g_0 are of arbitrary sign. Here the order parameter χ represents the local difference of the oil and water concentrations.

Elder et al. [6, 7, 1, 2] have proposed the so-called phase field crystal (PFC) model to describe the phenomenon of crystal growth on atomic length and diffusive time scales. The model is based on the free energy of the following form (known as the Brazovskii or the Swift-Hohenberg one, see [8])

$$f_{\mathrm{PFC}} = f_{\mathrm{PFC}}(\chi, \nabla^2\chi, \nabla^4\chi) = -\alpha\frac{\chi^2}{2} + \frac{\chi^4}{4} + \frac{\chi}{2}(1+\Delta)^2\chi, \tag{1.23}$$

where χ is an order parameter corresponding to atomic mass density, $\alpha = a(\theta_c - \theta)$, $a > 0$ is the parameter of the system periodicity, $\theta_c - \theta$ is the quench depth representing the control parameter with critical temperature θ_c and actual temperature θ.

With neglected gaussian random noise variable the PFC model is a conserved version of the simplest form of the Swift-Hohenberg equation:

$$\chi_t - \Delta\frac{\delta f_{\mathrm{PFC}}}{\delta \chi} = 0, \tag{1.24}$$

where

$$\frac{\delta f_{\mathrm{PFC}}}{\delta \chi} = (1-\alpha)\chi + \chi^3 + 2\Delta\chi + \Delta^2\chi.$$

Let us note that the following second order free energy density

$$f = f(\chi, \nabla\chi, \nabla^2\chi) = f_0(\chi) - |\nabla\chi|^2 + \frac{1}{2}|\Delta\chi|^2 \tag{1.25}$$

with

$$f_0(\chi) = (1-\alpha)\frac{\chi^2}{2} + \frac{\chi^4}{4},$$

which is a special case of (1.18) with $\varkappa_1 \equiv -2$ and $\varkappa_2 \equiv 1$, has the same first variation as (1.23), $\delta f/\delta\chi = \delta f_{\mathrm{PFC}}/\delta\chi$. Whence the PFC model may be considered as a particular nonviscous ($\beta = \gamma = 0$) variant of equation $(1.9)_1$.

1.6. **Relation to other results.** System (1.1)–(1.6) with $\beta = \gamma = 0$ and the Gompper et al. free energy (1.18), (1.22) has been studied in [18]. In a more general setting admitting the logarithmic type free energy f_0 and viscosity coefficients $\gamma = 0$, $\beta \geq 0$, system (1.1)–(1.6) has been recently addressed in [20] from the point of view of the existence of weak solutions. The behaviour of weak solutions in the case when $\varkappa_2$ is let tend to 0 has been studied there as well.

We mention also a sixth order Cahn-Hilliard type equation in two space dimensions which arises as a model for the faceting of a growing crystalline surface, derived by Savina et al. [19]. The model is based on a free energy of the form (1.25) with

$f_0 = f_0(\chi_x, \chi_y)$ where $\chi = \chi(x, y, t)$ describes the surface, $(x, y) \in \mathbb{R}^2$. The model differs from (1.1)–(1.2) with $\beta = \gamma = 0$ by the presence of a force-like term $|\nabla\chi|^2$ which is due to the deposition rate and causes that χ is not a conserved quantity. Such a model has been recently studied mathematically in [13, 14].

1.7. **Plan of the paper.** In Section 2 a thermodynamic background of system (1.1)–(1.2) is outlined. A more general formulation of a thermodynamically consistent sixth order system with a conserved order parameter is presented. In Section 3 notation and some auxiliary results are introduced. In Section 4 suitable a priori estimates are derived. They comprise energy estimates, finite-time estimates depending on T, and additional global estimates. In Section 5 the proof of the existence of solutions is presented. It is based on the Leray-Schauder fixed point theorem and the global a priori estimate which allows to extend the unique finite-time solution step by step on the infinite time interval.

2. Thermodynamic background.

2.1. **Sketch of derivation.** A general form of a thermodynamically consistent system governed by a second order gradient free energy, including (1.1)–(1.2) as a particular case, can be derived by employing the second law of thermodynamics in the form of the Müller-Liu entropy inequality with multipliers [16, 15].
The application of this approach in the case of gradient type systems requires a special procedure which has been described in [17] for the Cahn-Hilliard and Allen-Cahn equations accounting for elastic effects. Here we briefly sketch the procedure leading to models with a second order free energy. The details of the derivation will be presented elsewhere.

We consider a balance law (local form) for the order parameter χ:

$$\chi_t + \nabla \cdot j = 0 \quad \text{in } \Omega^T, \tag{2.1}$$

where j is the mass flux. We assume that j is given by the constitutive equation $j = \hat{j}(Y)$ with the set of constitutive variables

$$Y = (\chi, \nabla\chi, \dots, \nabla^4\chi, \chi_t, \nabla\chi_t)$$

accounting for inhomogeneous and viscous effects, expressed by the space and time-space derivatives, respectively. As explained in [17], in order to admit the free energy depending on $\nabla^m\chi$, $m \in \mathbb{N}$, the set of constitutive variables has to include $\nabla^{m-1}\chi_t$. Since our goal is to construct a model with the free energy depending at most on $\nabla^2\chi$ we have to admit $\nabla\chi_t$ as a constitutive variable.

Next we postulate the free energy inequality with a multiplier

$$f_t + \nabla \cdot \Phi + \lambda(\chi_t + \nabla \cdot j) \leq 0 \tag{2.2}$$

to be satisfied for all fields χ. Here $f = \hat{f}(Y)$ is the free energy, $\Phi = \hat{\Phi}(Y)$ is the free energy flux, and $\lambda = \hat{\lambda}(Y)$ is the multiplier conjugated with the balance equation (2.1). By algebraic operations the evaluation of (2.2) leads to a number of relations for f, Φ and λ as well as to a residual inequality.

The key point of the procedure is the postulate that the multiplier λ is an additional independent variable. This postulate originates from extended thermodynamic's idea of treating the Lagrange multipliers as privileged fields. Then, regarding the obtained algebraic relations we arrive at an extended system of equations with χ and $\mu \equiv -\lambda$ as independent variables, and with the constitutive relation for the free energy restricted to $f = \hat{f}(\chi, \nabla\chi, \nabla^2\chi)$. This system has the form:

$$
\begin{aligned}
&\chi_t + \nabla \cdot \boldsymbol{j} = 0 && \text{in } \Omega^T, \\
&\mu = \frac{\delta f}{\delta \chi} + A^0 - \nabla \cdot \boldsymbol{A}^1 && \text{in } \Omega^T,
\end{aligned} \tag{2.3}
$$

where

$$
\frac{\delta f}{\delta \chi} = f_{,\chi} - \nabla \cdot f_{,\nabla\chi} + \nabla^2 \cdot f_{,\nabla^2\chi} \quad (\nabla^2 \cdot = \nabla \cdot \nabla \cdot),
$$

and the variable μ is identified with the chemical potential.
The quantities, the scalar $A^0 = \hat{A}^0(\mathcal{Z})$ and the vectors $\boldsymbol{A}^1 = \hat{\boldsymbol{A}}^1(\mathcal{Z})$, $\boldsymbol{j} = \hat{\boldsymbol{j}}(\mathcal{Z})$, with the constitutive set $\mathcal{Z}$ given by

$$
\mathcal{Z} := (X; \omega), \quad X := (\chi_t, \nabla\chi_t, \nabla\mu), \quad \omega := (\chi, \nabla\chi, \dots, \nabla^4\chi, \mu),
$$

are determined as solutions to the residual dissipation inequality

$$
\sigma := \chi_t A^0 + \nabla\chi_t \cdot \boldsymbol{A}^1 - \nabla\mu \cdot \boldsymbol{j} \geq 0 \tag{2.4}
$$

to be satisfied for all variables $\mathcal{Z}$.

Inequality (2.4) represents the standard thermodynamic inequality

$$
\sigma = X \cdot J(X; \omega) \geq 0 \quad \text{for all} \quad (X; \omega),
$$

where $J = (A^0, \boldsymbol{A}^1, -\boldsymbol{j})$ is the thermodynamic flux, and the sets X and ω correspond to thermodynamic force and state variables, respectively. Moreover, $\sigma = \hat{\sigma}(X; \omega)$ represents the dissipation scalar.

Inequality (2.4) can be solved by applying the Edelen decomposition theorem [4] which asserts that there exists a dissipation potential $\mathcal{D} = \hat{\mathcal{D}}(X; \omega)$ which is nonnegative, convex in X and achieves its absolute minimum of zero at $X = 0$, such that

$$
\sigma = X \cdot \mathcal{D}_{,X}(X; \omega) \geq 0 \quad \text{for all} \quad (X; \omega).
$$

Thus,

$$
A^0 = \frac{\partial \mathcal{D}}{\partial \chi_t}, \quad \boldsymbol{A}^1 = \frac{\partial \mathcal{D}}{\partial \nabla\chi_t}, \quad -\boldsymbol{j} = \frac{\partial \mathcal{D}}{\partial \nabla\mu},
$$

so that (2.3) leads to system (1.20).

2.2. **Alternative representation.** To solve the residual inequality (2.4) we can apply instead of Edelen's theorem the linear map representation result due to Gurtin [12]. Then we arrive at an alternative formulation of the model with the second order free energy $f = \hat{f}(\chi, \nabla\chi, \nabla^2\chi)$. It has the form of system (2.3), where the quantities $A^0 = \hat{A}^0(X; \omega)$ and $\boldsymbol{A}^1 = \hat{\boldsymbol{A}}^1(X; \omega)$, $\boldsymbol{j} = \hat{\boldsymbol{j}}(X; \omega)$ are given by

$$
\begin{aligned}
A^0 &= \beta\chi_t + \boldsymbol{a} \cdot \nabla\chi_t + \boldsymbol{b} \cdot \nabla\mu, \\
\boldsymbol{A}^1 &= \boldsymbol{c}\chi_t + \boldsymbol{A}\nabla\chi_t + \boldsymbol{B}\nabla\mu, \\
-\boldsymbol{j} &= \boldsymbol{d}\chi_t + \boldsymbol{C}\nabla\chi_t + \boldsymbol{D}\nabla\mu.
\end{aligned} \tag{2.5}
$$

Here the constitutive moduli β (a scalar), $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ (four vectors), and $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$ (four matrices) may depend on the variables (X, ω) and are consistent with the inequality

$$\begin{bmatrix} \chi_t \\ \nabla\chi_t \\ \nabla\mu \end{bmatrix} \cdot \begin{bmatrix} \beta & a^T & b^T \\ c & A & B \\ d & C & D \end{bmatrix} \begin{bmatrix} \chi_t \\ \nabla\chi_t \\ \nabla\mu \end{bmatrix} \geq 0 \tag{2.6}$$

for all variables $(X;\omega)$. Inserting (2.5) into (2.3) yields the system

$$\begin{aligned} &\chi_t - \nabla\cdot(d\chi_t + C\nabla\chi_t) = \nabla\cdot(D\nabla\mu),\\ &\mu - b\cdot\nabla\mu + \nabla\cdot(B\nabla\mu) = \beta\chi_t + a\cdot\nabla\chi_t - \nabla\cdot(c\chi_t)\\ &\qquad - \nabla\cdot(A\nabla\chi_t) + f_{,\chi} - \nabla\cdot f_{,\nabla\chi} + \nabla^2\cdot f_{,\nabla^2\chi} \end{aligned} \tag{2.7}$$

with the moduli satisfying (2.6).

Equations (1.1)–(1.2) result from (2.7) by setting $a = b = c = d = 0$, $C = B = 0$ and $A = \gamma I$, $D = MI$ where I is the identity matrix.

It is of interest to note that system (2.7) with the restriction to the first order gradient free energy, $f = \hat{f}(\chi, \nabla\chi)$, has been derived by Efendiev and Miranville [5, eq. (2.10), (2.11)] in the framework of the Fried-Gurtin theory based on a microforce balance. We remark also that in the case of a second order gradient free energy the model formulated in [5] has a different, more complicated structure than (2.7).

3. Notation and auxiliary results.

3.1. **Notation.** Let $\Omega \subset \mathbb{R}^n$ be an open bounded subset of $\mathbb{R}^n$, $n \geq 1$, with a smooth boundary S, and $\Omega^T = \Omega\times(0,T)$. We introduce:

$$W_2^k(\Omega) = H^k(\Omega), \quad k\in\mathbb{N}\cup\{0\}$$

– the Sobolev space on Ω endowed with the standard norm $\|\cdot\|_{H^k(\Omega)}$;
$H^0(\Omega) = L_2(\Omega)$;

$$W_p^{kl,l}(\Omega^T) = L_p(0,T;W_p^{kl}(\Omega))\cap W_p^l(0,T;L_p(\Omega)), \quad k,l\in\mathbb{N}, \quad p\in[1,\infty)$$

– the Sobolev space on Ω^T with the finite norm

$$\|u\|_{W_p^{kl,l}(\Omega^T)} = \bigg(\sum_{|\alpha|+ka\leq kl}\int_{\Omega^T}|D_x^\alpha\partial_t^a u|^p dxdt\bigg)^{1/p};$$

$$W_p^{ks,s}(\Omega^T) = L_p(0,T;W_p^{ks}(\Omega))\cap W_p^s(0,T;L_p(\Omega)), \quad k\in\mathbb{N}, \quad s\in\mathbb{R}_+, \quad p\in[1,\infty)$$

– the Sobolev-Slobodecki space on Ω^T with the finite norm

$$\begin{aligned} \|u\|_{W_p^{ks,s}(\Omega^T)} = \bigg(&\sum_{|\alpha|+ka\leq[ks]}\int_{\Omega^T}|D_x^\alpha\partial_t^a u|^p dxdt\\ &+ \sum_{|\alpha|=[ks]}\int_0^T\int_\Omega\int_\Omega \frac{|D_x^\alpha u(x,t) - D_{x'}^\alpha u(x',t)|^p}{|x-x'|^{n+p(ks-[ks])}}dxdx'dt\\ &+ \int_\Omega\int_0^T\int_0^T \frac{|\partial_t^{[s]}u(x,t) - \partial_{t'}^{[s]}u(x,t')|^p}{|t-t'|^{1+p(s-[s])}}dtdt'dx\bigg)^{1/p}, \end{aligned}$$

where $[s]$ is the integer part of s.

By c we denote a generic positive constant which changes its value from formula to formula and depends at most on imbedding constants, constants of the problem and the regularity of the boundary.

By $\varphi = \varphi(\sigma_1, \dots, \sigma_k)$, $k \in \mathbb{N}$, we denote a generic function which is a positive, increasing function of its arguments $\sigma_1, \dots, \sigma_k$, and may change from formula to formula. Moreover, ε will denote an arbitrarily small positive constant.

3.2. **Imbeddings in Sobolev-Slobodecki spaces.** Following [22, 23] we introduce the fractional derivative norms. For $\mu \in (0,1)$ and $p \in [1,\infty)$ let

$$[u]_{\mu,p,\Omega^T,x} = \left(\int_0^T \int_\Omega \int_\Omega \frac{|u(x,t) - u(x',t)|^p}{|x - x'|^{n+p\mu}} dx dx' dt \right)^{1/p} \equiv \|\partial_x^\mu u\|_{L_p(\Omega^T)},$$

$$[u]_{\mu,\infty,\Omega^T,x} = \sup_{t\in(0,T)} \sup_{x,x'\in\Omega} \frac{|u(x,t) - u(x',t)|}{|x - x'|^\mu} \equiv \|\partial_x^\mu u\|_{L_\infty(\Omega^T)},$$

and

$$[u]_{\mu,p,\Omega^T,t} = \left(\int_\Omega \int_0^T \int_0^T \frac{|u(x,t) - u(x,t')|^p}{|t - t'|^{1+p\mu}} dt dt' dx \right)^{1/p} \equiv \|\partial_t^\mu u\|_{L_p(\Omega^T)},$$

$$[u]_{\mu,\infty,\Omega^T,t} = \sup_{x\in\Omega} \sup_{t,t'\in(0,T)} \frac{|u(x,t) - u(x,t')|}{|t - t'|^\mu} \equiv \|\partial_t^\mu u\|_{L_\infty(\Omega^T)}.$$

For simplicity we denote the fractional derivatives by $\partial_x^\mu u$ and $\partial_t^\mu u$.
We need the following results.

Theorem 3.1. *(see [3, Chap. 3, Sect. 10]). Let $u \in W_p^{ks,s}(\Omega^T)$, $\Omega \subset \mathbb{R}^n$, $n, k \in \mathbb{N}$, $s \in \mathbb{R}_+$, $p \in [1,\infty]$. Let*

$$\varkappa = \left(\frac{n+k}{p} - \frac{n}{q} - \frac{k}{r} + |\alpha| + ka \right) \frac{1}{ks} \le 1,$$

where $q, r \in [1,\infty]$, $\alpha = (\alpha_1, \dots, \alpha_n)$ be the multiindex, $\alpha_i \in \mathbb{N} \cup \{0\}$, $i = 1, \dots, n$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $a \in \mathbb{R}_+ \cup \{0\}$. Then

$$D_x^\alpha \partial_t^a u \in L_r(0,T; L_q(\Omega)), \quad D_x^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n},$$

and the following interpolation holds

$$\|D_x^\alpha \partial_t^a u\|_{L_r(0,T;L_q(\Omega))} \le \varepsilon^{1-\varkappa} \left(\|\partial_t^s u\|_{L_p(\Omega^T)} + \sum_{i=1}^n \|\partial_{x_i}^{ks} u\|_{L_p(\Omega^T)} \right) + c\varepsilon^{-\varkappa} \|u\|_{L_p(\Omega^T)},$$

where $\varepsilon \in \mathbb{R}_+$ and $q, r \ge p$. In the case either $q = \infty$ or $r = \infty$ the above inequality holds provided $\varkappa < 1$.
Furthermore, in the case $r = \infty$, $q < \infty$ and $\varkappa = 1$ we have the estimate

$$\|D_x^\alpha \partial_t^a u\|_{L_\infty(0,T;L_q(\Omega))} \le c\|u\|_{W_p^{ks,s}(\Omega^T)}.$$

Theorem 3.2. *(Direct boundary trace theorem) [22]. Let us assume that:*
(1) $\Omega \subset \mathbb{R}^n$ is a domain and S is either a boundary of Ω or a subdomain of Ω with $\dim S = n - 1$.
(2) $u \in W_p^{ks,s}(\Omega^T)$, $k \in \mathbb{N}$, $s \in \mathbb{R}_+$, $p \in (1,\infty)$, $S \in C^{ks}$.
Then there exists a function $\tilde{u} = u|_{S^T}$ such that $\tilde{u} \in W_p^{ks-1/p,s-1/kp}(S^T)$ and

$$\|\tilde{u}\|_{W_p^{ks-1/p,s-1/kp}(S^T)} \le c\|u\|_{W_p^{ks,s}(\Omega^T)},$$

where constant c does not depend on u.

Theorem 3.3. *(Inverse boundary trace theorem) [22, 23, Sect. 20]. Let assumption (1) of Theorem 3.2 be satisfied. Let $\tilde{u} \in W_p^{ks-1/p,s-1/kp}(S^T)$, $k \in \mathbb{N}$, $s \in \mathbb{R}_+$, $p \in (1,\infty)$, $S \in C^{ks}$. Then there exists a function u such that $u|_{S^T} = \tilde{u}$, $u \in W_p^{ks,s}(\Omega^T)$, and*

$$\|u\|_{W_p^{ks,s}(\Omega^T)} \leq c\|\tilde{u}\|_{W_p^{ks-1/p,s-1/kp}(S^T)},$$

where c does not depend on $\tilde{u}$.

Theorem 3.4. *(Direct initial trace theorem) [22]. Let $u \in W_p^{ks,s}(\Omega^T)$ $k \in \mathbb{N}$, $s \in \mathbb{R}_+$, $s > 1/p$, $p \in (1,\infty)$. Then $\tilde{u} = u|_{t=t_0}$, where $t_0 \in [0,T]$, belongs to $W_p^{ks-k/p}(\Omega)$, and*

$$\|\tilde{u}\|_{W_p^{ks-k/p}(\Omega)} \leq c\|u\|_{W_p^{ks,s}(\Omega^T)},$$

where c does not depend on u.

Theorem 3.5. *(Inverse initial trace theorem) [22]. Let $\tilde{u} \in W_p^{ks-k/p}(\Omega)$, $k \in \mathbb{N}$, $s \in \mathbb{R}_+$, $s > 1/p$, $p \in (1,\infty)$. Then there exists $u \in W_p^{ks,s}(\Omega^T)$ such that $u|_{t=t_0} = \tilde{u}$, $t_0 \in [0,T]$, and*

$$\|u\|_{W_p^{ks,s}(\Omega^T)} \leq c\|\tilde{u}\|_{W_p^{ks-k/p}(\Omega)},$$

where c does not depend on $\tilde{u}$.

3.3. **Auxiliary linear problems.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be an open bounded subset of $\mathbb{R}^n$, with a smooth boundary S. Let us consider the problem

$$\begin{aligned} \Delta\chi &= f \quad \text{in } \Omega, \\ \boldsymbol{n}\cdot\nabla\chi &= 0 \quad \text{on } S, \\ \fint_\Omega \chi dx &= \chi_m, \end{aligned} \tag{3.1}$$

where $\fint_\Omega \chi dx = \frac{1}{|\Omega|}\int_\Omega \chi dx$ and χ_m is a given constant. We recall

Lemma 3.1. *(see e.g. [18]). Let us assume that $f \in H^r(\Omega)$, $S \in C^{r+2}$, $r \in \mathbb{N}\cup\{0\}$, and the compatibility condition $\int_\Omega f dx = 0$ holds. Then there exists a unique solution $\chi \in H^{r+2}(\Omega)$ to (3.1) such that*

$$\|\chi\|_{H^{r+2}(\Omega)} \leq c(\|f\|_{H^r(\Omega)} + |\chi_m|), \tag{3.2}$$

where constant c depends at most on r and S.

Next, let us consider the fourth order elliptic problem

$$\begin{aligned} \Delta^2\chi &= f \quad && \text{in } \Omega, \\ \boldsymbol{n}\cdot\nabla\chi &= 0, \quad \boldsymbol{n}\cdot\nabla\Delta\chi = 0 && \text{on } S, \\ \fint_\Omega \chi dx &= \chi_m, \end{aligned} \tag{3.3}$$

where χ_m is a given constant. We have

Lemma 3.2. *(see e.g. [18]). Let us assume that $f \in H^r(\Omega)$, $S \in C^{r+4}$, $r \in \mathbb{N} \cup \{0\}$, and the compatibility condition $\int_\Omega f dx = 0$ holds. Then there exists a unique solution $\chi \in H^{r+4}(\Omega)$ to (3.3) such that*

$$\|\chi\|_{H^{r+4}(\Omega)} \le c(\|f\|_{H^r(\Omega)} + |\chi_m|), \tag{3.4}$$

where constant c depends at most on r and S.

In the sequel we shall need the solvability of the following linear problem

$$\begin{aligned} &\chi_t - \beta\Delta\chi_t + \gamma\Delta^2\chi_t - \varkappa\Delta^3\chi = F && \text{in } \Omega^T, \\ &\chi|_{t=0} = \chi_0 && \text{in } \Omega, \\ &\boldsymbol{n}\cdot\nabla\chi = 0, \quad \boldsymbol{n}\cdot\nabla\Delta\chi = 0 && \text{on } S^T, \\ &\boldsymbol{n}\cdot\nabla\Delta^2\chi = G && \text{on } S^T, \end{aligned} \tag{3.5}$$

where β, γ, $\varkappa$ are positive constants, and F, G, χ_0 given functions.
Since problem (3.5) is not parabolic in the sense of Petrovskii, the general theory of parabolic initial-boundary-value problems can not be applied. Hence, we prove the following result.

Theorem 3.6. *Let $\Omega \subset \mathbb{R}^3$ be bounded with boundary $S \in C^6$, $F \in L_2(\Omega^T)$, $G \in W_2^{1/2,1/4}(S^T)$, and $\chi_0 \in H^5(\Omega)$ satisfy the compatibility conditions*

$$\boldsymbol{n}\cdot\nabla\chi_0 = 0, \quad \boldsymbol{n}\cdot\nabla\Delta\chi_0 = 0 \quad \text{on } S. \tag{3.6}$$

Then there exists a unique solution to problem (3.5) such that $\chi \in L_2(0,T;H^6(\Omega)) \cap H^1(0,T;H^4(\Omega))$, and

$$\begin{aligned} &\|\chi\|_{L_2(0,T;H^6(\Omega))} + \|\chi_t\|_{L_2(0,T;H^4(\Omega))} \\ &\le c(\|F\|_{L_2(\Omega^T)} + \|G\|_{W_2^{1/2,1/4}(S^T)} + \|\chi_0\|_{H^5(\Omega)} + T^{1/2}\|\chi_0\|_{L_2(\Omega)}) \equiv cA_1, \end{aligned} \tag{3.7}$$

where constant c does not depend on T.

We prove this theorem in two steps. First we construct a weak solution and then show that for sufficiently smooth data this solution has the desired regularity.

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^3$ be bounded with the boundary $S \in C^4$, $F \in L_2(\Omega^T)$, $G \in L_2(S^T)$ and $\chi_0 \in H^3(\Omega)$ satisfy $\boldsymbol{n}\cdot\nabla\chi_0 = 0$ on S.*
Then there exists a weak solution to problem (3.5) in the following sense

$$\chi \in L_\infty(0,T;H^3(\Omega)) \cap H^1(0,T;H^2(\Omega)), \tag{3.8}$$

$$\chi(0) = \chi_0, \tag{3.9}$$

$$\int_\Omega (\chi_t\xi + \beta\nabla\chi_t\cdot\nabla\xi + \gamma\Delta\chi_t\Delta\xi + \varkappa\nabla\Delta\chi\cdot\nabla\Delta\xi)dx = \int_\Omega F\xi dx + \int_S G\xi dS \tag{3.10}$$

$\forall\xi \in H_N^4(\Omega) \equiv \{\xi \in H^4(\Omega) : \boldsymbol{n}\cdot\nabla\xi = 0, \ \ \boldsymbol{n}\cdot\nabla\Delta\xi = 0 \ \ \text{on } S, \ \ \text{a.e. } t \in (0,T)\}$.
Moreover,

$$\begin{aligned} &\|\chi\|_{L_\infty(0,T;H^3(\Omega))} + \|\chi_t\|_{L_2(0,T;H^2(\Omega))} \\ &\le c(\|F\|_{L_2(\Omega^T)} + \|G\|_{L^2(S^T)} + \|\chi_0\|_{H^3(\Omega)} + T^{1/2}\|\chi_0\|_{L_2(\Omega)}) \equiv cA_2. \end{aligned} \tag{3.11}$$

Proof. We will prove existence by the Galerkin method. To construct a basis of $H^4_N(\Omega)$ we introduce the fourth order elliptic operator

$$\mathcal{L} = \{L, \boldsymbol{n}\cdot\nabla|_S,\ \boldsymbol{n}\cdot\nabla\Delta|_S\},$$

where

$$L\varphi = \varphi - \beta\Delta\varphi + \gamma\Delta^2\varphi \quad \text{in } \Omega. \tag{3.12}$$

We note that $\mathcal{L}$ is unbounded, selfadjoint, linear operator and its inverse $\mathcal{L}^{-1}$ is compact, selfadjoint, linear operator. Thus by the Hilbert-Schmidt Theorem we can conclude what follows: There exist the eigenvalues $\{\bar\lambda_j\}_{j\in\mathbb{N}}$ and the eigenfunctions $\{\varphi_j\}_{j\in\mathbb{N}}$ of $\mathcal{L}^{-1}$, defined by the problem

$$\begin{aligned} L^{-1}\varphi_j &= \bar\lambda_j\varphi_j && \text{in } \Omega,\ j\in\mathbb{N},\\ \boldsymbol{n}\cdot\nabla\varphi_j &= 0 && \text{on } S,\\ \boldsymbol{n}\cdot\nabla\Delta\varphi_j &= 0 && \text{on } S. \end{aligned} \tag{3.13}$$

The eigenvalues $\bar\lambda_j$ are real and they can be ordered such that $|\bar\lambda_{j+1}|\le|\bar\lambda_j|$, $j\in\mathbb{N}$, and $\lim_{j\to\infty}\bar\lambda_j = 0$. Then $\mathcal{L}$ has an infinite set of eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$ that correspond to the set of eigenfunctions $\{\varphi_j\}_{j\in\mathbb{N}}$. They can be ordered such that $|\lambda_{j+1}|\ge|\lambda_j|$, $j\in\mathbb{N}$, and then $\lim_{j\to\infty}|\lambda_j|=\infty$. Moreover, the value $\lambda=0$ does not belong to the spectrum.

The set of eigenfunctions $\{\varphi_j\}_{j\in\mathbb{N}}$ forms an orthonormal basis for $L_2(\Omega)$ which is also orthogonal in $H^2(\Omega)$. The orthogonality in $H^2(\Omega)$ follows from the relations

(3.14)
$$\begin{aligned} &\lambda_j(\varphi_j,\varphi_k) = (\mathcal{L}\varphi_j,\varphi_k) = (\mathcal{L}^{1/2}\varphi_j,\mathcal{L}^{1/2}\varphi_k) = (\varphi_j,\mathcal{L}\varphi_k) = \lambda_k(\varphi_j,\varphi_k),\\ &(\mathcal{L}^{1/2}\varphi_j,\mathcal{L}^{1/2}\varphi_k) = 0 \quad \text{for } \lambda_j\ne\lambda_k,\\ &(\mathcal{L}^{1/2}\varphi_j,\mathcal{L}^{1/2}\varphi_k) = \int_\Omega(\varphi_j\varphi_k + \beta\nabla\varphi_j\cdot\nabla\varphi_k + \gamma\Delta\varphi_j\Delta\varphi_k)dx, \end{aligned}$$

and the fact that the norm generated by the scalar product $(\mathcal{L}^{1/2}\varphi,\mathcal{L}^{1/2}\varphi)$ is equivalent to the standard $H^2(\Omega)$-norm.

Finally, we note that the system $\{\varphi_j\}_{j\in\mathbb{N}}\subset C^4(\bar\Omega)$ is dense in $H^4_N(\Omega)$.

Given $m\in\mathbb{N}$ and the basis $\{\varphi_j\}_{j\in\mathbb{N}}$ we introduce now the Galerkin approximation corresponding to the weak formulation of (3.5):

$$\chi^{(m)}(x,t) = \sum_{i=1}^m \alpha_i^{(m)}(t)\varphi_j(x), \tag{3.15}$$

(3.16)
$$\begin{aligned} &\int_\Omega(\chi_t^{(m)}\xi^{(m)} + \beta\nabla\chi_t^{(m)}\cdot\nabla\xi^{(m)} + \gamma\Delta\chi_t^{(m)}\Delta\xi^{(m)} + \varkappa\nabla\Delta\chi^{(m)}\cdot\nabla\Delta\xi^{(m)})dx\\ &= \int_\Omega F\xi^{(m)}dx + \int_S G\xi^{(m)}dS\forall\xi^{(m)}\in V_m\equiv\operatorname{span}\{\varphi_1,\dots,\varphi_m\},\ \text{a.e. } t\in(0,T), \end{aligned}$$

$$\chi^{(m)}(0) = \chi_0^{(m)}, \tag{3.17}$$

where $\chi_0^{(m)}\in V_m$ is the projection of χ_0 on V_m such that

$$\chi_0^{(m)}\to\chi_0 \quad \text{weakly in } H^3(\Omega). \tag{3.18}$$

From (3.15)–(3.17) we obtain the system of ODE's

$$\begin{aligned}&\sum_{i=1}^{m}\left(\frac{d}{dt}\alpha_i^{(m)}(t)a_{ij}+\alpha_i^{(m)}(t)b_{ij}\right)=\int\limits_\Omega F(t)\varphi_j dx+\int\limits_S G(t)\varphi_j dS,\\ &\alpha_j(0)=\int\limits_\Omega \chi_0^{(m)}\varphi_j,\quad j=1,\dots,m,\end{aligned}\tag{3.19}$$

where

$$a_{ij}=\int\limits_\Omega(\varphi_i\varphi_j+\beta\nabla\varphi_i\cdot\nabla\varphi_j+\gamma\Delta\varphi_i\Delta\varphi_j)dx,\quad b_{ij}=\varkappa\int\limits_\Omega\nabla\Delta\varphi_i\cdot\nabla\Delta\varphi_j dx.$$

Due to the properties of the basis $\{\varphi_j\}$, the matrix $(a_{ij})_{i,j=1,\dots,m}$ is diagonal and invertible. Hence we conclude the existence of a solution $\alpha_i^{(m)}(t)$, $i=1,\dots,m$, to (3.19).

We derive now uniform with respect to m estimates on $\chi^{(m)}$. Setting $\xi^{(m)}=\chi^{(m)}$ in (3.16) yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\limits_\Omega(|\chi^{(m)}|^2+\beta|\nabla\chi^{(m)}|^2+\gamma|\Delta\chi^{(m)}|^2)dx\\ &\quad+\varkappa\int\limits_\Omega|\nabla\Delta\chi^{(m)}|^2dx=\int\limits_\Omega F\chi^{(m)}dx+\int\limits_S G\chi^{(m)}dS.\end{aligned}\tag{3.20}$$

Hence, after integrating with respect to time and applying the Young inequality to the two integrals on the right-hand side we obtain the first uniform estimate

$$\begin{aligned}&\|\chi^{(m)}(t)\|^2_{L_2(\Omega)}+\beta\|\nabla\chi^{(m)}(t)\|^2_{L_2(\Omega)}+\gamma\|\Delta\chi^{(m)}(t)\|^2_{L_2(\Omega)}\\ &+\frac{\varkappa}{2}\|\nabla\Delta\chi^{(m)}\|_{L_2(\Omega^t)}\le c(\|F\|^2_{L_2(\Omega^t)}+\|G\|^2_{L_2(S^t)})\\ &+c(1+t)\|\chi_0^{(m)}\|^2_{L_2(\Omega)}+\beta\|\nabla\chi_0^{(m)}\|^2_{L_2(\Omega)}+\gamma\|\Delta\chi_0^{(m)}\|_{L_2(\Omega)},\quad t\in(0,T).\end{aligned}\tag{3.21}$$

Next, setting $\xi^{(m)}=\chi_t^{(m)}$ in (3.16) yields

$$\begin{aligned}&\int\limits_\Omega(|\chi_t^{(m)}|^2+\beta|\nabla\chi_t^{(m)}|^2+\gamma|\Delta\chi_t^{(m)}|^2)dx+\frac{\varkappa}{2}\frac{d}{dt}\int\limits_\Omega|\nabla\Delta\chi^{(m)}|^2dx\\ &=\int\limits_\Omega F\chi_t^{(m)}dx+\int\limits_S G\chi_t^{(m)}dS.\end{aligned}\tag{3.22}$$

Again, integrating with respect to time and applying the Young inequality to the integrals on the right-hand side we obtain the second uniform estimate

$$\begin{aligned}&\varkappa\|\nabla\Delta\chi^{(m)}(t)\|^2_{L_2(\Omega)}+\|\chi_{t'}^{(m)}\|_{L_2(\Omega^t)}+\beta\|\nabla\chi_{t'}^{(m)}\|^2_{L_2(\Omega^t)}+\gamma\|\Delta\chi_{t'}^{(m)}\|^2_{L_2(\Omega^t)}\\ &\le c(\|F\|^2_{L_2(\Omega^t)}+\|G\|^2_{L_2(S^t)})+\varkappa\|\nabla\Delta\chi_0^{(m)}\|^2_{L_2(\Omega)},\quad t\in(0,T).\end{aligned}\tag{3.23}$$

From (3.21) and (3.23) it follows that there exists

$$\chi\in L_\infty(0,T;H^3(\Omega))\cap H^1(0,T;H^2(\Omega))\tag{3.24}$$

and a subsequence of solutions $\chi^{(m)}$ to (3.16) (denoted by the same index) such that as $m \to \infty$

$$\begin{aligned} &\chi^{(m)} \rightharpoonup \chi \quad \text{weakly}^* \text{ in } L_\infty(0,T;H^3(\Omega)),\\ &\chi_t^{(m)} \rightharpoonup \chi_t \quad \text{weakly in } L_2(0,T;H^2(\Omega)). \end{aligned} \tag{3.25}$$

Hence, by compactness arguments,

$$\chi^{(m)} \to \chi \quad \text{strongly in } C([0,T];H^2(\Omega)). \tag{3.26}$$

From (3.26) it follows that

$$\chi^m(0) = \chi_0^m \to \chi(0) \quad \text{strongly in} \quad H^2(\Omega), \tag{3.27}$$

which together with (3.18) implies that

$$\chi(0) = \chi_0. \tag{3.28}$$

Due to the convergences (3.25) we can immediately pass to the limit $m \to \infty$ in (3.16) expressed in the form

$$\begin{aligned} &\int\limits_{\Omega^T} (\chi_t^{(m)}\xi + \beta\nabla\chi_t^{(m)}\cdot\nabla\xi + \gamma\Delta\xi_t^{(m)}\Delta\xi + \varkappa\nabla\Delta\chi^{(m)}\cdot\nabla\Delta\xi)dxdt\\ &= \int\limits_{\Omega^T} F\xi dxdt + \int\limits_S G\xi dSdt \ \forall\xi \in L_2(0,T;V_m). \end{aligned} \tag{3.29}$$

To this end we follow the standard procedure. First we fix $m = m_0 \in \mathbb{N}$ in the space V_m, take subsequences (3.25) with $m \geq m_0$ and pass to the limit in (3.29) for a subsequence $m_0 \leq m \to \infty$. Next we pass to the limit $m_0 \to \infty$ in V_{m_0}. Taking into account the density of $\bigcup_{m\in\mathbb{N}} V_m$ in $H_N^4(\Omega)$ we arrive at (3.10).
A priori estimate (3.11) follows immediately from the uniform estimates (3.21), (3.23) and the weak convergences (3.25). Thus, the proof is complete. □

Proof of Theorem 3.6. Let χ be a weak solution to problem (3.5), constructed in Lemma 3.3. It will be of use to note that, setting $\xi = 1$ in (3.10), yields

$$\begin{aligned} &-\!\!\!\!\!\!\int\limits_\Omega \chi_t(t)dx = \frac{1}{|\Omega|}\bigg(\int\limits_\Omega F(t)dx + \int\limits_S G(t)dS\bigg) \equiv \bar{M}(t),\\ &-\!\!\!\!\!\!\int\limits_\Omega \chi(t)dx = -\!\!\!\!\!\!\int\limits_\Omega \chi_0 dx + \int\limits_0^t \bar{M}(t')dt' \equiv M(t). \end{aligned} \tag{3.30}$$

Let us consider the following problem which results from $(3.5)_1$, $(3.5)_2$ and $(3.5)_4$:

$$\begin{aligned} &\gamma\Delta^2\chi_t - \varkappa\Delta^3\chi = F - \chi_t + \beta\Delta\chi_t \equiv F_0 && \text{in } \Omega^T,\\ &\chi|_{t=0} = \chi_0 && \text{in } \Omega,\\ &\boldsymbol{n}\cdot\nabla\Delta^2\chi = G && \text{on } S^T. \end{aligned} \tag{3.31}$$

By (3.11) we have

$$\|F_0\|_{L_2(\Omega^T)} \leq \|F\|_{L_2(\Omega^T)} + c\|\chi_t\|_{L_2(0,T;H^2(\Omega))} \leq cA_2. \tag{3.32}$$

If we set

$$v = \Delta^2\chi \tag{3.33}$$

then (3.31) yields

$$
\begin{aligned}
&\gamma v_t - \varkappa \Delta v = F_0 && \text{in } \Omega^T,\\
&v|_{t=0} = v_0 = \Delta^2 \chi_0 && \text{in } \Omega,\\
&\boldsymbol{n} \cdot \nabla v = G && \text{on } S^T,
\end{aligned}
\tag{3.34}
$$

where $F_0 \in L_2(\Omega^T)$ and, by assumptions, $G \in W_2^{1/2,1/4}(S^T)$ and $v_0 \in H^1(\Omega)$. Thus, by virtue of the classical parabolic theory [22, 23] there exists a unique solution to problem (3.34) such that $v \in W_2^{2,1}(\Omega^T)$ and

$$
\begin{aligned}
\|v\|_{W_2^{2,1}(\Omega^T)} &\le c(\|F_0\|_{L_2(\Omega^T)} + \|G\|_{W_2^{1/2,1/4}(S^T)} + \|v_0\|_{H^1(\Omega)})\\
&\le c(A_2 + \|G\|_{W_2^{1/2,1/4}(S^T)} + \|\chi_0\|_{H^5(\Omega)})\\
&\le c(\|F\|_{L_2(\Omega^T)} + \|G\|_{W_2^{1/2,1/4}(S^T)} + \|\chi_0\|_{H^5(\Omega)}) \equiv cA_1
\end{aligned}
\tag{3.35}
$$

where constant c does not depend on T due to Theorem 3.1.1 from [24, Ch. 3].

Let us now consider the elliptic problem which follows from (3.33), $(3.5)_3$ and (3.30):

$$
\begin{aligned}
&\Delta^2 \chi = v && \text{in } \Omega, \quad \text{a.e. } t \in (0,T),\\
&\boldsymbol{n} \cdot \nabla \chi = 0, \quad \boldsymbol{n} \cdot \nabla \Delta \chi = 0 && \text{on } S,\\
&\fint_\Omega \chi dx = M.
\end{aligned}
\tag{3.36}
$$

Then by the elliptic estimate (see Lemma 3.2) we have

$$
\begin{aligned}
&\|\chi\|_{H^{r+4}(\Omega)} \le c(\|v\|_{H^r(\Omega)} + |M|), \quad r = 0, 2,\\
&\|\chi_t\|_{H^4(\Omega)} \le c(\|v_t\|_{L_2(\Omega)} + |\bar{M}|).
\end{aligned}
\tag{3.37}
$$

Consequently, from (3.35) and (3.37) it follows that

$$
\begin{aligned}
&\|\chi\|_{L_2(0,T;H^6(\Omega))} \le c(\|v\|_{L_2(0,T;H^2(\Omega))} + \|M\|_{L_2(0,T)}) \le cA_1,\\
&\|\chi_t\|_{L_2(0,T;H^4(\Omega))} \le c(\|v_t\|_{L_2(\Omega^T)} + \|\bar{M}\|_{L_2(0,T)}) \le cA_1
\end{aligned}
$$

which completes the proof. □

4. A priori estimates.

4.1. Energy estimates. First we notice that integrating (1.1) over Ω and using boundary condition (1.6) yields

$$
\frac{d}{dt}\int_\Omega \chi dx = 0, \tag{4.1}
$$

which shows that the spatial mean of χ is preserved,

$$
\fint_\Omega \chi(t) dx = \fint_\Omega \chi_0 dx =: \chi_m \quad \text{for all } t \in \mathbb{R}_+.
$$

Next we notice that on account of assumptions (1.7), (1.8) it follows that there exist positive constants c_{f_0} and $c_{\varkappa_1}$ such that

$$
f_0(\chi) \ge \frac{1}{2} a_{2k} \chi^{2k} - c_{f_0}, \quad \varkappa_1(\chi) \ge \frac{1}{2} b_{2l} \chi^{2l} - c_{\varkappa_1}. \tag{4.2}
$$

We have

Lemma 4.1. *(Energy estimate). Let us assume that* $\chi_0 \in H^2(\Omega)$, $\underset{\Omega}{\textstyle\fint} \chi_0 dx = \chi_m$. *Then a sufficiently regular solution to problem (1.1)–(1.6) satisfies*

$$
\begin{aligned}
&a_{2k}\|\chi\|^{2k}_{L_{2k}(\Omega)} + b_{2l}\|\chi^l\nabla\chi\|^2_{L_2(\Omega)} + \|\chi\|^2_{H^2(\Omega)} + \beta\|\chi_{t'}\|^2_{L_2(\Omega^t)}\\
&\quad + \gamma\|\nabla\chi_{t'}\|^2_{L_2(\Omega^t)} + \|\nabla\mu\|^2_{L_2(\Omega^t)} \le c_1 \quad \text{for } t \in \mathbb{R}_+,
\end{aligned} \tag{4.3}
$$

with a constant $c_1 = \varphi(\|\chi_0\|_{H^2(\Omega)}, |\chi_m|, \varkappa_2, c_{\varkappa_1}, c_{f_0}, a_{2k})$.

Proof. As shown in Section 1.4, any sufficiently regular solution to (1.1)–(1.6) satisfies energy equality (1.21). Integrating this equality in time and using (4.2) we obtain

$$
\begin{aligned}
&\int_\Omega \left(\frac{1}{2}a_{2k}\chi^{2k} + \frac{1}{4}b_{2l}\chi^{2l}|\nabla\chi|^2 + \frac{1}{2}\varkappa_2|\Delta\chi|^2\right)dx\\
&\quad + \int_{\Omega^t} (\beta\chi^2_{t'} + \gamma|\nabla\chi_{t'}|^2 + M|\nabla\mu|^2)dxdt'\\
&\le \int_\Omega |f(0)|dx + c_{f_0}|\Omega| + \frac{1}{2}c_{\varkappa_1}\int_\Omega |\nabla\chi|^2 dx.
\end{aligned} \tag{4.4}
$$

The last term on the right-hand side of (4.4) is controlled with the help of suitable interpolation inequalities and Lemma 3.1 to give

$$
\begin{aligned}
\|\nabla\chi\|^2_{L_2(\Omega)} &\le \varepsilon_1\|\nabla^2\chi\|^2_{L_2(\Omega)} + c(1/\varepsilon_1)\|\chi\|^2_{L_2(\Omega)}\\
&\le \varepsilon_1 c(\|\Delta\chi\|^2_{L_2(\Omega)} + \chi^2_m) + c(1/\varepsilon_1)(\varepsilon_2\|\chi\|^{2k}_{L_{2k}(\Omega)} + c(1/\varepsilon_2)),
\end{aligned}
$$

where ε_1, ε_2 are any positive constants. Choosing them such that $\varepsilon_1 c c_{\varkappa_1} \le \varkappa_2/2$, $\varepsilon_2 c(1/\varepsilon_1)c_{\varkappa_1} \le a_{2k}/2$, and noting that $\|f(0)\|_{L_1(\Omega)} \le \varphi(\|\chi_0\|_{H^2(\Omega)})$, we conclude estimate (4.3). □

Corollary 4.1. *In the sequel we shall use (4.3) in the following simplified form*

$$
\|\chi\|_{L_\infty(0,t;H^2(\Omega))} + \|\chi_{t'}\|_{L_2(0,t;H^1(\Omega))} + \|\nabla\mu\|_{L_2(\Omega^t)} \le \varphi(c_1) \quad \text{for } t \in \mathbb{R}_+. \tag{4.5}
$$

Corollary 4.2. *On account of (1.4), (1.5), integration of (1.2) gives*

$$
\int_\Omega \mu dx = \int_\Omega \left(f_{0,\chi} + \frac{1}{2}\varkappa_{1,\chi}|\nabla\chi|^2\right)dx.
$$

Hence, using (1.7), (1.8) and (4.5), we obtain

$$
\operatorname{esssup}_{t'\in[0,t]}\left|\int_\Omega \mu dx\right| \le \varphi(c_1). \tag{4.6}
$$

Moreover, by the Poincaré inequality, (4.5) and (4.6) imply that

$$
\|\mu\|_{L_2(0,t;H^1(\Omega))} \le \varphi(c_1)t^{1/2} \quad \text{for } t \in \mathbb{R}_+. \tag{4.7}
$$

Lemma 4.2. *Let us assume that* $\chi_0 \in H^3(\Omega)$, $\underset{\Omega}{\textstyle\fint} \chi_0 dx = \chi_m$. *Then a sufficiently regular solution to (1.1)–(1.6) satisfies*

$$
\begin{aligned}
&\varkappa_2^{1/2}\|\chi\|_{L_\infty(0,t;H^3(\Omega))} + b_{2l}^{1/2}\|\chi^l\Delta\chi\|_{L_\infty(0,t;L_2(\Omega))}\\
&\quad + \|\chi_{t'}\|_{L_2(\Omega^t)} + \beta^{1/2}\|\chi_{t'}\|_{L_2(0,t;H^1(\Omega))} + \gamma^{1/2}\|\chi_{t'}\|_{L_2(0,t;H^2(\Omega))}\\
&\le \varphi(c_1, \|\chi_0\|_{H^3(\Omega)}, |\chi_m|)t^{1/2} \quad \text{for } t \in \mathbb{R}_+.
\end{aligned} \tag{4.8}
$$

Proof. Multiplying (1.9) by χ_t, integrating over Ω, and integrating by parts the terms with β, γ, we obtain

$$\begin{aligned}
&\int_\Omega \chi_t^2 dx + \beta\int_\Omega |\nabla\chi_t|^2 dx + \gamma\int_\Omega |\Delta\chi_t|^2 dx \\
&= \int_\Omega \Delta\Big(f_{0,\chi} - \frac{1}{2}\varkappa_{1,\chi}|\nabla\chi|^2 + \varkappa_2\Delta^2\chi - \varkappa_1\Delta\chi\Big)\chi_t dx \equiv I.
\end{aligned} \tag{4.9}$$

We now examine the integral I. Integrating by parts and using (1.11)–(1.13) yields

$$\begin{aligned}
I &= -\int_\Omega \nabla\Big(f_{0,\chi} - \frac{1}{2}\varkappa_{1,\chi}|\nabla\chi|^2\Big)\cdot\nabla\chi_t dx - \int_\Omega \nabla(\varkappa_2\Delta^2\chi - \varkappa_1\Delta\chi)\cdot\nabla\chi_t dx \\
&\equiv I_1 + I_2,
\end{aligned}$$

where

$$I_1 = -\int_\Omega f_{0,\chi\chi}\nabla\chi\cdot\nabla\chi_t dx + \frac{1}{2}\int_\Omega \varkappa_{1,\chi\chi}|\nabla\chi|^2\nabla\chi\cdot\nabla\chi_t dx + \int_\Omega \varkappa_{1,\chi}\partial_i\chi\partial_{ij}^2\chi\partial_j\chi_t dx,$$

and

$$\begin{aligned}
I_2 &= \varkappa_2\int_\Omega \Delta^2\chi\Delta\chi_t dx - \int_\Omega \varkappa_1\Delta\chi\Delta\chi_t dx \\
&= -\frac{\varkappa_2}{2}\frac{d}{dt}\int_\Omega |\nabla\Delta\chi|^2 dx - \frac{1}{2}\frac{d}{dt}\int_\Omega \varkappa_1|\Delta\chi|^2 dx + \frac{1}{2}\int_\Omega \varkappa_{1,\chi}|\Delta\chi|^2\chi_t dx.
\end{aligned}$$

Inserting the expressions I_1, I_2 into (4.9) and integrating the resulting equality with respect to time we find that

$$\begin{aligned}
&\frac{\varkappa_2}{2}\int_\Omega |\nabla\Delta\chi|^2 dx + \frac{1}{2}\int_\Omega \varkappa_1|\Delta\chi|^2 dx + \int_{\Omega^t} \chi_{t'}^2 dx dt' + \beta\int_{\Omega^t} |\nabla\chi_{t'}|^2 dx dt' \\
&+ \gamma\int_{\Omega^t} |\Delta\chi_{t'}|^2 dx dt' \le \frac{1}{2}\int_{\Omega^t} |\varkappa_{1,\chi}|\,|\Delta\chi|^2|\chi_{t'}| dx dt' + \int_{\Omega^t} |f_{0,\chi\chi}|\,|\nabla\chi|\,|\nabla\chi_{t'}| dx dt' \\
&+ \frac{1}{2}\int_{\Omega^t} |\varkappa_{1,\chi\chi}|\,|\nabla\chi|^3\,|\nabla\chi_{t'}| dx dt' + \int_{\Omega^t} |\varkappa_{1,\chi}|\,|\nabla\chi|\,|\nabla^2\chi|\,|\nabla\chi_{t'}| dx dt' \\
&+ \frac{\varkappa_2}{2}\int_\Omega |\nabla\Delta\chi_0|^2 dx + \frac{1}{2}\int_\Omega |\varkappa_1(\chi_0)|\,|\Delta\chi_0|^2 dx \equiv \sum_{k=1}^{6} R_k.
\end{aligned}$$

On account of (4.5) the terms $R_1, \dots, R_4$ are estimated as follows:

$$\begin{aligned}
R_1 &\le \varphi(c_1)\int_0^t \|\Delta\chi\|_{L_2(\Omega)}^2\|\chi_{t'}\|_{L_\infty(\Omega)} dt' \le \varphi(c_1)\int_0^t \|\chi_{t'}\|_{L_\infty(\Omega)} dt' \\
&\le \varepsilon_1\int_{\Omega^t} |\Delta\chi_{t'}|^2 dx dt' + \varphi(1/\varepsilon_1, c_1)t,
\end{aligned}$$

$$R_2 \le \varphi(c_1)\int_0^t \|\nabla\chi\|_{L_2(\Omega)}\|\nabla\chi_{t'}\|_{L_2(\Omega)}dt' \le \varphi(c_1)\int_0^t \|\nabla\chi_{t'}\|_{L_2(\Omega)}dt'$$
$$\le \varphi(c_1)t^{1/2},$$
$$R_3 \le \varphi(c_1)\int_0^t \|\chi\|^3_{H^2(\Omega)}\|\nabla\chi_{t'}\|_{L_2(\Omega)}dt' \le \varphi(c_1)\int_0^t \|\nabla\chi_{t'}\|_{L_2(\Omega)}dt'$$
$$\le \varphi(c_1)t^{1/2},$$
$$R_4 \le \varphi(c_1)\int_0^t \|\nabla\chi\|_{L_4(\Omega)}\|\nabla^2\chi\|_{L_2(\Omega)}\|\nabla\chi_{t'}\|_{L_4(\Omega)}dt'$$
$$\le \varphi(c_1)\int_0^t \|\nabla\chi_{t'}\|_{L_4(\Omega)}dt' \le \varepsilon_2\int_{\Omega^t}|\Delta\chi_{t'}|^2dxdt' + \varphi(1/\varepsilon_2, c_1)t,$$

where we used the inequality $\|\chi_t\|_{H^2(\Omega)} \le c\|\Delta\chi_t\|_{L_2(\Omega)}$. Clearly,

$$R_5 + R_6 \le c\|\chi_0\|^2_{H^3(\Omega)}.$$

Putting the above estimates together and choosing ε_1, ε_2 sufficiently small, we obtain

$$\varkappa_2\int_\Omega |\nabla\Delta\chi|^2dx + \int_\Omega \varkappa_1|\Delta\chi|^2dx + \int_{\Omega^t}\chi^2_{t'}dxdt'$$
$$+\beta\int_{\Omega^t}|\nabla\chi_{t'}|^2dxdt' + \gamma\int_{\Omega^t}|\Delta\chi_{t'}|^2dxdt' \le \varphi(c_1)t + c\|\chi_0\|^2_{H^3(\Omega)}.$$

Hence, in view of $(4.2)_2$, recalling (4.5) and Lemma 3.1, we conclude (4.8). □

Corollary 4.3. *Recalling (1.2) it follows from (4.5), (4.7) and (4.8) that*

$$\begin{aligned}&\|\chi\|_{L_2(0,t;H^4(\Omega))} \le c(\|\Delta^3\chi\|_{L_2(\Omega^t)} + |\chi_m|t^{1/2})\\ &\le c\|\mu\|_{L_2(\Omega^t)} + c\Big\|f_{0,\chi} - \frac{1}{2}\varkappa_{1,\chi}|\nabla\chi|^2 - \varkappa_1\Delta\chi + \beta\chi_{t'} - \gamma\Delta\chi_{t'}\Big\|_{L_2(\Omega^t)}\\ &\quad + c\|\chi_m|t^{1/2} \le \varphi(c_1, \|\chi_0\|_{H^3(\Omega)}, |\chi_m|)t^{1/2} \quad \text{for } t\in\mathbb{R}_+.\end{aligned} \tag{4.11}$$

4.2. **Finite-time estimates.** Let $k\in\mathbb{N}_0\equiv\mathbb{N}\cup\{0\}$, $T>0$ and $\Delta_kT\equiv(kT,(k+1)T)$. We introduce the problem

$$\begin{aligned}&\chi_t - \beta\Delta\chi_t + \gamma\Delta^2\chi_t - \varkappa_2\Delta^3\chi\\ &= \Delta\Big(f_{0,\chi} - \frac{1}{2}\varkappa_{1,\chi}|\nabla\chi|^2 - \varkappa_1\Delta\chi\Big) \equiv F && \text{in } \Omega^{kT} := \Omega\times\Delta_kT,\\ &\chi|_{t=kT} = \chi(kT) && \text{in } \Omega,\\ &n\cdot\nabla\chi = 0,\quad n\cdot\nabla\Delta\chi = 0 && \text{on } S^{kT} := S\times\Delta_kT,\\ &n\cdot\nabla\Delta^2\chi = \frac{1}{2\varkappa_2}\varkappa_{1,\chi}n\cdot\nabla|\nabla\chi|^2\\ &\quad = \frac{1}{\varkappa_2}n\cdot\nabla\Big(\frac{1}{2}\varkappa_{1,\chi}|\nabla\chi|^2\Big) \equiv G && \text{on } S^{kT}.\end{aligned} \tag{4.12}$$

For $k=0$ problem (4.12) becomes (1.9)–(1.13).

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^3$ be bounded with boundary $S \in C^6$, $T > 0$, $k \in \mathbb{N}_0$, and the assumptions of Lemma 4.2 hold. Moreover, let*

$$\chi(kT) \in H^5(\Omega), \quad n \cdot \nabla\chi(kT) = 0, \quad n \cdot \nabla\Delta\chi(kT) = 0 \quad on \ S.$$

Then solutions to problem (4.12) satisfy the estimate

$$\begin{aligned} &\|\chi_{t'}\|_{L_2(kT,t;H^4(\Omega))} + \|\chi\|_{L_2(kT,t;H^6(\Omega))} \\ &\le \varphi(c_1, \|\chi(kT)\|_{H^3(\Omega)}, |\chi_m|)(t-kT)^{1/2} + c\|\chi(kT)\|_{H^5(\Omega)}, \end{aligned} \tag{4.13}$$

where $t \in (kT, (k+1)T)$, and constant c_1 is defined in Lemma 4.1.

Proof. We use arguments similar to that in the proof of Theorem 3.6. Let us write $(4.12)_1$, $(4.12)_2$ and $(4.12)_4$ in the form

$$\begin{aligned} &\gamma\Delta^2\chi_t - \varkappa_2\Delta^3\chi = \Delta\Big(f_{0,\chi} - \frac{1}{2}\varkappa_{1,\chi}|\nabla\chi|^2 - \varkappa_1\Delta\chi\Big) \\ &\quad - (\chi_t - \beta\Delta\chi_t) = F - (\chi_t - \beta\Delta\chi_t) \equiv F_0 && \text{in } \ \Omega^{kT}, \\ &\chi|_{t=kT} = \chi(kT) && \text{in } \ \Omega, \\ &n \cdot \nabla\Delta^2\chi = \frac{1}{\varkappa_2} n \cdot \nabla\Big(\frac{1}{2}\varkappa_{1,\chi}|\nabla\chi|^2\Big) \equiv G && \text{on } \ S^{kT}. \end{aligned} \tag{4.14}$$

If we set

$$v = \Delta^2\chi, \tag{4.15}$$

then (4.14) yields

$$\begin{aligned} \gamma v_t - \varkappa_2\Delta v &= F_0 && \text{in } \ \Omega^{kT}, \\ v|_{t=kT} &= \Delta^2\chi|_{t=kT} && \text{in } \ \Omega, \\ n \cdot \nabla v &= G && \text{on } \ S^{kT}. \end{aligned}$$

By the inverse boundary trace theorem it follows that for any $G \in W_2^{1/2,1/4}(S^{kT})$ there exists a function $u \in W_2^{2,1}(\Omega^{kT})$ such that $n \cdot \nabla u|_{S^{kT}} = G$ and

$$\|u\|_{W_2^{2,1}(\Omega\times(kT,t))} \le c\|G\|_{W_2^{1/2,1/4}(S\times(kT,t))}, \quad t \in \Delta_k T. \tag{4.16}$$

Let us introduce the function

$$w = v - u, \tag{4.17}$$

which satisfies the problem

$$\begin{aligned} &\gamma w_t - \varkappa_2\Delta w = F_0 - (\gamma u_t - \varkappa_2\Delta u) \equiv F_* && \text{in } \ \Omega^{kT}, \\ &w|_{t=kT} = v|_{t=kT} - u|_{t=kT} && \text{in } \ \Omega, \\ &n \cdot \nabla w = 0 && \text{on } \ S^{kT}. \end{aligned} \tag{4.18}$$

By the classical parabolic theory (see [22, 23]) and Theorem 3.1.1 from [24, Ch. 3] solutions to (4.18) satisfy

$$\begin{aligned} \|w\|_{W_2^{2,1}(\Omega^{kT,t})} &\le c(\|F_0\|_{L_2(\Omega^{kT,t})} + \|u\|_{W_2^{2,1}(\Omega^{kT,t})} + \|w(kT)\|_{H^1(\Omega)}) \\ &\le c(\|F_0\|_{L_2(\Omega^{kT,t})} + \|G\|_{W_2^{1/2,1/4}(S^{kT,t})} + \|w(kT)\|_{H^1(\Omega)}), \end{aligned} \tag{4.19}$$

where in the last inequality we used (4.16) and set for simplicity

$$\Omega^{kT,t} \equiv \Omega \times (kT,t), \quad S^{kT,t} \equiv S \times (kT,t), \quad t \in \Delta_k T.$$

From (4.19) and (4.17), using the trace theorem

$$\|u(kT)\|_{H^1(\Omega)} \le c\|u\|_{W_2^{2,1}(\Omega^{kT,t})},$$

it follows that

$$\begin{aligned}(4.20)\qquad \|v\|_{W_2^{2,1}(\Omega^{kT,t})} &\le \|w\|_{W_2^{2,1}(\Omega^{kT,t})} + \|u\|_{W_2^{2,1}(\Omega^{kT,t})}\\ &\le c(\|F_0\|_{L_2(\Omega^{kT,t})} + \|G\|_{W^{1/2,1/4}(S^{kT,t})} + \|v(kT)\|_{H^1(\Omega)}).\end{aligned}$$

By applying (4.3), (4.8) and (4.11) we have

$$(4.21)\qquad \|F_0\|_{L_2(\Omega^{kT,t})} \le \|F\|_{L_2(\Omega^{kT,t})} + \|\chi_t - \beta\Delta\chi_t\|_{L_2(\Omega^{kT,t})},$$

where

$$\begin{aligned}\|F\|_{L_2(\Omega^{kT,t})} &= \Big\| f_{0,\chi\chi\chi}|\nabla\chi|^2 + f_{0,\chi\chi}\Delta\chi - \frac{1}{2}\varkappa_{1,\chi\chi\chi}|\nabla\chi|^4 - 2\varkappa_{1,\chi\chi}\partial_i\chi\partial_j\chi\partial_{ij}^2\chi\\ &\quad - \frac{3}{2}\varkappa_{1,\chi\chi}|\nabla\chi|^2\Delta\chi - 3\varkappa_{1,\chi}\partial_i\chi\partial_i\Delta\chi - \varkappa_{1,\chi}|\nabla^2\chi|^2 - \varkappa_{1,\chi}(\Delta\chi)^2\\ &\quad - \varkappa_1\Delta^2\chi\Big\|_{L_2(\Omega^{kT,t})} \le \varphi(c_1)(\sup_t \|\nabla\chi\|_{L_4(\Omega)}\|\nabla\chi\|_{L_2(kT,t,L_4(\Omega))} + \|\Delta\chi\|_{L_2(\Omega^{kT,t})}\\ &\quad + \sup_t \|\nabla\chi\|_{L_6(\Omega)}^3\|\nabla\chi\|_{L_2(kT,t;L_\infty(\Omega))} + \sup_t \|\nabla\chi\|_{L_6(\Omega)}^2\|\nabla^2\chi\|_{L_2(kT,t;L_6(\Omega))}\\ &\quad + \sup_t \|\nabla\chi\|_{L_6(\Omega)}^2\|\nabla\Delta\chi\|_{L_2(kT,t;L_3(\Omega))} + \sup_t \|\nabla^2\chi\|_{L_2(\Omega)}\|\nabla^2\chi\|_{L_2(kT,t;L_\infty(\Omega))}\\ &\quad + \|\Delta^2\chi\|_{L_2(\Omega^{kT,t})}) \le \varphi(c_1, \|\chi(kT)\|_{H^3(\Omega)}, |\chi_m|)(t-kT)^{1/2}\end{aligned}$$

and

$$\|\chi_t - \beta\Delta\chi_t\|_{L_2(\Omega^{kT,t})} \le \varphi(c_1, \|\chi(kT)\|_{H^3(\Omega)}, |\chi_m|)(t-kT)^{1/2}.$$

Hence,

$$(4.22)\qquad \|F_0\|_{L_2(\Omega^{kT,t})} \le \varphi(c_1, \|\chi(kT)\|_{H^3(\Omega)}, |\chi_m|)(t-kT)^{1/2}.$$

To estimate the boundary term $\|G\|_{W^{1/2,1/4}(S^{kT,t})}$ we introduce a smooth extension of the normal n to S onto a neighbourhood of S. Then, by the inverse boundary trace theorem,

$$\|G\|_{W_2^{1/2,1/4}(S^{kT,t})} \le c\|G\|_{W_2^{1,1/2}(\Omega^{kT,t})},$$

where $W_2^{1/2,1/4}(S^{kT,t})$ is the space of traces of functions from $W_2^{1,1/2}(\Omega^{kT,t})$. We have

$$\begin{aligned}(4.23)\qquad \|G\|_{W_2^{1,1/2}(\Omega^{kT,t})} &= \frac{1}{2\varkappa_2}\|\varkappa_{1,\chi}n\cdot\nabla|\nabla\chi|^2\|_{L_2(kT,t;H^1(\Omega))}\\ &\quad + \frac{1}{2\varkappa_2}\|\varkappa_{1,\chi}n\cdot\nabla|\nabla\chi|^2\|_{W_2^{1/2}(kT,t;L_2(\Omega))} \equiv I,\end{aligned}$$

where, by (4.5), (4.8) and (4.11),

$$\begin{aligned}I &\le \varphi(c_1)(\|\,|\nabla\chi|^2|\nabla^2\chi| + |\nabla^3\chi|\,|\nabla\chi| + |\nabla^2\chi|^2\|_{L_2(\Omega^{kT,t})} + \|\,|\partial_t^{1/2}\chi|\,|\nabla^2\chi|\,|\nabla\chi|\\ &\quad + |\partial_t^{1/2}\nabla^2\chi|\,|\nabla\chi| + |\nabla^2\chi|\,|\partial_t^{1/2}\nabla\chi|\,\|_{L_2(\Omega^{kT,t})} + \|\,|\nabla^2\chi|\,|\nabla\chi|\,\|_{L_2(\Omega^{kT,t})})\\ &\le \varphi(c_1)\big(\sup_t \|\nabla\chi\|_{L_6(\Omega)}^2\|\nabla^2\chi\|_{L_2(kT,t;L_6(\Omega))}\\ &\quad + \sup_t \|\nabla\chi\|_{L_\infty(\Omega)}\|\nabla^3\chi\|_{L_2(kT,t;L_2(\Omega))} + \sup_t \|\nabla^2\chi\|_{L_2(\Omega)}\|\nabla^2\chi\|_{L_2(kT,t;L_\infty(\Omega))}\end{aligned}$$

$$
\begin{aligned}
&+\sup_t \|\nabla\chi\|_{L_\infty(\Omega)} \sup_t \|\nabla^2\chi\|_{L_3(\Omega)} \|\partial_t^{1/2}\chi\|_{L_2(kT,t;L_6(\Omega))}\\
&+\sup_t \|\nabla\chi\|_{L_\infty(\Omega)} \|\partial_t^{1/2}\nabla^2\chi\|_{L_2(kT,t;L_2(\Omega))}\\
&+\sup_t \|\nabla^2\chi\|_{L_3(\Omega)} \|\partial_t^{1/2}\nabla\chi\|_{L_2(kT,t;L_6(\Omega))}\\
&+\sup_t \|\nabla\chi\|_{L_\infty(\Omega)} \|\nabla^2\chi\|_{L_2(kT,t;L_2(\Omega))}\Big)\\
&\le \varphi(c_1, \|\chi(kT)\|_{H^3(\Omega)}, |\chi_m|)(t-kT)^{1/2}.
\end{aligned}
$$

Putting together (4.20), (4.22) and (4.23) yields

$$
\|v\|_{W_2^{2,1}(\Omega^{kT,t})} \le \varphi(c_1, \|\chi(kT)\|_{H^3(\Omega)}, |\chi_m|)(t-kT)^{1/2} + c\|v(kT)\|_{H^1(\Omega)}. \tag{4.24}
$$

Let us now consider the elliptic problem resulting from (4.15), $(4.12)_3$ and (4.1):

$$
\begin{aligned}
&\Delta^2\chi = v && \text{in } \Omega, \quad \text{a.e. } t\in\Delta_kT,\\
&n\cdot\nabla\chi = 0, \quad n\cdot\nabla\Delta\chi = 0 && \text{on } S,\\
&\fint_\Omega \chi dx = \chi_m.
\end{aligned}
\tag{4.25}
$$

Then, by the elliptic estimate, we have

$$
\begin{aligned}
&\|\chi\|_{H^{r+4}(\Omega)} \le c(\|v\|_{H^r(\Omega)} + |\chi_m|), \quad r = 0, 2,\\
&\|\chi_t\|_{H^4(\Omega)} \le c\|v_t\|_{L_2(\Omega)}.
\end{aligned}
$$

From the above estimates and (4.24) we conclude the desired estimate (4.13). □

4.3. **Global estimates.** We prepare uniform in time estimates which allow to extend the regular finite-time solution step by step up to infinity. Firstly we prove an uniform estimate in $H^3(\Omega)$.

Lemma 4.4. *Let the assumptions of Lemmas 4.1 and 4.2 be satisfied, $T>0$ and $k\in\mathbb{N}_0$ be given. Let χ be a solution to problem (4.12) such that*

$$
\chi \in L_\infty(\Delta_kT; H^3(\Omega)) \cap L_2(\Delta_kT; H^4(\Omega)), \quad \Delta_kT = (kT,(k+1)T).
$$

Then there exists a positive constant $\alpha = \alpha(\varkappa_2, c_{\varkappa_1}, \beta, \gamma)$ such that for any $k\in\mathbb{N}_0$

$$
\|\chi((k+1)T)\|_{H^3(\Omega)}^2 \le \frac{1}{\alpha}\varphi(c_1) + e^{-\alpha T}\|\chi(kT)\|_{H^3(\Omega)}^2. \tag{4.26}
$$

Moreover, for any $k\in\mathbb{N}$,

$$
\|\chi(kT)\|_{H^3(\Omega)}^2 \le \frac{1}{\alpha}\varphi(c_1)\frac{1}{1-e^{-\alpha T}} + e^{-\alpha kT}\|\chi(0)\|_{H^3(\Omega)}^2, \tag{4.27}
$$

where c_1 is defined in Lemma 4.1.

Proof. Multiplying $(4.12)_1$ by $-\Delta\chi$, integrating over Ω and integrating by parts using the boundary conditions $(4.12)_3$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega (|\nabla\chi|^2 + \beta|\Delta\chi|^2 + \gamma|\nabla\Delta\chi|^2)dx + \varkappa_2\int_S n\cdot\nabla\Delta^2\chi\Delta\chi dS\\
&+\varkappa_2\int_\Omega |\Delta^2\chi|^2 dx = -\int_\Omega \Delta\Big(f_{0,\chi} - \frac{1}{2}\varkappa_{1,\chi}|\nabla\chi|^2 - \varkappa_1\Delta\chi\Big)\Delta\chi dx \equiv R_1.
\end{aligned}
\tag{4.28}
$$

Next, integrating by parts in the right-hand side of (4.28) and using $(4.12)_3$ gives

$$\tag{4.29} \begin{aligned} R_1 = \int_S \frac{1}{2}\varkappa_{1,\chi} n \cdot \nabla(|\nabla\chi|^2)\Delta\chi dS + \int_\Omega \Big(f_{0,\chi\chi}\partial_i\chi - \frac{1}{2}\varkappa_{1,\chi\chi}\partial_i\chi|\nabla\chi|^2 \\ - \varkappa_{1,\chi}\partial_j\chi\partial_{ij}^2\chi - \varkappa_{1,\chi}\partial_i\chi\Delta\chi\varkappa_1\partial_i\Delta\chi\Big)\partial_i\Delta\chi dx. \end{aligned}$$

Now, using (4.29) in (4.28) and recalling $(4.12)_4$ we arrive at

$$\tag{4.30} \begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega (|\nabla\chi|^2 + \beta|\Delta\chi|^2 + \gamma|\nabla\Delta\chi|^2)dx + \varkappa_2\int_\Omega |\Delta^2\chi|^2 dx + \int_\Omega \varkappa_1|\nabla\Delta\chi|^2 dx \\ = \int_\Omega \Big(f_{0,\chi\chi}\partial_i\chi - \frac{1}{2}\varkappa_{1,\chi\chi}\partial_i\chi|\nabla\chi|^2 - \varkappa_{1,\chi}\partial_j\chi\partial_{ij}^2\chi - \varkappa_{1,\chi}\partial_i\chi\Delta\chi\Big)\partial_i\Delta\chi dx. \end{aligned}$$

On account of (4.5) the right-hand side of (4.30) is bounded by

$$\begin{aligned} &\varphi(c_1)\int_\Omega (|\nabla\chi| + |\nabla\chi|^3 + |\nabla\chi|\,|\nabla^2\chi|)|\nabla\Delta\chi|dx \\ &\le \varepsilon\|\nabla\Delta\chi\|_{L_6(\Omega)}^2 + c(1/\varepsilon)(\|\nabla\chi\|_{L_{6/5}(\Omega)}^2 + \|\nabla\chi\|_{L_{18/5}(\Omega)}^6 + \|\nabla\chi\|_{L_3(\Omega)}^2\|\nabla^2\chi\|_{L_2(\Omega)}^2) \\ &\le \varepsilon\|\nabla\Delta\chi\|_{L_6(\Omega)}^2 + \varphi(1/\varepsilon, c_1). \end{aligned}$$

Hence, for sufficiently small ε, we obtain

$$\tag{4.31} \begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega (|\nabla\chi|^2 + \beta|\Delta\chi|^2 + \gamma|\nabla\Delta\chi|^2)dx \\ + \varkappa_2\int_\Omega |\Delta^2\chi|^2 dx + \int_\Omega \varkappa_1|\nabla\Delta\chi|^2 dx \le \varphi(c_1). \end{aligned}$$

In further considerations we need the full norm $\|\chi\|_{H^3(\Omega)}^2$ under the time derivative. For this purpose we multiply $(4.12)_1$ by χ, integrate over Ω and integrate by parts several times using the boundary conditions $(4.12)_{3,4}$ to arrive at the equality

$$\tag{4.32} \begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega (\chi^2 + \beta|\nabla\chi|^2 + \gamma|\Delta\chi|^2)dx - \varkappa_2\int_S (n\cdot\nabla\Delta^2\chi)\chi dS \\ + \varkappa_2\int_\Omega |\nabla\Delta\chi|^2 dx = \int_\Omega \Delta\Big(f_{0,\chi} - \frac{1}{2}\varkappa_{1,\chi}|\nabla\chi|^2 - \varkappa_1\Delta\chi\Big)\chi dx \equiv R_2. \end{aligned}$$

Now we examine the right-hand side of (4.32). After integrating by parts and using $(4.12)_3$ it takes the form

$$\begin{aligned} R_2 = -\frac{1}{2}\int_S \varkappa_{1,\chi} n\cdot\nabla(|\nabla\chi|^2)\chi dS - \int_\Omega \nabla\Big(f_{0,\chi} - \frac{1}{2}\varkappa_{1,\chi}|\nabla\chi|^2\Big)\cdot\nabla\chi dx \\ + \int_\Omega \nabla(\varkappa_1\Delta\chi)\cdot\nabla\chi dx. \end{aligned}$$

In view of the boundary condition $(4.12)_4$ the first integral in R_2 cancels with the boundary integral on the left-hand side of (4.32). By (4.3) the second integral in

R_2 is bounded by

$$\varphi(c_1)\int_\Omega(|\nabla\chi|^2+|\nabla\chi|^4+|\nabla\chi|^2|\nabla^2\chi|)dx \le \varphi(c_1)+\|\nabla\chi\|^2_{L_4(\Omega)}\|\nabla^2\chi\|_{L_2(\Omega)} \le \varphi(c_1).$$

After integrating by parts and using $(4.12)_3$ the third integral in R_2 becomes equal to $-\int_\Omega \varkappa_1|\Delta\chi|^2dx$. Employing the above conclusions in (4.32) we infer

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega(\chi^2+\beta|\nabla\chi|^2+\gamma|\Delta\chi|^2)dx+\varkappa_2\int_\Omega|\nabla\Delta\chi|^2dx\\ +\int_\Omega\varkappa_1|\Delta\chi|^2dx\le\varphi(c_1).\end{aligned}\tag{4.33}$$

Adding (4.31) and (4.33) yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega(\chi^2+(1+\beta)|\nabla\chi|^2+(\beta+\gamma)|\Delta\chi|^2+\gamma|\nabla\Delta\chi|^2)dx\\ &\quad+\varkappa_2\int_\Omega(|\nabla\Delta\chi|^2+|\Delta^2\chi|^2)dx+\int_\Omega\varkappa_1(|\Delta\chi|^2+|\nabla\Delta\chi|^2)dx\le\varphi(c_1).\end{aligned}$$

Hence, recalling the assumption (4.2) on $\varkappa_1$ and estimate (4.3), we deduce

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega(\chi^2+(1+\beta)|\nabla\chi|^2+(\beta+\gamma)|\Delta\chi|^2+\gamma|\nabla\Delta\chi|^2)dx\\ &\quad+\varkappa_2\int_\Omega(|\nabla\Delta\chi|^2+|\Delta^2\chi|^2)dx+\frac{1}{2}b_{2l}\int_\Omega\chi^{2l}(|\Delta\chi|^2+|\nabla\Delta\chi|^2)dx\\ &\le\varphi(c_1)+c_{\varkappa_1}\int_\Omega|\nabla\Delta\chi|^2dx.\end{aligned}\tag{4.34}$$

The last integral on the right-hand side of (4.34) can be absorbed by the left-hand side since

$$\begin{aligned}c_{\varkappa_1}\int_\Omega|\nabla\Delta\chi|^2dx&\le\varepsilon\|\chi\|^2_{H^4(\Omega)}+c(1/\varepsilon)\|\Delta\chi\|^2_{L_2(\Omega)}\\ &\le\varepsilon(\|\Delta^2\chi\|^2_{L_2(\Omega)}+|\chi_m|^2)+c(1/\varepsilon)\varphi(c_1,c_{\varkappa_1})\end{aligned}\tag{4.35}$$

for any $\varepsilon>0$, where the interpolation inequality and Lemmas 3.2 and 4.1 have been used. Now, let us introduce the norm

$$\|\chi\|_{\tilde H^3(\Omega)}=\left[\int_\Omega(\chi^2+(1+\beta)|\nabla\chi|^2+(\beta+\gamma)|\Delta\chi|^2+\gamma|\nabla\Delta\chi|^2)dx\right]^{1/2}$$

which is equivalent to the norm $\|\chi\|_{H^3(\Omega)}$. Thus, there exist positive constants $\underline{a}$, $\overline{a}$ dependent on β, γ such that

$$\underline{a}\|\cdot\|_{H^3(\Omega)}\le\|\cdot\|_{\tilde H^3(\Omega)}\le\overline{a}\|\cdot\|_{H^3(\Omega)}.\tag{4.36}$$

Furthermore, by virtue of Lemmas 3.1 and 3.2, there exists a positive constant $\alpha=\alpha(\beta,\gamma,\varkappa_2)$ such that

$$\varkappa_2\int_\Omega(|\nabla\Delta\chi|^2+|\Delta^2\chi|^2)dx\ge\alpha(\|\chi\|^2_{\tilde H^3(\Omega)}+\|\chi\|^2_{H^4(\Omega)}-|\chi_m|^2).\tag{4.37}$$

On account of (4.35)–(4.37) we deduce from (4.34) the inequality

$$\frac{d}{dt}\|\chi\|^2_{\bar H^3(\Omega)} + \alpha(\|\chi\|^2_{\bar H^3(\Omega)} + \|\chi\|^2_{\bar H^4(\Omega)}) \le \varphi(c_1). \tag{4.38}$$

Hence, by the classical Gronwall lemma (multiplying (4.38) by $\exp(\alpha t)$ and integrating with respect to time from kT to $t \in (kT,(k+1)T]$), we obtain

$$\|\chi(t)\|^2_{\bar H^3(\Omega)} \le \frac{1}{\alpha}\varphi(c_1) + e^{-\alpha(t-kT)}\|\chi(kT)\|^2_{\bar H^3(\Omega)}.$$

Hence, in view of (4.36), estimate (4.26) follows. Moreover, for any $k \in \mathbb{N}$ we have

$$\begin{aligned}\|\chi(kT)\|^2_{H^3(\Omega)} &\le \frac{1}{\alpha}\varphi(c_1)\sum_{j=0}^{k-1} e^{-j\alpha T} + e^{-\alpha kT}\|\chi(0)\|^2_{H^3(\Omega)}\\ &\le \frac{1}{\alpha}\varphi(c_1)\frac{1}{1-e^{-\alpha T}} + e^{-\alpha kT}\|\chi(0)\|^2_{H^3(\Omega)},\end{aligned}$$

which shows (4.27) and thereby completes the proof. □

Corollary 4.4. *Let $A > 0$ be a constant such that*

$$\|\chi(0)\|^2_{H^3(\Omega)} \le A.$$

Moreover, let A and T be choosen so large that

$$\frac{1}{\alpha}\varphi(c_1) + e^{-\alpha T}A \le A.$$

Then Lemma 4.4 implies

$$\|\chi(kT)\|^2_{H^3(\Omega)} \le A \quad \text{for any } k \in \mathbb{N}_0. \tag{4.39}$$

We now prove an uniform estimate in $H^5(\Omega)$.

Lemma 4.5. *Let the assumptions of Lemmas 4.3 and 4.4 be satisfied, $T > 0$, $k \in \mathbb{N}_0$, and χ be a solution to problem (4.12) such that*

$$\chi \in L_\infty(\Delta_k T; H^5(\Omega)) \cap L_2(\Delta_k T; H^6(\Omega)), \quad \Delta_k T = (kT,(k+1)T). \tag{4.40}$$

Then there exists a positive constant $\alpha_* = \alpha_*(\varkappa_2, c_{\varkappa_1}, \beta, \gamma)$ such that for any $k \in \mathbb{N}_0$,

$$\|\chi((k+1)T)\|^2_{H^5(\Omega)} \le \varphi(c_1, A, |\chi_m|)T + e^{-\alpha_* T}\|\chi(kT)\|^2_{H^5(\Omega)}, \tag{4.41}$$

where constant A is defined in Corollary 4.4, and c_1 in Lemma 4.1.
Moreover, for any $k \in \mathbb{N}$

$$\|\chi(kT)\|^2_{H^5(\Omega)} \le \varphi(c_1, A, |\chi_m|)T\frac{1}{1-e^{-\alpha_* T}} + e^{-\alpha_* kT}\|\chi(0)\|^2_{H^5(\Omega)}. \tag{4.42}$$

Proof. Let us turn back to the proof of Lemma 4.3 and consider system (4.18) again. Multiplying $(4.18)_1$ by $-\Delta w$, integrating over Ω and applying the Young inequality we obtain an energy type estimate

$$\frac{d}{dt}\int_\Omega |\nabla w|^2 dx + \frac{\varkappa_2}{\gamma}\int_\Omega |\Delta w|^2 dx \le \frac{1}{\varkappa_2\gamma}\int_\Omega F_*^2 dx. \tag{4.43}$$

Now, adding the inequality (4.38) from the proof of Lemma 4.4 and (4.43) we obtain

$$\begin{aligned}&\frac{d}{dt}(\|\chi\|^2_{\bar H^3(\Omega)} + \|\nabla w\|^2_{L_2(\Omega)}) + \alpha_*(\|\chi\|^2_{\bar H^3(\Omega)} + \|\chi\|^2_{\bar H^4(\Omega)} + \|\Delta w\|^2_{L_2(\Omega)})\\ &\le \varphi(c_1) + \frac{1}{\varkappa_2\gamma}\|F_*\|^2_{L_2(\Omega)},\end{aligned} \tag{4.44}$$

with a positive constant $\alpha_* = \alpha_*(\varkappa_2, c_{\varkappa_1}, \beta, \gamma)$. Taking into account that

$$\|w\|_{H^2(\Omega)} \le c\Big(\|\Delta w\|_{L_2(\Omega)} + \Big|\fint_\Omega w dx\Big|\Big), \quad \text{and}$$

$$\Big|\fint_\Omega w dx\Big| \le \Big|\fint_\Omega v dx\Big| + \Big|\fint_\Omega u dx\Big| = \Big|\fint_\Omega u dx\Big|,$$

where the latter equality follows from the compatibility condition to (4.25), we conclude from (4.44) by the Gronwall lemma that

$$\|\chi(t)\|^2_{\bar H^3(\Omega)} + \|\nabla w(t)\|^2_{L_2(\Omega)} \le \int_{kT}^t \Big(\varphi(c_1) + \frac{1}{\varkappa_2\gamma}\|F_*\|^2_{L_2(\Omega)}$$
$$+ \alpha_*\Big|\fint_\Omega u dx\Big|^2\Big)dt' + e^{-\alpha_*(t-kT)}(\|\chi(kT)\|^2_{\bar H^3(\Omega)} + \|\nabla w(kT)\|^2_{L_2(\Omega)})$$

for $t \in (kT, (k+1)T]$. Hence, since $v = w + u$, it follows that

$$\begin{aligned} &\|\chi(t)\|^2_{\bar H^3(\Omega)} + \|\nabla v(t)\|^2_{L_2(\Omega)} \\ &\le c\Big[\int_{kT}^t \Big(\varphi(c_1) + \|F_*\|^2_{L_2(\Omega)} + \Big|\fint_\Omega u dx\Big|^2\Big)dt' \\ &\quad + \|\nabla u(t)\|^2_{L_2(\Omega)} + e^{-\alpha_*(t-kT)}(\|\chi(kT)\|^2_{\bar H^3(\Omega)} \\ &\quad + \|\nabla v(kT)\|^2_{L_2(\Omega)} + \|\nabla u(kT)\|^2_{L_2(\Omega)})\Big]. \end{aligned} \tag{4.45}$$

By the trace theorem and the estimate (4.16) we have

$$\begin{aligned} &\sup_{t'\in[kT,t]} \|\nabla u(t')\|_{L_2(\Omega)} \le c\|u\|_{W_2^{2,1}(\Omega\times(kT,t))} \\ &\le c\|G\|_{W_2^{1/2,1/4}(S\times(kT,t))}, \quad t \in (kT, (k+1)T]. \end{aligned} \tag{4.46}$$

Since $v = \Delta^2\chi$ the left-hand side of (4.45) gives the norm

$$\|\chi\|_{\bar H^5(\Omega)} = \|\chi\|_{\bar H^3(\Omega)} + \|\nabla\Delta^2\chi\|_{L_2(\Omega)}, \tag{4.47}$$

which is equivalent to the norm $\|\chi\|_{H^5(\Omega)}$. Thus there exist positive constants $\underline{b}$, $\bar b$ depending on β, γ, such that

$$\underline{b}\|\chi\|_{H^5(\Omega)} \le \|\chi\|_{\bar H^5(\Omega)} \le \bar b\|\chi\|_{H^5(\Omega)}. \tag{4.48}$$

Using (4.46) and (4.47) in (4.45) yields

$$\|\chi(t)\|^2_{\bar H^5(\Omega)} \le c\Big[\int_{kT}^t \Big(\varphi(c_1) + \|F_*\|^2_{L_2(\Omega)} + \Big|\fint_\Omega u dx\Big|^2\Big)dt'$$
$$+ \|G\|_{W_2^{1/2,1/4}(S\times(kT,t))} + e^{-\alpha_*(t-kT)}\|\chi(kT)\|^2_{\bar H^5(\Omega)}\Big].$$

Hence, since $F_* = F_0 - (\gamma u_t - \varkappa_2 \Delta u)$, applying estimates (4.16), (4.22) and (4.23) from the proof of Lemma 4.3 we obtain

$$\begin{aligned} \|\chi(t)\|^2_{\tilde{H}^5(\Omega)} &\le \varphi(c_1, \|\chi(kT)\|_{H^3(\Omega)}, |\chi_m|)(t-kT) \\ &\quad + e^{-\alpha_*(t-kT)}\|\chi(kT)\|^2_{\tilde{H}^5(\Omega)} \end{aligned} \tag{4.49}$$

for $t \in (kT, (k+1)T]$, $k \in \mathbb{N}_0$.
By Corollary 4.4, $\|\chi(kT)\|^2_{H^3(\Omega)} \le A$ for any $k \in \mathbb{N}_0$. Hence, in view of (4.48) we conclude from (4.49) the estimate (4.41), and consequently (4.42) as well. This completes the proof. □

We note two important implications of Lemma 4.5.

Corollary 4.5. *(Uniform estimate) Let $A_* > 0$ be a constant such that*

$$\|\chi(0)\|^2_{H^5(\Omega)} \le A_*.$$

Moreover, let A_* and T be choosen so that

$$\varphi(c_1, A, |\chi_m|)T + e^{-\alpha_* T} A_* \le A_*.$$

Then it follows from (4.41) that

$$\|\chi(kT)\|^2_{H^5(\Omega)} \le A_* \quad \text{for any} \quad k \in \mathbb{N}_0. \tag{4.50}$$

Corollary 4.6. *(Absorbing type estimate) Letting*

$$M = \varphi(c_1, A, |\chi_m|)T \frac{1}{1 - e^{-\alpha_* T}} > 0,$$

it follows from (4.42) that

$$\limsup_{k \to \infty} \|\chi(kT)\|^2_{H^5(\Omega)} < M.$$

Furthermore, we deduce from (4.42) that for any initial datum $\chi(0) \in H^5(\Omega)$ and any positive number M' satisfying $M' > M$, there exists a time moment t_*, given by $t_* = \frac{1}{\alpha_*} \ln(\|\chi(0)\|^2_{H^5(\Omega)}/(M' - M))$, such that

$$\|\chi(kT)\|^2_{H^5(\Omega)} < M' \quad \text{for all} \quad kT \ge t_*.$$

5. Existence.

5.1. **Finite-time existence.** We prove the existence of solutions to problem (4.12) by the Leray-Schauder fixed point theorem, recalled here in one of its equivalent formulations for the reader's convenience.

Theorem 5.1. *(Leray-Schauder) Let X be a Banach space. Assume that $\Phi : [0,1] \times X \to X$ is a map with the following properties:*

(i) For any fixed $\tau \in [0,1]$ the map is completely continuous;
(ii) For every bounded subset $\mathcal{B}$ of X, the family of maps $\Phi(\cdot, \xi) : [0,1] \to X$, $\xi \in \mathcal{B}$, is uniformly equicontinuous;
(iii) $\Phi(0, \cdot)$ has precisely one fixed point in X;
(iv) There is a bounded subset $\mathcal{B}$ of X such that any fixed point in X of $\Phi(\tau, \cdot)$ is contained in $\mathcal{B}$ for every $\tau \in [0,1]$.

Then $\Phi(1, \cdot)$ has at least one fixed point.

Let $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\Delta_k T = (kT, (k+1)T)$, $\Omega^{kT} = \Omega \times \Delta_k T$ and $s \in (0,1]$. We introduce the spaces

$$Y^{6,1}(\Omega^{kT}) = L_2(\Delta_k T; H^6(\Omega)) \cap H^1(\Delta_k T; H^4(\Omega)),$$
$$Y^{6s,s}(\Omega^{kT}) = L_2(\Delta_k T; H^{6s}(\Omega)) \cap H^s(\Delta_k T; H^{4s}(\Omega))$$

endowed with the norms

$$\|u\|_{Y^{6,1}(\Omega^{kT})} = \|u\|_{L_2(\Delta_k T; H^6(\Omega))} + \|u\|_{H^1(\Delta_k T; H^4(\Omega))},$$
$$\|u\|_{Y^{6s,s}(\Omega^{kT})} = \|u\|_{L_2(\Delta_k T; H^{6s}(\Omega))} + \|u\|_{H^s(\Delta_k T; H^{4s}(\Omega))},$$

respectively.
It will be of use to note that if $s < 1$ then the imbedding

$$Y^{6,1}(\Omega^{kT}) \subset Y^{6s,s}(\Omega^{kT}) \quad \text{is compact.}$$

We now introduce the map

$$[0,1] \times Y^{6s,s}(\Omega^{kT}) \ni (\tau, \tilde{\chi}) \mapsto \Phi(\tau, \tilde{\chi}) = \chi \in Y^{6,1}(\Omega^{kT}),$$

by means of the following problem

$$\tag{5.1} \begin{aligned} &\chi_t - \beta\Delta\chi_t + \gamma\Delta^2\chi_t - \varkappa_2\Delta^3\chi \\ &\quad = \tau\Big[\Delta\Big(f_{0,\tilde{\chi}} - \frac{1}{2}\varkappa_{1,\tilde{\chi}}|\nabla\tilde{\chi}|^2 - \varkappa_1\Delta\tilde{\chi}\Big)\Big] \equiv \tau F(\tilde{\chi}) && \text{in } \Omega^{kT}, \\ &\chi|_{t=kT} = \tau\chi(kT) && \text{in } \Omega, \\ &\boldsymbol{n}\cdot\nabla\chi = 0, \quad \boldsymbol{n}\cdot\nabla\Delta\chi = 0 && \text{on } S^{kT}, \\ &\boldsymbol{n}\cdot\nabla\Delta^2\chi = \frac{\tau}{2\varkappa_2}\varkappa_{1,\tilde{\chi}}\boldsymbol{n}\cdot\nabla|\nabla\tilde{\chi}|^2 \equiv \tau G(\tilde{\chi}) && \text{on } S^{kT}, \end{aligned}$$

where $k \in \mathbb{N}_0$ is fixed.
First we prove that the map Φ is well-defined.

Lemma 5.1. *Let $s \in (\frac{11}{12}, 1)$, $k \in \mathbb{N}_0$, $T > 0$ be given. Then for any $\tilde{\chi} \in Y^{6s,s}(\Omega^{kT})$ and $\chi(kT) \in H^5(\Omega)$ satisfying the compatibility conditions*

$$\tag{5.2} \boldsymbol{n}\cdot\nabla\chi(kT) = 0, \quad \boldsymbol{n}\cdot\nabla\Delta\chi(kT) = 0 \quad \text{on } S,$$

there exists a unique solution $\chi \in Y^{6,1}(\Omega^{kT})$ to problem (5.1) such that

$$\tag{5.3} \|\chi\|_{Y^{6,1}(\Omega^{kT})} \le \varphi(\|\tilde{\chi}\|_{Y^{6s,s}(\Omega^{kT})}, \|\chi(kT)\|_{H^5(\Omega)}).$$

Proof. For simplicity, let

$$\tilde{A} \equiv \|\tilde{\chi}\|_{Y^{6s,s}(\Omega^{kT})}.$$

By Theorem 3.6 we have

$$\tag{5.4} \begin{aligned} \|\chi\|_{Y^{6,1}(\Omega^{kT})} &\le c_T[\|F(\tilde{\chi})\|_{L_2(\Omega^{kT})} \\ &\quad + \|G(\tilde{\chi})\|_{W_2^{1/2,1/4}(S^{kT})} + \|\chi(kT)\|_{H^5(\Omega)} + T^{1/2}\|\chi(kT)\|_{L_2(\Omega)}]. \end{aligned}$$

The norms on the right-hand side of the above inequality can be estimated in a similar way as in Lemma 4.3 Repeating the estimates in (4.21), (4.23) and using the following imbeddings

$$\tag{5.5} \begin{aligned} &\|\tilde{\chi}\|_{L_\infty(\Delta_k T; H^2(\Omega))} + \|\nabla\tilde{\chi}\|_{L_\infty(\Omega^{kT})} + \|\nabla^2\tilde{\chi}\|_{L_2(\Delta_k T; L_\infty(\Omega))} \\ &\quad + \|\nabla^3\tilde{\chi}\|_{L_2(\Delta_k T; L_3(\Omega))} + \|\Delta^2\tilde{\chi}\|_{L_2(\Omega^{kT})} + \|\partial_t^{1/2}\tilde{\chi}\|_{L_2(\Delta_k T; H^2(\Omega))} \le c\tilde{A}, \end{aligned}$$

which hold true for $s \in (\frac{11}{12}, 1)$, we obtain

$$\tag{5.6} \|F(\tilde{\chi})\|_{L_2(\Omega^{kT})} + \|G(\tilde{\chi})\|_{W_2^{1/2,1/4}(S^{kT})} \le \varphi(\tilde{A}).$$

Hence, by (5.4) we conclude (5.3). □

Corollary 5.1. *For $s \in \left(\frac{11}{12}, 1\right)$ the map $\Phi : Y^{6s,s}(\Omega^{kT}) \to Y^{6,1}(\Omega^{kT})$ is compact because the imbedding $Y^{6,1}(\Omega^{kT}) \subset Y^{6s,s}(\Omega^{kT})$ is compact.*

We now show the continuity of the map Φ. For a fixed $\tau \in [0,1]$, let $\chi_1 = \Phi(\tau, \tilde{\chi}_1)$ and $\chi_2 = \Phi(\tau, \tilde{\chi}_2)$ be two solutions of problem (5.1) corresponding to $\tilde{\chi}_1$ and $\tilde{\chi}_2$ from a bounded subset of $Y^{6s,s}(\Omega^{kT})$ such that

$$\|\tilde{\chi}_i\|_{Y^{6s,s}(\Omega^{kT})} \le \tilde{A}, \quad i = 1,2. \tag{5.7}$$

Introducing the differences

$$K = \chi_1 - \chi_2, \quad \tilde{K} = \tilde{\chi}_1 - \tilde{\chi}_2,$$

we see that they satisfy the problem

$$\begin{aligned} & K_t - \beta\Delta K_t + \gamma\Delta^2 K_t - \varkappa_2\Delta^3 K \\ & \quad = \tau(F(\tilde{\chi}_1) - F(\tilde{\chi}_2)) \equiv \tau\tilde{F}(\tilde{\chi}_1, \tilde{\chi}_2, \tilde{K}) && \text{in } \Omega^{kT}, \\ & K|_{t=kT} = 0 && \text{in } \Omega, \\ & n \cdot \nabla K = 0, \quad n \cdot \nabla\Delta K = 0 && \text{on } S^{kT}, \\ & n \cdot \nabla\Delta^2 K = \tau(G(\tilde{\chi}_1) - G(\tilde{\chi}_2)) \equiv \tau\tilde{G}(\tilde{\chi}_1, \tilde{\chi}_2, \tilde{K}) && \text{on } S^{kT}. \end{aligned} \tag{5.9}$$

Using similar arguments as in the proof of Lemma 4.3 from [18] we conclude after straightforward calculations the following

Lemma 5.2. *(Continuity of Φ) For any $\tilde{\chi}_1, \tilde{\chi}_2 \in Y^{6s,s}(\Omega^{kT})$, $s \in \left(\frac{11}{12}, 1\right)$, satisfying (5.7), and for any $\tau \in [0,1]$, the unique solution $K \in Y^{6,1}(\Omega^{kT})$ to problem (5.9) enjoys the estimate*

$$\|K\|_{Y^{6,1}(\Omega^{kT})} \le \tau\varphi(\tilde{A})\|\tilde{K}\|_{Y^{6s,s}(\Omega^{kT})}. \tag{5.10}$$

Corollary 5.2. *The continuity of the map Φ with respect to τ is evident (see Lemma 4.4 from [18]).*

Corollary 5.3. *By virtue of Theorem 3.6 problem (5.1) with $\tau = 0$ has the unique solution $\chi = 0$.*

Corollary 5.4. *It follows from Lemma 4.3 that for any fixed $k \in \mathbb{N}_0$ and $T > 0$, there exists a bounded subset $\mathcal{B}$ of $Y^{6s,s}(\Omega^{kT})$, given by*

$$\begin{aligned} \mathcal{B} \equiv \{\chi \in Y^{6,1}(\Omega^{kT}) : & \|\chi\|_{Y^{6,1}(\Omega^{kT})} \\ & \le \varphi(c_1, \|\chi(kT)\|_{H^3(\Omega)}, |\chi_m|)T^{1/2} + c\|\chi(kT)\|_{H^5(\Omega)}\} \end{aligned}$$

such that any fixed point of $\Phi(1,\cdot)$ is contained in $\mathcal{B}$. It is clear that the same property holds for any $\tau \in [0,1]$.

Due to the above results we obtain

Theorem 5.2. *(Finite-time existence). Let $\Omega \subset \mathbf{R}^3$ be bounded with boundary $S \in C^6$. Let us assume that for $k \in \mathbb{N}_0$, $\chi(kT) \in H^5(\Omega)$ satisfies the conditions $n \cdot \nabla\chi(kT) = 0$, $n \cdot \nabla\Delta\chi(kT) = 0$ on S. Then there exists a unique solution to problem (4.12) such that*

$$\chi \in Y^{6,1}(\Omega^{kT}), \tag{5.11}$$

satisfying estimates (4.3), (4.13) and (4.42).

Proof. From Lemmas 5.1, 5.2 and Corollaries 5.1–5.4 we infer that the map Φ fulfills the assumptions of the Leray-Schauder fixed point theorem. Hence, there exists at least one fixed point of $\Phi(1,\cdot)$ in the space $Y^{6s,s}(\Omega^{kT})$, $s \in (\frac{11}{12},1)$.
By the regularity properties (5.3) of this map it follows that the fixed point belongs to the space $Y^{6,1}(\Omega^{kT})$. Clearly, in view of the definition of $\Phi(1,\cdot)$ this means that problem (4.12) has a solution in this space, i.e. (5.11) holds. From Corollary 5.4 it follows that the solution satisfies estimates (4.3) and (4.13). Moreover, (5.11) ensures that the assumption (4.40) in Lemma 4.5 is satisfied. Therefore, estimate (4.42) holds as well. The uniqueness statement, holding true for any solution satisfying $\chi \in L_\infty(\Delta_k T; H^2(\Omega))$, $k \in \mathbb{N}_0$, can be proved in the same way as in [18]. This completes the proof. □

Corollary 5.5. *As* $\chi \in Y^{6,1}(\Omega^{kT})$ *we have* $\chi \in C^0([kT,(k+1)T]; H^5(\Omega))$, $k \in \mathbb{N}_0$.

5.2. **Global existence (completing the proof of Theorem 1.1).** The global existence results by applying Theorem 5.2 on the subsequent intervals $(kT,(k+1)T)$, $k \in \mathbb{N}_0$. More precisely, starting with $k = 0$ and $\chi(0) = \chi_0 \in H^5(\Omega)$ such that $\boldsymbol{n} \cdot \nabla\chi_0 = 0$, $\boldsymbol{n} \cdot \nabla\Delta\chi_0 = 0$ on S, and $\|\chi(0)\|_{H^5(\Omega)} \le A_*$, we get by Theorem 5.2 the existence of a unique solution to problem (1.9)–(1.13) (equivalent to (4.12) with $k = 0$) such that $\chi \in Y^{6,1}(\Omega^T)$, satisfying the estimate

$$\|\chi\|_{Y^{6,1}(\Omega\times(0,T))} \le \varphi(A_*, c_1, T) \tag{5.12}$$

with constant c_1 defined in Lemma 4.1 and A_* in Corollary 4.5.
Since $\chi \in C^0([kT,(k+1)T]; H^5(\Omega))$ we have $\boldsymbol{n} \cdot \nabla\chi(T) = 0$ and $\boldsymbol{n} \cdot \nabla\Delta\chi(T) = 0$ on S. Moreover, from Corollary 4.5 it follows that $\|\chi(T)\|_{H^5(\Omega)} \le A_*$.
Now we apply Theorem 5.2 on the interval $(T,2T)$ and obtain estimate (5.12) on $(T,2T)$. Repeating the procedure on the intervals $(kT,(k+1)T)$, $k \in \mathbb{N}$, $k \ge 2$, we conclude that there exists a unique solution defined on $\mathbb{R}_+$, such that

$$\chi \in Y^{6,1}(\Omega \times (kT,(k+1)T)),$$

satisfying

$$\|\chi\|_{Y^{6,1}(\Omega\times(kT,(k+1)T))} \le \varphi(A_*, c_1, T) \tag{5.13}$$

with the same $\varphi(A_*, c_1, T)$ as in (5.12).
This proves $(1.15)_1$ and (1.17). Clearly, $\chi|_{t=0} = \chi_0$. The constant mean value of χ follows from (4.1). Hence, $(1.15)_2$ is satisfied. Finally, energy estimate (1.16) results from Lemma 4.1. Thereby the proof of Theorem 1.1 is completed.

Acknowledgements. The authors are very grateful to the Referees for extremely helpful comments which allowed to correct an earlier version of this paper.

REFERENCES

[1] J. Berry, K. R. Elder, M. Grant, *Simulation of an atomistic dynamic field theory for monatomic liquids: Freezing and glass formation*, Phys. Rev. E **77** (2008), 061506.
[2] J. Berry, M. Grant, K. R. Elder, *Diffusive atomistic dynamics of edge dislocations in two dimensions*, Phys. Rev. E **73** (2006), 031609.
[3] O. V. Besov, V. P. Il'in, S. M. Nikolskij, *Integral Representation of Functions and Theorems of Imbeddings*, Nauka, Moscow, 1975 (in Russian).
[4] D. G. B. Edelen, *On the existence of symmetry relations and dissipation potentials*, Arch. Ration. Mech. Anal. **51** (1973), 218–227.
[5] M. Efendiev, A. Miranville, *New models of Cahn-Hilliard-Gurtin equations*, Continuum Mech. Thermodyn. 16 (2004), 441–451.

[6] K. R. Elder, M. Grant, *Modeling elastic and plastic deformations in nonequilibrium processing using phase field crystals*, Phys. Rev. E. **70** (2004), 051605.
[7] K.R. Elder, M. Katakowski, M. Haataja, M. Grant, *Modeling elasticity in crystal growth*, Phys. Rev. Lett. **88**, No 24 (2002), 245701.
[8] P. Galenko, D. Danilov, V. Lebedev, *Phase-field-crystal and Swift-Hohenberg equations with fast dynamics*, Phys. Rev. E **79** (2009), 051110(11).
[9] G. Gompper, M. Kraus, *Ginzburg-Landau theory of ternary amphiphilic systems. I. Gaussian interface fluctuations*, Phys. Rev. E **47** No 6 (1993), 4289–4300.
[10] G. Gompper, M. Kraus, *Ginzburg-Landau theory of ternary amphiphilic systems. II. Monte Carlo simulations*, Phys. Rev. E **47** No 6 (1993), 4301–4312.
[11] G. Gompper, S. Zschocke, *Ginzburg-Landau theory of oil-water-surfactant mixtures*, Phys. Rev. A **46** No 8 (1992), 4836–4851.
[12] M. E. Gurtin, *Generalized Ginzburg-Landau and Cahn-Hilliard equations based on a microforce balance*, Phys. D **92** (1996), 178–192.
[13] M. D. Korzec, P. Nayar, P. Rybka, *Global weak solutions to a sixth order Cahn-Hilliard type equation*, to appear, 2011.
[14] M. D. Korzec, P. Rybka, *On a higher order convective Cahn-Hilliard type equation*, to appear, 2011.
[15] I. S. Liu, *Method of Lagrange multipliers for exploitation of the entropy principle*, Arch. Rational Mech. Anal. **46** (1972), 131–148.
[16] I. Müller, *Thermodynamics*, Pitman, London, 1985.
[17] I. Pawłow, *Thermodynamically consistent Cahn-Hilliard and Allen-Cahn models in elastic solids*, Discrete Contin. Dyn. Syst. **15**, No 4 (2006), 1169–1191.
[18] I. Pawłow, W. M. Zajączkowski, *A sixth order Cahn-Hilliard type equation arising in oil-water-surfactant mixtures*, Commun. Pure Appl. Anal. **10**, No 6 (2011), 1823–1847.
[19] T. V. Savina, A. A. Golovin, S. H. Davis, A. A. Nepomnyashchy, P. W. Voorhees, *Faceting of a growing crystal surface by surface diffusion*, Phys. Rev. E **67** (2003), 021606.
[20] G. Schimperna, I. Pawłow, *On a class of Cahn-Hilliard models with nonlinear diffusion*, to appear, 2011.
[21] I. Singer-Loginova, H. M. Singer, *The phase field technique for modeling multiphase materials*, Rep. Prog. Phys. 71 (2008), 106501 (32 pp).
[22] V. A. Solonnikov, *A priori estimates for solutions of second order parabolic equations*, Trudy Mat. Inst. Steklov **70** (1964), 133–212 (in Russian).
[23] V. A. Solonnikov, *Boundary value problems for linear parabolic systems of differential equations of general type*, Trudy Mat. Inst. Steklov **83** (1965), 1–162 (in Russian).
[24] W. von Wahl, *The Equations of Navier-Stokes and Abstract Parabolic Equations*, Braunschweig, 1985.

Received xxxx 20xx; revised xxxx 20xx.

E-mail address: pawlow@ibspan.waw
E-mail address: wz@impan.gov.pl