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Initial-boundary-value problem for the sixth order Cahn-Hilliard type equation arising in oil-water-surfactant systems

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Abstract. An initial-boundary-value problem for the sixth order Cahn--Hilliard type equation in 3-D is studied. The problem describes dynamics of phase transitions in ternary oil-water-surfactant systems. It is based on the Landau-Ginzburg theory proposed for such systems by G. Gompper et al. We prove that the problem under consideration is well-posed in the sense that it admits the unique large-time, regular solution which depends continuously on the initial datum.

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1. Introduction

1.1. Motivation and goal

In this paper we study an initial-boundary-value problem for the sixth order Cahn-Hilliard type equation in 3-D. The problem describes dynamics of phase transitions in ternary oil-water-surfactant systems in which three phases, microemulsion, almost pure oil and almost pure water can coexist in equilibrium. Surfactant is a surface active molecule which has amphiphilic character; one part of it is hydrophilic (water-loving) and the other lipophilic (fat-lowing). Such molecule is called amphiphile. When a small amount of amphiphilic molecules is added to a phase separated mixture of oil and water, a homogeneous microemulsion phase forms.

Microemulsion is macroscopically a single-phase structured fluid. It consists of homogeneous regions of oil and water which form a complicated, intertwinned network with a typical length scale of a few hundred Ångstrom units. This is possible because of their amphiphilic character the surfactant molecules form a monolayer at the interface between oil and water regions and thereby reduce the interfacial tension. In result a phase with an extensive amount of internal interface can become stable. Among the most striking properties of microemulsions are that they can coexist in three phase equilibrium with two other phases – almost pure oil and almost pure water – and that the tension of the interfaces between pairs of these coexisting phases is very low, typically the fraction 10^{-2} or 10^{-3} of the tension of the oil-water without surfactant.

Our study is motivated by the second order Landau-Ginzburg free energy (involving second order space derivatives) proposed and justified experimentally in a line of papers by G. Gompper et al. [8–13]. It has been demonstrated that such free energy for a conserved order parameter is able to capture many of the essential static properties of the ternary oil-water-surfactant systems.

Assuming the Gompper et al. free energy we propose a conserved evolution model for the oil-water-surfactant system. The model has the form of an IBVP for the sixth order Cahn-Hilliard type equation and results as a direct extension of the classical fourth order Cahn Hilliard theory. We prove that such model is well-posed in the sense that it admits a unique large-time regular solution which depends continuously on the initial datum.

Higher order extensions of the Cahn-Hilliard equation arise also in other physical problems and attract recently a mathematical attention.

In [14] stationary solutions to one-dimensional sixth order convective Cahn-Hilliard type equation arising in epitaxially growing nanostructures have been analysed; also other related references have been indicated. In a very recent reference [4] the sixth order Cahn-Hilliard type equation in 2-D describing faceting of growing crystal surface [16] has been studied from the point of view of the existence of global in time weak solutions.

We mention also that the sixth order Cahn-Hilliard type equation with a similar structure as considered in the present paper arises as a conserved phase-field-crystal (PFC) model. It has been developed in [6, 2, 1] as an efficient tool to simulate materials on the microscopic scale. This model is based on a second order free energy. In case of a nonconserved evolution of the order parameter the corresponding fourth order parabolic equation is known as the Swift-Hohenberg (SH) equation. For a detailed discussion of PFC and SH equations and their hyperbolic type extensions as well as related references we refer to the recent paper [7].

1.2. The Gompper et al. free energy

The free energy functional proposed by G. Gompper et al. [8–13] has the form

$$\mathcal{F}\{\chi\} = \int\limits_{\Omega} f(\chi, \nabla\chi, \nabla^2\chi)dx, \quad \Omega \subset \mathbb{R}^3, \tag{1.1}$$

with the density

$$f(\chi, \nabla\chi, \nabla^2\chi) = f_0(\chi) + \frac{1}{2}\varkappa_1(\chi)|\nabla\chi|^2 + \frac{1}{2}\varkappa_2(\Delta\chi)^2.$$

Here χ is the scalar order parameter which is proportional to the local difference of the oil and water concentrations. The properties of the amphiphile and its concentration enter model (1.1) implicitly via the form of the functions $f_0(\chi)$ and $\varkappa_1(\chi)$ as well as the magnitude of constant $\varkappa_2 > 0$.

The funciton $f_0(\chi)$ is the volumetric free energy with three local minima at $\chi = \chi_0$, $\chi = \chi_w$ and $\chi = 0$ corresponding to oil-rich, water-rich and microemulsion phases, respectively. In the absence of amphiphilic molecules $f_0(\chi)$ has two minima at $\chi - \chi_0$ and $\chi = \chi_w$. When amphiphile is added to the oil-water system a third minimum of f_0 appears at $\chi = 0$, which describes a homogeneous microemulsion phase. The value of the

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microemulsion minimum depends on the amphiphile concentration: $f_0(0)$ is low for high amphiphile concentration and high for small amphiphile concentration.

In [8, 10] the following sixth order approximation of f_0 is used

$$f_0(\chi) = \omega(\chi - \chi_o)^2(\chi^2 + h_0)(\chi - \chi_w)^2 \tag{1.2}$$

where parameter $h_0 \in \mathbb{R}$ measures the deviation from oil-water-microemulsion coexistence and ω is a positive constant. In case of oil-water symmetry, $-\chi_o = \chi_w = \chi_{bulk} = 1$, $\omega = 1$, and then

$$f_0(\chi) = (\chi + 1)^2(\chi^2 + h_0)(\chi - 1)^2. \tag{1.3}$$

In the absence of amphiphile the first gradient coefficient $\varkappa_1$ is a positive constant. When amphiphile is added to the sytem a minimum of $\varkappa_1(\chi)$ develops at $\chi = 0$. For strong amphiphiles and with increasing their concentration $\varkappa_1(\chi)$ becomes negative at the microemulsion phase. In [8, 10, 13] the coefficient $\varkappa_1$ is approximated by the quadratic function

$$\varkappa_1(\chi) = g_0 + g_2\chi^2 \tag{1.4}$$

with constants g_0 of arbitrary sign and g_2 positive.

The second gradient coefficient is a positive constant

$$\varkappa_2 = \text{const} > 0. \tag{1.5}$$

The Gompper et al. free energy (1.1)–(1.5) is justified by scattering experiments. We recall after [9, 11, 13] that the scattering intensity $S(q)$ – which is the Fourier transform of the order parameter correlation function – corresponding to the free energy (1.1) has the form

$$S(q) \sim \frac{1}{\varkappa_2 q^4 + \varkappa_1(\chi_b)q^2 + f_0''(\chi_b)}$$

where $\chi_b \in \{\chi_o, 0, \chi_w\}$. Experiments show a peak at $q = 0$ in the oil-rich and water-rich phases, but at $q \neq 0$ in the microemulsion. This requires $\varkappa_2$ to be positive and $\varkappa_1$ to be positive in the oil and water phases but negative in the microemulsion.

1.3. Conserved evolution system

Like in the classical Cahn-Hilliard theory the order parameter χ in (1.1) is a conserved quantity. Thus it satisfies the conservation law

$$\chi_t + \nabla \cdot \boldsymbol{j} = 0 \tag{1.6}$$

with the mass flux $\boldsymbol{j}$ given by the constitutive equation

$$\boldsymbol{j} = -M\nabla\mu. \tag{1.7}$$

Here $M > 0$ is the mobility and μ represents the chemical potential differences (shortly called the chemical potential) between the phases. In accord with the Cahn-Hilliard theory the chemical potential is given as the first variation of the free energy functional with respect to the order parameter:

$$\mu = \frac{\delta f}{\delta \chi}. \tag{1.8}$$

The first variation $\delta f/\delta\chi$ is defined by the condition that

$$\frac{d}{d\lambda}\int_{\Omega} f(\chi + \lambda\zeta, \nabla\chi + \lambda\nabla\zeta, \nabla^2\chi + \lambda\nabla^2\zeta)dx\big|_{\lambda=0} =: \int_{\Omega} \frac{\delta f}{\delta\chi}\zeta dx \tag{1.9}$$

is to hold for all test functions $\zeta \in C_0^\infty(\Omega)$. In case of free energy (1.1) ($\varkappa_2 =$ const) this leads to the following expressions for μ and $\nabla\mu$:

$$\begin{aligned}\mu &= f_{0,\chi}(\chi) + \frac{1}{2}\varkappa_{1,\chi}(\chi)|\nabla\chi|^2 - \nabla\cdot(\varkappa_1(\chi)\nabla\chi) + \varkappa_2\Delta^2\chi \\ &= f_{0,\chi}(\chi) - \frac{1}{2}\varkappa_{1,\chi}|\nabla\chi|^2 - \varkappa_1(\chi)\Delta\chi + \varkappa_2\Delta^2\chi,\end{aligned} \tag{1.10}$$

and

$$\begin{aligned}\nabla\mu = f_{0,\chi\chi}(\chi)\nabla\chi - \frac{1}{2}\varkappa_{1,\chi\chi}(\chi)|\nabla\chi|^2\nabla\chi - \frac{1}{2}\varkappa_{1,\chi}(\chi)\nabla(|\nabla\chi|^2) \\ - \varkappa_{1,\chi}(\chi)\Delta\chi\nabla\chi - \varkappa_1(\chi)\nabla\Delta\chi + \varkappa_2\nabla\Delta^2\chi.\end{aligned} \tag{1.11}$$

Above and hereafter we use the notation $\chi_t = \partial\chi/\partial t$, $f_{,\chi}(\chi) = df(\chi)/d\chi$, vectors are denoted by bold letters, the dot $\cdot$ means the scalar product and $\nabla\cdot$ stands for the divergence.

Combining (1.6)–(1.10) we get the following conserved evolution system

$$\begin{aligned}&\chi_t - \nabla\cdot(M\nabla\mu) = 0 && \text{in } \Omega^T = \Omega\times(0,T), \\ &\mu = f_{0,\chi}(\chi) + \frac{1}{2}\varkappa_{1,\chi}(\chi)|\nabla\chi|^2 \\ &\quad \nabla\cdot(\varkappa_1(\chi)\nabla\chi) + \varkappa_2\Delta^2\chi && \text{in } \Omega^T\end{aligned} \tag{1.12}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with the boundary S, occupied by the ternary mixture, and $(0,T)$ is the time interval. We complement this system by the initial condition at $t=0$

$$\chi(0)=\chi_0 \quad \text{in} \ \ \Omega, \tag{1.13}$$

and by the following Neumann type boundary conditions

$$\begin{aligned} &\boldsymbol{n}\cdot\nabla\chi=0 && \text{on} \ \ S^T=S\times(0,T),\\ &\boldsymbol{n}\cdot\nabla\Delta\chi=0 && \text{on} \ \ S^T,\\ &\boldsymbol{n}\cdot\nabla\mu=0 && \text{on} \ \ S^T \end{aligned} \tag{1.14}$$

where $\boldsymbol{n}$ denotes the outward unit normal to S. The conditions $(1.14)_{1,2}$ are "natural" for the functional (1.1) (see derivation of energy identity in Lemma 3.1). In view of (1.7), the condition $(1.14)_3$ represents the mass isolation at the boundary S. System (1.12) can be also considered with other boundary conditions.

Combining equations $(1.12)_1$ and $(1.12)_2$, and taking into account that by $(1.17)_{1,2}$,

$$\boldsymbol{n}\cdot\nabla\mu=\boldsymbol{n}\cdot\Big[-\frac{1}{2}\varkappa_{1,\chi}(\chi)\nabla(|\nabla\chi|^2)+\varkappa_2\nabla\Delta^2\chi\Big],$$

system (1.12)–(1.14) can be formulated in the form of the IBVP for the sixth order Cahn-Hilliard type equation:

$$\begin{aligned} &\chi_t-M\varkappa_2\Delta^3\chi\\ &=M\Delta\Big[f_{0,\chi}(\chi)-\frac{1}{2}\varkappa_{1,\chi}(\chi)|\nabla\chi|^2-\varkappa_1(\chi)\Delta\chi\Big] \quad \text{in} \ \ \Omega^T, \end{aligned} \tag{1.15}$$

$$\chi(0)=\chi_0 \quad \text{in} \ \ \Omega, \tag{1.16}$$

$$\begin{aligned} &\boldsymbol{n}\cdot\nabla\chi=0, \qquad \boldsymbol{n}\cdot\nabla\Delta\chi=0,\\ &\varkappa_2\boldsymbol{n}\cdot\nabla\Delta^2\chi=\frac{1}{2}\varkappa_{1,\chi}(\chi)\boldsymbol{n}\cdot\nabla(|\nabla\chi|^2) \quad \text{on} \ \ S^T. \end{aligned} \tag{1.17}$$

It is of interest to note that in contrast to the classical fourth order Cahn-Hilliard IBVP (the case $\varkappa_2=0$ and $\varkappa_1=\text{const}>0$) system (1.15)–(1.17) involves the nonlinear boundary condition.

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1.4. Assumptions and main result

We consider system (1.12)–(1.14) (in equivalent form (1.15)–(1.17)) under the following assumptions:

(A1) $\Omega \subset \mathbb{R}^3$ is a bounded domain with a boundary S of class C^6; $T > 0$ is an arbitrary fixed time;

(A2) the free energy density $f(\chi, \nabla\chi, \nabla^2\chi)$ is defined in (1.1) where:

(i) $f_0(\chi)$ is a sixth order polynomial satisfying the condition

$$f_0(\chi) \geq c\chi^6 - \tilde{c} \text{ for all } \chi \in \mathbb{R} \text{ with constants } c > 0 \text{ and } \tilde{c} \geq 0; \tag{1.18}$$

(ii) $\varkappa_1(\chi) = g_0 + g_2\chi^2$ with constants $g_0 \in \mathbb{R}$ and $g_2 > 0$;

(iii) $\varkappa_2 > 0$ and $M > 0$ are constants.

Clearly, $f_0(\chi)$ given by (1.2) satisfies (1.18). We have the following

Theorem 1.1. *(Existence) Let assumptions (A1), (A2) hold and the initial datum χ_0 be such that*

$$\chi_0 \in H^3(\Omega), \tag{1.19}$$

and $\chi_t(0)$, computed from equation (1.15), satisfies

$$\begin{aligned} \chi_t(0) = M\varkappa_2\Delta^3\chi_0 + M\Delta\Big[f_{0,\chi_0}(\chi_0) + \frac{1}{2}\varkappa_{1,\chi_0}(\chi_0)|\nabla\chi_0|^2 \\ - \nabla\cdot(\varkappa_{1,\chi_0}(\chi_0)\nabla\chi_0)\Big] \in L_2(\Omega). \end{aligned} \tag{1.20}$$

Moreover, the following compatibility conditions hold on S:

(1.21)

$$\boldsymbol{n}\cdot\nabla\chi_0 = 0, \quad \boldsymbol{n}\cdot\nabla\Delta\chi_0 = 0, \quad \varkappa_2\boldsymbol{n}\cdot\nabla\Delta^2\chi_0 = \frac{1}{2}\varkappa_{1,\chi_0}(\chi_0)\boldsymbol{n}\cdot\nabla(|\nabla\chi_0|^2).$$

Then for any $T > 0$ problem (1.12)–(1.14) admits a strong solution $\chi \in W_2^{6,1}(\Omega^T)$ satisfying the estimate

$$\|\chi\|_{W_2^{6,1}(\Omega^T)} \leq c \tag{1.22}$$

with a constant $c = \varphi(\|\chi_0\|_{H^3(\Omega)}, \|\chi_t(0)\|_{L_2(\Omega)}, T)$, where $\varphi(\cdot)$ is a positive, increasing function of its arguments.

Theorem 1.2. *(Continuous dependence on χ_0)*
Let $\chi_1, \chi_2 \in L_\infty(0,T; H^2(\Omega))$ be two solutions to problem (1.12)–(1.14) corresponding to the initial data $\chi_{10}, \chi_{20} \in H^2(\Omega)$, respectively. Then

$$\|\chi_1(t) - \chi_2(t)\|_{L_2(\Omega)} \leq \|\chi_{10} - \chi_{20}\|_{L_2(\Omega)}c^{ct} \quad \text{for } t \in (0,T), \tag{1.23}$$

where

$$c = \varphi(\|\bar{\chi}\|_{L_\infty(0,T;H^2(\Omega))}), \quad \bar{\chi} = (\chi_1, \chi_2), \quad \textit{and}$$

$$\|\bar{\chi}\|_{L_\infty(0,T;H^2(\Omega))} = \sum_{i=1}^{2} \|\chi_i\|_{L_\infty(0,T;H^2(\Omega))}.$$

Corollary 1.3. *(Uniqueness) Theorem 1.2 implies that a solution $\chi \in L_\infty(0,T;H^2(\Omega))$ to problem (1.12)–(1.14) is uniquely defined. In particular, the regular solution χ in Theorem 1.1 is uniquely defined.*

The results of this paper were announced without proofs in [16].

2. Notation and auxiliary results

2.1. Notation

Let $\Omega \subset \mathbb{R}^n$ be an open bounded subset of $\mathbb{R}^n$, $n \geq 1$, with a smooth boundary S, and $\Omega^T = \Omega \times (0,T)$. We denote:
$W_2^k(\Omega) = H^k(\Omega)$, $k \in \mathbb{N} \cup \{0\}$ – the Sobolev space on Ω with the finite norm

$$\|u\|_{H^k(\Omega)} = \Bigg(\sum_{|\alpha| \leq k} \int_\Omega |D_x^\alpha u|^2 dx \Bigg)^{1/2},$$

$H^0(\Omega) \equiv L_2(\Omega)$;

$$W_p^{kl,l}(\Omega^T) = L_p(0,T;W_p^{kl}(\Omega)) \cap W_p^l(0,T;L_p(\Omega)), \quad k \in \mathbb{N}, \quad l \in \mathbb{N},$$

$p \in [1,\infty)$ – the Sobolev space on Ω^T with the finite norm

$$\|u\|_{W_p^{kl,l}(\Omega^T)} = \Bigg(\sum_{|\alpha|+ka \leq kl} \int_{\Omega^T} |D_x^\alpha \partial_t^a u|^p dx dt \Bigg)^{1/p};$$

$W_p^{ks,s}(\Omega^T) = L_p(0,T;W_p^{ks}(\Omega)) \cap W_p^s(0,T;L_p(\Omega))$, $k \in \mathbb{N}$, $s \in \mathbb{R}_+$, $p \in [1,\infty)$ – the Sobolev-Slobodecki space on Ω^T with the finite norm

$$\|u\|_{W_p^{ks,s}(\Omega^T)} = \Bigg(\sum_{|\alpha|+ka \leq [ks]} \int_{\Omega^T} |D_x^\alpha \partial_t^a u|^p dx dt$$

$$+ \sum_{|\alpha|=[ks]} \int_0^T \int_\Omega \int_\Omega \frac{|D_x^\alpha u(x,t) - D_{x'}^\alpha u(x',t)|^p}{|x-x'|^{n+p(ks-[ks])}} dx dx' dt$$

$$+ \int_\Omega \int_0^T \int_0^T \frac{|\partial_t^{[s]} u(x,t) - \partial_{t'}^{[s]} u(x,t')|^p}{|t-t'|^{1+p(s-[s])}} dt dt' dx \Bigg)^{1/p}$$

where $[s]$ is the integer part of s.

By c we denote a generic positive constant which changes its value from formula to formula and depends at most on imbedding constants, constants of the considered problem and the regularity of the boundary.

By $\varphi = \varphi(\sigma_1, \dots, \sigma_k)$, $k \in \mathbb{N}$, we denote a generic function which is a positive increasing function of its arguments $\sigma_1, \dots, \sigma_k$, and may change its form from formula to formula.

2.2. Imbeddings in Sobolev-Slobodecki spaces

In accord with [18, 19, Sect. 12] we define the fractional derivatives norms. For $\mu \in (0,1)$ and $p \in [1,\infty)$ let

$$
\begin{aligned}
[u]_{\mu,p,\Omega^T,x} &= \bigg(\int\limits_0^T \int\limits_\Omega \int\limits_\Omega \frac{|u(x,t)-u(x',t)|^p}{|x-x'|^{n+p\mu}} dx dx' dt\bigg)^{1/p} \\
&\equiv \|\partial_x^\mu u\|_{L_p(\Omega^T)}, \\
[u]_{\mu,\infty,\Omega^T,x} &= \sup_{t\in(0,T)} \sup_{x,x'\in\Omega} \frac{|u(x,t)-u(x',t)|}{|x-x'|^\mu} \\
&\equiv \|\partial_x^\mu u\|_{L_\infty(\Omega^T)},
\end{aligned}
$$

and

$$
\begin{aligned}
[u]_{\mu,p,\Omega^T,t} &= \bigg(\int\limits_\Omega \int\limits_0^T \int\limits_0^T \frac{|u(x,t)-u(x,t')|^p}{|t-t'|^{1+p\mu}} dt dt' dx\bigg)^{1/p} \\
&\equiv \|\partial_t^\mu u\|_{L_p(\Omega^T)}, \\
[u]_{\mu,\infty,\Omega^T,t} &= \sup_{\Omega} \sup_{t,t'\in(0,T)} \frac{|u(x,t)-u(x,t')|}{|t-t'|^\mu} \\
&\equiv \|\partial_t^\mu u\|_{L_\infty(\Omega^T)}.
\end{aligned}
$$

For simplicity later on we use the notation $\partial_x^\mu u$ and $\partial_t^\mu u$.

We need the following results.

Theorem 2.1. *(see [3, Chap. 3, Sect. 10]) Let $u \in W_p^{ks,s}(\Omega^T)$, $\Omega \subset \mathbb{R}^n$, $s \in \mathbb{R}_+$, $p \in [1,\infty]$. Let*

$$
\varkappa = \left(\frac{n+k}{p} - \frac{n+k}{q} + |\alpha| + ka\right)\frac{1}{ks} \le 1,
$$

where $q \in [1,\infty]$, $\alpha = (\alpha_1, \dots, \alpha_n)$ is the multiindex, $\alpha_i \in \mathbb{N} \cup \{0\}$, $i = 1, \dots, n$, $a \in \mathbb{N} \cup \{0\}$, $|\alpha| = \alpha_1 + \dots + \alpha_n$. Then

$$
D_x^\alpha \partial_t^a u \in L_q(\Omega^T), \quad D_x^\alpha = \partial_{x_1}^{\alpha_1}, \dots, \partial_{x_n}^{\alpha_n},
$$

and the interpolation holds

$$\|D_x^\alpha \partial_t^a\|_{L_q(\Omega^T)} \le \varepsilon^{1-\varkappa}\Big(\|\partial_t^s u\|_{L_p(\Omega^T)} + \sum_{i=1}^n \|\partial_{x_i}^{ks} u\|_{L_p(\Omega^T)}\Big) + c\varepsilon^{-\varkappa}\|u\|_{L_p(\Omega^T)}, \tag{2.1}$$

where $\varepsilon \in \mathbb{R}_+$ *and* $q \ge p$.
In case $q = \infty$, *(2.1) holds provided* $\varkappa < 1$.

Theorem 2.2. *(direct boundary trace theorem, see [18])*
(1) Let $\Omega \subset \mathbb{R}^n$ *be a domain and* S *be either a boundary of* Ω *or a subdomain of* Ω *with dim* $S = n - 1$.
(2) Let $u \in W_p^{ks,s}(\Omega^T)$, $s \in \mathbb{R}_+$, $k \in \mathbb{N}$, $p \in [1,\infty)$, $S \in C^{ks}$.
Then there exists a function $\tilde u = u|_{S^T}$ *such that*

$$\tilde u \in W_p^{ks-1/p,s-1/kp}(S^T)$$

and

$$\|\tilde u\|_{W_p^{ks-1/p,s-1/kp}(S^T)} \le c\|u\|_{W_p^{ks,s}(\Omega^T)}, \tag{2.2}$$

where c *does not depend on* u.

Theorem 2.3. *(inverse boundary trace theorem, see [18, 19, Sect. 20]). Let assumption (1) of Theorem 2.2 hold. Let* $\tilde u \in W_p^{ks-1/p,s-1/kp}(S^T)$, $s \in \mathbb{R}_+$, $k \in \mathbb{N}$, $p \in [1,\infty)$, $S \in C^{ks}$. *Then there exists a function* u *such that* $u|_{S^T} = \tilde u$, $u \in W_p^{kl,l}(\Omega^T)$ *and*

$$\|u\|_{W_p^{ks,s}(\Omega^T)} \le c\|\tilde u\|_{W_p^{ks-1/p,s-1/kp}(S^T)}, \tag{2.3}$$

where c *does not depend on* $\tilde u$.

Theorem 2.4. *(direct initial trace theorem) (see [18]) Let* $u \in W_p^{ks,s}(\Omega^T)$, $s \in \mathbb{R}_+$, $k \in \mathbb{N}$, $s > 1/p$, $p \in (1,\infty)$. *Then* $\tilde u = u|_{t=t_0}$ *where* $t_0 \in [0,T]$, *belongs to* $W_p^{ks-k/p}(\Omega)$ *and*

$$\|\tilde u\|_{W_p^{ks-k/p}(\Omega)} \le c\|u\|_{W_p^{ks,s}(\Omega^T)}, \tag{2.4}$$

where c *does not depend on* u.

Theorem 2.5. *(inverse initial trace theorem) (see [18])*
Let $\tilde u \in W_p^{ks-k/p}(\Omega)$, $s \in \mathbb{R}_+$, $k \in \mathbb{N}$, $s > 1/p$, $p \in (1,\infty)$. *Then there exists* $u \in W_p^{ks,s}(\Omega^T)$ *such that* $u|_{t=t_0} = \tilde u$, $t_0 \in [0,T]$, *and*

$$\|u\|_{W_p^{ks,s}(\Omega^T)} \le c\|\tilde u\|_{W_p^{ks-k/p}(\Omega)}, \tag{2.5}$$

where c *does not depend on* $\tilde u$.

2.3. Estimates for second and fourth order elliptic problems

For further purposes we prepare two lemmas providing estimates for second and fourth order linear elliptic problems. These estimates represent some specialized versions of the well-known general elliptic estimate, see [15, Chap. 2, Thm. 5.1].

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be an open bounded subset of $\mathbb{R}^n$, with a smooth boundary S. Let us consider the problem

$$
\begin{aligned}
&\Delta\chi = f && \text{in } \Omega,\\
&\boldsymbol{n}\cdot\nabla\chi = 0 && \text{on } S,\\
&⨍_\Omega \chi dx = \chi_m = \text{const}
\end{aligned}
\tag{2.6}
$$

where

$$
⨍_\Omega \chi dx = \frac{1}{|\Omega|}\int_\Omega \chi dx.
$$

Lemma 2.6. *Let $r \in \mathbb{N}\cup\{0\}$ and $f \in H^r(\Omega)$. Then there exists a unique solution $\chi \in H^{2+r}(\Omega)$ to (2.6) such that*

$$
\|\chi\|_{H^{2+r}(\Omega)} \leq c(\|f\|_{H^r(\Omega)} + |\chi_m|). \tag{2.7}
$$

Proof. Let us recall the elliptic estimate [15, Chap. 2, Thm. 5]

$$
\|\chi\|_{H^{2+r}(\Omega)} \leq c(\|\Delta\chi\|_{H^r(\Omega)} + \|\chi\|_{H^{(2+r)-1}(\Omega)}). \tag{2.8}
$$

Since, by the interpolation,

$$
\|\chi\|_{H^{(2+r)-1}(\Omega)} \leq \varepsilon_1\|\chi\|_{H^{2+r}(\Omega)} + c(1/\varepsilon_1)\|\chi\|_{L_2(\Omega)}, \quad \varepsilon_1 > 0,
$$

(2.8) implies that the solution to problem (2.6) satisfies

$$
\|\chi\|_{H^{2+r}(\Omega)} \leq c(\|f\|_{H^r(\Omega)} + \|\chi\|_{L_2(\Omega)}). \tag{2.9}
$$

Multiplying $(2.6)_1$ by χ, integrating over Ω, using the Young and Poincaré inequalities leads to

$$
\|\chi\|_{L_2(\Omega)} + \|\nabla\chi\|_{L_2(\Omega)} \leq c(\|f\|_{L_2(\Omega)} + |\chi_m|). \tag{2.10}
$$

Now applying (2.10) in (2.9) we conclude inequality (2.7). □

Next, let us consider the fourth order elliptic problem

$$
\begin{aligned}
&\Delta^2\chi = f && \text{in } \Omega,\\
&\boldsymbol{n}\cdot\nabla\chi = 0 && \text{on } S,\\
&\boldsymbol{n}\cdot\nabla\Delta\chi = 0 && \text{on } S,\\
&\fint_\Omega \chi dx = \chi_m = \text{const.}
\end{aligned}
\tag{2.11}
$$

Lemma 2.7. *Let $r \in \mathbb{N}\cup\{0\}$ and $f \in H^r(\Omega)$. Then there exists a unique solution $\chi \in H^{4+r}(\Omega)$ to (2.11) such that*

$$
\|\chi\|_{H^{4+r}(\Omega)} \le c(\|f\|_{H^r(\Omega)} + |\chi_m|). \tag{2.12}
$$

Proof. In accord with the elliptic estimate [15, Chap. 2, Thm. 5.1],

$$
\|\chi\|_{H^{4+r}(\Omega)} \le c(\|\Delta^2\chi\|_{H^r(\Omega)} + \|\chi\|_{H^{(4+r)-1}(\Omega)}). \tag{2.13}
$$

Since, by the interpolation,

$$
\|\chi\|_{H^{(4+r)-1}(\Omega)} \le \varepsilon_1\|\chi\|_{H^{4+r}(\Omega)} + c(1/\varepsilon_1)\|\chi\|_{L_2(\Omega)}, \quad \varepsilon_1 > 0,
$$

(2.13) implies that the solution to problem (2.11) satisfies

$$
\|\chi\|_{H^{4+r}(\Omega)} \le c(\|f\|_{H^r(\Omega)} + \|\chi\|_{L_2(\Omega)}). \tag{2.14}
$$

In addition, we prove the estimate

$$
\|\chi\|_{L_2(\Omega)} + \|\nabla\chi\|_{L_2(\Omega)} \le c(\|f\|_{L_2(\Omega)} + |\chi_m|). \tag{2.15}
$$

To this end multiplying $(2.11)_1$ by χ, integrating over Ω and twice by parts using boundary conditions $(2.11)_{2,3}$, and applying the Young inequality we get

$$
\int_\Omega (\Delta\chi)^2 dx \le \varepsilon_2 \int_\Omega \chi^2 dx + c(1/\varepsilon_2)\int_\Omega f^2 dx, \quad \varepsilon_2 > 0. \tag{2.16}
$$

Now, considering the auxiliary artificial problem (with $\Delta\chi$ on the right-hand side of $(2.17)_1$ treated as given)

$$
\begin{aligned}
&\Delta\chi = \Delta\chi && \text{in } \Omega,\\
&\boldsymbol{n}\cdot\nabla\chi = 0 && \text{on } S,\\
&\fint_\Omega \chi dx = \chi_m,
\end{aligned}
\tag{2.17}
$$

and recalling the inequality (2.10), yields

$$\|\chi\|^2_{L_2(\Omega)} + \|\nabla\chi\|^2_{L_2(\Omega)} \le c\int_\Omega (\Delta\chi)^2 dx + c\chi_m^2. \tag{2.18}$$

Inserting (2.16) into (2.18) and choosing ε_2 sufficiently small we arrive at (2.15). Now, using (2.15) in (2.14) gives (2.12). □

2.4. Linear parabolic problem of the sixth order

We recall the solvability result for the sixth order linear parabolic problem which is of crucial importance for the proof of Theorem 1.1.

Lemma 2.8. *(see [15, 19]) Let us consider the linear IBVP*

(2.19)
$$\begin{aligned} &\chi_t - \Delta^3\chi = F && in\ \Omega^T = \Omega\times(0,T),\\ &\chi(0) = \chi_0 && in\ \Omega,\\ &\boldsymbol{n}\cdot\nabla\chi = 0,\quad \boldsymbol{n}\cdot\nabla\Delta\chi = 0,\quad \boldsymbol{n}\cdot\nabla\Delta^2\chi = G && on\ S^T = S\times(0,T), \end{aligned}$$

where $\Omega\subset\mathbb{R}^n$ is a domain with a boundary S of class C^6. Assume that

$$F\in L_2(\Omega^T),\quad G\in W_2^{1/2,1/12}(S^T),\quad \chi_0\in W_2^3(\Omega)\equiv H^3(\Omega). \tag{2.20}$$

Moreover, let the following compatibility conditions hold on S

$$\boldsymbol{n}\cdot\nabla\chi_0 = 0,\quad \boldsymbol{n}\cdot\nabla\Delta\chi_0 = 0,\quad \boldsymbol{n}\cdot\nabla\Delta^2\chi_0 = G(0) \tag{2.21}$$

with the last two in the weak sense. Then for any $T>0$ problem (2.19) has the unique solution $\chi\in W_2^{6,1}(\Omega^T)$ satisfying the estimate

$$\|\chi\|_{W_2^{6,1}(\Omega^T)} \le c(\|F\|_{L_2(\Omega^T)} + \|G\|_{W_2^{1/2,1/12}(S^T)} + \|\chi_0\|_{H^3(\Omega)}). \tag{2.22}$$

3. A priori estimates

We begin with noting the conservation property of system (1.12)–(1.14):

$$\frac{d}{dt}\int_\Omega \chi dx = 0 \tag{3.1}$$

which follows from $(1.12)_1$ and $(1.14)_3$. It shows that the mean value of χ is preserved, i.e.,

$$⨍_\Omega \chi(t)dx = ⨍_\Omega \chi_0 dx \equiv \chi_m \quad \text{for}\quad t\ge 0. \tag{3.2}$$

Next we derive the energy identity for (1.12)–(1.14).

Lemma 3.1. *(Energy identity) Let χ be a sufficiently regular solution to system (1.12)–(1.14). Then the following identity is satisfied*

$$\begin{aligned}\frac{d}{dt}\int\limits_\Omega \Big[& f_0(\chi)+\frac{1}{2}\varkappa_1(\chi)|\nabla\chi|^2 \\ & +\frac{1}{2}\varkappa_2(\Delta\chi)^2\Big]dx+M\int\limits_\Omega|\nabla\mu|^2dx=0 \quad \text{for} \quad t\geq 0. \end{aligned} \tag{3.3}$$

Proof. Multiplying $(1.12)_1$ by μ, integrating over Ω and by parts using boundary condition $(1.14)_3$ leads to

$$\int\limits_\Omega \chi_t\mu dx+M\int\limits_\Omega|\nabla\mu|^2dx=0. \tag{3.4}$$

Further, multiplying $(1.12)_2$ by $-\chi_t$ and integrating over Ω gives

$$\begin{aligned}-\int\limits_\Omega \mu\chi_t dx+\int\limits_\Omega\Big[& f_{0,\chi}(\chi)\chi_t+\frac{1}{2}\varkappa_{1,\chi}(\chi)|\nabla\chi|^2\chi_t \\ & -\nabla\cdot(\varkappa_1(\chi)\nabla\chi)\chi_t+\varkappa_2\Delta^2\chi\chi_t\Big]dx=0.\end{aligned} \tag{3.5}$$

Since

$$\int\limits_\Omega f_{0,\chi}\chi_t dx=\frac{d}{dt}\int\limits_\Omega f_0(\chi)dx,$$

and on account of boundary conditions $(1.14)_1$ and $(1.14)_2$,

$$\begin{aligned}&\int\limits_\Omega\Big[\frac{1}{2}\varkappa_{1,\chi}(\chi)|\nabla\chi|^2\chi_t-\nabla\cdot(\varkappa_1(\chi)\nabla\chi)\chi_t\Big]dx \\ &=\int\limits_\Omega\Big[\frac{1}{2}\varkappa_{1,\chi}(\chi)|\nabla\chi|^2\chi_t+\varkappa_1(\chi)\nabla\chi\cdot\nabla\chi_t\Big]dx \\ &=\frac{1}{2}\frac{d}{dt}\int\limits_\Omega\varkappa_1(\chi)|\nabla\chi|^2,\end{aligned} \tag{3.6}$$

$$\begin{aligned}&\varkappa_2\int\limits_\Omega\Delta^2\chi\chi_t dx=-\varkappa_2\int\limits_\Omega\nabla\Delta\chi\cdot\nabla\chi_t dx=\varkappa_2\int\limits_\Omega\Delta\chi\Delta\chi_t dx \\ &=\frac{\varkappa_2}{2}\frac{d}{dt}\int\limits_\Omega|\Delta\chi|^2dx,\end{aligned}$$

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the identity (3.5) becomes

$$(3.7)\qquad -\int_\Omega \mu\chi_t + \frac{d}{dt}\int_\Omega \left[f_0(\chi) + \frac{1}{2}\varkappa_1(\chi)|\nabla\chi|^2 + \frac{1}{2}\varkappa_2|\Delta\chi|^2\right]dx.$$

Now, adding (3.4) and (3.7) by sides yields the assertion. □

From identity (3.3) we deduce

Lemma 3.2. *(Energy estimate) Let assumptions (A1), (A2) hold and* $\chi_0 \in H^2(\Omega)$. *Then a solution to (1.12)–(1.14) satisfies*

$$(3.8)\qquad \begin{aligned} &\|\chi\|^2_{H^2(\Omega)} + g_2\int_\Omega \chi^2|\nabla\chi|^2dx + \int_\Omega \chi^6 dx + \int_{\Omega^t}|\nabla\mu|^2dxdt' \\ &\le c(D_0+1) \quad \text{for} \quad t\in[0,T] \end{aligned}$$

where

$$D_0 = \int_\Omega \left[f_0(\chi_0) + \frac{1}{2}(|g_0| + g_2\chi_0^2)|\nabla\chi_0|^2 + \frac{1}{2}\varkappa_2|\Delta\chi_0|^2\right]dx.$$

Proof. Integrating (3.3) with respect to time from 0 to $t\in[0,T]$ and taking into account condition (1.18) in (A2) (i), we get

$$(3.9)\qquad \begin{aligned} &\int_\Omega \left[\chi^6 + \frac{1}{2}(g_0 + g_2\chi^2)|\nabla\chi|^2 + \frac{1}{2}\varkappa_2(\Delta\chi)^2\right]dx + \int_{\Omega^t}|\nabla\mu|^2dxdt' \\ &\le D_0 + c \quad \text{for} \quad t\in[0,T], \end{aligned}$$

where D_0 is defined in (3.8). Hence,

$$(3.10)\qquad \begin{aligned} &\int_\Omega [\chi^6 + g_2\chi^2|\nabla\chi|^2 + \varkappa_2(\Delta\chi)^2]dx + \int_{\Omega^t}|\nabla\mu|^2dxdt' \\ &\le c|g_0|\int_\Omega |\nabla\chi|^2dx + D_0 + c. \end{aligned}$$

Let us consider now the artificial elliptic problem

$$(3.11)\qquad \begin{aligned} &\Delta\chi = \Delta\chi && \text{in} \quad \Omega, \\ &\boldsymbol{n}\cdot\nabla\chi = 0 && \text{on} \quad S, \\ &\fint_\Omega \chi dx = \fint_\Omega \chi_0 dx \equiv \chi_m \end{aligned}$$

with the right-hand side of $(3.11)_1$ treated as given. Note that conditions $(3.11)_{2,3}$ are ensured by $(1.14)_1$ and (3.2). Then, in accord with Lemma 2.6,

$$\|\chi\|_{H^2(\Omega)} \le c(\|\Delta\chi\|_{L_2(\Omega)} + |\chi_m|). \tag{3.12}$$

By virtue of (3.12), applying the interpolations

$$\begin{aligned}\int_\Omega |\nabla\chi|^2 dx &\le \varepsilon_1 \|\chi\|^2_{H^2(\Omega)} + c(1/\varepsilon_1)\|\chi\|^2_{L_2(\Omega)} \\ &\le \varepsilon_1 \|\chi\|^2_{H^2(\Omega)} + c(1/\varepsilon_1)(\varepsilon_2\|\chi\|^6_{L_6(\Omega)} + c(1/\varepsilon_2)),\end{aligned}$$

with $\varepsilon, \varepsilon_2 > 0$, and choosing ε_1, ε_2 sufficiently small we deduce from (3.10) inequality (3.8). □

Corollary 3.3. *From (3.8) it follows that*

$$\|\chi\|_{L_\infty(0,T;H^2(\Omega))} + \|\nabla\mu\|_{L_2(\Omega^T)} \le \varphi(c_1) \tag{3.13}$$

where $c_1 = \|\chi_0\|_{H^2(\Omega)}$. *Note that constant* c_1 *is independent of* T. *By virtue of the imbedding, (3.13) implies that*

$$\|\chi\|_{L_\infty(\Omega^T)} \le \varphi(c_1), \quad \|\nabla\chi\|_{L_\infty(0,T;L_6(\Omega))} \le \varphi(c_1). \tag{3.14}$$

Corollary 3.4. *On account of boundary conditions* $(1.4)_{1,2}$, *integration of* $(1.12)_2$ *gives*

$$\int_\Omega \mu dx = \int_\Omega \Big[f_{0,\chi}(\chi) + \frac{1}{2}\varkappa_{1,\chi}(\chi)|\nabla\chi|^2\Big] dx. \tag{3.15}$$

Hence, using (3.14), it follows that

$$\operatorname*{ess\,sup}_{t\in[0,T]} \Big|\int_\Omega \mu dx\Big| \le \varphi(c_1). \tag{3.16}$$

Now, by the Poincaré inequality, estimates (3.13) and (3.16) imply that

$$\|\mu\|_{L_2(0,T;H^1(\Omega))} \le \varphi(c_1). \tag{3.17}$$

Our goal now is to estimate the separate terms of $\nabla\mu$ given by formula (1.11).

Lemma 3.5. *(Estimate on $\nabla\Delta\chi$) Let the assumptions of Lemma 3.2 be satisfied. Then*

$$\|\chi\|_{L_\infty(0,T;L_2(\Omega))} + \|\chi\Delta\chi\|_{L_2(\Omega^T)} + \|\nabla\Delta\chi\|_{L_2(\Omega^T)} \le \varphi(c_1, T). \tag{3.18}$$

Proof. Multiplying $(1.12)_1$ by χ, integrating over Ω and using boundary condition $(1.14)_3$ gives

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \chi^2 dx + M\int_\Omega \nabla\mu\cdot\nabla\chi dx = 0. \tag{3.19}$$

Inserting formula $(1.10)_2$ for μ, (3.19) becomes

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega \chi^2 dx + M\int_\Omega \nabla\Big(f_{0,\chi}(\chi) - \frac{1}{2}\varkappa_{1,\chi}(\chi)|\nabla\chi|^2 - \varkappa_1(\chi)\Delta\chi\\ &\quad + \varkappa_2\Delta^2\chi\Big)\cdot\nabla\chi dx = 0.\end{aligned} \tag{3.20}$$

On account of boundary conditions $(1.14)_1$ and $(1.14)_2$ integration by parts yields

$$\begin{aligned}&\int_\Omega \nabla\Big(f_{0,\chi}(\chi) - \frac{1}{2}\varkappa_{1,\chi}(\chi)|\nabla\chi|^2 - \varkappa_1(\chi)\Delta\chi\Big)\cdot\nabla\chi dx\\ &= -\int_\Omega \Big(f_{0,\chi}(\chi) - \frac{1}{2}\varkappa_{1,\chi}(\chi)|\nabla\chi|^2 - \varkappa_1(\chi)\Delta\chi\Big)\Delta\chi dx\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}&\varkappa_2\int_\Omega \nabla(\Delta^2\chi)\cdot\nabla\chi dx = -\varkappa_2\int_\Omega \Delta^2\chi\Delta\chi dx\\ &= -\varkappa_2\int_\Omega \nabla\cdot(\nabla\Delta\chi)\Delta\chi dx = \varkappa_2\int_\Omega |\nabla\Delta\chi|^2 dx.\end{aligned} \tag{3.22}$$

Inserting (3.21) and (3.22) into (3.20) and using (A2) (ii), (iii) we arrive at the identity

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega \chi^2 dx + M\varkappa_2\int_\Omega |\nabla\Delta\chi|^2 dx + Mg_2\int_\Omega \chi^2(\Delta\chi)^2 dx\\ &= -Mg_0\int_\Omega (\Delta\chi)^2 dx + M\int_\Omega f_{0,\chi}(\chi)\Delta\chi dx\\ &\quad - Mg_2\int_\Omega \chi|\nabla\chi|^2\Delta\chi dx \equiv I_1 + I_2 + I_3.\end{aligned} \tag{3.23}$$

On account of (3.13) and $(3.14)_1$,

$$I_1 \le cc_1, \quad I_2 \le c\|f_{0,\chi}(\chi)\|_{L_2(\Omega)}\|\Delta\chi\|_{L_2(\Omega)} \le \varphi(c_1).$$

Moreover,

$$I_3 \le \varepsilon_1 \int\limits_\Omega \chi^2(\Delta\chi)^2 dx + c(1/\varepsilon_1) \int\limits_\Omega |\nabla\chi|^4 dx, \quad \varepsilon_1 > 0,$$

where by virtue of $(3.14)_2$ the second integral is bounded by $\varphi(c_1)$. Combining the above estimates in (3.23), choosing ε_1 small and integrating the resulting inequality with respect to time we get

$$\int\limits_\Omega \chi^2 dx + \int\limits_{\Omega^T} \chi^2(\Delta\chi)^2 dxdt + \int\limits_{\Omega^T} |\nabla\Delta\chi|^2 dxdt \le \int\limits_\Omega \chi_0^2 dx + \varphi(c_1)T.$$

Hence, by definition of c_1, estimate (3.18) follows. □

Corollary 3.6. *Considering for $t \in (0,T)$ the artificial elliptic problem*

$$\begin{aligned} &\Delta\chi = \Delta\chi && \text{in } \Omega, \\ &\boldsymbol{n}\cdot\nabla\chi = 0 && \text{on } S, \\ &⨍\limits_\Omega \chi dx = ⨍\limits_\Omega \chi_0 dx \equiv \chi_m, \end{aligned}$$

and recalling Lemma 2.6, it follows that

$$\|\chi\|_{L_2(0,T;H^3(\Omega))} \le c(\|\Delta\chi\|_{L_2(0,T;H^1(\Omega))} + T^{1/2}|\chi_m|).$$

Hence, by virtue of estimates (3.13) and (3.18),

$$\|\chi\|_{L_2(0,T;H^3(\Omega))} \le c_1 T^{1/2} + \varphi(c_1,T) + T^{1/2}|\chi_m| \equiv \varphi(c_1,\chi_m,T). \tag{3.24}$$

Lemma 3.7. *(Estimate on $\nabla\Delta^2\chi$) Let the assumptions of Lemma 3.2 be satisfied. Then*

$$\|\nabla\Delta^2\chi\|_{L_2(\Omega^T)} \le \varphi(c_1,\chi_m,T). \tag{3.25}$$

Proof. Using (1.11) we have

$$\begin{aligned} \|\nabla\Delta^2\chi\|_{L_2(\Omega^T)} &\le c(\|\nabla\mu\|_{L_2(\Omega^T)} + \|f_{0,\chi\chi}(\chi)\nabla\chi\|_{L_2(\Omega^T)} \\ &\quad + \|\varkappa_{1,\chi\chi}(\chi)|\nabla\chi|^2\nabla\chi\|_{L_2(\Omega^T)} + \|\varkappa_{1,\chi}(\chi)\nabla(|\nabla\chi|^2)\|_{L_2(\Omega^T)} \\ &\quad + \|\varkappa_{1,\chi}(\chi)\Delta\chi\nabla\chi\|_{L_2(\Omega^T)} + \|\varkappa_1(\chi)\nabla\Delta\chi\|_{L_2(\Omega^T)}) \\ &\equiv \sum_{k=1}^6 I_k. \end{aligned} \tag{3.26}$$

Using (3.13) and (3.14),

$$I_1 + I_2 + I_3 \le \varphi(c_1)T^{1/2}.$$

Further, by (3.14) and Hölder's inequality

$$\begin{aligned} I_4 + I_5 &\le \varphi(c_1)\| \, |\nabla\chi| \, |\nabla^2\chi| \, \|_{L_2(\Omega^T)} \\ &\le \varphi(c_1)\bigg(\int_0^T \|\nabla\chi\|^2_{L_6(\Omega)} \|\nabla^2\chi\|^2_{L_3(\Omega)} dt\bigg)^{1/2} \\ &\le \varphi(c_1)\bigg(\int_0^T \|\nabla^2\chi\|^2_{L_3(\Omega)} dt\bigg)^{1/2}. \end{aligned}$$

Applying the interpolation and estimates (3.24), (3.13) we have

$$\begin{aligned} &\|\nabla^2\chi\|_{L_2(0,T;L_3(\Omega))} \le c(\|\nabla^2\chi\|_{L_2(0,T;H^1(\Omega))} + \|\nabla^2\chi\|_{L_2(0,T;L_2(\Omega))}) \\ &\le \varphi(c_1, \chi_m, T). \end{aligned}$$

Hence,

$$I_4 + I_5 \le \varphi(c_1, \chi_m, T).$$

Finally, by $(3.14)_1$ and (3.18),

$$I_6 \le \varphi(c_1, \chi_m, T).$$

After using the above estimates in (3.26) we conclude (3.25). □

Corollary 3.8. *Considering for $t \in (0,T)$ the artificial elliptic problem*

$$\begin{aligned} &\Delta^2\chi = \Delta^2\chi && \text{in } \Omega, \\ &\boldsymbol{n}\cdot\nabla\chi = 0, \quad n\cdot\nabla\Delta\chi = 0 && \text{on } S, \\ &\fint_\Omega \chi dx = \fint_\Omega \chi_0 dx \equiv \chi_m, \end{aligned}$$

and recalling Lemma 2.7, it follows that

$$\|\chi\|_{L_2(0,T;H^5(\Omega))} \le c(\|\Delta^2\chi\|_{L_2(0,T;H^1(\Omega))} + T^{1/2}|\chi_m|).$$

Hence, since $\fint_\Omega \Delta^2\chi dx = 0$, the Poincaré inequality implies that

$$\|\chi\|_{L_2(0,T;H^5(\Omega))} \le c(\|\nabla\Delta^2\chi\|_{L_2(\Omega^T)} + T^{1/2}|\chi_m|).$$

This upon the use of (3.25) gives the estimate

$$\|\chi\|_{L_2(0,T;H^5(\Omega))} \le \varphi(c_1, \chi_m, T). \tag{3.27}$$

To get estimates on χ_t we consider system (1.12)–(1.14) rewritten in the form (1.15)–(1.17) and differentiate equation (1.15) with respect to time. We proceed in such a way because due to the nonlinear boundary condition $(1.17)_3$ the direct approach based on multiplying (1.15) by χ_t turns out to be unsuccessful.

Lemma 3.9. *(Estimates on χ_t) Let the assumptions of Lemma 3.2 be satisfied and, in addition, let $\chi_t(0)$ computed from equation (1.15) be such that*

$$\begin{aligned}\chi_t(0) \equiv M\Delta\Big(f_{0,\chi_0}(\chi_0) - \frac{1}{2}\varkappa_{1,\chi}(\chi_0)|\nabla\chi_0|^2 - \varkappa_1(\chi_0)\Delta\chi_0 \\ + \varkappa_2\Delta^2\chi_0\Big) \in L_2(\Omega).\end{aligned} \tag{3.28}$$

Then

$$\begin{aligned}&\int\limits_\Omega \chi_t^2 dx + \int\limits_{\Omega^t} |\nabla\Delta\chi_{t'}|^2 dx dt' + \int\limits_{\Omega^t} \chi^2|\Delta\chi_{t'}|^2 dx dt' \\ &\le \varphi(c_1, c_2, \chi_m, T) \quad \text{for } t\in(0,T)\end{aligned} \tag{3.29}$$

where $c_2 = \|\chi_t(0)\|_{L_2(\Omega)}$.

Proof. Differentiating equation (1.15) with respect to time, multiplying by χ_t and integrating over Ω gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\limits_\Omega \chi_t^2 dx - M\int\limits_\Omega \Delta\Big(\varkappa_2\Delta^2\chi - \frac{1}{2}\varkappa_{1,\chi}|\nabla\chi|^2\Big)_{,t}\chi_t dx \\ &- M\int\limits_\Omega \Delta(f_{0,\chi} - \varkappa_1\Delta\chi)_{,t}\chi_t dx = 0.\end{aligned} \tag{3.30}$$

Let us write the second integral in (3.30) in the form

$$\begin{aligned}&-M\int\limits_\Omega \nabla\cdot\Big[\nabla\Big(\varkappa_2\Delta^2\chi - \frac{1}{2}\varkappa_{1,\chi}|\nabla\chi|^2\Big)_{,t}\chi_t\Big]dx \\ &+M\int\limits_\Omega \nabla\Big(\varkappa_2\Delta^2\chi - \frac{1}{2}\varkappa_{1,\chi}|\nabla\chi|^2\Big)_{,t}\cdot\nabla\chi_t dx \equiv I_1 + I_2.\end{aligned}$$

In view of boundary conditions $(1.17)_{1,3}$ the integral I_1 vanishes, and

$$\begin{aligned}I_2 = M&\int\limits_\Omega \nabla\cdot\Big[\Big(\varkappa_2\Delta^2\chi - \frac{1}{2}\varkappa_{1,\chi}|\nabla\chi|^2\Big)_{,t}\nabla\chi_t\Big]dx \\ -M&\int\limits_\Omega \Big(\varkappa_2\Delta^2\chi - \frac{1}{2}\varkappa_{1,\chi}|\nabla\chi|^2\Big)_{,t}\Delta\chi_t dx \equiv I_2^1 + I_2^2.\end{aligned}$$

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Here, in view of boundary condition $(1.17)_1$ the integral I_2^1 vanishes, and I_2^2 we write as

$$I_2^2 = -M\varkappa_2 \int\limits_\Omega \nabla\cdot(\Delta\chi_t\nabla\Delta\chi_t)dx + M\varkappa_2 \int\limits_\Omega \nabla\Delta\chi_t\cdot\nabla\Delta\chi_t dx$$
$$+ \frac{M}{2}\int\limits_\Omega (\varkappa_{1,\chi}|\nabla\chi|^2)_{,t}\Delta\chi_t dx \equiv J_0 + J_1 + J_2.$$

By boundary conditions $(1.17)_2$, J_0 vanishes,

$$J_1 = M\varkappa_2 \int\limits_\Omega |\nabla\Delta\chi_t|^2 dx,$$

and

$$J_2 = \frac{M}{2}\int\limits_\Omega (\varkappa_{1,\chi\chi}\chi_t|\nabla\chi|^2 + 2\varkappa_{1,\chi}\nabla\chi\cdot\nabla\chi_t)\Delta\chi_t dx.$$

On account of $(3.14)_1$,

$$|J_2| \le \varphi(c_1)\int\limits_\Omega (|\chi_t|\,|\nabla\chi|^2 + |\nabla\chi|\,|\nabla\chi_t|)|\Delta\chi_t|dx \equiv J_2^1.$$

Applying the Hölder and Young inequalities, and using estimate $(3.14)_2$,

$$\begin{aligned} J_2^1 &\le \varphi(c_1)\|\Delta\chi_t\|_{L_6(\Omega)}\|\chi_t\|_{L_2(\Omega)}\|\nabla\chi\|^2_{L_6(\Omega)} \\ &\quad + \varphi(c_1)\|\Delta\chi_t\|_{L_6(\Omega)}\|\nabla\chi\|_{L_6(\Omega)}\|\nabla\chi_t\|_{L_{3/2}(\Omega)} \\ &\le \varepsilon_1\|\Delta\chi_t\|^2_{L_6(\Omega)} + \varphi(1/\varepsilon_1)(\|\chi_t\|^2_{L_2(\Omega)} + \|\nabla\chi_t\|^2_{L_{3/2}(\Omega)}) \\ &\equiv J_2^2, \quad \varepsilon_1 > 0. \end{aligned}$$

Further, on account of the interpolation

$$\|\nabla\chi_t\|^2_{L_{3/2}(\Omega)} \le \varepsilon_2\|\nabla^2\chi_t\|^2_{L_2(\Omega)} + \varphi(1/\varepsilon_2)\|\chi_t\|^2_{L_2(\Omega)}, \quad \varepsilon_2 > 0, \tag{3.31}$$

we get

$$\begin{aligned} |J_2| \le J_2^2 &\le \varepsilon_3\|\Delta\chi_t\|^2_{H^1(\Omega)} + \varphi(1/\varepsilon_3)\|\chi_t\|^2_{L_2(\Omega)} \\ &\le \varepsilon_4\|\nabla\Delta\chi_t\|^2_{L_2(\Omega)} + \varphi(1/\varepsilon_4)\|\chi_t\|^2_{L_2(\Omega)}, \quad \varepsilon_3 > 0, \ \varepsilon_4 > 0, \end{aligned}$$

where in the latter estimate we used the Poincaré inequality with $\int\limits_\Omega \Delta\chi_t dx = 0$.

Let us examine now the third integral in (3.30) by splitting it into two terms:

$$M\int\limits_\Omega \Delta(\varkappa_1\Delta\chi)_{,t}\chi_t dx - M\int\limits_\Omega \Delta(f_{0,\chi})_{,t}\chi_t dx \equiv K_1 + K_2.$$

We write K_1 in the form

$$K_1 = M\int\limits_\Omega \nabla\cdot[\nabla(\varkappa_1\Delta\chi)_{,t}\chi_t]dx - M\int\limits_\Omega \nabla(\varkappa_1\Delta\chi)_{,t}\cdot\nabla\chi_t dx \equiv K_1^1 + K_1^2.$$

On account of boundary conditions $(1.17)_{1,2}$,

$$\begin{aligned} K_1^1 &= M\int\limits_S n\cdot\nabla(\varkappa_1\Delta\chi)_{,t}\chi_t dS \\ &= M\int\limits_S (\varkappa_{1,\chi} n\cdot\nabla\chi\Delta\chi + \varkappa_1 n\cdot\nabla\Delta\chi)_{,t}\chi_t dS = 0. \end{aligned}$$

By $(1.17)_1$,

$$\begin{aligned} K_1^2 &= -M\int\limits_\Omega \nabla\cdot[(\varkappa_1\Delta\chi)_{,t}\nabla\chi_t]dx + M\int\limits_\Omega (\varkappa_1\Delta\chi)_{,t}\Delta\chi_t dx \\ &= M\int\limits_\Omega (\varkappa_1\Delta\chi)_{,t}\Delta\chi_t dx \\ &= M\int\limits_\Omega [\varkappa_{1,\chi}\chi_t\Delta\chi\Delta\chi_t + \varkappa_1(\Delta\chi_t)^2]dx, \end{aligned}$$

which after recalling the form of $\varkappa_1(\chi)$ (see (A2) (ii)) gives

$$K_1^2 = M\int\limits_\Omega [2g_2\chi\chi_t\Delta\chi\Delta\chi_t + (g_0 + g_2\chi^2)(\Delta\chi_t)^2]dx.$$

Hence,

$$K_1^2 = Mg_2\int\limits_\Omega \chi^2(\Delta\chi_t)^2 dx + K_1^3$$

where

$$K_1^3 = M\int\limits_\Omega [2g_2\chi\chi_t\Delta\chi\Delta\chi_t + g_0(\Delta\chi_t)^2]dx.$$

Using the Hölder and Young inequalities along with estimates $(3.14)_1$ and $(3.13)_1$ we get

$$\begin{aligned}|K_1^3| &\le c_1\|\chi_t\|_{L_3(\Omega)}\|\Delta\chi\|_{L_2(\Omega)}\|\Delta\chi_t\|_{L_6(\Omega)}+c\|\Delta\chi_t\|^2_{L_2(\Omega)}\\ &\le \varepsilon_5\|\Delta\chi_t\|^2_{L_6(\Omega)}+\varphi(1/\varepsilon_5)\|\chi_t\|^2_{L_3(\Omega)}+c\|\Delta\chi_t\|^2_{L_2(\Omega)}.\end{aligned}$$

Now we apply the following interpolations together with the Poincaré inequality accounting for $\underset{\Omega}{⨍}\Delta\chi_t dx=0$:

$$\begin{aligned}&\|\chi_t\|^2_{L_3(\Omega)}\le\varepsilon_6\|\nabla\Delta\chi_t\|^2_{L_2(\Omega)}+\varphi(1/\varepsilon_6)\|\chi_t\|^2_{L_2(\Omega)},\\ &\|\Delta\chi_t\|^2_{L_2(\Omega)}\le\varepsilon_7\|\nabla\Delta\chi_t\|^2_{L_2(\Omega)}+\varphi(1/\varepsilon_7)\|\chi_t\|^2_{L_2(\Omega)},\quad \varepsilon_6,\varepsilon_7>0.\end{aligned}\tag{3.32}$$

Then we get

$$|K_1^3|\le\varepsilon_8\|\nabla\Delta\chi_t\|^2_{L_2(\Omega)}+\varphi(1/\varepsilon_8)\|\chi_t\|^2_{L_2(\Omega)},\quad \varepsilon_8>0.$$

It remains to examine the integral K_2. After performing differentiation,

$$\begin{aligned}K_2&=-M\int_\Omega(f_{0,\chi\chi\chi}|\nabla\chi|^2+f_{0,\chi\chi}\Delta\chi)_{,t}\chi_t dx\\ &=-M\int_\Omega(f_{0,\chi\chi\chi\chi}\chi_t|\nabla\chi|^2+2f_{0,\chi\chi\chi}\nabla\chi\cdot\nabla\chi_t+f_{0,\chi\chi\chi}\chi_t\Delta\chi\\ &\quad+f_{0,\chi\chi}\Delta\chi_t)\chi_t dx.\end{aligned}$$

Recalling $(3.14)_1$,

$$\begin{aligned}|K_2|\le\varphi(c_1)\int_\Omega&[(\chi_t)^2|\nabla\chi|^2+|\nabla\chi|\,|\nabla\chi_t|\,|\chi_t|+(\chi_t)^2|\Delta\chi|\\ &+|\Delta\chi_t|\,|\chi_t|]dx\equiv K_2^1+K_2^2+K_2^3+K_2^4.\end{aligned}$$

Now, using (3.13), (3.14) and the interpolations in a similar way as above we get

$$\begin{aligned}K_1^1&\le\varphi(c_1)\|\chi_t\|^2_{L_3(\Omega)}\|\nabla\chi\|^2_{L_6(\Omega)}\le\varphi(c_1)\|\chi_t\|^2_{L_3(\Omega)}\\ &\le\varepsilon_9\|\nabla\Delta\chi_t\|^2_{L_2(\Omega)}+\varphi(1/\varepsilon_9,c_1)\|\chi_t\|^2_{L_2(\Omega)},\quad\varepsilon_9>0,\end{aligned}$$

$$\begin{aligned}K_2^2&\le\varphi(c_1)\|\nabla\chi\|_{L_6(\Omega)}\|\nabla\chi_t\|_{L_3(\Omega)}\|\chi_t\|_{L_2(\Omega)}\\ &\le\varepsilon_{10}\|\nabla\Delta\chi_t\|^2_{L_2(\Omega)}+\varphi(1/\varepsilon_{10},c_1)\|\chi_t\|^2_{L_2(\Omega)},\quad\varepsilon_{10}>0,\end{aligned}$$

$$K_2^3 \le \varphi(c_1)\|\Delta\chi\|_{L_2(\Omega)}\|\chi_t\|^2_{L_4(\Omega)} \le \varphi(c_1)\|\chi_t\|^2_{L_4(\Omega)}$$
$$\le \varepsilon_{11}\|\nabla\Delta\chi_t\|^2_{L_2(\Omega)} + \varphi(1/\varepsilon_{11}, c_1)\|\chi_t\|^2_{L_2(\Omega)}, \quad \varepsilon_{11} > 0,$$
$$K_2^4 \le \varphi(c_1)\|\Delta\chi_t\|_{L_2(\Omega)}\|\chi_t\|_{L_2(\Omega)}$$
$$\le \varepsilon_{12}\|\nabla\Delta\chi_t\|^2_{L_2(\Omega)} + \varphi(1/\varepsilon_{12}, c_1)\|\chi_t\|^2_{L_2(\Omega)}, \quad \varepsilon_{12} > 0.$$

In result,

$$|K_2| \le \varepsilon_{13}\|\nabla\Delta\chi_t\|^2_{L_2(\Omega)} + \varphi(1/\varepsilon_{13}, c_1)\|\chi_t\|^2_{L_2(\Omega)}, \quad \varepsilon_{13} > 0.$$

Summarizing the above estimates in (3.30) leads to the inequality

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega \chi_t^2 dx + M\varkappa_2 \int_\Omega |\nabla\Delta\chi_t|^2 dx + Mg_2 \int_\Omega \chi^2(\Delta\chi_t)^2 dx \\ &\le \varepsilon_{14}\|\nabla\Delta\chi_t\|^2_{L_2(\Omega)} + \varphi(1/\varepsilon_{14}, c_1)\|\chi_t\|^2_{L_2(\Omega)} \quad \text{for } t \in (0,T), \end{aligned} \tag{3.33}$$

where $\varepsilon_4 + \varepsilon_8 + \varepsilon_{13} \le \varepsilon_{14}$. Hence, choosing ε_{14} sufficiently small and applying the Gronwall lemma we conclude (3.29). □

Corollary 3.10. *Since*

$$\|\chi_t\|_{H^3(\Omega)} \le c\|\Delta\chi_t\|_{H^1(\Omega)} \le c\|\nabla\Delta\chi_t\|_{L_2(\Omega)},$$

which follows from the elliptic property and the Poincaré inequality $\left(\fint_\Omega \chi_t dx = 0, \ \fint_\Omega \Delta\chi_t dx = 0\right)$, *(3.29) implies the estimate*

$$\|\chi_t\|_{L_\infty(0,T;L_2(\Omega))} + \|\chi_t\|_{L_2(0,T;H^3(\Omega))} \le \varphi(c_1, c_2, \chi_m, T). \tag{3.34}$$

Moreover, combining (3.27) and (3.34) we deduce that

$$\|\chi\|_{W_2^{5,1}(\Omega^T)} \le \varphi(c_1, c_2, \chi_m, T). \tag{3.35}$$

Estimate (3.35) allows to apply the parabolic theory (see Lemma 2.8) to system (1.15)–(1.17) expressed as

(3.36)

$$\begin{aligned} &\chi_t - M\varkappa_2\Delta^3\chi = M\Delta\left[f_{0,\chi}(\chi) - \frac{1}{2}\varkappa_{1,\chi}(\chi)|\nabla\chi|^2 - \varkappa_1(\chi)\Delta\chi\right] \\ &\qquad\qquad \equiv F(\chi) && \text{in } \Omega^T, \\ &\chi(0) = \chi_0 && \text{in } \Omega, \\ &\boldsymbol{n}\cdot\nabla\chi = 0, \quad \boldsymbol{n}\cdot\nabla\Delta\chi = 0 && \text{on } S^T, \\ &\boldsymbol{n}\cdot\nabla\Delta^2\chi = \frac{1}{2\varkappa_2}\varkappa_{1,\chi}(\chi)\boldsymbol{n}\cdot\nabla(|\nabla\chi|^2) \equiv G(\chi) && \text{on } S^T. \end{aligned}$$

Lemma 3.11. *(Estimate in $W_2^{6,1}(\Omega^T)$) Let the assumptions of Lemma 3.9 be satisfied, and $\chi_0 \in H^3(\Omega)$. Then a solution to problem (3.36) satisfies the estimate*

$$\|\chi\|_{W_2^{6,1}(\Omega^T)} \le \varphi(c_1, c_2, c_3, \chi_m, T) \tag{3.37}$$

where $c_3 = \|\chi_0\|_{H^3(\Omega)}$.

Proof. By virtue of Lemma 2.8, if

$$F(\chi) \in L_2(\Omega^T), \quad G(\chi) \in W_2^{1/2,1/12}(S^T), \quad \chi_0 \in H^3(\Omega),$$

and the compatibility conditions hold on S

$$\boldsymbol{n} \cdot \nabla\chi_0 = 0, \quad \boldsymbol{n} \cdot \nabla\Delta\chi_0 = 0, \quad \boldsymbol{n} \cdot \nabla\Delta^2\chi_0 = G(\chi_0)$$

(with the last two in the weak sense) then problem (3.36) has the unique solution $\chi \in W_2^{6,1}(\Omega^T)$ satisfying the estimate

$$\|\chi\|_{W_2^{6,1}(\Omega^T)} \le c(\|F(\chi)\|_{L_2(\Omega^T)} + \|G(\chi)\|_{W_2^{1/2,1/12}(S^T)} + \|\chi_0\|_{H^3(\Omega)}). \tag{3.38}$$

By (3.35),

$$A \equiv \|\chi\|_{W_2^{5,1}(\Omega^T)} \le \varphi(c_1, c_2, \chi_m, T).$$

We estimate the first two norms on the right-hand side of (3.38) by A. Let us note that on account of the imbeddings (see Sect. 2.2) the following holds:

$$\begin{aligned} &\|\chi\|_{L_\infty(\Omega^T)} \le cA, \\ &\|\nabla\chi\|_{L_q(\Omega^T)} \le cA \quad \text{for any } \ q < \infty, \\ &\|\nabla^2\chi\|_{L_q(\Omega^T)} \le cA \quad \text{for } \ 2 \le q \le 8, \\ &\|\nabla^3\chi\|_{L_q(\Omega^T)} \le cA \quad \text{for } \ 2 \le q \le 4, \\ &\|\nabla^4\chi\|_{L_q(\Omega^T)} \le cA \quad \text{for } \ 2 \le q \le 8/3. \end{aligned} \tag{3.39}$$

Moreover,

$$\begin{aligned} &\|\partial_t^{1/6}\chi\|_{L_\infty(\Omega^T)} \le cA, \\ &\|\partial_t^{1/6}\nabla\chi\|_{L_q(\Omega^T)} \le cA \quad \text{for } \ 2 \le q \le 48/5, \\ &\|\partial_t^{1/6}\nabla^2\chi\|_{L_q(\Omega^T)} \le cA \quad \text{for } \ 2 \le q \le 48/11 \end{aligned} \tag{3.40}$$

where the notation of fractional derivatives is used (see Sec. 2.2).

After differentiating,

$$F(\chi) = M\bigg[f_{0,\chi\chi\chi}|\nabla\chi|^2 + f_{0,\chi\chi}\Delta\chi - \frac{1}{2}\varkappa_{1,\chi\chi\chi}|\nabla\chi|^4 - \varkappa_{1,\chi\chi}\nabla(|\nabla\chi|^2)\cdot\nabla\chi - \frac{3}{2}\varkappa_{1,\chi\chi}|\nabla\chi|^2\Delta\chi - \frac{1}{2}\varkappa_{1,\chi}\Delta(|\nabla\chi|^2) - 2\varkappa_{1,\chi}\nabla\Delta\chi\cdot\nabla\chi - \varkappa_{1,\chi}(\Delta\chi)^2 - \varkappa_1\Delta^2\chi\bigg].$$

On account of the assumptions of f_0 and $\varkappa_1$ (see (A2) (i), (ii)), using (3.39) we get

$$\begin{aligned}\|F(\chi)\|_{L_2(\Omega^T)} &\le c\| \,|\chi|^3|\nabla\chi|^2 + \chi^4|\Delta\chi| + |\nabla^2\chi|\,|\nabla\chi|^2 \\ &\quad + |\chi|\,|\nabla^2\chi|^2 + |\chi|\,|\nabla\Delta\chi|\,|\nabla\chi| \\ &\quad + (1+|\chi|)|\Delta^2\chi|\,\|_{L_2(\Omega^T)} \\ &\le cA^4(\|\nabla\chi\|^2_{L_4(\Omega^T)} + \|\Delta\chi\|_{L_2(\Omega)} + \|\nabla^2\chi\|_{L_4(\Omega^T)}\|\nabla\chi\|^2_{L_8(\Omega^T)} \\ &\quad + \|\nabla^2\chi\|^2_{L_4(\Omega^T)} + \|\nabla\Delta\chi\|_{L_4(\Omega^T)}\|\nabla\chi\|_{L_4(\Omega^T)} + \|\Delta^2\chi\|_{L_2(\Omega^T)}) \\ &\le \varphi(A).\end{aligned} \tag{3.41}$$

To estimate the boundary term on the right-hand side of (3.38) we introduce a smooth extension of the outward unit normal $\boldsymbol{n}$ to S onto a neighbourhood of S. Then by the inverse boundary trace theorem (see Theorem 2.3)

$$\|G(\chi)\|_{W_2^{1/2,1/12}(S^T)} \le c\|G(\chi)\|_{W_2^{1,1/6}(\Omega^T)} \tag{3.42}$$

where $W_2^{1/2,1/12}(S^T)$ is the space of traces of functions from $W_2^{1,1/6}(\Omega^T)$. We have

$$\begin{aligned}\|G(\chi)\|_{W_2^{1,1/6}(\Omega^T)} &= \frac{1}{2\varkappa_2}\|\varkappa_{1,\chi}\boldsymbol{n}\cdot\nabla(|\nabla\chi|^2)\|_{W_2^{1,1/6}(\Omega^T)} \\ &= \frac{1}{2\varkappa_2}\|\varkappa_{1,\chi}\boldsymbol{n}\cdot\nabla(|\nabla\chi|^2)\|_{L_2(0,T;W_2^1(\Omega))} \\ &\quad + \frac{1}{2\varkappa_2}\|\varkappa_{1,\chi}\boldsymbol{n}\cdot\nabla(|\nabla\chi|^2)\|_{W_2^{1/6}(0,T;L_2(\Omega))} \equiv I_1 + I_2.\end{aligned} \tag{3.43}$$

On account of assumption on $\varkappa_1$, smoothness of S and the bounds (3.39) the term I_1 is estimated as follows:

$$I_1 \le cA(\|\,|\nabla\chi|\,|\nabla^2\chi|\,\|_{L_2(\Omega^T)} + \|\,|\nabla\chi|^2|\nabla^2\chi|\,\|_{L_2(\Omega^T)} + \|\,|\nabla^2\chi|^2\|_{L_2(\Omega^T)} + \|\,|\nabla\chi|\,|\nabla^3\chi|\,\|_{L_2(\Omega^T)}) \equiv \sum_{k=1}^4 I_1^k,$$

where

$$
\begin{aligned}
I_1^1 &\le cA\|\nabla\chi\|_{L_4(\Omega^T)}\|\nabla^2\chi\|_{L_4(\Omega^T)} \le \varphi(A),\\
I_1^2 &\le cA\|\nabla\chi\|_{L_8(\Omega^T)}^2\|\nabla^2\chi\|_{L_4(\Omega^T)} \le \varphi(A),\\
I_1^3 &\le cA\|\nabla^2\chi\|_{L_4(\Omega^T)}^2 \le \varphi(A),\\
I_1^4 &\le cA\|\nabla\chi\|_{L_4(\Omega^T)}\|\nabla^3\chi\|_{L_4(\Omega^T)} \le \varphi(A).
\end{aligned}
$$

Hence,

$$
I_1 \le \varphi(A). \tag{3.44}
$$

Finally, using (3.39) and (3.40),

$$
\begin{aligned}
I_2 \le c\|\,|\partial_t^{1/6}\chi|\,|\nabla\chi|\,|\nabla^2\chi|\,\|_{L_2(\Omega^T)} + cA\|\,|\partial_t^{1/6}\nabla\chi|\,|\nabla^2\chi|\,\|_{L_2(\Omega^T)}\\
+ cA\|\,|\nabla\chi|\,|\partial_t^{1/6}\nabla^2\chi|\,\|_{L_2(\Omega^T)} \equiv \sum_{k=1}^3 I_2^k,
\end{aligned}
$$

where

$$
\begin{aligned}
I_2^1 &\le \|\partial_t^{1/6}\chi\|_{L_\infty(\Omega^T)}\|\nabla\chi\|_{L_4(\Omega^T)}\|\nabla^2\chi\|_{L_4(\Omega^T)} \le \varphi(A),\\
I_2^2 &\le cA\|\partial_t^{1/6}\nabla\chi\|_{L_4(\Omega^T)}\|\nabla^2\chi\|_{L_4(\Omega^T)} \le \varphi(A),\\
I_2^3 &\le cA\|\nabla\chi\|_{L_4(\Omega^T)}\|\partial_t^{1/6}\nabla^2\chi\|_{L_4(\Omega^T)} \le \varphi(A).
\end{aligned}
$$

Hence,

$$
I_2 \le \varphi(A). \tag{3.45}
$$

After summarizing (3.42)–(3.45), we conclude that

$$
\|G(\chi)\|_{W_2^{1/2,1/12}(S^T)} \le \varphi(A). \tag{3.46}
$$

Consequently, by (3.38) it follows that

$$
\|\chi\|_{W_2^{6,1}(\Omega^T)} \le \varphi(A) + \|\chi_0\|_{H^3(\Omega)} \le \varphi(c_1, c_2, c_3, \chi_m, T)
$$

which proves the assertion. □

4. Proof of Theorem 1.1 (Existence)

We apply the Leray-schauder fixed point theorem in the following formulation (see e.g. [5]):

Theorem 4.1. *(Leray-Schauder) Let X be a Banach space. Assume that $\Phi: [0,1]\times X \to X$ is a map with the following properties:*

(i) for any fixed $\tau\in[0,1]$ the map is completely continuous;

(ii) For every bounded subset $\mathcal{B}$ of X, the family of maps $\Phi(\cdot,\xi): [0,1]\to X$, $\xi\in\mathcal{B}$, is uniformly equicontinuous;

(iii) $\Phi(0,\cdot)$ has precisely one fixed point in X;

(iv) There is a bounded subset $\mathcal{B}$ of X such that any fixed point in X of $\Phi(\tau,\cdot)$ is contained in $\mathcal{B}$ for every $\tau\in[0,1]$.

Then $\Phi(1,\cdot)$ has at least one fixed point.

We choose as the solution space the Sobolev-Slobodecki space

$$X = W_2^{6s,s}(\Omega^T),\quad s\in(0,1),\quad \Omega\subset\mathbb{R}^3, \tag{4.1}$$

with the finite norm

$$\begin{aligned}\|\chi\|_{W_2^{6s,s}(\Omega^T)} = \Bigg(&\sum_{|\alpha|+6a\le[6s]}\int_{\Omega^T} |D_x^\alpha \partial_t^a \chi|^2 dxdt\\ &+\sum_{|\alpha|=[6s]}\int_0^T\int_\Omega\int_\Omega \frac{|D_x^\alpha\chi(x,t)-D_{x'}^\alpha\chi(x',t)|^2}{|x-x'|^{3+2(6s-[6s])}}dxdx'dt\\ &+\int_\Omega\int_0^T\int_0^T \frac{|\partial_t^{[s]}\chi(x,t)-\partial_{t'}^{[s]}\chi(x,t')|^2}{|t-t'|^{1+2(s-[s])}}dtdt'dx\Bigg)^{1/2}.\end{aligned}$$

The parameter $s\in(0,1)$ will be specified below in Lemma 4.1.

The solution map

(4.2)
$$\Phi(\tau,\cdot):\ W_2^{6s,s}(\Omega^T)\ni\tilde\chi\mapsto\chi\in W_2^{6,1}(\Omega^T)\subset W_2^{6s,s}(\Omega^T),\quad \tau\in[0,1],$$

is defined by means of the following initial-boundary value problem

$$\begin{aligned}&\chi_t - M\varkappa_2\Delta^3\chi = \tau M\Delta\Big[f_{0,\tilde\chi}(\tilde\chi)\\ &\qquad -\frac12\varkappa_{1,\tilde\chi}(\tilde\chi)|\nabla\tilde\chi|^2-\varkappa_1(\tilde\chi)\Delta\tilde\chi\Big]\equiv\tau F(\tilde\chi) && \text{in } \Omega^T,\\ &\chi(0)=\tau\chi_0 && \text{in } \Omega,\\ &\boldsymbol{n}\cdot\nabla\chi=0,\quad \boldsymbol{n}\cdot\nabla\Delta\chi=0 && \text{on } S^T,\\ &\boldsymbol{n}\cdot\nabla\Delta^2\chi=\tau\frac{1}{2\varkappa_2}\varkappa_{1,\tilde\chi}(\tilde\chi)\boldsymbol{n}\cdot\nabla(|\nabla\tilde\chi|^2)\equiv\tau G(\tilde\chi) && \text{on } S^T\end{aligned}\tag{4.3}$$

where $\tau\in[0,1]$.

Clearly, χ defined as a fixed point of $\Phi(1,\cdot)$ is a solution to problem (1.15)–(1.17).

We prove first that the map $\Phi(\tau,\cdot)$ is well-defined.

Lemma 4.1. *Let the solution map $\Phi(\tau,\cdot)$ be defined by (4.2), (4.3) and the solution space be $W_2^{6s,s}(\Omega^T)$ with $s\in(11/12,1)$. Then for any $\tilde\chi\in W_2^{6s,s}(\Omega^T)$ and $\chi_0\in H^3(\Omega)$ satisfying the compatibility conditions*

$$\boldsymbol{n}\cdot\nabla\chi_0=0,\quad \boldsymbol{n}\cdot\nabla\Delta\chi_0=0,\quad \boldsymbol{n}\cdot\nabla\Delta^2\chi_0=G(\tilde\chi(0))\quad on\ \ S, \tag{4.4}$$

with the last two in the weak sense, there exists a unique solution $\chi\in W_2^{6,1}(\Omega^T)$ to problem (4.3) such that

$$\|\chi\|_{W_2^{6,1}(\Omega^T)}\le\varphi(\|\tilde\chi\|_{W_2^{6s,s}(\Omega^T)},\|\chi_0\|_{H^3(\Omega)}). \tag{4.5}$$

Proof. Let $\tilde\chi\in W_2^{6s,s}(\Omega^T)$ where $s\in(11/12,1)$ and let

$$\tilde A\equiv\|\tilde\chi\|_{W_2^{6s,s}(\Omega^T)}.$$

We proceed in the same manner as in Lemma 3.11. By virtue of Lemma 2.8 problem (4.3) has the unique solution $\chi\in W_2^{6,1}(\Omega^T)$ provided that $\tau F(\tilde\chi)\in L_2(\Omega^T)$, $\tau G(\tilde\chi)\in W_2^{1/2,1/12}(S^T)$, $\tau\chi_0\in H^3(\Omega)$ and compatibility conditions (4.4) hold. Then

$$\|\chi\|_{W_2^{6,1}(\Omega^T)}\le c(\tau\|F(\tilde\chi)\|_{L_2(\Omega^T)}+\tau\|G(\tilde\chi)\|_{W_2^{1/2,1/12}(S^T)}+\tau\|\chi_0\|_{H^3(\Omega)}). \tag{4.6}$$

We estimate the first two norms on the right-hand side of (4.6) by the norm $\tilde A$. If the parameter $s\in(11/12,1)$ then on account of the imbeddings (see Sect. 2.2) we have

$$\begin{aligned}&\|\tilde\chi\|_{L_\infty(\Omega^T)}\le c\tilde A,\qquad \|\nabla\tilde\chi\|_{L_\infty(\Omega^T)}\le c\tilde A,\\ &\|\nabla^2\tilde\chi\|_{L_q(\Omega^T)}\le c\tilde A\quad\text{for}\ \ 2\le q\le 9,\\ &\|\nabla^3\tilde\chi\|_{L_q(\Omega^T)}\le c\tilde A\quad\text{for}\ \ 2\le q\le 9/2,\\ &\|\nabla^4\tilde\chi\|_{L_q(\Omega^T)}\le c\tilde A\quad\text{for}\ \ 2\le q\le 3.\end{aligned} \tag{4.7}$$

Moreover,

$$\begin{aligned}&\|\partial_t^{1/6}\tilde\chi\|_{L_\infty(\Omega^T)}\le c\tilde A,\\ &\|\partial_t^{1/6}\nabla\tilde\chi\|_{L_q(\Omega^T)}\le c\tilde A\quad\text{for}\ \ 2\le q\le 9,\\ &\|\partial_t^{1/6}\nabla^2\tilde\chi\|_{L_q(\Omega^T)}\le c\tilde A\quad\text{for}\ \ 2\le q\le 9/2.\end{aligned} \tag{4.8}$$

Repeating estimates (3.41)–(3.46) from Lemma 3.11 and using (4.7), (4.8) we conclude that

$$\begin{aligned} &\|F(\tilde\chi)\|_{L_2(\Omega^T)} \le \varphi(\tilde A), \\ &\|G(\tilde\chi)\|_{W_2^{1/2,1/12}(S^T)} \le c\|G(\tilde\chi)\|_{W_2^{1,1/6}(\Omega^T)} \le \varphi(\tilde A). \end{aligned} \tag{4.9}$$

Then, by (4.6),

$$\|\chi\|_{W_2^{6,1}(\Omega^T)} \le \varphi(\tilde A) + c\|\chi_0\|_{H^3(\Omega)} \le \varphi(\|\tilde\chi\|_{W_2^{6s,s}(\Omega^T)}, \|\chi_0\|_{H^3(\Omega)}) \tag{4.10}$$

for any $\tau \in [0,1]$. This proves the assertion. □

We check that the map $\Phi(\tau,\cdot)$ defined by (4.2), (4.3) satisfies the assumptions of Theorem 4.1.

(i) *Complete continuity*

From (4.2) it follows immediately

Corollary 4.2. *Since for $s<1$, the imbedding $W_2^{6,1}(\Omega^T) \subset W_2^{6s,s}(\Omega^T)$ is compact, the map $\Phi(\tau,\cdot)$ takes bounded subsets in $W_2^{6s,s}(\Omega^T)$ into precompact subsets in $W_2^{6s,s}(\Omega^T)$.*

Thus, to show the complete continuity of the map $\Phi(\tau,\cdot)$ it remains to prove its continuity.

For a fixed $\tau \in [0,1]$, let $\chi_1 = \Phi(\tau,\tilde\chi_1)$ and $\chi_2 = \Phi(\tau,\tilde\chi_2)$ be two solutions of problem (4.3) corresponding to $\tilde\chi_1$ and $\tilde\chi_2$ from a bounded subset of $W_2^{6s,s}(\Omega^T)$, such that

$$\|\tilde\chi_k\|_{W_2^{6s,s}(\Omega^T)} \le \tilde B, \quad k=1,2. \tag{4.11}$$

Denoting the differences

$$K = \chi_1 - \chi_2, \quad \tilde K = \tilde\chi_1 - \tilde\chi_2, \tag{4.12}$$

and subtracting by sides the corresponding equations for χ_1 and χ_2 we can see that K satisfies the following problem:

$$\begin{aligned} &K_t - M\varkappa_2\Delta^3 K = \tau M\Delta[(f_{0,\tilde\chi_1}(\tilde\chi_1) - f_{0,\tilde\chi_2}(\tilde\chi_2)) \\ &\quad -\frac{1}{2}(\varkappa_{1,\tilde\chi_1}(\tilde\chi_1)|\nabla\tilde\chi_1|^2 - \varkappa_{1,\tilde\chi_2}(\tilde\chi_2)|\nabla\tilde\chi_2|^2) \\ &\quad -(\varkappa_1(\tilde\chi_1)\Delta\tilde\chi_1 - \varkappa_1(\tilde\chi_2)\Delta\tilde\chi_2)] \equiv \tau\tilde F(\tilde\chi_1,\tilde\chi_2) \quad &&\text{in } \Omega^T, \\ &K(0)=0 &&\text{in } \Omega, \\ &\boldsymbol{n}\cdot\nabla K = 0, \quad \boldsymbol{n}\cdot\nabla\Delta K = 0 &&\text{on } S^T, \\ &\boldsymbol{n}\cdot\nabla\Delta^2 K = \tau\frac{1}{2\varkappa_2}[\varkappa_{1,\tilde\chi_1}(\tilde\chi_1)\boldsymbol{n}\cdot\nabla(|\nabla\tilde\chi_1|^2) \\ &\quad -\varkappa_{1,\tilde\chi_2}(\tilde\chi_2)\boldsymbol{n}\cdot\nabla(|\nabla\tilde\chi_2|^2)] \equiv \tau\tilde G(\tilde\chi_1,\tilde\chi_2) &&\text{on } S^T. \end{aligned} \tag{4.13}$$

Analysis similar to that in the proof of Lemma 4.1 yields

Lemma 4.3. *(Continuity of* Φ*) For any* $\tilde\chi_1,\tilde\chi_2 \in W_2^{6s,s}(\Omega^T)$, $s\in(11/12,1)$, *satisfying (4.11), and for any* $\tau\in[0,1]$, *the unique solution* $K\in W_2^{6,1}(\Omega^T)$ *to problem (4.13) obeys the estimate*

$$\|K\|_{W_2^{6,1}(\Omega^T)} \le \tau\varphi(\tilde B)\|\tilde K\|_{W_2^{6s,s}(\Omega^T)}. \tag{4.14}$$

Proof. First, let us note that by virtue of the imbeddings the following estimates hold for $\tilde\chi_k$, $k=1,2$ (compare (4.7), (4.8))

$$\begin{aligned}
&\|\tilde\chi_k\|_{L_\infty(\Omega^T)} + \|\nabla\tilde\chi_k\|_{L_\infty(\Omega^T)} + \|\partial_t^{1/6}\tilde\chi_k\|_{L_\infty(\Omega^T)}\\
&\quad \le c\|\tilde\chi_k\|_{W_2^{6s,s}(\Omega^T)} \le c\tilde B,\\
&\|\nabla^2\tilde\chi_k\|_{L_q(\Omega^T)} + \|\partial_t^{1/6}\nabla\tilde\chi_k\|_{L_q(\Omega^T)}\\
&\quad \le c\|\tilde\chi_k\|_{W_2^{6s,s}(\Omega^T)} \le c\tilde B \qquad \text{for } 2\le q\le 9,\\
&\|\nabla^3\tilde\chi_k\|_{L_q(\Omega^T)} + \|\partial_t^{1/6}\nabla^2\tilde\chi_k\|_{L_q(\Omega^T)}\\
&\quad \le c\|\tilde\chi_k\|_{W_2^{6s,s}(\Omega^T)} \le c\tilde B \qquad \text{for } 2\le q\le 9/2,\\
&\|\nabla^4\tilde\chi_k\|_{L_q(\Omega^T)} \le c\|\tilde\chi_k\|_{W_2^{6s,s}(\Omega^T)} \le c\tilde B \qquad \text{for } 2\le q\le 3.
\end{aligned} \tag{4.15}$$

From Lemma 2.8 it follows that if $\tau\tilde F(\tilde\chi_1,\tilde\chi_2)\in L_2(\Omega^T)$, $\tau\tilde G(\tilde\chi_1,\tilde\chi_2)\in W_2^{1/2,1/12}(S^T)$ and the compatibility condition

$$\tilde G(\tilde\chi_1(0),\tilde\chi_2(0)) = 0 \tag{4.16}$$

holds on S, then there exists a unique solution to problem (4.13) such that

$$\|K\|_{W_2^{6,1}(\Omega^T)} \le c(\tau\|\tilde F(\tilde\chi_1,\tilde\chi_2)\|_{L_2(\Omega^T)} + \tau\|\tilde G(\tilde\chi_1,\tilde\chi_2)\|_{W_2^{1/2,1/12}(S^T)}). \tag{4.17}$$

Let us note that condition (4.16) holds true since $\tilde\chi_1$ and $\tilde\chi_2$ satisfy (4.4).

We proceed to estimate the norms on the right-hand side of (4.17). Using the assumptions on f_0 and $\varkappa_1$ we write $\tilde F(\tilde\chi_1,\tilde\chi_2)$ in the form

$$\tilde F(\tilde\chi_1,\tilde\chi_2) = \sum_{i=1}^{3}\tilde F_i(\tilde\chi_1,\tilde\chi_2) \tag{4.18}$$

where

$$\begin{aligned}
\tilde F_1(\tilde\chi_1,\tilde\chi_2) &\equiv M\Delta[f_{0,\tilde\chi_1}(\tilde\chi_1) - f_{0,\tilde\chi_2}(\tilde\chi_2)]\\
&= M[f_{0,\xi\xi\xi\xi}(\xi)|\nabla\tilde\chi_1|^2\tilde K + f_{0,\tilde\chi_2\tilde\chi_2\tilde\chi_2}(\tilde\chi_2)(\nabla\tilde\chi_1+\nabla\tilde\chi_2)\cdot\nabla\tilde K\\
&\quad + f_{0,\eta\eta\eta}(\eta)\Delta\tilde\chi_1\tilde K + f_{0,\tilde\chi_2\tilde\chi_2}(\tilde\chi_2)\Delta\tilde K]
\end{aligned}$$

with some $\xi, \eta \in (\tilde{\chi}_1, \tilde{\chi}_2)$,

$$\begin{aligned}
\tilde{F}_2(\tilde{\chi}_1, \tilde{\chi}_2) &\equiv -\frac{1}{2} M\Delta[\varkappa_{1,\tilde{\chi}_1}(\tilde{\chi}_1)|\nabla\tilde{\chi}_1|^2 - \varkappa_{1,\tilde{\chi}_2}(\tilde{\chi}_2)|\nabla\tilde{\chi}_2|^2] \\
&= -\frac{1}{2} M g_2[|\nabla\tilde{\chi}_1|^2\Delta\tilde{K} + 2\nabla(|\nabla\tilde{\chi}_1|^2)\cdot\nabla\tilde{K} + \Delta(|\nabla\tilde{\chi}_1|^2)\tilde{K} \\
&\quad + \Delta\tilde{\chi}_2(\nabla\tilde{\chi}_1 + \nabla\tilde{\chi}_2)\cdot\nabla\tilde{K} + 2\nabla\tilde{\chi}_2\cdot((\nabla^2\tilde{\chi}_1 + \nabla^2\tilde{\chi}_2)\nabla\tilde{K}) \\
&\quad + 2\nabla\tilde{\chi}_2\cdot(\nabla^2\tilde{K}(\nabla\tilde{\chi}_1 + \nabla\tilde{\chi}_2)) + \tilde{\chi}_2\nabla\Delta(\tilde{\chi}_1 + \tilde{\chi}_2)\cdot\nabla\tilde{K} \\
&\quad + 2\tilde{\chi}_2(\nabla^2\tilde{\chi}_1 + \nabla^2\tilde{\chi}_2)\cdot\nabla^2\tilde{K} + \tilde{\chi}_2(\nabla\tilde{\chi}_1 + \nabla\tilde{\chi}_2)\cdot\nabla\Delta\tilde{K}],
\end{aligned}$$

$$\begin{aligned}
\tilde{F}_3(\tilde{\chi}_1, \tilde{\chi}_2) &\equiv -M\Delta[\varkappa_1(\tilde{\chi}_1)\Delta\tilde{\chi}_1 - \varkappa_1(\tilde{\chi}_2)\Delta\tilde{\chi}_2] \\
&= -Mg_2[(\Delta\tilde{\chi}_1 + \Delta\tilde{\chi}_2)\Delta\tilde{\chi}_1\tilde{K} + 2(\nabla\tilde{\chi}_1 + \nabla\tilde{\chi}_2)\cdot\nabla\Delta\tilde{\chi}_1\tilde{K} \\
&\quad + 2\Delta\tilde{\chi}_1(\nabla\tilde{\chi}_1 + \nabla\tilde{\chi}_2)\cdot\nabla\tilde{K} + (\tilde{\chi}_1 + \tilde{\chi}_2)\Delta^2\tilde{\chi}_1\tilde{K} \\
&\quad + 2(\tilde{\chi}_1 + \tilde{\chi}_2)\nabla\Delta\tilde{\chi}_1\cdot\nabla\tilde{K} + (\tilde{\chi}_1 + \tilde{\chi}_2)\Delta\tilde{\chi}_1\Delta\tilde{K} \\
&\quad + \Delta\tilde{\chi}_2\Delta\tilde{K} + 2\nabla\tilde{\chi}_2\cdot\nabla\Delta\tilde{K} + \tilde{\chi}_2\Delta^2\tilde{K}] - Mg_0\Delta^2\tilde{K}.
\end{aligned}$$

On account of assumption on $\varkappa_1$ the boundary term takes the form

$$\begin{aligned}
\tilde{G}(\tilde{\chi}_1, \tilde{\chi}_2) &\equiv \frac{1}{2\varkappa_2}[(\varkappa_{1,\tilde{\chi}_1}(\tilde{\chi}_1) - \varkappa_{1,\tilde{\chi}_2}(\tilde{\chi}_2))\boldsymbol{n}\cdot\nabla(|\nabla\tilde{\chi}_1|^2) \\
&\quad + \varkappa_{1,\tilde{\chi}_2}(\tilde{\chi}_2)\boldsymbol{n}\cdot\nabla(|\nabla\tilde{\chi}_1|^2 - |\nabla\tilde{\chi}_2|^2)] \\
&= \frac{g_2}{2\varkappa_2}[\boldsymbol{n}\cdot\nabla(|\nabla\tilde{\chi}_1|^2)\tilde{K} + \tilde{\chi}_2\boldsymbol{n}\cdot((\nabla^2\tilde{\chi}_1 + \nabla^2\tilde{\chi}_2)\nabla\tilde{K}) \\
&\quad + \tilde{\chi}_2\boldsymbol{n}\cdot(\nabla^2\tilde{K}(\nabla\tilde{\chi}_1 + \nabla\tilde{\chi}_2))].
\end{aligned} \tag{4.19}$$

Using (4.15) and the analogous imbeddings for $\tilde{K}$ we estimate the particular terms $\tilde{F}_i$ to obtain

$$\sum_{i=1}^{3}\|\tilde{F}_i(\tilde{\chi}_1, \tilde{\chi}_2)\|_{L_2(\Omega^T)} \le \varphi(\tilde{B})\|\tilde{K}\|_{W_2^{6s,s}(\Omega^T)}. \tag{4.20}$$

The boundary term (4.17) is estimated as follows:

$$\begin{aligned}
&\|\tilde{G}(\tilde{\chi}_1, \tilde{\chi}_2)\|_{W_2^{1/2,1/12}(S^T)} \le c\|\tilde{G}(\tilde{\chi}_1, \tilde{\chi}_2)\|_{W_2^{1,1/6}(\Omega^T)} \\
&= c(\|\tilde{G}(\tilde{\chi}_1, \tilde{\chi}_2)\|_{L_2(0,T;W_2^1(\Omega))} + \|\tilde{G}(\tilde{\chi}_1, \tilde{\chi}_2)\|_{L_2(\Omega;W^{1/6}(0,T))}) \\
&\equiv I_1 + I_2
\end{aligned}$$

where

$$\begin{aligned}
I_1 &\le c[\|\boldsymbol{n}\cdot\nabla(|\nabla\tilde{\chi}_1|^2)\tilde{K}\|_{L_2(0,T;W_2^1(\Omega))} \\
&\quad + \|\tilde{\chi}_2\boldsymbol{n}\cdot((\nabla^2\tilde{\chi}_1 + \nabla^2\tilde{\chi}_2)\nabla\tilde{K})\|_{L_2(0,T;W_2^1(\Omega))} \\
&\quad + \|\tilde{\chi}_2\boldsymbol{n}\cdot(\nabla^2\tilde{K}(\nabla\tilde{\chi}_1 + \nabla\tilde{\chi}_2))\|_{L_2(0,T;W_2^1(\Omega))}] \\
&\equiv \sum_{j=1}^{3} I_1^j,
\end{aligned}$$

$$
\begin{aligned}
I_2 &\le c[\|\boldsymbol{n}\cdot\nabla(|\nabla\tilde{\chi}_1|^2)\tilde{K}\|_{L_2(\Omega;W^{1/6}(0,T))} \\
&\quad+\|\tilde{\chi}_2\boldsymbol{n}\cdot(\nabla^2\tilde{\chi}_1+\nabla^2\tilde{\chi}_2)\nabla\tilde{K})\|_{L_2(\Omega;W^{1/6}(0,T))} \\
&\quad+\|\tilde{\chi}_2\boldsymbol{n}\cdot(\nabla^2\tilde{K}(\nabla\tilde{\chi}_1+\nabla\tilde{\chi}_2))\|_{L_2(\Omega;W^{1/6}(0,T))}] \\
&\equiv\sum_{j=1}^{3}I_2^j.
\end{aligned}
$$

Again, on account of (4.15) and the analogous imbeddings for $\tilde{K}$ after the straightforward calculations we have

$$
\|\tilde{G}(\tilde{\chi}_1,\tilde{\chi}_2)\|_{W_2^{1/2,1/12}(S^T)}\le\sum_{i=1}^{2}\sum_{j=1}^{3}I_i^j\le\varphi(\tilde{B})\|\tilde{K}\|_{W_2^{6s,s}(\Omega^T)}. \tag{4.21}
$$

Substituting (4.20) and (4.21) into (4.17) implies the desired estimate (4.14). □

Concluding, Corollary 4.2 and Lemma 4.3 prove that assumption (i) of Theorem 4.1 is satisfied.

(ii) *Uniform equicontinuity*

Let us consider the family of maps $\Phi(\cdot,\xi):\ [0,1]\to W_2^{6,1}(\Omega^T)$ with ξ in a bounded subset of $W_2^{6s,s}(\Omega^T)$, $s\in(11/12,1)$:

$$
\xi\in\mathcal{B}\equiv\{\xi\in W_2^{6s,s}(\Omega^T):\ \|\xi\|_{W_2^{6s,s}(\Omega^T)}\le c_{\mathcal{B}}\}. \tag{4.22}
$$

Let $\chi_1=\Phi(\tau_1,\xi)$, $\chi_2=\Phi(\tau_2,\xi)$, $\xi\in\mathcal{B}$, $\tau_1,\tau_2\in[0,1]$ be two families of solutions to problem (4.3) corresponding to parameters τ_1 and τ_2, respectively. For a fixed $\xi\in\mathcal{B}$ let us denote the difference

$$
H=\chi_1-\chi_2
$$

which satisfies the following problem

$$
\begin{aligned}
&H_t-M\varkappa_2\Delta^3H=(\tau_1-\tau_2)F(\xi) && \text{in } \Omega^T,\\
&H(0)=(\tau_1-\tau_2)\chi_0 && \text{on } \Omega,\\
&\boldsymbol{n}\cdot\nabla H=0,\quad \boldsymbol{n}\cdot\nabla\Delta H=0 && \text{on } S^T,\\
&\boldsymbol{n}\cdot\nabla\Delta^2H=(\tau_1-\tau_2)G(\xi) && \text{on } S^T.
\end{aligned} \tag{4.23}
$$

Lemma 4.4. *(Uniform equicontinuity) For any $\xi\in\mathcal{B}$ and any $\tau_1,\tau_2\in[0,1]$ the solution $H\in W_2^{6,1}(\Omega^T)$ to problem (4.23) satisfies the estimate*

$$
\|H\|_{W_2^{6,1}(\Omega^T)}\le\varphi(c_{\mathcal{B}},\|\chi_0\|_{H^3(\Omega)})|\tau_1-\tau_2|. \tag{4.24}
$$

Proof. We use Lemma 2.8 which implies that the unique solution $H \in W_2^{6,1}(\Omega^T)$ to problem (4.23) satisfies

$$
(4.25) \qquad \begin{aligned} \|H\|_{W_2^{6,1}(\Omega^T)} &\le c(|\tau_1 - \tau_2| \, \|F(\xi)\|_{L_2(\Omega^T)} \\ &+ |\tau_1 - \tau_2| \, \|G(\xi)\|_{W_2^{1/2,1/12}(S^T)} + |\tau_1 - \tau_2| \, \|\chi_0\|_{H^3(\Omega)}). \end{aligned}
$$

Recalling estimates (4.9) we have

$$
(4.26) \qquad \begin{aligned} &\|F(\xi)\|_{L_2(\Omega^T)} \le \varphi(\|\xi\|_{W_2^{6s,s}(\Omega^T)}) \le \varphi(c_{\mathcal{B}}), \\ &\|G(\xi)\|_{W_2^{1/2,1/12}(S^T)} \le \varphi(\|\xi\|_{W_2^{6s,s}(\Omega^T)} \le \varphi(c_{\mathcal{B}}) \end{aligned}
$$

for any $\xi \in \mathcal{B}$. Thus, (4.25) and (4.26) imply (4.24). □

Lemma 4.4 proves the uniform equicontinuity of the family of maps $\Phi(\cdot,\xi): [0,1] \to W_2^{6s,s}(\Omega^T)$, $\xi \in \mathcal{B}$.

(iii) *Uniqueness for $\tau = 0$*

By Lemma 2.8 for $\tau = 0$ problem (4.3) has the unique solution $\chi \equiv 0$.

(iv) *A priori bound*

It follows from Lemma 3.11 that there exists a bounded subset $\mathcal{B}$ of $W_2^{6s,s}(\Omega^T)$, given by

$$
\mathcal{B} \equiv \{\chi \in W_2^{6,1}(\Omega^T): \ \|\chi\|_{W_2^{6,1}(\Omega^T)} \le c_4 = \varphi(\|\chi_0\|_{H^3(\Omega)}, \|\chi_t(0)\|_{L_2(\Omega)}, T),
$$

such that any fixed point of $\Phi(1,\cdot)$ is contained in $\mathcal{B}$. It is clear that the same property holds for any $\tau \in [0,1]$, so assumption (iv) of Theorem 4.1 is satisfied.

In conclusion, we deduce from Theorem 4.1 the existence of at least one fixed point of the map $\Phi(1,\cdot)$ in the space $W_2^{6s,s}(\Omega^T)$, $s \in (11/12, 1)$. By the regularity properties (4.5) of the solution map it follows that the fixed point belongs to the space $W_2^{6,1}(\Omega^T)$. Clearly, in view of the definition of the map $\Phi(1,\cdot)$ this means that IBVP (1.15)–(1.17) (equivalent to (1.12)–(1.14)) has a solution $\chi \in W_2^{6,1}(\Omega^T)$ satisfying estimate (1.22). Thereby the proof of Theorem 1.1 is completed. □

5. Proof of Theorem 1.2 (Continuous dependence on χ_0)

Let $\chi_1, \chi_2 \in L_\infty(0,T;H^2(\Omega))$ be two solutions to problem (1.12)–(1.14) corresponding to $\chi_{10}, \chi_{20} \in H^2(\Omega)$, respectively. Denoting

$$
K = \chi_1 - \chi_2, \quad K_0 = \chi_{10} - \chi_{20} \quad \text{and} \quad N = \mu_1 - \mu_2
$$

where $\mu_i = \mu(\chi_i)$, we have

$$
(5.1)\qquad
\begin{aligned}
&K_t - \nabla\cdot(M\nabla N) = 0 && \text{in } \Omega^T,\\
&N = f_{0,\chi_1}(\chi_1) - f_{0,\chi_2}(\chi_2) - \frac{1}{2}(\varkappa_{1,\chi_1}(\chi_1)|\nabla\chi_1|^2\\
&\quad - \varkappa_{1,\chi_2}(\chi_2)|\nabla\chi_2|^2) - (\varkappa_1(\chi_1)\Delta\chi_1 - \varkappa_1(\chi_2)\Delta\chi_2)\\
&\quad + \varkappa_2\Delta^2 K && \text{in } \Omega^T,\\
&K(0) = K_0 && \text{in } \Omega,\\
&\boldsymbol{n}\cdot\nabla K = 0,\quad \boldsymbol{n}\cdot\nabla\Delta K = 0 && \text{on } S^T,\\
&\boldsymbol{n}\cdot\nabla N = 0 && \text{on } S^T.
\end{aligned}
$$

Multiplying $(5.1)_1$ by K, integrating over Ω and twice by parts we get

$$
(5.2)\qquad \frac{1}{2}\frac{d}{dt}\int_\Omega K^2 dx - M\int_\Omega N\Delta K dx = 0.
$$

After inserting the identity $(5.1)_2$ for N, and taking into account that $\varkappa_1 = g_0 + g_2\chi^2$, (5.2) takes the form

$$
(5.3)\qquad
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega K^2 dx + M\varkappa_2\int_\Omega |\nabla\Delta K|^2 dx\\
&= M\int_\Omega (f_{0,\chi_1}(\chi_1) - f_{0,\chi_2}(\chi_2))\Delta K dx\\
&\quad - \frac{1}{2}Mg_2\int_\Omega (K|\nabla\chi_1|^2 + \chi_2\nabla(\chi_1+\chi_2)\cdot\nabla K)\Delta K dx\\
&\quad - M\int_\Omega (g_0|\Delta K|^2 + g_2 K(\chi_1+\chi_2)\Delta\chi_1\Delta K + g_2\chi_2^2|\Delta K|^2)dx.
\end{aligned}
$$

Hence, in view of the fact that $f_0 \in C^2$ and $\chi_1, \chi_2 \in L_\infty(0,T;H^2(\Omega))$ it follows that

$$
(5.4)\qquad
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega K^2 dx + M\varkappa_2\int_\Omega |\nabla\Delta K|^2 dx\\
&\le \varphi(\|\bar\chi\|_{L_\infty(0,T;H^2(\Omega))})\|K\|^2_{H^2(\Omega)}
\end{aligned}
$$

where $\bar\chi = (\chi_1,\chi_2)$, $\|\bar\chi\|_{L_\infty(0,T;H^2(\Omega))} = \sum_{i=1}^2 \|\chi_i\|_{L_\infty(0,T;H^2(\Omega))}$.
Now, applying the interpolation

$$
\|K\|^2_{H^2(\Omega)} \le \varepsilon\|\nabla\Delta K\|^2_{L_2(\Omega)} + c(1/\varepsilon)\|K\|^2_{L_2(\Omega)}, \quad \varepsilon > 0,
$$

we deduce from (5.4) that

$$\frac{d}{dt}\int\limits_{\Omega} K^2 dx + \int\limits_{\Omega} |\nabla\Delta K|^2 dx \leq \varphi(\|\bar{\chi}\|_{L_\infty(0,T;H^2(\Omega))})\|K\|^2_{L_2(\Omega)}.$$

Hence, by the Gronwall inequality, it follows that

$$\|K(t)\|^2_{L_2(\Omega)} \leq \|K_0\|^2_{L_2(\Omega)} e^{\varphi(\|\bar{\chi}\|_{L_\infty(0,T;H^2(\Omega))})t} \quad \text{for} \quad t \in [0,T),$$

which proves the theorem. □

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