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I. Pawłow, W.M. Zajączkowski

Instytut Badań Systemowych
Polska Akademia Nauk

Systems Research Institute
Polish Academy of Sciences

POLSKA AKADEMIA NAUK

Instytut Badań Systemowych

ul. Newelska 6

01-447 Warszawa

tel.: (+48) (22) 3810100

fax: (+48) (22) 3810105

Kierownik Pracowni zgłaszający pracę:
Doc. dr hab. inż. Antoni Żochowski

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Kelvin-Voigt type thermoviscoelastic system. Existence of regular solutions

Dedicated to Professor Mitsuharu Ôtani on the occasion of his sixtieth birthday

IRENA PAWŁOW
Systems Research Institute, Polish Academy of Sciences
Newelska 6, 01-447 Warsaw, Poland
and Institute of Mathematics and Cryptology, Cybernetics Faculty,
Military University of Technology,
Kaliskiego 2, 00-908 Warsaw, Poland
e-mail: pawlow@ibspan.waw.pl
WOJCIECH M. ZAJĄCZKOWSKI
Institute of Mathematics, Polish Academy of Sciences
Śniadeckich 8, 00-956 Warsaw, Poland
and Institute of Mathematics and Cryptology, Cybernetics Faculty,
Military University of Technology,
Kaliskiego 2, 00-908 Warsaw, Poland
E-mail:wz@impan.gov.pl

Abstract. In this paper we address the question of the existence and uniqueness of large-time regular solutions to a 3-D Kelvin-Voigt thermoviscoelastic system at small strains.

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1 Introduction

This article is concerned with the existence and uniqueness of large-time regular solutions to a classical 3-D thermoviscoelastic system at small strains. The system describes materials which have the properties both of elasticity and viscosity. Such materials are usually referred to as Kelvin-Voigt type.

As noted in the recent paper by Roubíček [13] – and according to our best knowledge as well – the existence of large-time solutions to a thermoviscoelastic system with constant specific heat and heat conductivity is, in spite of great effort through many decades, still open in dimensions $n \geq 2$. In dimension $n = 1$ it was established in the pioneering papers by Slemrod [15], Dafermos [5] and Dafermos-Hsiao [6].

The local in time existence and global uniqueness of a weak solution to 3-D thermoviscoelastic system with constant specific heat and heat conductivity has been proved by Bonetti-Bonfanti [3]. Other known results on multidimensional thermoviscoelasticity deal with a modified energy equation. Modifications involve either nonconstant specific heat or nonconstant heat conductivity. Thermoviscoelastic system with temperature-dependent specific heat has been addressed by Blanchard-Guibé [2] where the existence of large-time, weak-renormalized solutions has been proved, and recently in [13] where the existence of a very weak solution has been established.

In a more general setting allowing for large strains 3-D thermoviscoelastic system with temperature-dependent specific heat has been studied by Shibata [14] under small data assumption.

For thermoviscoelastic problems with modified heat conductivity we refer to Eck-Jarušek-Krbec [8] and the references therein.

In the present paper we consider thermoviscoelastic system with specific heat linearly increasing with temperature and with constant heat conductivity. Such setting is a particular case of systems addressed in [2] and [13].

The novelty of the existence result presented in this paper concerns the regularity of a 3-D large-time solution corresponding to appropriately smooth but arbitrary in size initial data. The proof of the existence theorem is based on the successive approximation method. The key regularity estimates are derived with the help of the parabolic theory in anisotropic Sobolev spaces $W^{2,1}_{p,p_0}(\Omega^T)$, $\Omega^T = \Omega \times (0,T)$, $p, p_0 \in (1,\infty)$, with mixed norm with respect to space and time variables. Such framework has been previously applied by the authors [12] to the thermoviscoelastic system arising in shape memory alloys. It turned out to be advantageous in the procedure of deriving regularity estimates.

As known, in deriving a priori estimates for a solution of a system of balance laws it is common to begin with estimates arising from the conservation of a total energy. Such estimates provide L_∞-time regularity for the conserved quantities. To take advantage of such time regularity in deriving subsequent regularity estimates it is desirable to work in Sobolev spaces with mixed norms, $W^{2,1}_{p,p_0}(\Omega^T)$, where space exponent p is determined by the energy structure and time exponent p_0 may be arbitrarily large. This is the idea behind using the framework of Sobolev spaces with a mixed norm to the thermoviscoelastic system under consideration.

The theory of IBVP's in Sobolev spaces with a mixed norm is the subject of recent theoretical studies. We apply general results due to Krylov [9] and Denk-Hieber-Prüss [7].

The system under consideration has the following form:

$$u_{tt} - \nabla \cdot [A_1\varepsilon_t + A_2(\varepsilon - \theta\alpha)] = b, \tag{1.1}$$

$$c_v\theta\theta_t - k\Delta\theta = -\theta(A_2\alpha)\cdot\varepsilon_t + (A_1\varepsilon_t)\cdot\varepsilon_t + g \quad \text{in} \quad \Omega^T = \Omega\times(0,T), \tag{1.2}$$

where

$$\varepsilon \equiv \varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T), \quad \varepsilon_t \equiv \varepsilon(u_t) = \frac{1}{2}(\nabla u_t + (\nabla u_t)^T).$$

Here $\Omega \subset \mathbb{R}^3$ is a bounded domain occupied by a body in a fixed reference configuration, and $(0,T)$ is the time interval. The system is completed by appropriate, boundary and initial conditions. Here we assume

$$u = 0, \quad n\cdot\nabla\theta = 0 \quad \text{on} \quad S^T = S\times(0,T), \tag{1.3}$$

$$u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1, \quad \theta|_{t=0} = \theta_0 \quad \text{in} \quad \Omega, \tag{1.4}$$

where S is the boundary of Ω and n the unit outward normal to S.

The field $u : \Omega^T \to \mathbb{R}^3$ is the displacement, $\theta : \Omega^T \to \mathbb{R}_+ = (0,\infty)$ is the absolute temperature, second order tensors $\varepsilon = (\varepsilon_{ij})$ and $\varepsilon_t = ((\varepsilon_t)_{ij})$ denote respectively the linearized strain and the strain rate.

Equation (1.1) is the linear momentum balance with the stress tensor given by a linear thermoviscoelastic law of the Kelvin-Voigt type (cf. [8], Chap. 5.4)

$$S = A_1\varepsilon_t + A_2(\varepsilon - \theta\alpha).$$

The fourth order tensors $A_1 = ((A_1)_{ijkl})$ and $A_2 = ((A_2)_{ijkl})$ are respectively the linear viscosity and elasticity tensors, defined by

$$\varepsilon \mapsto A_p\varepsilon = \lambda_p tr\varepsilon I + 2\mu_p\varepsilon, \quad p = 1,2, \tag{1.5}$$

where λ_1, μ_1 are the viscosity coefficients, and λ_2, μ_2 are the Lamé constants, both λ_1, μ_1 and λ_2, μ_2 with values within elasticity range

$$\mu_p > 0, \quad 3\lambda_p + 2\mu_p > 0, \quad p = 1,2; \tag{1.6}$$

I is the identity tensor.
The second order symmetric tensor $\alpha = (\alpha_{ij})$ with constant α_{ij}, represents the thermal expansion. The vector field $b : \Omega^T \to \mathbb{R}^3$ is the external body force.

Above and hereafter the summation convention over the repeated indices is used, vectors and tensors are denoted by bold letters, and the dot denotes the inner product of tensors, e.g.

$$(A\varepsilon)\cdot\varepsilon = A_{ijkl}\varepsilon_{kl}\varepsilon_{ij}.$$

Equation (1.2) is the energy balance in which the linear Fourier law for the heat flux

$$q = -k\nabla\theta$$

with constant heat conductivity $k > 0$, and temperature-dependent specific heat, $c_v\theta$ with $c_v > 0$, have been adopted. The first two terms on the right-hand side of (1.2) represent heat sources created by the deformation of the material and by the viscosity. The field $g : \Omega^T \to \mathbb{R}$ is the external heat source.

The boundary conditions in (1.3) mean that the body is fixed at the boundary S and thermally isolated. The initial conditions (1.4) prescribe displacement, velocity and temperature at $t = 0$.

Let us introduce the linear viscosity and elasticity operators, $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$, defined by

$$u \mapsto \boldsymbol{Q}_p u = \nabla \cdot (\boldsymbol{A}_p \boldsymbol{\varepsilon}(u)) = \mu_p \Delta u + (\lambda_p + \mu_p)\nabla(\nabla \cdot u), \quad p = 1, 2, \tag{1.7}$$

with domains $D(\boldsymbol{Q}_p) = \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_0(\Omega)$.
Due to conditions (1.6) the operators $\boldsymbol{Q}_p$ are strongly elliptic. With the use of $\boldsymbol{Q}_p$ system (1.1), (1.2) takes the form

$$\begin{aligned} &u_{tt} - \boldsymbol{Q}_1 u = \boldsymbol{Q}_2 u - \nabla \cdot (\boldsymbol{A}_2 \boldsymbol{\alpha} \theta) + b, \\ &c_v \theta \theta_t - k \Delta \theta = -\theta(\boldsymbol{A}_2 \boldsymbol{\alpha}) \cdot \boldsymbol{\varepsilon}_t + (\boldsymbol{A}_1 \boldsymbol{\varepsilon}_t) \cdot \boldsymbol{\varepsilon}_t + g \quad \text{in} \ \ \Omega^T, \end{aligned} \tag{1.8}$$

with boundary and initial conditions (1.3), (1.4).

We prove the following existence and uniqueness result in Sobolev space with a mixed norm.

Theorem 1 *Let $T > 0$ and the numbers $p, p_0, q, q_0 \in (1, \infty)$ satisfy the conditions*

$$\frac{3}{p} + \frac{2}{p_0} \le 1, \quad \frac{3}{q} + \frac{2}{q_0} < 1 \quad p \le q, \quad p_0 \le q_0. \tag{1.9}$$

Moreover, let

$$\begin{aligned} &(u_0, u_1, \theta_0) \in (\boldsymbol{W}^2_p(\Omega) \cap \boldsymbol{H}^1_0(\Omega)) \times (\boldsymbol{B}^{2-2/p_0}_{p,p_0}(\Omega) \cap \boldsymbol{H}^1_0(\Omega)) \\ &\qquad \times (B^{2-2/q_0}_{q,q_0}(\Omega) \cap H^1(\Omega)) =: \mathcal{U}, \\ &(b, g) \in \boldsymbol{L}_{p,p_0}(\Omega^T) \times (L_1(0,T; L_\infty(\Omega)) \cap L_{q,q_0}(\Omega^T)) =: \mathcal{V} \end{aligned} \tag{1.10}$$

and

$$\begin{aligned} &0 < \underline{\theta} \le \theta_0 \le \bar{\theta} \quad a.e.\ in\ \ \Omega, \\ &g \ge 0 \qquad\qquad\ \ a.e.\ in\ \ \Omega^T, \end{aligned}$$

where $\underline{\theta}$, $\bar{\theta}$ are positive constants.
Then there exists a unique solution (u, θ) to system (1.1)–(1.4) such that

$$(u_t, \theta) \in \boldsymbol{W}^{2,1}_{p,p_0}(\Omega^T) \times W^{2,1}_{q,q_0}(\Omega^T), \tag{1.11}$$

satisfying

$$\begin{aligned} &\|u_t\|_{\boldsymbol{W}^{2,1}_{p,p_0}(\Omega^T)} + \|\theta\|_{W^{2,1}_{q,q_0}(\Omega^T)} \le c, \\ &0 < \theta_* \le \theta \le \theta^* \quad a.e.\ in\ \ \Omega^T, \end{aligned} \tag{1.12}$$

with constants c, θ_*, θ^* depending on

$$\begin{aligned} &\|(u_0, u_1, \theta_0)\|_{\mathcal{U}} = \|u_0\|_{\boldsymbol{W}^2_p(\Omega)} + \|u_1\|_{\boldsymbol{B}^{2-2/p_0}_{p,p_0}(\Omega)} + \|\theta_0\|_{B^{2-2/q_0}_{q,q_0}(\Omega)}, \\ &\|(b, g)\|_{\mathcal{V}} = \|b\|_{\boldsymbol{L}_{p,p_0}(\Omega)} + \|g\|_{L_{q,q_0}(\Omega)}, \end{aligned}$$

and exponentially on T.

2 Auxiliary results

By Sobolev space with a mixed norm $W^{k,k/2}_{p,p_0}(\Omega^T)$, $\Omega \subset \mathbb{R}^n$, $k, k/2 \in N \cup \{0\} \equiv N_0$, $p, p_0 \in [1, \infty]$, we dentoe a completion of $c^\infty(\Omega^T)$-functions under the finite norm

$$\|u\|_{W^{k,k/2}_{p,p_0}(\Omega^T)} = \left(\sum_{|\alpha|+2a\le k} \int_\Omega |D^\alpha_x \partial^a_t u|^p dx \right)^{p_0/p} dt \Bigg)^{1/p_0}, \tag{2.1}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multiindex, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha_i \ge 0$, $i = 1, \dots, n$, $D^\alpha_x = \partial^{\alpha_1}_{x_1} \dots \partial^{\alpha_n}_{x_n}$.

By Besov space $B^\lambda_{p,p_0}(\Omega)$, $\Omega \subset \mathbb{R}^n$, $\lambda \in \mathbb{R}_+$, $p, p_0 \in [1, \infty]$, we denote a set of functions with the finite norm

$$\begin{aligned} \|u\|_{B^\lambda_{p,p_0}(\Omega)} &= \|u\|_{L_p(\Omega)} \\ &+ \left(\sum_{i=1}^n \int_{\mathbb{R}_+} \frac{\|\Delta^m_i(h,\Omega)\partial^l_{x_i} u\|^{p_0}_{L_p(\Omega)}}{h^{1+(\lambda-l)p_0}} dh \right)^{1/p_0}, \end{aligned} \tag{2.2}$$

where $m > \lambda - l > 0$, $m, l \in N_0$ and $\Delta^k_i(h,\Omega)u(x)$ is a finite difference of function $u(x)$ of the order k with respect to x_i:

$$\Delta^1(h,\Omega)u \equiv \Delta_i(h,\Omega)u = u(x_1, \dots, x_{i-1}, x_i + h, x_{i+1}, \dots, x_n) - u(x_1, \dots, x_n),$$

and $\Delta_i(h,\Omega)u = 0$ for $x + h \notin \Omega$,

$$\Delta^k_i(h,\Omega)u = \Delta_i(h,\Omega)\Delta^{k-1}_i(h,\Omega)u, \quad k \in N.$$

We recall from [4] the trace and the inverse trace theorems for Sobolev spaces with a mixed norm.

Lemma 1 *Let $u \in W^{k,k/2}_{p,p_0}(\Omega \times \mathbb{R}_+)$. Then $u(x,t_0) = u(x,t)_{t=t_0} \in B^{k-2/p_0}_{p,p_0}(\Omega)$ and*

$$\|u(\cdot,t_0)\|_{B^{k-2/p_0}_{p,p_0}(\Omega)} \le c\|u\|_{W^{k,k/2}_{p,p_0}(\Omega\times\mathbb{R}_+)}. \tag{2.3}$$

Moreover, for a given $v \in B^{k-2/p_0}_{p,p_0}(\Omega)$, there exists a function $\tilde v \in W^{k,k/2}_{p,p_0}(\Omega \times \mathbb{R}_+)$ such that $\tilde v(x,t)|_{t=t_0} = v(x)$ and

$$\|\tilde v\|_{W^{k,k/2}_{p,p_0}(\Omega\times\mathbb{R}_+)} \le c\|v\|_{B^{k-2/p_0}_{p,p_0}(\Omega)}. \tag{2.4}$$

We use theorems of imbeddings between Besov spaces and Besov and Sobolev spaces from [1, 11].

Let us consider the parabolic non-diagonal problem

$$\begin{aligned} &u_t - Qu = f && \text{in } \Omega^T = \Omega \times (0,T), \\ &u = 0 && \text{on } S^T = S \times (0,T), \\ &u|_{t=0} = u_0 && \text{in } \Omega, \end{aligned} \tag{2.5}$$

where $\Omega \subset \mathbb{R}^3$, $S = \partial\Omega$ and

$$Qu = \mu\Delta u + \nu\nabla(\nabla \cdot u) \tag{2.6}$$

with positive constants μ, ν.

The next lemma plays a key role in the proof of Theorem 1. This lemma generalizes the result by Krylov [9] from the single parabolic equation to the parabolic system (2.5).

Lemma 2 [7, 9, 12]

(i) Assume that $f \in L_{p,p_0}(\Omega^T)$, $u_0 \in B^{2-2/p_0}_{p,p_0}(\Omega)$, $p, p_0 \in (1,\infty)$, $S \in C^2$. If $2 - 2/p_0 - 1/p > 0$ the compatibility condition $u_0|_S = 0$ is assumed. Then there exists a unique solution to problem (2.5) such that $u \in W^{2,1}_{p,p_0}(\Omega^T)$ and

$$\|u\|_{W^{2,1}_{p,p_0}(\Omega^T)} \le c(\|f\|_{L_{p,p_0}(\Omega^T)} + \|u_0\|_{B^{2-2/p_0}_{p,p_0}(\Omega)}) \tag{2.7}$$

with the Calderon-Zygmund estimate constant c dependent on Ω, T, S, p, p_0.

(ii) Assume that $f = \nabla \cdot g + b$, $g, b \in L_{p,p_0}(\Omega^T)$, $u_0 \in B^{2-2/p_0}_{p,p_0}(\Omega)$. Assume the compatibility condition

$$u_0|_S = 0 \quad if \quad 1 - 2/p_0 - 1/p > 0.$$

Then there exists a unique solution to (2.5) such that $u \in W^{1,1/2}_{p,p_0}(\Omega^T)$ and

$$\begin{aligned} \|u\|_{W^{1,1/2}_{p,p_0}(\Omega^T)} \le c(&\|g\|_{L_{p,p_0}(\Omega^T)} + \|b\|_{L_{p,p_0}(\Omega^T)} \\ &+ \|u_0\|_{B^{2-2/p_0}_{p,p_0}(\Omega)}) \end{aligned} \tag{2.8}$$

with the Calderon-Zygmund estimate constant c.

To prove Theorem 1 we need also the following regularity result which is a special case of general results in [7], Theorem 2.3.

Lemma 3 [7] *Let us consider the problem*

$$\begin{aligned} &\theta_t - \varrho\Delta\theta = g \quad in \ \ \Omega^T, \\ &n \cdot \nabla\theta = 0 \qquad on \ \ S^T, \\ &\theta|_{t=0} = \theta_0 \qquad in \ \ \Omega, \end{aligned} \tag{2.9}$$

where $\varrho(x,t)$ is a continuous function on Ω^T such that $\inf_\Omega \varrho > 0$. Assume that $g \in L_{p,p_0}(\Omega^T)$, $\theta_0 \in B^{2-2/p_0}_{p,p_0}(\Omega)$, $p, p_0 \in (1,\infty)$, $S \in C^2$ and the corresponding compatibility conditions are satisfied. Then there exists a unique solution to problem (2.9) such that $\theta \in W^{2,1}_{p,p_0}(\Omega^T)$ and

$$\|\theta\|_{W^{2,1}_{p,p_0}(\Omega^T)} \le c(\|g\|_{L_{p,p_0}(\Omega^T)} + \|\theta_0\|_{B^{2-2/p_0}_{p,p_0}(\Omega)}) \tag{2.10}$$

with constant c dependent on Ω, T, S, $\inf_{\Omega^T} \varrho$ and $\|\varrho\|_{C(\Omega^T)}$.

3 A priori estimates

In this section we outline the derivation of a priori estimates for solutions of problem (1.1)–(1.4) on an arbitrary time interval $(0, T)$. The estimates are essential for the proof of the long time existence.

The main result of this section is the following

Theorem 2 *Assume that*

$$(u_0, u_1, \theta_0) \in (W_p^2(\Omega) \cap H_0^1(\Omega)) \times (B_{p,p_0}^{2-2/p_0}(\Omega) \cap H_0^1(\Omega)) \times B_{q,q_0}^{2-2/q_0}(\Omega) =: \mathcal{U}_7,$$
$$(b, g) \in L_{p,p_0}(\Omega^T) \times L_1(0, T; L_\infty(\Omega)) \cap L_{q,q_0}(\Omega^T) =: \mathcal{V}_7,$$
$$0 < \underline{\theta} \le \theta_0 < \bar{\theta} < \infty.$$

Assume

$$\frac{3}{p} + \frac{2}{p_0} \le 1, \quad \frac{3}{q} + \frac{2}{q_0} < 1, \quad p \le q, \quad p_0 \le q_0.$$

Then solutions to problem (1.1)–(1.4) satisfy the estimate

$$\|u_t\|_{W_{p,p_0}^{2,1}(\Omega^T)} + \|\theta\|_{W_{q,q_0}^{2,1}(\Omega^T)} \le \varphi(\|(u_0, u_1, \theta_0)\|_{\mathcal{U}_7}, \|(b, g)\|_{\mathcal{V}_7}). \tag{3.1}$$

where φ is an increasing positive function of its arguments.

Above and hereafter we shall use φ as a generic function so it changes from formula to formula.

Theorem 2 is proved by means of the series of lemmas.

In the first lemma the lower bound for θ is established. To get such a bound we assume that $\theta \ge 0$ a.e. in Ω^T. This property will be proved in Section 4 by applying the method of successive approximation and prolonging a local solution step by step in time.

Lemma 4 *(Lower bound on θ) Assume that $\theta \ge 0$, $g \ge 0$ a.e. in Ω^T and $\theta_0 \ge \underline{\theta} > 0$, where $\underline{\theta}$ is a constant. Then there exists a positive constant c such that*

$$\theta(t) \ge \underline{\theta} \exp(-cT) \equiv \theta_*, \quad t \in [0, T]. \tag{3.2}$$

The next lemma provides energy estimates which arise from the conservation of the total energy.

Lemma 5 *(Energy estimates) Assume that $\theta_0 \ge \underline{\theta}$,*

$$(u_0, u_1, \theta_0) \in H_0^1(\Omega) \times L_2(\Omega) \times L_2(\Omega) =: \mathcal{U}_0,$$
$$(b, g) \in L_{2,1}(\Omega^T) \times L_1(\Omega^T) =: \mathcal{V}_0.$$

Then solutions to (1.1)–(1.4) satisfy the estimate

$$\begin{aligned} &\|u_t\|_{L_\infty(0,T;L_2(\Omega))} + \|\varepsilon\|_{L_\infty(0,T;L_2(\Omega))} \\ &\quad + \|\theta\|_{L_\infty(0,T;L_2(\Omega))} \le \varphi(\|u_0, u_1, \theta_0\|_{\mathcal{U}_0}, \|b, g\|_{\mathcal{V}_0}). \end{aligned} \tag{3.3}$$

Further procedure consists in iterative improvement of energy estimates. The main tools applied in this procedure are the results on linear parabolic systems in Sobolev spaces with a mixed norm, stated in Lemmas 2 and 3.

First, using Lemma 2.2 (ii) and recalling energy estimates (3.3) we directly deduce

Lemma 6 *Assume that* $\theta \geq \underline{\theta} > 0$, $g \geq 0$ *and*

$$(u_0, u_1, \theta_0) \in H_0^1(\Omega) \times B_{2,\sigma}^{2-2/\sigma}(\Omega) \times L_2(\Omega) =: \mathcal{U}_1(2,\sigma),$$
$$(b, g) \in L_{2,\sigma}(\Omega^T) \times L_1(\Omega^T) =: \mathcal{V}_1(2,\sigma), \quad \sigma \in (1,\infty).$$

Then solutions to (1.1)-(1.4) satisfy the estimate

$$\|\varepsilon_t\|_{L_{2,\sigma}(\Omega^T)} \leq \varphi(\|u_0, u_1, \theta_0\|_{\mathcal{U}_1(2,\sigma)}, \|(b,g)\|_{\mathcal{V}_1(2,\sigma)}, T). \tag{3.4}$$

Testing equation $(1.8)_1$ by Qu_t and $(1.8)_2$ by θ in conjunction with interpolation inequalities allows to prove

Lemma 7 *Assume that* $\theta_0 \geq \underline{\theta} > 0$, $g \geq 0$,

$$(u_0, u_1, \theta_0) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times (B_{2,\sigma}^{2-2/\sigma}(\Omega) \cap H_0^1(\Omega)) \times L_3(\Omega) \equiv \mathcal{U}_2(2,\sigma),$$
$$(b, g) \in L_{2,\sigma}(\Omega^T) \times L_{2,1}(\Omega^T) \equiv \mathcal{V}_2(2,\sigma), \quad \sigma \in (1,\infty).$$

Then solutions to (1.1)-(1.4) satisfy

$$\begin{aligned} &\|\varepsilon_t\|_{L_\infty(0,T;L_2(\Omega))} + \|\nabla\varepsilon_t\|_{L_2(\Omega^T)} \\ &\quad + \|\theta\|_{L_\infty(0,T;L_3(\Omega))} + \|\nabla\theta\|_{L_2(\Omega^T)} \leq \varphi(\|(u_0, u_1, \theta_0)\|_{\mathcal{U}_2(2,\sigma)}, \\ &\quad \|(b,g)\|_{\mathcal{V}_2(2,\sigma)}, T). \end{aligned} \tag{3.5}$$

Further estimates on ε_t and θ follow by testing energy equation $(1.8)_2$ by θ_t and afterwards applying Lemma 2 (i).

Lemma 8 *Assume that* $\theta_0 \geq \underline{\theta}$, $g \geq 0$,

$$(u_0, u_1, \theta_0) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times (B_{2,\sigma}^{2-2/\sigma}(\Omega) \cap H_0^1(\Omega)) \times H^1(\Omega) \equiv \mathcal{U}_3(2,\sigma),$$
$$(b, g) \in L_{2,\sigma}(\Omega^T) \times L_2(\Omega^T) \equiv \mathcal{V}_3(2,\sigma).$$

Then

$$\begin{aligned} &\|\varepsilon_t\|_{W_{2,\sigma}^{1,1/2}(\Omega^T)} + \|\nabla\theta\|_{L_\infty(0,T;L_2(\Omega))} + \|\theta_t\|_{L_2(\Omega^T)} \\ &\leq \varphi(\|(u_0, u_1, \theta_0)\|_{\mathcal{U}_3(2,\sigma)}, \|(b,g)\|_{\mathcal{V}_3(2,\sigma)}, T), \quad \sigma \in (4,\infty). \end{aligned} \tag{3.6}$$

The subsequent steps concern improvement of estimates on θ. The final goal is to prove the continuity of θ and then to apply the existence result stated in Lemma 3.

Lemma 9 *Assume that $\theta_0 \geq \underline{\theta} > 0$, $g \geq 0$,*

$$\begin{aligned}&(u_0, u_1, \theta_0) \in (W_p^1(\Omega) \cap H^2(\Omega) \cap H_0^1(\Omega)) \times (B_{p,\sigma}^{2-2/\sigma}(\Omega) \cap H_0^1(\Omega)) \times H^1(\Omega)\\ &\equiv \mathcal{U}_4(p,\sigma),\\ &(b, g) \in L_{p,\sigma}(\Omega^T) \times L_2(\Omega^T) \equiv \mathcal{V}_4(p,\sigma), \quad p, \sigma \in (1,\infty).\end{aligned}$$

Moreover, let $\theta \in L_{p,\sigma}(\Omega^T)$, $p, \sigma \in (1,\infty)$. Then the inequality

$$\begin{aligned}\|u_t\|_{W_{p,\sigma}^{1,1/2}(\Omega^T)} &\leq c\|\theta\|_{L_{p,\sigma}(\Omega^T)}\\ &+ \varphi(\|(u_0, u_1, \theta_0)\|_{\mathcal{U}_4(p,\sigma)}, \|(b,g)\|_{\mathcal{V}_4(p,\sigma)}, T)\end{aligned} \tag{3.7}$$

holds.

By testing energy equation $(1.8)_2$ by θ^r, $r > 1$, and using estimate (3.7) we prove

Lemma 10 *Assume that $\theta_0 \geq \underline{\theta} > 0$, $g > 0$,*

$$\begin{aligned}&(u_0, u_1, \theta_0) \in (W_p^1(\Omega) \cap H^2(\Omega) \cap H_0^1(\Omega)) \times (B_{p,\sigma}^{2-2/\sigma}(\Omega) \cap H_0^1(\Omega))\\ &\quad \times (L_p(\Omega) \times H^1(\Omega)) \equiv \mathcal{U}_5(p,\sigma),\\ &(b, g) \in L_{p,\sigma}(\Omega^T) \times (L_{p,\sigma}(\Omega^T) \cap L_2(\Omega^T)) \equiv \mathcal{V}_5(p,\sigma),\\ &p, \sigma \in (1,\infty).\end{aligned}$$

Assume also that the inequality

$$\|\varepsilon_t\|_{L_r(\Omega^T)} \leq c(\|\theta\|_{L_r(\Omega^T)} + 1) \tag{3.8}$$

holds for $r \in (1,\infty)$.
Then

$$\|\theta\|_{L_\infty(0,T;L_r(\Omega))} \leq \varphi(\|u_0, u_1, \theta_0)\|_{\mathcal{U}_5(r,r)}, \|(b,g)\|_{\mathcal{V}_5(r,r)}, T). \tag{3.9}$$

Lemma 10 has important consequences. Using (3.9) in (3.8) yields

$$\|\varepsilon_t\|_{L_{p,\sigma}(\Omega^T)} \leq \varphi(\|(u_0, u_1, \theta_0)\|_{\mathcal{U}_5(p,\sigma)}, \|(b,g)\|_{\mathcal{V}_5(p,\sigma)}, T), \tag{3.10}$$

for $(p, \sigma) \in (1,\infty)$.
Next, Lemma 10 implies

$$\begin{aligned}&\|\theta\|_{W_s^{2,1}(\Omega^T)} + \|\nabla\theta\|_{L_{p',s}(\Omega^T)}\\ &\leq \varphi(\|(u_0, u_1, \theta_0)\|_{\mathcal{U}_5(r,r)}, \|(b,g)\|_{\mathcal{V}_5(r,r)}, T)\end{aligned} \tag{3.11}$$

where $s < 2$ but close to 2 and $s \leq p' < 6$.
On account of the bound on $\nabla\theta$ in (3.11) we deduce

Lemma 11 *Assume that* $\theta_0 \geq \underline{\theta} > 0$, $g \geq 0$,

$$
\begin{aligned}
&(u_0, u_1, \theta_0) \in (W_p^2 \cap H^1(\Omega)) \times (B_{p,\sigma}^{2-2/\sigma}(\Omega) \cap H_0^1(\Omega)) \times (L_p(\Omega) \cap H^1(\Omega)) \\
&\quad \equiv \mathcal{U}_6(p, \sigma), \\
&(b, g) \in L_{p,\sigma}(\Omega^T) \times L_{p,\sigma}(\Omega^T) \equiv \mathcal{V}_6(p, \sigma), \quad p, \sigma \in (1, \infty).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\|\varepsilon_t\|_{W_{p',s}^{1,1/2}(\Omega^T)} \leq c\|u_t\|_{W_{p',s}^{2,1}(\Omega^T)} \\
&\leq \varphi(T, \|(u_0, u_1, \theta_0)\|_{\mathcal{U}_6(p',s)}, \|(b, g)\|_{\mathcal{V}_6(p',s)})
\end{aligned}
\tag{3.12}
$$

for $s < 2$ *close to 2,* $s \leq p' < 6$.

The above lemma allows us to conclude the following key estimates on ε_t and θ:

$$
\begin{aligned}
&\|\varepsilon_t\|_{L_2(0,T;L_\infty(\Omega))} + \|\theta\|_{L_\infty(\Omega^T)} + \|\theta\|_{W_2^{2,1}(\Omega^T)} \\
&\leq \varphi(T, \|(u_0, u_1, \theta_0)\|_{\mathcal{U}_6(p',s)}, \|(b, g)\|_{\mathcal{V}_6(p',s)}).
\end{aligned}
\tag{3.13}
$$

Due to estimates (3.13) we can use the parabolic De Giorgi method in the way presented in [10], Chap. II. 7 to deduce

Lemma 12 *Let the assumptions of Lemma 3.8 hold. Let* $\theta_0 \in C^\alpha(\Omega)$, $\alpha \in (0, 1)$. *Then*

$$
\theta \in C^{\alpha', \alpha'/2}(\Omega^T), \quad 0 < \alpha' \leq \alpha.
\tag{3.14}
$$

Thanks to the Hölder continuity of θ we can apply Lemma 3 on heat equation with continuous coefficient to conclude the estimate on θ in $W_{q,q_0}^{2,1}(\Omega^T)$-norm and subsequently the estimate on u_t in $W_{p,p_0}^{2,1}(\Omega^T)$-norm. This way we complete the proof of Theorem 2.

4 Existence

To prove the existence of solutions to problem (1.1)–(1.4) we use the following method of successive approximations:

$$
\begin{aligned}
&u_{tt}^{n+1} - \nabla \cdot (A_1 \varepsilon(u_t^{n+1})) = \nabla \cdot [A_2 \varepsilon(u^n) - (A_2 \alpha)\theta^n] + b && \text{in } \Omega^T, \\
&c_v \theta_0 \theta_t^{n+1} - k \Delta \theta^{n+1} = c_v(\theta_0 - \theta^n)\theta_t^{n+1} \\
&\qquad - \theta^n (A_2 \alpha)\varepsilon(u_t^n) + (A_1 \varepsilon(u_t^n)) \cdot \varepsilon(u_t^n) + g && \text{in } \Omega^T, \\
&u^{n+1} = 0 && \text{on } S^T, \\
&n \cdot \nabla \theta^{n+1} = 0 && \text{on } S^T, \\
&u^{n+1}|_{t=0} = u_0, \quad u_t^{n+1}|_{t=0} = u_1 && \text{in } \Omega, \\
&\theta^{n+1}|_{t=0} = \theta_0 && \text{in } \Omega,
\end{aligned}
\tag{4.1}
$$

where u^n, θ^n are treated as given.

Moreover, the approximation (u^0, θ^0) is constructed by an extension of the initial data in such a way that

$$u^0|_{t=0} = u_0, \quad u^0_t|_{t=0} = u_1, \quad \theta^0|_{t=0} = \theta_0 \quad \text{in } \Omega \tag{4.2}$$

and

$$u^0 = 0, \quad n \cdot \nabla\theta^0 = 0 \quad \text{on } S^T. \tag{4.3}$$

First we show that the sequence $\{u^n, \theta^n\}$ is uniformly bounded.

Lemma 13 *Assume*

$$D = \|u_0\|_{W^2_p(\Omega)} + \|u_1\|_{B^{2-2/p_0}_{p,p_0}(\Omega)} + \|\theta_0\|_{B^{2-2/q_0}_{q,q_0}(\Omega)} + \|b\|_{L_{p,p_0}(\Omega^\tau)}$$
$$+ \|g\|_{L_{q,q_0}(\Omega^\tau)} < \infty, \quad p \geq q, \quad 2q \geq p, \quad p_0 \geq q_0, \quad 2q_0 \geq p_0,$$

$\frac{3}{q} + \frac{2}{q_0} \leq 1$, $\frac{3}{p} + \frac{2}{p_0} \leq 1$. *Assume that τ is sufficiently small. Then there exists a constant A such that*

$$\|u^n\|_{W^{2,1}_{p,p_0}(\Omega^\tau)} + \|\theta^n\|_{W^{2,1}_{q,q_0}(\Omega^\tau)} \leq A \tag{4.4}$$

where A is independent of n but depends on D, p, q, p_0, q_0.

To show convergence of the sequence $\{u^n, \theta^n\}$ we introduce the differences

$$\begin{aligned} \mathcal{U}^n(t) &= u^n(t) - u^{n-1}(t), \\ \vartheta^n(t) &= \theta^n(t) - \theta^{n-1}(t), \end{aligned} \tag{4.5}$$

which are solutions to the following problems

$$\begin{aligned} &\mathcal{U}^{n+1}_{tt} - \nabla \cdot (A_1 \varepsilon(\mathcal{U}^{n+1}_t)) = \nabla \cdot (A_2 \varepsilon(\mathcal{U}^n)) - \nabla \cdot (A_2 \alpha \vartheta^n), \\ &\mathcal{U}^{n+1}|_{t=0} = 0, \quad \mathcal{U}^{n+1}_t|_{t=0} = 0, \quad \mathcal{U}^{n+1}|_S = 0, \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} c_v \theta^n \vartheta^{n+1}_{,t} - k\Delta\vartheta^{n+1} &= -c_v \vartheta^n \theta^n_t \\ &\quad - \theta^n (A_2\alpha) \cdot \varepsilon(\mathcal{U}^n_t) - \vartheta^n (A_2 \alpha) \cdot \varepsilon(u^n_t) \\ &\quad + (A_1 \varepsilon(\mathcal{U}^n_t)) \cdot \varepsilon(u^n_t) + (A_1 \varepsilon(u^n_t)) \cdot \varepsilon(\mathcal{U}^n_t), \\ \vartheta^{n+1}|_{t=0} &= 0, \quad n \cdot \nabla\vartheta^{n+1}|_S = 0. \end{aligned} \tag{4.7}$$

Let us introduce the quantity

$$Y^n(\tau) = \|\mathcal{U}^n\|_{W^{2,1}_{p,p_0}(\Omega^\tau)} + \|\vartheta^n\|_{W^{2,1}_{q,q_0}(\Omega^\tau)}. \tag{4.8}$$

We have the following

Lemma 14 *Let the assumptions of Lemma 13 hold. Assume that* $q = 2\bar{q}$, $q_0 = 2\bar{q}_0$, $\frac{3}{q} + \frac{2}{\bar{q}_0} - \frac{3}{p} - \frac{2}{\bar{p}_0} < 1$, $\frac{3}{\bar{q}} + \frac{2}{\bar{q}_0} < 4$, $\frac{3}{\bar{p}} + \frac{2}{\bar{p}_0} - \frac{3}{2\bar{q}} - \frac{2}{2\bar{q}_0} < 1$, $\frac{3}{p} + \frac{2}{p_0} - \frac{3}{q} - \frac{2}{q_0} < 1$. *Then there exists a constant d which depends on* D, p, p_0, q, q_0, $\bar{p}$, $\bar{p}_0$ *such that*

$$Y^{n+1}(\tau) \le d\tau^{\alpha} Y^n(\tau), \tag{4.9}$$

where $n \in N_0 \equiv N \cup \{0\}$, $Y^0 = \|u^0\|_{W^{2,1}_{\bar{p},\bar{p}_0}(\Omega^\tau)} + \|\theta^0\|_{W^{2,1}_{\bar{q},\bar{q}_0}(\Omega^\tau)}$ *and* $\alpha > 0$.

Lemmas 13 and 14 imply the existence of local solutions to problem (1.1)–(1.4).

Lemma 15 *Let the assumptions of Lemma 13 and 14 hold. Then there exists a local solution to problem (1.1)–(1.4) such that* $u_t \in W^{2,1}_{p,p_0}(\Omega^\tau)$, $\theta \in W^{2,1}_{q,q_0}(\Omega^\tau)$ *and*

$$\begin{aligned} &\|u_t\|_{W^{2,1}_{p,p_0}(\Omega^\tau)} + \|\theta\|_{W^{2,1}_{q,q_0}(\Omega^\tau)} \\ &\le \varphi(\|(u_0, u_1, \theta_0)\|_{\mathcal{U}}, \|(b, g)\|_{\mathcal{V}}), \end{aligned} \tag{4.10}$$

where φ *is an increasing positive function and*

$$\begin{aligned} &\|(u_0, u_1, \theta_0)\|_{\mathcal{U}} = \|u_0\|_{W^2_p(\Omega)} + \|u_1\|_{B^{2-2/p_0}_{p,p_0}(\Omega)} + \|\theta_0\|_{B^{2-2/q_0}_{q,q_0}(\Omega)}, \\ &\|(b, q)\|_{\mathcal{V}} \equiv \|b\|_{L_{p,p_0}(\Omega^\tau)} + \|g\|_{L_{q,q_0}(\Omega^\tau)}. \end{aligned} \tag{4.11}$$

To prove the existence of solutions on the interval $[0, T]$ which appears in Theorem 3.1 we choose $p, p_0, q, q_0,$ so large that $\mathcal{U}_7 = \mathcal{U}$.
Then Theorem 2 implies the estimate

$$\|u_t\|_{W^{2,1}_{p,p_0}(\Omega^T)} + \|\theta\|_{W^{2,1}_{q,q_0}(\Omega^T)} \le \varphi(\|(u_0, u_1, \theta_0)\|_{\mathcal{U}}, \|(b, g)\|_{\mathcal{V}}). \tag{4.12}$$

From (4.12) we obtain for any $t \in (0, T]$ the estimate

$$\begin{aligned} &\|u_t(t)\|_{B^{2-2/p_0}_{p,p_0}(\Omega)} + \|\theta(t)\|_{B^{2-2/q_0}_{q,q_0}(\Omega)} \\ &\le \varphi(\|(u_0, u_1, \theta_0)\|_{\mathcal{U}}, \|(b, g)\|_{\mathcal{V}}). \end{aligned} \tag{4.13}$$

To use (4.13) for a prolongation of the local solution described by Lemma 15 we need to estimate $\|u(t)\|_{W^2_p(\Omega)}$ in terms of the right-hand side of (4.13). Since

$$u(t) = \int_0^t u_{t'}(t')dt' + u(0),$$

we have

$$\|u(t)\|_{W^2_p(\Omega)} \le t^{1/p_0'} \left(\int_0^t \|u_t(t')\|^{p_0}_{W^2_p(\Omega)} dt' \right)^{1/p_0} + \|u(0)\|_{W^2_p(\Omega)}$$

where $\frac{1}{p_0} + \frac{1}{p_0'} = 1$.

Hence, (4.13) implies

$$\|(u(t), u_t(t), \theta(t))\|_{\mathcal{U}} \le \varphi(\|(u_0, u_1, \theta_0)\|_{\mathcal{U}}, \|(b, g)\|_{\mathcal{V}}). \tag{4.14}$$

In view of (4.14) we can divide interval $[0, T]$ on subintervals of the length τ, where τ is described by Lemmas 13, 14. Hence we can start from $t = k\tau$, $k \in N$, and prove the existence in $[k\tau, (k+1)\tau]$ by Lemma 15.
Repeating the argument on any subintervals $[k\tau, (k+1)\tau]$, $k = 0, 1, \ldots, [\frac{T}{\tau}]$ we prove the existence of solutions described in Theorem 1.
This concludes the proof of Theorem 1.

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