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Abstract. In this paper we prove the existence and uniqueness of a global in time, regular solution to the Cahn-Hilliard system coupled with viscoelasticity. The system arises as a model, regularized by a viscous damping, of phase separation process in a binary deformable alloy quenched below a critical temperature.
The key tool in the analysis are estimates of absorbing type with the property of exponentially time-decreasing contribution of the initial data. Such estimates allow not only to prolong the solution step by step on the infinite time interval but also to conclude the existence of an absorbing set.
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1. Introduction

In this paper we study the issue of the existence and uniqueness of global in time, regular solutions to the Cahn-Hilliard system coupled with viscoelasticity. The system arises as a model, regularized by adding a viscous damping, of phase separation process in a binary deformable alloy quenched below a critical temperature.

In recent years Cahn-Hilliard systems accounting for elastic effects, known to have a pronounced impact on the phase separation process, have been the subject of many modelling, mathematical and numerical studies, see e.g. [MirSchim06], [BarPaw05], [PawZaj07b] for up to date references. A general setting of the Cahn-Hilliard system coupled with elasticity, accounting for additional anisotropic, heterogeneous and kinetic effects, was introduced by Gurtin [Gur96] within the frame of his thermodynamical theory based on a microforce balance. Since the mechanical equilibrium is usually attained on a much faster timescale than the diffusion, in most of the studies a quasi-stationary approximation of the elasticity system, leading to a problem of elliptic-parabolic type, was used, see e.g. Garcke [Gar03], [Gar05], Bonetti et al. [BCDGSS02], Miranville and associates [CarMirPR99], [CarMir00], [Mir00], [Mir01a], [Mir01b].

At the initial stages of phase separation process a formation of the microstructure is on a very fast timescale, thus nonstationary effects may gain importance. The Cahn-Hilliard system with nonstationary elasticity leads to a problem of hyperbolic-parabolic type. It was studied in [CarMirP00], [Mir01a], [BarPaw05], [PawZaj07a], [PawZaj07b], [PawZaj07d] where the existence and properties of weak solutions were examined, and in [PawZaj06], [PawZaj07c] where the existence of strong solutions was proved on a finite time interval in 1-D and 3-D cases. The main difficulties we encountered in the analysis of such problem come from the 3-D setting and the hyperbolic nature of the elasticity system. We underline that the regularity estimates obtained in [PawZaj07c] depend exponentially on time, thus are not useful for the long-time analysis of the problem.

In view of the importance of the long-time analysis and the question of approaching equilibrium states from an arbitrary initial state, in the present paper we investigate the existence of global in time solutions and establish estimates of absorbing type which for sufficiently large time moments are independent of the initial conditions. We solve this question for the Cahn-Hilliard system coupled with elasticity regularized by adding a linear viscoelastic damping. From the physical point of view, adding such term provides an additional mechanical dissipation to the system, and from the mathematical point of view, replaces the hyperbolic system by the one with a hidden parabolic structure, see e.g. Rybka [Ryb92].

The central part of the paper, new in comparison with the previous authors paper [PawZaj07c], constitute estimates of absorbing type which allow not only to prolong the strong solution step by step on the infinite time interval but also to conclude the existence of an absorbing set. The latter property may be of interest in a long-time analysis of the problem.

The system under consideration has the following form:

$$\begin{aligned} &\boldsymbol{u}_{tt} - \nabla \cdot [W_{,\varepsilon}(\varepsilon(\boldsymbol{u}), \chi) + \nu \boldsymbol{A}\varepsilon(\boldsymbol{u}_t)] = \boldsymbol{b} && \text{in } \Omega^T = \Omega \times (0,T), \\ &\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \quad \boldsymbol{u}_t|_{t=0} = \boldsymbol{u}_1 && \text{in } \Omega, \\ &\boldsymbol{u} = \boldsymbol{0} && \text{on } S^T = S \times (0,T), \end{aligned} \tag{1.1}$$

$$\begin{aligned} &\chi_t - \Delta\mu = 0 && \text{in } \Omega^T, \\ &\chi|_{t=0} = \chi_0 && \text{in } \Omega, \\ &\boldsymbol{n} \cdot \nabla\mu = 0 && \text{on } S^T, \end{aligned} \tag{1.2}$$

$$\begin{aligned} &\mu = -\gamma\Delta\chi + \psi'(\chi) + W_{,\chi}(\varepsilon(\boldsymbol{u}), \chi) && \text{in } \Omega^T, \\ &\boldsymbol{n} \cdot \nabla\chi = 0 && \text{on } S^T. \end{aligned} \tag{1.3}$$

Here $\Omega \subset \mathbb{R}^3$ is a bounded domain with a smooth boundary S, occupied by a solid body in a reference configuration with constant mass density $\varrho = 1$; $\boldsymbol{n}$ is the outward unit normal to S and $T > 0$ is an arbitrary fixed time. Since the objective of this paper is to prove the global existence of a solution on $\mathbb{R}_+ = (0, \infty)$, problem (1.1)-(1.3) will be in fact considered on the time intervals $[kT, (k+1)T]$ with $k \in \mathbb{N} \cup \{0\}$.

The body under consideration is a binary $a - b$ alloy which driven by thermomechanical effects undergoes phase separation process. Such process appears when the alloy is cooled sufficiently fast below a critical temperature. Here we assume that temperature is constant below a critical value.

The unknowns are the fields $\boldsymbol{u}$, χ and μ, where $\boldsymbol{u} : \Omega^T \to \mathbb{R}^3$ is the displacement vector, $\chi : \Omega^T \to \mathbb{R}$ is the order parameter (phase ratio) and $\mu : \Omega^T \to \mathbb{R}$ is the chemical potential difference between the components, shortly referred to as the chemical potential. In case of a binary $a - b$ alloy the order parameter is related to the volumetric fraction of one of the two phases characterized by different crystalline structures of the components. We shall identify $\chi = -1$ with the phase a and $\chi = 1$ with the phase b.

The second order tensor

$$\varepsilon = \varepsilon(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)$$

denotes the linearized strain tensor. The function

$$W(\varepsilon(\boldsymbol{u}), \chi) = \frac{1}{2}(\varepsilon(\boldsymbol{u}) - \bar{\varepsilon}(\chi)) \cdot \boldsymbol{A}(\varepsilon(\boldsymbol{u}) - \bar{\varepsilon}(\chi)) \tag{1.4}$$

represents the elastic energy. The corresponding derivatives

$$W_{,\varepsilon}(\varepsilon(\boldsymbol{u}), \chi) = \boldsymbol{A}(\varepsilon(\boldsymbol{u}) - \bar{\varepsilon}(\chi)) \tag{1.5}$$

and

$$W_{,\chi}(\varepsilon(\boldsymbol{u}), \chi) = -\bar{\varepsilon}'(\chi) \cdot \boldsymbol{A}(\varepsilon(\boldsymbol{u}) - \bar{\varepsilon}(\chi))$$

denote respectively the stress tensor and the elastic part of the chemical potential. The fourth order tensor $\boldsymbol{A} = (A_{ijkl})$ stands for the elasticity tensor given by

$$\varepsilon(\boldsymbol{u}) \mapsto \boldsymbol{A}\varepsilon(\boldsymbol{u}) = \bar{\lambda} tr \varepsilon(\boldsymbol{u}) \boldsymbol{I} + 2\bar{\mu}\varepsilon(\boldsymbol{u}) \tag{1.6}$$

where $\boldsymbol{I} = (\delta_{ij})$ is the identity tensor, and $\bar{\lambda}$, $\bar{\mu}$ are the Lamé constants with values within the elasticity range (see (2.1)). Since $\boldsymbol{A}$ is assumed constant, (1.4) refers to an isotropic, homogeneous body with the same elastic properties of the phases.

The second order tensor $\bar{\varepsilon}(\chi)$ denotes the eigenstrain, i.e. the stress free strain corresponding to the phase ratio χ, defined by

$$\bar{\varepsilon}(\chi) = (1 - z(\chi))\bar{\varepsilon}_a + z(\chi)\bar{\varepsilon}_b \tag{1.7}$$

with $\bar{\varepsilon}_a, \bar{\varepsilon}_b$ denoting constant eigenstrains of phases a, b, and $z : \mathbb{R} \to [0, 1]$ being a sufficiently smooth interpolation function satisfying

$$z(\chi) = 0 \quad \text{for} \quad \chi \leq -1 \quad \text{and} \quad z(\chi) = 1 \quad \text{for} \quad \chi \geq 1. \tag{1.8}$$

The term $\nu \boldsymbol{A}\varepsilon(\boldsymbol{u}_t)$, with $\nu = \text{const} > 0$, represents a visous stress tensor; ν being a viscosity coefficient.

The function $\psi(\chi)$ denotes the chemical energy of the material at zero stress, assumed here in the standard double-well form

$$\psi(\chi) = \frac{1}{4}(1 - \chi^2)^2 \tag{1.9}$$

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with two equal minima at $\chi = -1$ and $\chi = 1$ corresponding to the pure phases of the material.

System (1.1)–(1.3) represents respectively the linear momentum balance, the mass balance and a generalized equation for the chemical potential which in Gurtin's theory [Gur96] is identified with a microforce balance. The free energy density underlying (1.1)–(1.3) has the Landau-Ginzburg-Cahn-Hilliard form

$$f(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi, \nabla\chi) = W(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi) + \psi(\chi) + \frac{\gamma}{2}|\nabla\chi|^2 \tag{1.10}$$

with the three terms on the right-hand side representing respectively the elastic, chemical and interfacial energy with a positive constant $\gamma > 0$ related to the surface tension.

The remaining quantities in (1.1)–(1.3) have the following meaning: $\boldsymbol{b} : \Omega^T \to \mathbb{R}^3$ is an external body force, and $\boldsymbol{u}_0, \boldsymbol{u}_1 : \Omega \to \mathbb{R}^3$, $\chi_0 : \Omega \to \mathbb{R}$ are the initial conditions respectively for the dispacement, the velocity and the order parameter. The homogeneous boundary conditions are chosen for the sake of simplicity. The condition $(1.1)_3$ means that the body is fixed at the boundary S, $(1.2)_3$ reflects the mass isolation at S, and $(1.3)_2$ is the natural boundary condition for the free energy density (1.10).

We remark that the polynomial (1.9) is commonly used as a simplest approximation of the physically realistic so-called regular solution form

$$\psi(\chi) = (1+\chi)\log(1+\chi) + (1-\chi)\log(1-\chi) + \alpha(1+\chi)(1-\chi)$$

where α is a positive constant. Such form – on the contrary to (1.9) – accounts for the physical constraint $\chi \in [-1,1]$ insuring that the order parameter attains physically meaningful values for all times. Another way to account for such a constraint, often used in mathematical literature on the subject (see e.g. [BCDGSS02]) is to augment (1.10) by the indicator function $I_{[-1,1]}(\chi)$ of the interval $[-1,1]$. Both approaches with the logarithmic energy and the indicator function lead to much more involved mathematical problems with singularities. In the present paper, assuming $\psi(\chi)$ to be polynomial (1.9) we cannot a priori guarantee that $\chi \in [-1,1]$. We can only prove that $\|\chi(t)\|_{L_\infty(\Omega)} \le c$ for all $t \in [0,\infty)$ with a constant c in explicitly computed form, depending on the data and absolute constants.

Let us introduce now a simplified formulation of (1.1)–(1.3) which results after taking into account the constitutive equations (1.4)–(1.7). Let $\boldsymbol{Q}$ denote the linear elasticity operator defined by

$$\boldsymbol{u} \mapsto \boldsymbol{Q}\boldsymbol{u} = \nabla \cdot \boldsymbol{A}\varepsilon(\boldsymbol{u}) = \bar{\mu}\Delta\boldsymbol{u} + (\bar{\lambda} + \bar{\mu})\nabla(\nabla \cdot \boldsymbol{u}) \tag{1.11}$$

with the domain $D(\boldsymbol{Q}) = \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}_0^1(\Omega)$. Let us define also the auxiliary quantities

$$(1.12) \qquad \boldsymbol{B} = -\boldsymbol{A}(\bar{\boldsymbol{\varepsilon}}_b - \bar{\boldsymbol{\varepsilon}}_a), \quad D = -\boldsymbol{B} \cdot (\bar{\boldsymbol{\varepsilon}}_b - \bar{\boldsymbol{\varepsilon}}_a), \quad E = -\boldsymbol{B} \cdot \bar{\boldsymbol{\varepsilon}}_a.$$

With such notation

$$(1.13) \qquad \begin{aligned} &W_{,\varepsilon}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi) = \boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{A}\bar{\boldsymbol{\varepsilon}}_a + z(\chi)\boldsymbol{B}, \\ &W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi) = z'(\chi)(\boldsymbol{B} \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}) + Dz(\chi) + E), \end{aligned}$$

so that system (1.1)–(1.3) can be recast into the form

$$(1.14) \qquad \begin{aligned} &\boldsymbol{u}_{tt} - \boldsymbol{Q}\boldsymbol{u} - \nu\boldsymbol{Q}\boldsymbol{u}_t = z'(\chi)\boldsymbol{B}\nabla\chi + \boldsymbol{b} && \text{in } \Omega^T, \\ &\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \quad \boldsymbol{u}_t|_{t=0} = \boldsymbol{u}_1 && \text{in } \Omega, \\ &\boldsymbol{u} = \boldsymbol{0} && \text{on } S^T, \end{aligned}$$

$$(1.15) \qquad \begin{aligned} &\chi_t - \Delta\mu = 0 && \text{in } \Omega^T, \\ &\chi|_{t=0} = \chi_0 && \text{in } \Omega, \\ &\boldsymbol{n} \cdot \nabla\mu = 0 && \text{on } S^T, \end{aligned}$$

$$(1.16) \qquad \begin{aligned} &\mu = -\gamma\Delta\chi + \psi'(\chi) + z'(\chi)(\boldsymbol{B} \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}) + Dz(\chi) + E) && \text{in } \Omega^T, \\ &\boldsymbol{n} \cdot \nabla\chi = 0 && \text{on } S^T. \end{aligned}$$

Let us note that the combined systems (1.15) and (1.16) yield the boundary value problem for the Cahn-Hilliard equation

$$(1.17) \qquad \begin{aligned} &\chi_t + \gamma\Delta^2\chi = \Delta[\psi'(\chi) + z'(\chi)(\boldsymbol{B} \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}) + Dz(\chi) + E)] && \text{in } \Omega^T, \\ &\chi|_{t=0} = \chi_0 && \text{in } \Omega, \\ &\boldsymbol{n} \cdot \nabla\chi = 0 && \text{on } S^T, \\ &\boldsymbol{n} \cdot \nabla[-\gamma\Delta\chi + \psi'(\chi) + z'(\chi)(\boldsymbol{B} \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}) + Dz(\chi) + E)] = 0 && \text{on } S^T, \end{aligned}$$

coupled with the elasticity system (1.14). We note that the problems are coupled not only through the right-hand sides but also through the boundary conditions.

In our analysis of system (1.14)–(1.16) we use standard energy methods combined with differentiation of the system with respect to time variable.

The paper is organized as follows: in Section 2 we present the main assumptions and results, stated in Theorems 2.1–2.3. Theorem 2.1 asserts the global in time existence of regular solutions, Theorem 2.2 provides an absorbing estimate for such solutions, and Theorem 2.3 asserts their uniqueness. In Section 3 we derive energy istimates of absorbing type which have the property of exponentially time-decreasing contribution of the initial data. Such estimates are based on a suitably modified total energy of the system. In Section 4, with the help of time-differentiation procedure we derive regularity estimates of absorbing type. Sections 5, 6 and 7 provide respectively the proofs of Theorems 2.1, 2.2 and 2.3.

Throughout the paper, in order to examine the contribution of various parameters in the estimates, we shall record all constants in their explicitly computed form.

For further use we collect here some frequently used inequalities and record the arising absolute positive constants:

— the Korn inequality

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq d_1^{-1/2}\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{\boldsymbol{L}_2(\Omega)} \quad \text{for} \quad \boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega); \tag{1.18}$$

— the Poincaré inequality

$$\int_\Omega \left|\chi - \fint_\Omega \chi dx'\right|^2 dx \leq d_2\|\nabla\chi\|^2_{\boldsymbol{L}_2(\Omega)} \quad \text{for} \quad \chi \in H^1(\Omega); \tag{1.19}$$

where $\fint_\Omega \chi dx = \frac{1}{|\Omega|}\int_\Omega \chi dx$, $|\Omega| = \text{meas }\Omega$, denotes the mean value of χ;

— the Poincaré-Friedrichs inequality

$$\|\boldsymbol{u}\|_{\boldsymbol{L}_2(\Omega)} \leq d_3^{1/2}\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}_2(\Omega)} \quad \text{for} \quad \boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega); \tag{1.20}$$

— the Sobolev imbedding

$$\|\chi\|_{L_6(\Omega)} \leq d_4^{1/2}\|\chi\|_{H^1(\Omega)} \quad \text{for} \quad \chi \in H^1(\Omega); \tag{1.21}$$

— the elliptic property of the Laplace operator with the homogeneous Neumann boundary condition (see e.g. [LU73], Chap. III 8)

$$\|\chi\|^2_{H^2(\Omega)} \leq d_5\|\Delta\chi\|^2_{L_2(\Omega)} + \left|\fint_\Omega \chi dx\right|^2 \tag{1.22}$$

for $\chi \in H^2_N(\Omega) := \{\chi \in H^2(\Omega) : \boldsymbol{n}\cdot\nabla\chi = 0 \text{ on } S\}$;

— the Sobolev imbedding
(1.23)
$$\|\chi\|^2_{L_\infty(\Omega)} + \|\nabla\chi\|^2_{L_2(\Omega)} \le d_6\|\Delta\chi\|^2_{L_2(\Omega)} + \left|\fint_\Omega \chi dx\right|^2 \quad \text{for } \chi \in H^2_N(\Omega);$$

— the interpolation inequality (see e.g. [BIN96], Chap. III, Sec. 10)
(1.24)
$$\|\chi\|_{L_q(\Omega)} \le \delta^{1-\varkappa}\|D^l\chi\|_{L_p(\Omega)} + d_7\delta^{-\varkappa}\|\chi\|_{L_p(\Omega)} \quad \text{for } \chi \in W^l_p(\Omega),$$

where $1 \le p \le q \le \infty$, $\varkappa = \left(\frac{3}{p} - \frac{3}{q}\right)\frac{1}{l} < 1$, $\delta > 0$.

We use the following notations:

$\boldsymbol{x} = (x_i)_{i=1,2,3}$ the material point,

$f_{,i} = \dfrac{\partial f}{\partial x_i}$, $f_t = \dfrac{df}{dt}$ the material space and time derivatives,

$$\boldsymbol{\varepsilon} = (\varepsilon_{ij})_{i,j=1,2,3}, \quad W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\chi) = \left(\frac{\partial W(\boldsymbol{\varepsilon},\chi)}{\partial \varepsilon_{ij}}\right)_{i,j=1,2,3},$$

$$W_{,\chi}(\boldsymbol{\varepsilon},\chi) = \frac{\partial W(\boldsymbol{\varepsilon},\chi)}{\partial \chi}, \quad \psi'(\chi) = \frac{d\psi(\chi)}{d\chi}.$$

For simplicity, whenever there is no danger of confusion, the arguments $(\boldsymbol{\varepsilon},\chi)$ are omitted. The specification of tensor indices is omitted as well. Vector- and tensor-valued mappings are denoted by bold letters.

The summation convention over repeated indices is used, as well as the notation: for vectors $\boldsymbol{a} = (a_i)$, $\tilde{\boldsymbol{a}} = (\tilde{a}_i)$ and tensors $\boldsymbol{B} = (B_{ij})$, $\tilde{\boldsymbol{B}} = (\tilde{B}_{ij})$, $\boldsymbol{A} = (A_{ijkl})$, we write

$$\boldsymbol{a}\cdot\tilde{\boldsymbol{a}} = a_i\tilde{a}_i, \qquad \boldsymbol{B}\cdot\tilde{\boldsymbol{B}} = B_{ij}\tilde{B}_{ij}, \quad \boldsymbol{AB} = (A_{ijkl}B_{kl}),$$
$$|\boldsymbol{a}| = (a_ia_i)^{1/2}, \quad |\boldsymbol{B}| = (B_{ij}B_{ij})^{1/2}.$$

The symbols ∇ and $\nabla\cdot$ denote the gradient and the divergence operators with respect to the material point $\boldsymbol{x}$. For the divergence of a tensor field we use the convention of the contraction over the last index, e.g. $\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{x}) = (\varepsilon_{ij,j}(\boldsymbol{x}))$.

We use the standard Sobolev spaces notation $H^m(\Omega) = W^m_2(\Omega)$ for $m \in \mathbb{N}$. Besides,

$$H^1_0(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } S\},$$
$$H^2_N(\Omega) = \{v \in H^2(\Omega) : \boldsymbol{n}\cdot\nabla v = 0 \text{ on } S\},$$

where $\boldsymbol{n}$ is the outward unit normal to $S = \partial\Omega$, denote the subspaces respectively of $H^1(\Omega)$ and $H^2(\Omega)$, with the standard norms of $H^1(\Omega)$ and $H^2(\Omega)$.

By bold letters we denote the spaces of vector- or tensor-valued functions, e.g. $\boldsymbol{L}_2(\Omega) = (L_2(\Omega))^n$, $\boldsymbol{H}^1(\Omega) = (H^1(\Omega))^n$, $n \in \mathbb{N}$, if there is no confusion we do not specify dimension n. Moreover, we write

$$\|\boldsymbol{a}\|_{L_2(\Omega)} = \| \, |a| \, \|_{L_2(\Omega)}, \quad \|\boldsymbol{a}\|_{\boldsymbol{H}^1(\Omega)} = \| \, |\boldsymbol{a}| \, \|_{L_2(\Omega)} + \| \, |\nabla a| \, \|_{L_2(\Omega)}$$

for the corresponding norms of a vector-valued function $\boldsymbol{a}(\boldsymbol{x}) = (a_i(\boldsymbol{x}))$; similarly for tensor-valued functions.

As common, the symbol $(\cdot,\cdot)$ denotes the scalar product in $L_2(\Omega)$. For simplicity, we use the same symbol to denote scalar products in $\boldsymbol{L}_2(\Omega) = (L_2(\Omega))^n$.

2. Assumptions and main results

System (1.1)–(1.3) (in simplified form (1.14)–(1.16)) is studied under the following assumptions:

(A1) $\Omega \subset \mathbb{R}^3$ is a bounded domain with the boundary S of class at least C^2; $T > 0$ is an arbitrary fixed time.

(A2) The coefficients of the operator $\boldsymbol{Q}$ (see (1.11)) satisfy

$$\bar{\mu} > 0, \quad 3\bar{\lambda} + 2\bar{\mu} > 0 \quad \text{(elasticity range)}. \tag{2.1}$$

These two conditions assure the following:

(i) the elasticity tensor is coercive and bounded

$$c_*|\varepsilon|^2 \le \boldsymbol{\varepsilon} \cdot \boldsymbol{A}\boldsymbol{\varepsilon} \le c^*|\boldsymbol{\varepsilon}|^2 \quad \text{for all} \quad \boldsymbol{\varepsilon} \in \boldsymbol{S}^2 \tag{2.2}$$

where $\boldsymbol{S}^2$ denotes the set of symmetric second order tensors in $\mathbb{R}^3$, and

$$c_* = \min\{3\bar{\lambda} + 2\bar{\mu}, \ 2\bar{\mu}\}, \quad c^* = \max\{3\bar{\lambda} + 2\bar{\mu}, \ 2\bar{\mu}\};$$

(ii) The operator $\boldsymbol{Q}$ is strongly elliptic and satisfies the estimate (see [Nec67], Lemma 3.2):

$$\underline{c}_Q\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} \le \|\boldsymbol{Q}\boldsymbol{u}\|_{\boldsymbol{L}_2(\Omega)} \quad \text{for} \quad \boldsymbol{u} \in D(\boldsymbol{Q}) = \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}_0^1(\Omega) \tag{2.3}$$

with positive constant $\underline{c}_Q$ depending on Ω. Since

$$\|\boldsymbol{Q}\boldsymbol{u}\|_{\boldsymbol{L}_2(\Omega} \le \bar{c}_Q\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)}, \quad \bar{c}_Q > 0,$$

it follows that the norms $\|\boldsymbol{Q}\boldsymbol{u}\|_{\boldsymbol{L}_2(\Omega)}$ and $\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)}$ are equivalent on $D(\boldsymbol{Q})$;

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(iii) The operator $\boldsymbol{Q}$ is self-adjoint on $D(\boldsymbol{Q})$:

$$(\boldsymbol{Qu},\boldsymbol{v})_{L_2(\Omega)} = -\bar{\mu}(\nabla\boldsymbol{u},\nabla\boldsymbol{v})_{L_2(\Omega)} - (\bar{\lambda}+\bar{\mu})(\nabla\cdot\boldsymbol{u},\nabla\cdot\boldsymbol{v})_{L_2(\Omega)} \\ = (\boldsymbol{u},\boldsymbol{Qv})_{L_2(\Omega)} \quad \text{for} \quad \boldsymbol{u},\boldsymbol{v}\in D(\boldsymbol{Q}),$$

$-\boldsymbol{Q}$ is positive on $D(\boldsymbol{Q})$:

$$(-\boldsymbol{Qu},\boldsymbol{u})_{L_2(\Omega)} = \bar{\mu}\|\nabla\boldsymbol{u}\|^2_{L_2(\Omega)} + (\bar{\lambda}+\bar{\mu})\|\nabla\cdot\boldsymbol{u}\|^2_{L_2(\Omega)} \geq 0 \\ \text{for} \quad \boldsymbol{u}\in D(\boldsymbol{Q}).$$

Hence, there exists a fractional power $\boldsymbol{Q}^{1/2}$ with the domain $D(\boldsymbol{Q}^{1/2}) = \boldsymbol{H}^1_0(\Omega)$ which satisfies

$$(\boldsymbol{Q}^{1/2}\boldsymbol{u},\boldsymbol{Q}^{1/2}\boldsymbol{v})_{L_2(\Omega)} = (-\boldsymbol{Qu},\boldsymbol{v})_{L_2(\Omega)} = (\boldsymbol{u},-\boldsymbol{Qv})_{L_2(\Omega)} \\ \text{for} \quad \boldsymbol{u},\boldsymbol{v}\in D(\boldsymbol{Q}). \tag{2.4}$$

For later purpose let us note that by inequalities (2.2) and (1.18) it follows that

$$\begin{aligned} \|\boldsymbol{Q}^{1/2}\boldsymbol{u}\|^2_{L_2(\Omega)} &= \bar{\mu}\|\nabla\boldsymbol{u}\|^2_{L_2(\Omega)} + (\bar{\lambda}+\bar{\mu})\|\nabla\cdot\boldsymbol{u}\|^2_{L_2(\Omega)} \\ &= (\boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{u}))_{L_2(\Omega)} \\ &\geq c_*\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|^2_{L_2(\Omega)} \geq c_* d_1\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)}. \end{aligned} \tag{2.5}$$

The next assumption postulates the presence of a viscous damping.
(A3) The mechanical viscosity coefficient $\nu = \text{const} > 0$.

Further three assumptions concern the ingredients of the free energy $f(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi,\nabla\chi)$ in (1.10).
(A4) The elastic energy $W(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi)$ is given by (1.4) with $\boldsymbol{A}$ and $\bar{\boldsymbol{\varepsilon}}(\chi)$ defined by (1.6) and (1.7). The interpolation function $z:\mathbb{R}\to[0,1]$ in (1.7) is of class C^2 with the property (1.8). Hence,

$$0\leq z(\chi)\leq 1 \quad \text{and} \quad |z'(\chi)|+|z''(\chi)|\leq c \quad \text{for all} \quad \chi\in\mathbb{R}. \tag{2.6}$$

The auxiliary quantities $\boldsymbol{B}, D$ and E are defined in (1.12).
(A5) The function $\psi(\chi)$ is given by (1.9), hence

$$\psi'(\chi) = \chi^3-\chi, \quad \psi''(\chi) = 3\chi^2-1, \quad \psi'''(\chi) = 6\chi. \tag{2.7}$$

(A6) The interfacial energy coefficient is strictly positive, $\gamma = \text{const} > 0$.

Let us note that, in view of (1.13), it follows from (A4) that there exist positive constants a_1, a_2 such that

$$|W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi)| \leq a_1(|\boldsymbol{\varepsilon}(\boldsymbol{u})|+1), \\ |W_{,\varepsilon}(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi) \leq a_2(|\boldsymbol{\varepsilon}(\boldsymbol{u})|+1) \tag{2.8}$$

for all $\varepsilon(\boldsymbol{u}) \in \boldsymbol{S}^2$ and $\chi \in \mathbb{R}$. Moreover, by (2.2) and the Young inequality, it holds

$$W(\varepsilon(\boldsymbol{u}),\chi) \geq \frac{c_*}{2}|\varepsilon(\boldsymbol{u}) - \bar{\varepsilon}(\chi)|^2 \geq \frac{c_*}{2}\left(\frac{1}{2}|\varepsilon(\boldsymbol{u})|^2 - |\bar{\varepsilon}(\chi)|^2\right), \tag{2.9}$$

and

$$\psi(\chi) \geq \frac{1}{8}\chi^4 - \frac{1}{4}. \tag{2.10}$$

We note also that due to assumption (A4) there exist positive constants a_3, a_4, a_5 and a_6 such that

$$\begin{aligned}
&|\bar{\varepsilon}(\chi)| \leq a_3,\\
&|\bar{\varepsilon}'(\chi)| + |\bar{\varepsilon}'(\chi)\chi| \leq a_4,\\
&|z'(\chi)\boldsymbol{B}| + |z''(\chi)\boldsymbol{B}| \leq a_5,\\
&|z''(\chi)(Dz(\chi)+E)| + |z'(\chi)(Dz(\chi)+E)| + |Dz'(\chi)| \leq a_6
\end{aligned} \tag{2.11}$$

for all $\chi \in \mathbb{R}$.

The next assumption concerns the initial data. We introduce, in addition to

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{u}_t(0) = \boldsymbol{u}_1, \quad \chi(0) = \chi_0 \quad \text{in } \Omega,$$

the initial conditions for $\boldsymbol{u}_{tt}(0)$ and $\chi_t(0)$, calculated from equations $(1.14)_1$, $(1.15)_1$, $(1.16)_1$ in terms of $\boldsymbol{u}_0$, $\boldsymbol{u}_1$ and χ_0:

$$\begin{aligned}
\boldsymbol{u}_2 &:= \boldsymbol{u}_{tt}(0) = \boldsymbol{Q}\boldsymbol{u}_0 + \nu\boldsymbol{Q}\boldsymbol{u}_1 + z'(\chi_0)\boldsymbol{B}\nabla\chi_0 + \boldsymbol{b}(0),\\
\chi_1 &:= \chi_t(0) = \Delta\mu(0) = -\gamma\Delta^2\chi_0 + \Delta[\psi'(\chi_0) + z'(\chi_0)(\boldsymbol{B}\cdot\varepsilon(\boldsymbol{u}_0)\\
&\qquad + Dz(\chi_0) + E)] \quad \text{in } \Omega.
\end{aligned}$$

We assume that

(A7)

$$\begin{aligned}
&\boldsymbol{u}_0, \boldsymbol{u}_1 \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_0(\Omega), \quad \boldsymbol{u}_2 \in \boldsymbol{H}^1_0(\Omega),\\
&\chi_0 \in H^2_N(\Omega) = \{\chi \in H^2(\Omega) : \boldsymbol{n}\cdot\nabla\chi = 0 \quad \text{on } S\},\\
&\fint_\Omega \chi_0 dx =: \chi_m, \quad \chi_1 \in L_2(\Omega).
\end{aligned}$$

Let us note that (A7) implies that $\boldsymbol{u}_0 \in \boldsymbol{H}^3(\Omega) \cap \boldsymbol{H}^1_0(\Omega)$, $\chi_0 \in H^4(\Omega) \cap H^2_N(\Omega)$.

Finally, we require

(A8) $\boldsymbol{b} \in L_1((0,\infty); \boldsymbol{L}_2(\Omega)) \cap W^1_\infty((0,\infty); \boldsymbol{L}_2(\Omega))$.

We state now the main results of the paper.

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Theorem 2.1. *Global existence on* $[0,\infty)$.
Let assumptions (A1)–(A8) hold true. Then there exists a solution $(\boldsymbol{u},\chi,\mu)$ *to problem (1.1)–(1.3) on* $[0,\infty)$ *such that*

$$
\begin{aligned}
&\boldsymbol{u}\in C^1([0,\infty);\boldsymbol{H}^2(\Omega)\cap\boldsymbol{H}^1_0(\Omega))\cap C^2([0,\infty);\boldsymbol{H}^1_0(\Omega)),\\
&\chi\in C([0,\infty);H^2_N(\Omega))\cap C^1([0,\infty);L_2(\Omega)),\\
&\mu\in C([0,\infty);H^2_N(\Omega)),\quad \fint_\Omega \chi(t)dx=\chi_m \quad \textit{for}\quad t\in[0,\infty),\\
&\boldsymbol{u}_t\in L_2((0,\infty);\boldsymbol{H}^1_0(\Omega)),\quad \nabla\mu\in L_2((0,\infty);\boldsymbol{L}_2(\Omega)),
\end{aligned}
\tag{2.12}
$$

(2.13)
$$\boldsymbol{u}(0)=\boldsymbol{u}_0,\quad \boldsymbol{u}_t(0)=\boldsymbol{u}_1,\quad \boldsymbol{u}_{tt}(0)=\boldsymbol{u}_2,\quad \chi(0)=\chi_0,\quad \chi_t(0)=\chi_1\quad in\quad \Omega.$$

Moreover, for any fixed number $T>0$ *and* $k\in\mathbb{N}\cup\{0\}$,

$$\boldsymbol{u}_{tt}\in L_2(kT,(k+1)T;\boldsymbol{H}^2(\Omega)),\qquad \chi_t\in L_2(kT,(k+1)T;H^2_N(\Omega)). \tag{2.14}$$

The solution satisfies the following estimates uniform in time

$$
\begin{aligned}
&\|\boldsymbol{u}\|_{C([0,\infty);\boldsymbol{H}^1_0(\Omega))}+\|\boldsymbol{u}_t\|_{C([0,\infty);\boldsymbol{L}_2(\Omega))}+\|\chi\|_{C([0,\infty);H^1(\Omega))}\\
&\quad+\|\boldsymbol{u}_t\|_{L_2((0,\infty);\boldsymbol{H}^1_0(\Omega))}+\|\nabla\mu\|_{L_2((0,\infty);\boldsymbol{L}_2(\Omega))}\le c_0,
\end{aligned}
\tag{2.15}
$$

$$
\begin{aligned}
&\|\boldsymbol{u}\|_{C^1([0,\infty);\boldsymbol{H}^2(\Omega))}+\|\boldsymbol{u}_{tt}\|_{C([0,\infty);\boldsymbol{H}^1_0(\Omega))}+\|\chi\|_{C([0,\infty);H^2_N(\Omega))}\\
&\quad+\|\chi_t\|_{C([0,\infty);L_2(\Omega))}+\|\mu\|_{C([0,\infty);H^2_N(\Omega))}\le c
\end{aligned}
\tag{2.16}
$$

with positive constants

$$
\begin{aligned}
c_0&=c_0(\|\boldsymbol{u}_0\|_{\boldsymbol{H}^1(\Omega)},\ \|\boldsymbol{u}_1\|_{\boldsymbol{L}_2(\Omega)},\ \|\chi_0\|_{H^1(\Omega)},\ \|\boldsymbol{b}\|_{L_1((0,\infty);\boldsymbol{L}_2(\Omega))}),\\
c&=c(\|\boldsymbol{u}_0\|_{\boldsymbol{H}^2(\Omega)},\ \|\boldsymbol{u}_1\|_{\boldsymbol{H}^2(\Omega)},\ \|\boldsymbol{u}_2\|_{\boldsymbol{H}^1_0(\Omega)},\ \|\chi_0\|_{H^2_N(\Omega)},\ \|\chi_1\|_{L_2(\Omega)},\\
&\qquad \|\boldsymbol{b}\|_{W^1_\infty((0,\infty);\boldsymbol{L}_2(\Omega))}),
\end{aligned}
$$

and estimates depending on T

(2.17)
$$\sup_{k\in\mathbb{N}\cup\{0\}}\left(\|\boldsymbol{u}_{tt}\|_{L_2(kT;(k+1)T;\boldsymbol{H}^2(\Omega))}+\|\chi_t\|_{L_2(kT,(k+1)T;H^2_N(\Omega))}\right)\le c(T^{1/2}+1)$$

with constant c *as above.*

Remark 2.1. By virtue of the imbedding $H^2(\Omega)\subset L_\infty(\Omega)$, (2.14) implies that

$$\|\chi(t)\|_{L_\infty(\Omega)}\le c\quad \text{for all}\quad t\in[0,\infty)$$

with constant c determined explicitly by estimates (5.16) and (5.12).

The next theorem provides an absorbing estimate. Let $N(t):[0,\infty)\to[0,\infty)$ be a function defined as a linear combination with appropriately chosen coefficients of the following norms (see (4.57))

$$\begin{array}{lll} \|\boldsymbol{Q}\boldsymbol{u}(t)\|^2_{\boldsymbol{L}_2(\Omega)}, & \|\boldsymbol{Q}^{1/2}\boldsymbol{u}_t(t)\|^2_{\boldsymbol{L}_2(\Omega)}, & \|\boldsymbol{Q}\boldsymbol{u}_t(t)\|^2_{\boldsymbol{L}_2(\Omega)}, \\ \|\boldsymbol{Q}^{1/2}\boldsymbol{u}_{tt}(t)\|^2_{\boldsymbol{L}_2(\Omega)}, & \|\chi(t)\|^2_{L_2(\Omega)}, & \|\Delta\chi(t)\|^2_{L_2(\Omega)}, \\ \|\chi_t(t)\|^2_{L_2(\Omega)}, & & \end{array}$$

and a modified total energy $F(t):[0,\infty)\to[0,\infty)$ given by

$$\begin{aligned} G(t)=\int_\Omega\bigg[&\frac{1}{2}|\boldsymbol{u}_t(t)|^2+W(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\chi(t))+\psi(\chi(t))+\frac{\gamma}{2}|\nabla\chi(t)|^2\\ &+\frac{\nu c_* d_1}{2}\Big(\boldsymbol{u}_t(t)\cdot\boldsymbol{u}(t)+\frac{\nu}{2}\boldsymbol{\varepsilon}(\boldsymbol{u}(t))\cdot\boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t))\Big)\bigg]dx \end{aligned}$$

with constants c_*, d_1 from (2.2) and (1.18).
The constructed function $N(t)$ satisfies for sufficiently large times t the following bound (see (6.8))

$$\begin{aligned} N(t)\geq c_{7a}(&\|\boldsymbol{u}(t)\|^2_{\boldsymbol{H}^2(\Omega)}+\|\boldsymbol{u}_t(t)\|^2_{\boldsymbol{H}^2(\Omega)}+\|\boldsymbol{u}_{tt}(t)\|^2_{\boldsymbol{H}^1(\Omega)}\\ &+\|\chi(t)\|^2_{H^2(\Omega)}+\|\chi_t(t)\|^2_{L_2(\Omega)})-c'_{8a} \end{aligned} \tag{2.18}$$

with positive constants c_{7a}, c'_{8a} independent of the initial condition $N(0)$.

Theorem 2.2. *Absorbing estimate*
Let the assumptions of Theorem 2.1 be satisfied and the function $N(t):[0,\infty)\to[0,\infty)$ be as above. Then there exist positive numbers A_{2a}, β_{4a} (see (6.6)) independent of the initial conditions such that

$$N(t)\leq A_{2a}(1-e^{-\beta_{4a}t})+N(0)e^{-\beta_{4a}t}\quad \textit{for all}\quad t\geq t_1, \tag{2.19}$$

where t_1 is a time moment dependent on the intial condition $G(0)$ (see (6.3)). Moreover, for any positive number A'_2 satisfying $A'_2>A_{2a}$, there exists a time moment $t_*=\max\{t_1,t_2\}$ with t_2 (see (6.10)) depending on the initial condition $N(0)$ and A'_2 such that

$$\begin{aligned} c_{7a}(&\|\boldsymbol{u}(t)\|^2_{\boldsymbol{H}^2(\Omega)}+\|\boldsymbol{u}_t(t)\|^2_{\boldsymbol{H}^2(\Omega)}+\|\boldsymbol{u}_{tt}(t)\|^2_{\boldsymbol{H}^1(\Omega)}\\ &+\|\chi(t)\|^2_{H^2(\Omega)}+\|\chi_t(t)\|^2_{L_2(\Omega)})<A'_2+c'_{8a}\quad \textit{for all}\quad t\geq t_*, \end{aligned} \tag{2.20}$$

where c_{7a} and c'_{8a} are positive numbers independent of $N(0)$.

The uniqueness result is stated in

Theorem 2.3. *Uniqueness*
Let the assumptions of Theorem 2.1 be satisfied and in addition

$$z(\cdot) \text{ be of class } C^3 \text{ with } |z'''(\chi)| \le c \text{ for all } \chi \in \mathbb{R}. \tag{2.21}$$

Then the solution $(\boldsymbol{u}, \chi, \mu)$ *in Theorem 2.1 is unique.*

3. Energy estimates of absorbing type

In this section we derive energy estimates with exponentially time-decreasing contribution from the initial data. We call such estimates of absorbing type since they allow not only to prolong a solution step by step on the infinite time interval but also to conclude the existence of an absorbing set in energy norms. In the next section we combine the energy estimates with additional regularity estimates to conclude more refined estimates of absorbing type. Such estimates will allow us to conclude the existence of a regular solution on the infinite time interval and an absorbing set in higher norms.

For the clarity we present only formal derivation of the estimates which can be made rigorous with the help of a Faedo-Galerkin approximation and passing to the limit with approximation by standard arguments, for example in a similar fashion as in [PawZaj07b], [PawZaj07c].

Throughout this section we use the physical form (1.1)–(1.3) of the system. Moreover, we assume that hypotheses (A1)–(A6) are satisfied.

3.1. Energy estimates

We begin with noting the conservation property

$$\frac{d}{dt}\int_\Omega \chi dx = 0, \tag{3.1}$$

which follows from equations $(1.2)_1$ and $(1.2)_3$. It shows that the mean value of χ is preserved, i.e.

$$\fint_\Omega \chi dx = \fint_\Omega \chi_0 dx \equiv \chi_m \quad \text{for} \quad t \ge 0. \tag{3.2}$$

Next we derive the energy identity for system (1.1)–(1.3). It follows by testing elasticity system $(1.1)_1$ by $\boldsymbol{u}_t$, mass balance $(1.2)_1$ by μ and the chemical potential equation $(1.3)_1$ by $-\chi_t$, and summing up the resulting equalities.

Let $F : [0,\infty) \to [0,\infty)$ be the function defined by

$$
\begin{aligned}
F(t) &= \int\limits_\Omega \Big[\frac{1}{2}|\boldsymbol{u}_t(t)|^2 + f(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \chi(t), \nabla\chi(t))\Big] dx \\
&= \int\limits_\Omega \Big[\frac{1}{2}|\boldsymbol{u}_t(t)|^2 + W(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \chi) + \psi(\chi(t)) + \frac{\gamma}{2}|\nabla\chi(t)|^2\Big] dx,
\end{aligned} \tag{3.3}
$$

denoting the total energy of system (1.1)–(1.3), i.e. the sum of the kinetic, elastic, chemical and interfacial energy. We have

Lemma 3.1. *Let $(\boldsymbol{u}, \chi, \mu)$ be a sufficiently regular solution to problem (1.1)–(1.3), and $F(t)$ be given by 3.3). Then the following identity is satisfied*

$$
\begin{aligned}
&\frac{d}{dt}F(t) + \nu \int\limits_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}_t(t)) \cdot \boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_t(t)) dx + \int\limits_\Omega |\nabla\mu(t)|^2 dx \\
&= \int\limits_\Omega \boldsymbol{b}(t) \cdot \boldsymbol{u}_t(t) dx \quad for \quad t \geq 0.
\end{aligned} \tag{3.4}
$$

Proof. Multiplying $(1.1)_1$ by $\boldsymbol{u}_t(t)$, integrating over Ω and by parts, using boundary condition $(1.1)_3$, it follows that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\limits_\Omega |\boldsymbol{u}_t|^2 dx + \int\limits_\Omega W_{,\varepsilon}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi) \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_t) dx \\
&+ \nu \int\limits_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}_t) \cdot \boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_t) dx = \int\limits_\Omega \boldsymbol{b} \cdot \boldsymbol{u}_t dx.
\end{aligned} \tag{3.5}
$$

Further, testing $(1.2)_1$ by $\mu(t)$, integrating over Ω and by parts, using $(1.2)_3$, yields

$$
\int\limits_\Omega \chi_t \mu dx + \int\limits_\Omega |\nabla\mu|^2 dx = 0. \tag{3.6}
$$

Finally, testing $(1.3)_1$ by $-\chi_t(t)$, integrating by parts and using $(1.3)_2$, leads to

(3.7)

$$
-\int\limits_\Omega \mu\chi_t + \frac{\gamma}{2}\frac{d}{dt}\int\limits_\Omega |\nabla\chi|^2 dx + \int\limits_\Omega \psi'(\chi)\chi_t dx + \int\limits_\Omega W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi)\chi_t dx = 0.
$$

Summing up (3.5)–(3.7) gives identity (3.4) and completes the proof. □

From identity (3.4) we deduce the following energy estimate.

Lemma 3.2. *Let (A1)-(A6) hold, $F(t)$ is given by (3.3) and $\boldsymbol{b} \in L_1(0,T;\boldsymbol{L}_2(\Omega))$. Then*

$$\begin{aligned}
&\frac{1}{2}\|\boldsymbol{u}_t\|^2_{\boldsymbol{L}_2(\Omega)} + \frac{\gamma}{2}\|\nabla\chi\|^2_{\boldsymbol{L}_2(\Omega)} + \frac{1}{8}\|\chi\|^4_{L_4(\Omega)} + \frac{c_* d_1}{4}\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} \\
&\quad + \nu c_* d_1 \int_0^t \|\boldsymbol{u}_{t'}\|^2_{\boldsymbol{H}^1(\Omega)} dt' + \int_0^t \|\nabla\mu\|^2_{\boldsymbol{L}_2(\Omega)} dt' \\
&\le F(t) + \nu c_* d_1 \int_0^t \|\boldsymbol{u}_{t'}\|^2_{\boldsymbol{H}^1(\Omega)} dt' + \int_0^t \|\nabla\mu\|^2_{\boldsymbol{L}_2(\Omega)} dt' + c_1' \\
&\le 2F(0) + \frac{3}{2}\|\boldsymbol{b}\|^2_{L_1(0,T;\boldsymbol{L}_2(\Omega))} + c_1' = c_0 \quad \text{for} \quad \mathrm{t} \in [0,\mathrm{T}]
\end{aligned} \tag{3.8}$$

with positive constant $c_1' = \frac{|\Omega|}{2}\left(c_* a_3^2 + \frac{1}{2}\right)$.

Proof. We apply the Hölder inequality to the right-hand side of (3.4), use the definition of F and condition (2.2) to conclude

(3.9)

$$\frac{d}{dt}F + \nu c_*\|\boldsymbol{\varepsilon}(\boldsymbol{u}_t)\|^2_{L_2(\Omega)} + \|\nabla\mu\|^2_{\boldsymbol{L}_2(\Omega)} \le \sqrt{2}\|\boldsymbol{b}\|_{\boldsymbol{L}_2(\Omega)}\sqrt{F} \quad for \quad t \in (0,T).$$

Hence,

$$\frac{d}{dt}\sqrt{F} \le \frac{1}{\sqrt{2}}\|\boldsymbol{b}\|_{\boldsymbol{L}_2(\Omega)}. \tag{3.10}$$

Integrating (3.10) with respect to time from 0 to $t \in [0,T]$ gives

$$\sqrt{F(t)} \le \frac{1}{\sqrt{2}}\|\boldsymbol{b}\|_{L_1(0,t;\boldsymbol{L}_2(\Omega))} + \sqrt{F(0)}. \tag{3.11}$$

Further, using (3.11) in (3.9) and integrating the result with respect to time from 0 to $t \in [0,T]$ we get

$$\begin{aligned}
&F(t) + \nu c_* \int_0^t \|\boldsymbol{\varepsilon}(\boldsymbol{u}_{t'}\|^2_{L_2(\Omega)} dt' + \int_0^t \|\nabla\mu\|^2_{\boldsymbol{L}_2(\Omega)} dt' \\
&\le \|\boldsymbol{b}\|_{L_1(0,t;\boldsymbol{L}_2(\Omega))}(\|\boldsymbol{b}\|_{L_1(0,t;\boldsymbol{L}_2(\Omega))} + \sqrt{2F(0)}) + F(0) \\
&\le 2F(0) + \frac{3}{2}\|\boldsymbol{b}\|^2_{L_1(0,t;\boldsymbol{L}_2(\Omega))}.
\end{aligned} \tag{3.12}$$

Now we note that, on account of (2.9), (2.10), $(2.11)_1$ and (1.18),

(3.13)

$$\begin{aligned}
F(t) &\ge \int_\Omega \left[\frac{1}{2}|\boldsymbol{u}_t|^2 + \frac{\gamma}{2}|\nabla\chi|^2 + \frac{1}{8}\chi^4 + \frac{c_*}{4}|\boldsymbol{\varepsilon}(\boldsymbol{u})|^2 - \frac{1}{4} - \frac{c_*}{2}a_3^2\right]dx \\
&\ge \frac{1}{2}\|\boldsymbol{u}_t\|^2_{\boldsymbol{L}_2(\Omega)} + \frac{\gamma}{2}\|\nabla\chi\|^2_{\boldsymbol{L}_2(\Omega)} + \frac{1}{8}\|\chi\|^4_{L_4(\Omega)} + \frac{c_* d_1}{4}\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} - c_1'
\end{aligned}$$

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for $t \in [0,T]$, with constant c_1' defined in (3.8). From (3.12) and (3.13) we conclude (3.8). This completes the proof. $\square$

Remark 3.1. Estimates (3.8) are independent of time horizon T (depend on time only through the norm $\|\boldsymbol{b}\|_{L_1(0,T;\boldsymbol{L}_2(\Omega))}$). Thanks to this property they can be used in the proof of the global existence of a weak solution by prolonging a local solution step by step on the intervals $[kT,(k+1)T]$, $k \in \mathbb{N} \cup \{0\}$ (see e.g.]PawZaj07d], Thm 2.2).

Since our aim in this paper is to prove not only the global existence of a regular solution but also the absorbing set, we derive below more refined energy estimates of absorbing type. Such estimates are independent of T and – on the contrary to (3.8) – enjoy the property of exponentially time-decreasing influence of the initial data.

3.3. Energy estimates of absorbing type

We derive a differential inequality which will allow to deduce energy estimates of absorbing type. Such inequality has the form (see (3.17))

$$\frac{d}{dt}G(t) + \beta_1 G(t) + \text{ nonnegative terms } \leq \Lambda_1 \|\boldsymbol{b}(t)\|^2_{\boldsymbol{L}_2(\Omega)} + \Lambda_2$$

for $t \in (0,T)$, where $G(t)$ is an appropriately modified total energy $F(t)$ and β_1, Λ_1, Λ_2 are positive constants. The derivation of such inequality is based on the three identities: the energy identity (3.4), the identity resulting from testing the chemical potential equation $(1.3)_1$ by χ (see (3.19)), and the identity following by testing the elasticity system $(1.1)_1$ by $\boldsymbol{u}$ (see (3.31)).

Let $G[0,\infty) \to [0,\infty)$ be the function defined by

$$G(t) = F(t) + \frac{\nu c_* d_1}{2} \int\limits_\Omega \Big(\boldsymbol{u}_t(t)\cdot\boldsymbol{u}(t) + \frac{\nu}{2}\boldsymbol{\varepsilon}(\boldsymbol{u}(t))\cdot\boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t))\Big)dx$$

$$(3.14)\qquad = \int\limits_\Omega \Big[\frac{1}{2}|\boldsymbol{u}_t(t)|^2 + W(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\chi(t)) + \psi(\chi(t)) + \frac{\gamma}{2}|\nabla\chi(t)|^2 + \frac{\nu c_* d_1}{2}\Big(\boldsymbol{u}_t(t)\cdot\boldsymbol{u}(t) + \frac{\nu}{2}\boldsymbol{\varepsilon}(\boldsymbol{u}(t))\cdot\boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t))\Big)\Big]dx.$$

This function will be shown (see (3.47)–(3.51)) to satisfy the bound (3.15)

$$G(t) \geq \int_\Omega \left[\frac{1}{4}|\boldsymbol{u}_t(t)|^2 + W(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \chi(t)) + \psi(\chi(t)) + \frac{\gamma}{2}|\nabla\chi(t)|^2\right] dx$$
$$= F(t) - \frac{1}{4}\int_\Omega |\boldsymbol{u}_t(t)|^2 dx \geq 0.$$

We shall prove the following

Lemma 3.3. *Let $G(t)$ be defined by (3.14), and $\sup_{0\leq t\leq T} \|\boldsymbol{b}(t)\|_{\boldsymbol{L}_2(\Omega)} < \infty$. There exists a positive constant*

$$\beta_1 = \min\left\{\frac{\nu c_* d_1}{8}, \frac{1}{d'}, \frac{1}{4}\sqrt{\frac{c_* d_1}{2}}, \frac{d_1}{8\nu}\right\} \quad \text{with } d_1' = \frac{d_2^2}{\gamma}, \tag{3.16}$$

such that solutions of problem (1.1)–(1.3) satisfy the differential inequality

$$\begin{aligned}&\frac{d}{dt}G(t) + \beta_1 G(t) + \frac{\nu c_* d_1}{8}\|\boldsymbol{u}_t(t)\|^2_{\boldsymbol{H}^1(\Omega)} + \frac{1}{2}\|\nabla\mu(t)\|^2_{\boldsymbol{L}_2(\Omega)}\\ &\leq \Lambda_1\|\boldsymbol{b}(t)\|^2_{\boldsymbol{L}_2(\Omega)} + \Lambda_2 \quad \text{for } t \in (0,T),\end{aligned} \tag{3.17}$$

with positive constants Λ_1, Λ_2 given by

$$\begin{aligned}\Lambda_1 &= 2\nu + \frac{1}{2\nu c_* d_1},\\ \Lambda_2 &= |\Omega|\left[\frac{1}{d'}\left(\frac{3}{2} + \frac{\gamma}{2d_2}\chi_m^2\right) + \frac{27\chi_m^4}{4d'} + \frac{\nu c_*^2 a_3^2 d_1}{8}\right]\\ &\quad + |\boldsymbol{A}|^2\left[\nu d_1 a_3^2 + \frac{16a_4^2|\Omega|(\chi_m^2+1)}{\nu c_*^2 d_1 (d')^2}\right].\end{aligned} \tag{3.18}$$

Proof. Multiplying $(1.3)_1$ by χ, integrating over Ω and by parts using $(1.3)_2$, gives

$$\gamma\int_\Omega |\nabla\chi|^2 dx + \int_\Omega [\psi'(\chi)\chi + W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi)\chi] dx = \int_\Omega \mu\chi dx. \tag{3.19}$$

Using the equality

$$\int_\Omega \mu\chi dx = \int_\Omega (\mu - \fint_\Omega \mu dx')\chi dx + \int_\Omega \mu dx \fint_\Omega \chi dx, \tag{3.20}$$

and next applying the Young and the Poincaré inequality (1.19) to the first integral on the right-hand side of (3.20) and recalling the mean value property (3.2), we have

$$\Big|\int_\Omega \mu\chi dx\Big| \le \frac{\delta_1}{2}\|\chi\|^2_{L_2(\Omega)} + \frac{d_2}{2\delta_1}\|\nabla\mu\|^2_{L_2(\Omega)} + \Big|\int_\Omega \mu dx\Big||\chi_m|, \quad \delta_1 > 0. \tag{3.21}$$

On account of (3.21)we conclude from (3.19) that

$$\begin{aligned}&\gamma\int_\Omega |\nabla\chi|^2 dx + \int_\Omega [\psi'(\chi)\chi + W_{,\chi}(\varepsilon(\boldsymbol{u}),\chi)\chi]dx\\ &\le \frac{\delta_1}{2}\|\chi\|^2_{L_2(\Omega)} + \frac{d_2}{2\delta_1}\|\nabla\mu\|^2_{L_2(\Omega)} + \Big|\int_\Omega \mu dx\Big||\chi_m|.\end{aligned} \tag{3.22}$$

Further, since

$$\begin{aligned}&\int_\Omega \Big|\chi - \fint_\Omega \chi dx'\Big|^2 dx = \int_\Omega \Big(\chi^2 - 2\chi\fint_\Omega \chi dx' + \Big|\fint_\Omega \chi dx'\Big|^2\Big)dx\\ &= \int_\Omega \chi^2 dx - |\Omega|\Big|\fint_\Omega \chi dx\Big|^2,\end{aligned}$$

it follows, by the Poincaré inequality (1.19) and the property (3.2), that

$$\begin{aligned}\|\chi\|^2_{L_2(\Omega)} &= \int_\Omega \Big|\chi - \fint_\Omega \chi dx'\Big|^2 dx + |\Omega|\Big|\fint_\Omega \chi dx\Big|^2\\ &\le d_2\|\nabla\chi\|^2_{L_2(\Omega)} + |\Omega|\chi_m^2.\end{aligned} \tag{3.23}$$

Due to (3.23), setting $\delta_1 = \frac{\gamma}{d_2}$, (3.22) yields

$$\begin{aligned}&\frac{\gamma}{2}\int_\Omega |\nabla\chi|^2 dx + \int_\Omega [\psi'(\chi)\chi + W_{,\chi}(\varepsilon(\boldsymbol{u}),\chi)\chi]dx\\ &\le \frac{d_2^2}{2\gamma}\|\nabla\mu\|^2_{L_2(\Omega)} + \frac{\gamma}{2d_2}|\Omega|\chi_m^2 + \Big|\int_\Omega \mu dx\Big||\chi_m|.\end{aligned} \tag{3.24}$$

Let us turn now to the energy identity (3.4). In view of (3.3) and (2.2) we have

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \Big[\frac{1}{2}(|\boldsymbol{u}_t|^2 + \gamma|\nabla\chi|^2) + \psi(\chi) + W(\varepsilon(\boldsymbol{u}),\chi)\Big]dx\\ &+ \nu c_*\|\varepsilon(\boldsymbol{u}_t)\|^2_{L_2(\Omega)} + \|\nabla\mu\|^2_{L_2(\Omega)} \le \int_\Omega \boldsymbol{b}\cdot\boldsymbol{u}_t dx.\end{aligned} \tag{3.25}$$

By the Hölder and Young inequalities together with Korn inequality (1.18),

$$
\begin{aligned}
\Big|\int_\Omega \boldsymbol{b}\cdot\boldsymbol{u}_t dx\Big| &\le \frac{\delta_2}{2}\|\boldsymbol{u}_t\|^2_{\boldsymbol{L}_2(\Omega)} + \frac{1}{2\delta_2}\|\boldsymbol{b}\|^2_{\boldsymbol{L}_2(\Omega)}\\
&\le \frac{\delta_2}{2d_1}\|\varepsilon(\boldsymbol{u}_t)\|^2_{\boldsymbol{L}_2(\Omega)} + \frac{1}{2\delta_2}\|\boldsymbol{b}\|^2_{\boldsymbol{L}_2(\Omega)}, \quad \delta_2>0.
\end{aligned}
$$

Hence, setting $\delta_2 = \nu c_* d_1$, it follows from (3.25) that

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \Big[\frac{1}{2}(|\boldsymbol{u}_t|^2 + \gamma|\nabla\chi|^2) + \psi(\chi) + W(\varepsilon(\boldsymbol{u}),\chi)\Big]dx\\
&+\frac{\nu c_*}{2}\|\varepsilon(\boldsymbol{u}_t)\|^2_{\boldsymbol{L}_2(\Omega)} + \|\nabla\mu\|^2_{L_2(\Omega)} \le \frac{1}{2\nu c_* d_2}\|\boldsymbol{b}\|^2_{\boldsymbol{L}_2(\Omega)}.
\end{aligned}
\tag{3.26}
$$

Let us multiply now (3.26) by the constant

$$
d' \equiv \frac{d_2^2}{\gamma} > 0,
\tag{3.27}
$$

and then add by sides with (3.24) to get

$$
\begin{aligned}
&d'\frac{d}{dt}\int_\Omega \Big[\frac{1}{2}(|\boldsymbol{u}_t|^2 + \gamma|\nabla\chi|^2) + \psi(\chi) + W(\varepsilon(\boldsymbol{u}),\chi)\Big]dx\\
&+\int_\Omega\Big[\frac{\gamma}{2}|\nabla\chi|^2 + \psi'(\chi)\chi + W_{,\chi}(\varepsilon(\boldsymbol{u}),\chi)\chi\Big]dx\\
&+\frac{\nu d' c_*}{2}\|\varepsilon(\boldsymbol{u}_t)\|^2_{\boldsymbol{L}_2(\Omega)} + \frac{d'}{2}\|\nabla\mu\|^2_{L_2(\Omega)}\\
&\le \frac{\gamma}{2d_2}|\Omega|\chi_m^2 + \Big|\int_\Omega \mu dx\Big||\chi_m| + \frac{d'}{2\nu c_* d_1}\|\boldsymbol{b}\|^2_{\boldsymbol{L}_2(\Omega)}.
\end{aligned}
\tag{3.28}
$$

Noting that

$$
\int_\Omega \psi'(\chi)\chi dx = \int_\Omega (\chi^4-\chi^2)dx,
$$

and

$$
\int_\Omega \psi(\chi)dx = \frac{1}{4}\int_\Omega(\chi^4-\chi^2)dx + \frac{1}{4}\int_\Omega(1-\chi^2)dx,
$$

we have

$$
\begin{aligned}
\int_\Omega \psi(\chi)dx &= \frac{1}{4}\int_\Omega \psi'(\chi)\chi dx + \frac{1}{4}\int_\Omega(1-\chi^2)dx\\
&\le \frac{1}{4}\int_\Omega \psi'(\chi)\chi dx + \frac{|\Omega|}{4}.
\end{aligned}
\tag{3.29}
$$

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Using (3.29) in (3.28) and dividing the result by $d' > 0$ we arrive at

$$
\begin{aligned}
&\frac{d}{dt}\int\limits_\Omega \left[\frac{1}{2}(|\boldsymbol{u}_t|^2+\gamma|\nabla\chi|^2)+\psi(\chi)+W(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi)\right]dx\\
&+\frac{1}{d'}\int\limits_\Omega \left[\frac{\gamma}{2}|\nabla\chi|^2+4\psi(\chi)+W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi)\chi\right]dx\\
&+\frac{\nu c_*}{2}\|\boldsymbol{\varepsilon}(\boldsymbol{u}_t)\|^2_{L_2(\Omega)}+\frac{1}{2}\|\nabla\mu\|^2_{L_2(\Omega)}\\
&\le \frac{|\chi_m|}{d'}\left|\int\limits_\Omega \mu dx\right|+R_1^2(t),
\end{aligned}
\tag{3.30}
$$

where

$$
R_1^2(t)=\frac{1}{2\nu c_* d_1}\|\boldsymbol{b}(t)\|^2_{L_2(\Omega)}+\frac{|\Omega|}{d'}\left(1+\frac{\gamma}{2d_2}\chi_m^2\right).
$$

Let us consider now the elasticity system (1.1). Multiplying $(1.1)_1$ by $\boldsymbol{u}(t)$, integrating over Ω and by parts using $(1.1)_3$, we get

$$
\begin{aligned}
&\frac{d}{dt}\int\limits_\Omega \boldsymbol{u}_t\cdot\boldsymbol{u}dx+\int\limits_\Omega W_{,\varepsilon}(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi)\cdot\boldsymbol{\varepsilon}(\boldsymbol{u})dx\\
&+\nu\int\limits_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_t)dx=\int\limits_\Omega \boldsymbol{b}\cdot\boldsymbol{u}dx+\int\limits_\Omega |\boldsymbol{u}_t|^2dx.
\end{aligned}
\tag{3.31}
$$

Usung the equality

$$
W_{,\varepsilon}(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi)\cdot\boldsymbol{\varepsilon}(\boldsymbol{u})=2W(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi)+\bar{\boldsymbol{\varepsilon}}(\chi)\cdot\boldsymbol{A}(\boldsymbol{\varepsilon}(\boldsymbol{u})-\bar{\boldsymbol{\varepsilon}}(\chi)),
$$

we write (3.31) in the form

$$
\begin{aligned}
&\frac{d}{dt}\int\limits_\Omega \left(\boldsymbol{u}_t\cdot\boldsymbol{u}+\frac{\nu}{2}\boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u})\right)dx+2\int\limits_\Omega W(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi)dx\\
&=-\int\limits_\Omega \bar{\boldsymbol{\varepsilon}}(\chi)\cdot\boldsymbol{A}(\boldsymbol{\varepsilon}(\boldsymbol{u})-\bar{\boldsymbol{\varepsilon}}(\chi))dx+\int\limits_\Omega \boldsymbol{b}\cdot\boldsymbol{u}dx+\int\limits_\Omega |\boldsymbol{u}_t|^2dx.
\end{aligned}
\tag{3.32}
$$

On account of the following inequalities (due to (2.9), $(2.11)_1$ and (1.18)):

$$
\begin{aligned}
&\int\limits_\Omega W(\boldsymbol{\varepsilon}(u),\chi)dx\ge\frac{c_*}{2}\int\limits_\Omega |\boldsymbol{\varepsilon}(\boldsymbol{u})-\bar{\boldsymbol{\varepsilon}}(\chi)|^2dx,\\
&\int\limits_\Omega W(\boldsymbol{\varepsilon}(u),\chi)dx\ge\frac{c_*}{2}\int\limits_\Omega \left(\frac{1}{2}|\boldsymbol{\varepsilon}(\boldsymbol{u})|^2-|\bar{\boldsymbol{\varepsilon}}(\chi)|^2\right)dx\\
&\ge\frac{c_*}{4}d_1\|\boldsymbol{u}\|^2_{H^1(\Omega)}-\frac{c_*}{2}a_3^2|\Omega|,
\end{aligned}
\tag{3.33}
$$

and

$$\left|\int_\Omega \bar{\varepsilon}(\chi)\cdot \boldsymbol{A}(\varepsilon(\boldsymbol{u})-\bar{\varepsilon}(\chi))dx\right|$$
$$\le \frac{\delta_3}{2}\int_\Omega |\varepsilon(\boldsymbol{u})-\bar{\varepsilon}(\chi)|^2dx + \frac{1}{2\delta_3}a_3^2|\boldsymbol{A}|^2, \quad \delta_3>0,$$

$$\left|\int_\Omega \boldsymbol{b}\cdot\boldsymbol{u}dx\right| \le \frac{\delta_4}{2}\|\boldsymbol{u}\|^2_{\boldsymbol{L}_2(\Omega)} + \frac{1}{2\delta_4}\|\boldsymbol{b}\|^2_{\boldsymbol{L}_2(\Omega)}, \quad \delta_4>0,$$

(3.32) yields

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \Big(\boldsymbol{u}_t\cdot\boldsymbol{u} + \frac{\nu}{2}\varepsilon(\boldsymbol{u})\cdot\boldsymbol{A}\varepsilon(\boldsymbol{u})\Big)dx + \frac{c_*}{4}\int_\Omega |\varepsilon(\boldsymbol{u})-\bar{\varepsilon}(\chi)|^2dx\\
&\quad + \frac{c_*d_1}{8}\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} + \int_\Omega W(\varepsilon(\boldsymbol{u}),\chi)dx\\
&\le \frac{\delta_3}{2}\int_\Omega |\varepsilon(\boldsymbol{u})-\bar{\varepsilon}(\chi)|^2dx + \frac{1}{2\delta_3}a_3^2|\boldsymbol{A}|^2 + \frac{\delta_4}{2}\|\boldsymbol{u}\|^2_{\boldsymbol{L}_2(\Omega)}\\
&\quad + \frac{1}{2\delta_4}\|\boldsymbol{b}\|^2_{\boldsymbol{L}_2(\Omega)} + \frac{c_*a_3^2}{4}|\Omega| + \|\boldsymbol{u}_t\|^2_{\boldsymbol{L}_2(\Omega)}.
\end{aligned} \tag{3.34}$$

Hence, setting $\delta_3 = c_*/4$, $\delta_4 = c_*d_1/8$, we get

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \Big(\boldsymbol{u}_t\cdot\boldsymbol{u} + \frac{\nu}{2}\varepsilon(\boldsymbol{u})\cdot\boldsymbol{A}\varepsilon(\boldsymbol{u})\Big)dx + \frac{c_*}{8}\int_\Omega |\varepsilon(\boldsymbol{u})-\bar{\varepsilon}(\chi)|^2dx\\
&\quad + \frac{c_*d_1}{16}\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} + \int_\Omega W(\varepsilon(\boldsymbol{u}),\chi)dx \le \|\boldsymbol{u}_t\|^2_{\boldsymbol{L}_2(\Omega)} + R_2^2(t),
\end{aligned} \tag{3.35}$$

where

$$R_2^2(t) = \frac{4}{c_*d_1}\|\boldsymbol{b}(t)\|^2_{\boldsymbol{L}_2(\Omega)} + \frac{2a_3^2}{c_*}|\boldsymbol{A}|^2 + \frac{c_*a_3^2}{4}|\Omega|.$$

Now we multiply (3.35) by a constant $\delta_5>0$ (to be chosen later on) and add by sides with (3.30) to arrive, after using Korn's inequality (1.18), at

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \Big[\frac{1}{2}(|\boldsymbol{u}_t|^2+\gamma|\nabla\chi|^2)+\psi(\chi)+W(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi)\\
&\quad+\delta_5\Big(\boldsymbol{u}_t\cdot\boldsymbol{u}+\frac{\nu}{2}\boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u})\Big)\Big]dx\\
&\quad+\frac{1}{d'}\int_\Omega\Big[\frac{\gamma}{2}|\nabla\chi|^2+4\psi(\chi)+\frac{\delta_5 c_* d'}{8}|\boldsymbol{\varepsilon}(\boldsymbol{u})-\bar{\boldsymbol{\varepsilon}}(\chi)|^2\\
&\quad+\frac{\delta_5 c_* d_1 d'}{16}(|\boldsymbol{u}|^2+|\nabla\boldsymbol{u}|^2)+\delta_5 d' W(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi)\Big]dx\\
&\quad+\frac{\nu c_* d_1}{2}\|\boldsymbol{u}_t\|^2_{\boldsymbol{H}^1(\Omega)}+\frac{1}{2}\|\nabla\mu\|^2_{L_2(\Omega)}\\
&\le\frac{1}{d'}\Big|\int_\Omega W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi)\chi dx\Big|+\frac{|\chi_m|}{d'}\Big|\int_\Omega\mu dx\Big|+\delta_5\|\boldsymbol{u}_t\|^2_{\boldsymbol{L}_2(\Omega)}\\
&\quad+R_1^2(t)+\delta_5 R_2^2(t).
\end{aligned}\tag{3.36}
$$

Our goal now is to estimate the first two integrals on the right-hand side of (3.36). For the first one, on account of $(2.11)_2$, we have

$$
\begin{aligned}
\frac{1}{d'}\Big|\int_\Omega W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi)\chi dx\Big| &\le\frac{1}{d'}a_4|\boldsymbol{A}|\int_\Omega|\boldsymbol{\varepsilon}(\boldsymbol{u})-\bar{\boldsymbol{\varepsilon}}(\chi)|dx\\
&\le\frac{1}{d'}a_4|\boldsymbol{A}||\Omega|^{1/2}\|\boldsymbol{\varepsilon}(\boldsymbol{u})-\bar{\boldsymbol{\varepsilon}}(\chi)\|_{\boldsymbol{L}_2(\Omega)}.
\end{aligned}\tag{3.37}
$$

For the second integral, using the identity

$$
\int_\Omega\mu dx=\int_\Omega(\psi'(\chi)+W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi))dx,\tag{3.38}
$$

we have

$$
\Big|\int_\Omega\mu dx\Big|\le\Big|\int_\Omega\psi'(\chi)dx\Big|+a_4|\boldsymbol{A}||\Omega|^{1/2}\|\boldsymbol{\varepsilon}(\boldsymbol{u})-\bar{\boldsymbol{\varepsilon}}(\chi)\|_{\boldsymbol{L}_2(\Omega)}.
$$

With the use of (3.2),

$$
\Big|\int_\Omega\psi'(\chi)dx\Big|=\Big|\int_\Omega(\chi^3-\chi)dx\Big|=\Big|\int_\Omega\chi^3dx-|\Omega|\chi_m\Big|\le\int_\Omega|\chi^3|dx+|\Omega||\chi_m|.
$$

Futher, by the Young inequality

$$
\int_\Omega|\chi|^3dx\le\frac{3}{4}\delta_6^{4/3}\int_\Omega\chi^4dx+\frac{1}{4\delta_6^4}|\Omega|,\quad\delta_6>0,
$$

and the fact that (see (2.10)) $\chi^4 \le 8\psi(\chi) + 2$, we deduce that

$$\left| \int_\Omega \psi'(\chi) dx \right| \le 6\delta_6^{4/3} \int_\Omega \psi(\chi) dx + \left(\frac{3}{2}\delta_6^{4/3} + \frac{1}{4\delta_6^4} \right) |\Omega|.$$

Consequently,

$$\begin{aligned} \left| \int_\Omega \mu dx \right| &\le a_4 |\boldsymbol{A}| |\Omega|^{1/2} \|\boldsymbol{\varepsilon}(\boldsymbol{u}) - \bar{\boldsymbol{\varepsilon}}(\chi)\|_{\boldsymbol{L}_2(\Omega)} \\ &+ 6\delta_6^{4/3} \int_\Omega \psi(\chi) dx + \left(\frac{3}{2}\delta_6^{4/3} + \frac{1}{4\delta_6^4} \right) |\Omega|. \end{aligned} \tag{3.39}$$

Using estimates (3.37) and (3.39) in (3.36), and then choosing constants δ_5 and δ_6 so that

$$\delta_5 \le \frac{\nu c_* d_1}{4}, \quad \frac{6|\chi_m|}{d'} \delta_6^{4/3} = \frac{2}{d'}, \quad \text{so} \quad \delta_6 = \frac{1}{(3|\chi_m|)^{3/4}}, \tag{3.40}$$

we conclude the inequality

$$\begin{aligned} &\frac{d}{dt} \int_\Omega \left[\frac{1}{2}(|\boldsymbol{u}_t|^2 + \gamma |\nabla \chi|^2) + \psi(\chi) + W(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi) \right. \\ &\left. + \delta_5 \left(\boldsymbol{u}_t \cdot \boldsymbol{u} + \frac{\nu}{2} \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{A} \boldsymbol{\varepsilon}(\boldsymbol{u}) \right) \right] dx \\ &+ \frac{1}{d'} \int_\Omega \left[\frac{\gamma}{2} |\nabla \chi|^2 + 2\psi(\chi) + \frac{\delta_5 c_* d'}{8} |\boldsymbol{\varepsilon}(\boldsymbol{u}) - \bar{\boldsymbol{\varepsilon}}(\chi)|^2 \right. \\ &\left. + \frac{\delta_5 c_* d_1 d'}{16} (|\boldsymbol{u}|^2 + |\nabla \boldsymbol{u}|^2) + \delta_5 d' W(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi) \right] dx \\ &+ \frac{\nu c_* d_1}{4} \|\boldsymbol{u}_t\|^2_{\boldsymbol{H}^1(\Omega)} + \frac{1}{2} \|\nabla \mu\|^2_{L_2(\Omega)} \\ &\le \frac{1}{d'} a_4 |\boldsymbol{A}| |\Omega|^{1/2} (|\chi_m| + 1) \|\boldsymbol{\varepsilon}(\boldsymbol{u}) - \bar{\boldsymbol{\varepsilon}}(\chi)\|_{\boldsymbol{L}_2(\Omega)} + R_3^2(t), \end{aligned} \tag{3.41}$$

where

$$R_3^2(t) = R_1^2(t) + \delta_5 R_2^2(t) + \frac{2 + 27\chi_m^4}{4d'} |\Omega|.$$

Finally, using the estimate

$$\begin{aligned} &\frac{1}{d'} a_4 |\boldsymbol{A}| |\Omega|^{1/2} (|\chi_m| + 1) \|\boldsymbol{\varepsilon}(\boldsymbol{u}) - \bar{\boldsymbol{\varepsilon}}(\chi)\|_{\boldsymbol{L}_2(\Omega)} \\ &\le \frac{\delta_7}{2} \|\boldsymbol{\varepsilon}(\boldsymbol{u}) - \bar{\boldsymbol{\varepsilon}}(\chi)\|^2_{\boldsymbol{L}_2(\Omega)} + \frac{1}{\delta_7} \frac{a_4^2 |\boldsymbol{A}|^2 |\Omega| (\chi_m^2 + 1)}{(d')^2} \end{aligned}$$

and setting $\delta_7 = \frac{\delta_5 c_*}{8}$, inequality (3.41) is reduced to

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \Big[\frac{1}{2}(|\boldsymbol{u}_t|^2+\gamma|\nabla\chi|^2)+\psi(\chi)+W(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi)\\
&+\delta_5\Big(\boldsymbol{u}_t\cdot\boldsymbol{u}+\frac{\nu}{2}\boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u})\Big)\Big]\,dx\\
&+\frac{1}{d'}\int_\Omega\Big[\frac{\gamma}{2}|\nabla\chi|^2+2\psi(\chi)+\frac{\delta_5 c_* d'}{16}|\boldsymbol{\varepsilon}(\boldsymbol{u})-\bar{\boldsymbol{\varepsilon}}(\chi)|^2\\
&+\frac{\delta_5 c_* d_1 d'}{16}(|\boldsymbol{u}|^2+|\nabla\boldsymbol{u}|^2)+\delta_5 d' W(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi)\Big]\,dx\\
&+\frac{\nu c_* d_1}{4}\|\boldsymbol{u}_t\|^2_{\boldsymbol{H}^1(\Omega)}+\frac{1}{2}\|\nabla\mu\|^2_{L_2(\Omega)}\le R_4^2(t),
\end{aligned}
\tag{3.42}
$$

where

$$
R_4^2(t)=R_3^2(t)+\frac{8}{\delta_5 c_*}\frac{a_4^2|A|^2|\Omega|(\chi_m^2+1)}{(d')^2}.
$$

Let us define the function

$$
\begin{aligned}
G_{\delta_5}(t)=&\int_\Omega\Big[\frac{1}{2}(|\boldsymbol{u}_t|^2+\gamma|\nabla\chi|^2)+\psi(\chi)+W(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi)\\
&+\delta_5\Big(\boldsymbol{u}_t\cdot\boldsymbol{u}+\frac{\nu}{2}\boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u})\Big)\Big]\,dx
\end{aligned}
\tag{3.43}
$$

with constant $\delta_5 > 0$ to be selected below.

Now, let us choose constant $\delta_8 > 0$ in such a way that the sum of the second and the half of the third term on the left-hand side of (3.42) is bounded from below by $\delta_8 G_{\delta_5}(t)$, more precisely so that to satisfy the bound

$$
\begin{aligned}
\delta_8 G_{\delta_5}(t)\le&\frac{1}{d'}\int_\Omega\Big[\frac{\gamma}{2}|\nabla\chi|^2+2\psi(\chi)+\frac{\delta_5 c_* d'}{16}|\boldsymbol{\varepsilon}(\boldsymbol{u})-\bar{\boldsymbol{\varepsilon}}(\chi)|^2\\
&+\frac{\delta_5 c_* d_1 d'}{16}(|\boldsymbol{u}|^2+|\nabla\boldsymbol{u}|^2)+\delta_5 d' W(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi)\\
&+\frac{\nu c_* d_1 d'}{8}(|\boldsymbol{u}_t|^2+|\nabla\boldsymbol{u}_t|^2)\Big]\,dx.
\end{aligned}
\tag{3.44}
$$

The bound (3.44) holds true provided the following conditions:

$$\frac{\delta_8}{2}|\boldsymbol{u}_t|^2 \le \frac{\nu c_* d_1}{16}|\boldsymbol{u}_t|^2, \qquad \text{so} \quad \delta_8 \le \frac{\nu c_* d_1}{8};$$
$$\frac{\delta_8}{2}\gamma|\nabla\chi|^2 \le \frac{\gamma}{2d'}|\nabla\chi|^2, \qquad \text{so} \quad \delta_8 \le \frac{1}{d'};$$
$$\delta_8\psi(\chi) \le \frac{2}{d'}\psi(\chi), \qquad \text{so} \quad \delta_8 \le \frac{2}{d'};$$
$$\delta_8 W(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi) \le \delta_5 W(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi), \qquad \text{so} \quad \delta_8 \le \delta_5;$$
$$\delta_8\delta_5 \boldsymbol{u}_t\cdot\boldsymbol{u} \le \delta_8\delta_5\left(\frac{\delta_9}{2}|\boldsymbol{u}_t|^2 + \frac{1}{2\delta_9}|\boldsymbol{u}|^2\right) \le \frac{\nu c_* d_1}{16}|\boldsymbol{u}_t|^2 + \frac{\delta_5 c_* d_1}{16}|\boldsymbol{u}|^2,$$
$$\text{so, for example,} \quad \delta_9 = \frac{\nu c_* d_1}{8\delta_8\delta_5} \quad \text{and} \quad \delta_8^2 \le \frac{\nu c_*^2 d_1^2}{64\delta_5};$$
$$\delta_8\delta_5\frac{\nu}{2}\boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) \le \frac{\delta_5 c_* d_1}{16}|\nabla\boldsymbol{u}|^2, \qquad \text{so, by (2.2),} \quad \delta_8 \le \frac{d_1}{8\nu}.$$

Consequently, choosing

$$\delta_8 = \min\left\{\frac{\nu c_* d_1}{8},\ \frac{1}{d'},\ \delta_5,\ \frac{c_* d_1}{8}\sqrt{\frac{\nu}{\delta_5}},\ \frac{d_1}{8\nu}\right\}, \tag{3.45}$$

we conclude from (3.42) the inequality

$$\frac{d}{dt}G_{\delta_5}(t) + \delta_8 G_{\delta_5}(t) + \frac{\nu c_* d_1}{8}\|\boldsymbol{u}_t\|^2_{H^1(\Omega)} + \frac{1}{2}\|\nabla\mu\|^2_{L_2(\Omega)} \le R_4^2(t). \tag{3.46}$$

Finally, we choose constant $\delta_5 > 0$ in such a way to satisfy the bound

$$G_{\delta_5}(t) \ge \int\limits_\Omega \left[\frac{1}{4}|\boldsymbol{u}_t|^2 + W(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi) + \psi(\chi) + \frac{\gamma}{2}|\nabla\chi|^2\right]dx. \tag{3.47}$$

To this purpose let us note that since (see (2.2), (1.18))

$$\int\limits_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u})dx \ge c_* d_1\int\limits_\Omega |\boldsymbol{u}|^2 dx, \tag{3.48}$$

it follows with the help of the Young inequality that

$$\begin{aligned} G_{\delta_5}(t) \ge \int\limits_\Omega &\left[\frac{1}{2}|\boldsymbol{u}_t|^2 + W(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi) + \psi(\chi) + \frac{\gamma}{2}|\nabla\chi|^2\right. \\ &\left. - \delta_5\left(\frac{\delta_{10}}{2}|\boldsymbol{u}_t|^2 + \frac{1}{2\delta_{10}}|\boldsymbol{u}|^2\right) + \delta_5\frac{\nu c_* d_1}{2}|\boldsymbol{u}|^2\right]dx, \end{aligned} \tag{3.49}$$

where $\delta_{10} > 0$. Hence, choosing $\frac{\delta_5\delta_{10}}{2} = \frac{1}{4}$ and $\frac{\delta_5}{2\delta_{10}} = \frac{\delta_5 \nu c_* d_1}{2}$, that is

$$(3.50) \qquad \delta_{10} = \frac{1}{\nu c_* d_1} \quad \text{and} \quad \delta_5 = \frac{\nu c_* d_1}{2},$$

inequality (3.49) guarantees the bound (3.47). Besides,

$$(3.51) \qquad G_{\delta_5}(t) \equiv G(t)$$

with $G(t)$ defined in (3.14). Hence, (3.47) implies the property (3.15). Moreover, with the above choice of δ_5 the condition (3.45) reduces to

$$(3.52) \qquad \beta_1 \equiv \delta_8 = \min\left\{\frac{\nu c_* d_1}{8}, \frac{1}{d'}, \frac{1}{4}\sqrt{\frac{c_* d_1}{2}}, \frac{d_1}{8\nu}\right\}$$

which yields (3.16). Consequently, inequality (3.46) takes the form

$$(3.53) \qquad \begin{aligned} &\frac{d}{dt}G(t) + \beta_1 G(t) + \frac{\nu c_* d_1}{8}\|\boldsymbol{u}_t(t)\|^2_{H^1(\Omega)} + \frac{1}{2}\|\nabla\mu(t)\|^2_{L_2(\Omega)} \\ &\le R_4^2(t) \equiv \Lambda_1\|\boldsymbol{b}(t)\|^2_{L_2(\Omega)} + \Lambda_2, \end{aligned}$$

where

$$\begin{aligned} R_4^2(t) &= R_3^2(t) + \frac{16a_4^2|\boldsymbol{A}|^2|\Omega|(\chi_m^2+1)}{\nu c_*^2 d_1 (d')^2}, \\ R_3^2(t) &= R_1^2(t) + \frac{\nu c_* d_1}{2}R_2^2(t) + \frac{2+27\chi_m^4}{4d'}|\Omega|, \\ R_2^2(t) &= \frac{4}{c_* d_1}\|\boldsymbol{b}(t)\|^2_{L_2(\Omega)} + \frac{2a_3^2|\boldsymbol{A}|^2}{c_*} + \frac{c_* a_3^2|\Omega|}{4}, \\ R_1^2(t) &= \frac{1}{2\nu c_* d_1}\|\boldsymbol{b}(t)\|^2_{L_2(\Omega)} + \frac{|\Omega|}{d'}\left(1 + \frac{\gamma}{2d_2}\chi_m^2\right). \end{aligned}$$

This shows (3.17), (3.18) and thereby completes the proof. □

From Lemma 3.3 we deduce an absorbing estimate on the interval $[kT, (k+1)T]$, $k = 0$.

Lemma 3.4. *Let $G(t)$ be defined by (3.14), $G(0) < \infty$, and*

$$(3.54) \qquad b_{10} = \sup_{t\in[0,T]} \|\boldsymbol{b}(t)\|_{L_2(\Omega)} < \infty.$$

The function $G(t)$ is Lipschitz continuous on $[0, T]$ and satisfies the following estimates:

$$(3.55) \qquad G(t) \le A_{10}(1 - e^{-\beta_1 t}) + G(0)e^{-\beta_1 t},$$

and

$$G(t) + \beta_1 \int_0^t G(t')dt' + \frac{\nu c_* d_1}{8} \int_0^t \|\boldsymbol{u}_{t'}\|^2_{\boldsymbol{H}^1(\Omega)} dt'$$

(3.56)

$$+ \frac{1}{2} \int_0^t \|\nabla \mu\|^2_{L_2(\Omega)} dt' \le G(0) + \beta_1 A_{10} t \quad for \quad t \in [0,T],$$

with positive constant

$$A_{10} = \frac{1}{\beta_1}(\Lambda_1 b_{10}^2 + \Lambda_2),$$

and constants $\beta_1, \Lambda_1, \Lambda_2$ from Lemma 3.3.

Proof. From (3.17) it follows that for any $t_1, t_2 \in [0,T]$,

$$|G(t_1) - G(t_2)| \le (\Lambda_1 \sup_{t \in [0,T]} \|\boldsymbol{b}(t)\|^2_{\boldsymbol{L}_2(\Omega)} + \Lambda_2)|t_1 - t_2| \le \beta_1 A_{10} |t_1 - t_2|,$$

which shows Lipschitz continuity of $G(t)$.

Estimate (3.55) results on account of the classical Gronwall lemma (multiplying (3.17) by $\exp(\beta_1 t)$ and integrating from $t = 0$ to $t \in [0,T]$). Estimate (3.56) follows directly by integrating (3.17) from $t = 0$ to $t \in [0,T]$. □

In view of the bound

(3.57)

$$G(t) \ge \frac{1}{4}\|\boldsymbol{u}_t(t)\|^2_{\boldsymbol{L}_2(\Omega)} + \frac{\gamma}{2}\|\nabla\chi(t)\|^2_{L_2(\Omega)} + \frac{1}{8}\|\chi(t)\|^4_{L_4(\Omega)} + \frac{c_*}{4}\|\varepsilon(\boldsymbol{u}(t))\|^2_{L_2(\Omega)} - c_1',$$

where $c_1' = \frac{|\Omega|}{2}\left(c_* a_3^2 + \frac{1}{2}\right)$, resulting from (3.15) and (3.13), we deduce from (3.55) the following

Corollary 3.1. *Let (A1)–(A6) hold, $G(t)$ be given by (3.14), $G(0) < \infty$, and $\boldsymbol{b} \in L_\infty(\mathbb{R}_+; \boldsymbol{L}_2(\Omega))$. Then*

(3.58)

$$\frac{1}{4}\|\boldsymbol{u}_t(t)\|^2_{\boldsymbol{L}_2(\Omega)} + \frac{\gamma}{2}\|\nabla\chi(t)\|^2_{L_2(\Omega)} + \frac{1}{8}\|\chi(t)\|^2_{L_4(\Omega)} + \frac{c_*}{4}\|\varepsilon(\boldsymbol{u}(t))\|^2_{L_2(\Omega)} \le G(t) + c_1'$$

$$\le A_{10} + G(0) + c_1' \le A_1 + G(0) + c_1' \equiv c_1 \quad for \quad t \in [0,T],$$

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where

$$A_1 = \frac{1}{\beta_1}(\Lambda_1 \sup_{t\in\mathbb{R}_+} \|\boldsymbol{b}(t)\|^2_{L_2(\Omega)} + \Lambda_2),$$
$$c_1' = \frac{|\Omega|}{2}\left(c_* a_3^2 + \frac{1}{2}\right).$$

Remark 3.2. Estimate (3.58) depends on the initial conditon $G(0)$. To prove global existence we shall consider problem (1.14)–(1.16) on the subsequent time intervals $[kT, (k+1)T]$, $k \in \mathbb{N} \cup \{0\}$. Of key importance will be the fact that constant c_1 is independent of the time step k (see (5.3)).

In Section 6, dealing with the absorbing set property, we shall use another version of estimate (3.58) which is independent of the initial condition and holds for sufficiently large t.

4. Regularity estimates of absorbing type

In this section we derive a differential inequality which allow us to deduce regularity estimates of absorbing type. The inequality has the form (see (4.50))

$$\begin{aligned}&\frac{d}{dt}N(t) + \beta_4 N(t) + \text{ nonnegative terms}\\ &\le \Lambda_3\|\boldsymbol{b}(t)\|^2_{L_2(\Omega)} + \Lambda_4\|\boldsymbol{b}_t(t)\|^2_{L_2(\Omega)} + \Lambda_5 \quad \text{for } \ t \in (0,T),\end{aligned}$$

where $N(t)$ is an appropriately constructed nonnegative function (see (4.57)), being a linear combination of the modified energy $G(t)$ in (3.14) and the norms

$$\|\boldsymbol{Q}\boldsymbol{u}(t)\|^2_{L_2(\Omega)}, \quad \|\boldsymbol{Q}^{1/2}\boldsymbol{u}_t(t)\|^2_{L_2(\Omega)}, \quad \|\boldsymbol{Q}\boldsymbol{u}_t(t)\|^2_{L_2(\Omega)}, \quad \|\boldsymbol{Q}^{1/2}\boldsymbol{u}_{tt}(t)^2_{L_2(\Omega)},$$
$$\|\chi(t)\|^2_{L_2(\Omega)}, \qquad \|\Delta\chi(t)\|^2_{L_2(\Omega)}, \qquad \|\chi_t(t)\|^2_{L_2(\Omega)}$$

with coefficients depending on the constant c_1 from energy estimate (3.58). Moreover, $\beta_4, \Lambda_3, \Lambda_4$ and Λ_5 are positive constants depending on c_1 as well.

The derivation of such inequality is based on differentiating system (1.14)–(1.16) with respect to time variable. The procedure consists of four main steps. In the first step (see Lemma 4.1) we derive a differential inequality for elasticity system (1.14). The right-hand side of this inequality includes the terms $\|\nabla\chi(t)\|^2_{L_2(\Omega)} + \|\chi_t(t)\nabla\chi(t)\|^2_{L_2(\Omega)} + \|\nabla\chi_t(t)\|^2_{L_2(\Omega)}$ which arise due to differentiation in time.

In the second step we consider system (1.15), (1.16) rewritten in the Cahn-Hilliard form (1.17). We derive a differential inequality (see Lemma

4.3) which is appropriate to handle the above mentioned terms in the previous inequality.

In the third step we combine the inequalities from the first and the second step to obtain a differential inequality with higher order terms (see Lemma 4.4). The right-hand side of this inequality includes the term $\|\varepsilon(\boldsymbol{u}_t)(t)\|^2_{\boldsymbol{L}_2(\Omega)}$.

To absorb this term, in the fourth step (see Lemma 4.5) we combine the latter inequality with the differential inequality for the energy $G(t)$, derived in Lemma 3.3.

In this section we use the simplified formulation (1.14)–(1.15) of system (1.1)–(1.3). Moreover, we assume that hypotheses (A1)–(A6) are satisfied.

4.1. Estimates for $\boldsymbol{u}$

Lemma 4.1. *Assume that*

$$\begin{aligned}\sup_{0\le t\le T}(\|\nabla\chi(t)\|_{\boldsymbol{L}_2(\Omega)} + \|\chi_t(t)\nabla\chi(t)\|_{\boldsymbol{L}_2(\Omega)} + \|\nabla\chi_t(t)\|_{\boldsymbol{L}_2(\Omega)}) < \infty,\\ \sup_{0\le t\le T}(\|\boldsymbol{b}(t)\|_{\boldsymbol{L}_2(\Omega)} + \|\boldsymbol{b}_t(t)\|_{\boldsymbol{L}_2(\Omega)}) < \infty.\end{aligned} \tag{4.1}$$

Let $H : [0,\infty) \to [0,\infty)$ be the function defined by

$$\begin{aligned}H(t) = (\nu+1)\|\boldsymbol{Q}\boldsymbol{u}(t)\|^2_{\boldsymbol{L}_2(\Omega)} + \|\boldsymbol{Q}^{1/2}\boldsymbol{u}_t(t)\|^2_{\boldsymbol{L}_2(\Omega)}\\ + \frac{16d_3^2}{\nu\bar{\mu}^2}(\|\boldsymbol{Q}\boldsymbol{u}_t(t)\|^2_{\boldsymbol{L}_2(\Omega)} + \|\boldsymbol{Q}^{1/2}\boldsymbol{u}_{tt}(t)\|^2_{\boldsymbol{L}_2(\Omega)}).\end{aligned} \tag{4.2}$$

Then there exists a positive constant

$$\beta_2 = \min\left\{\frac{1}{\nu+1},\ \frac{\nu\bar{\mu}}{4d_3},\ \frac{\nu^2\bar{\mu}^2}{32d_3^2}\right\} \tag{4.3}$$

such that solutions $\boldsymbol{u}$ of system (1.14) satisfy the differential inequality

$$\begin{aligned}&\frac{d}{dt}H(t) + \beta_2 H(t) + \|\boldsymbol{u}_{tt}(t)\|^2_{\boldsymbol{L}_2(\Omega)} + \frac{8d_3^2}{\bar{\mu}^2}\|\boldsymbol{Q}\boldsymbol{u}_{tt}(t)\|^2_{\boldsymbol{L}_2(\Omega)}\\ &\le a_5^2\left(3+\frac{2}{\nu}\right)\|\nabla\chi(t)\|^2_{\boldsymbol{L}_2(\Omega)} + \frac{48a_5^2d_3^2}{\nu^2\bar{\mu}^2}(\|\chi_t(t)\nabla\chi(t)\|^2_{\boldsymbol{L}_2(\Omega)}\\ &+ \|\nabla\chi_t(t)\|^2_{\boldsymbol{L}_2(\Omega)}) + \left(3+\frac{2}{\nu}\right)\|\boldsymbol{b}(t)\|^2_{\boldsymbol{L}_2(\Omega)} + \frac{48d_3^2}{\nu^2\bar{\mu}^2}\|\boldsymbol{b}_t(t)\|^2_{\boldsymbol{L}_2(\Omega)}\end{aligned} \tag{4.4}$$

for $t \in (0,T)$.

Prior to presenting the proof of this lemma we prepare

Lemma 4.2. *Let the operator $\boldsymbol{Q}$ be defined by (1.11). Then the following estimates hold true*

$$\text{(4.5)} \qquad \|\boldsymbol{Q}^{1/2}\boldsymbol{u}\|^2_{L_2(\Omega)} \le \frac{d_3}{\bar{\mu}}\|\boldsymbol{Q}\boldsymbol{u}\|^2_{L_2(\Omega)}, \quad \|\boldsymbol{u}\|^2_{L_2(\Omega)} \le \frac{d_3^2}{\bar{\mu}^2}\|\boldsymbol{Q}\boldsymbol{u}\|^2_{L_2(\Omega)}$$

for all $\boldsymbol{u} \in D(\boldsymbol{Q})$.

Proof. Let us consider the elliptic problem

$$\text{(4.6)} \qquad \begin{aligned} &\bar{\mu}\Delta\boldsymbol{u} + (\bar{\lambda} + \bar{\mu})\nabla(\nabla\cdot\boldsymbol{u}) = \boldsymbol{Q}\boldsymbol{u} && \text{in } \Omega, \\ &\boldsymbol{u} = \boldsymbol{0} && \text{on } S, \end{aligned}$$

where the right-hand side $\boldsymbol{Q}\boldsymbol{u}$ is treated as a given datum. Multiplying $(4.6)_1$ by $\boldsymbol{u}$, integrating by parts and recalling the definition of $\boldsymbol{Q}^{1/2}$ (see (2.4)) gives

$$\text{(4.7)} \quad \int_\Omega |Q^{1/2}\boldsymbol{u}|^2 dx \equiv \int_\Omega [\bar{\mu}|\nabla\boldsymbol{u}|^2 + (\bar{\lambda} + \bar{\mu})|\nabla\cdot\boldsymbol{u}|^2]dx = -\int_\Omega \boldsymbol{Q}\boldsymbol{u}\cdot\boldsymbol{u}dx.$$

With the help of the Hölder and Young inequalities together with the Poincaré-Friedrichs inequality (1.20) applied to the right-hand side of (4.7), we conclude that

$$\text{(4.8)} \qquad \begin{aligned} &\|\boldsymbol{Q}^{1/2}\boldsymbol{u}\|^2_{L_2(\Omega)} = \bar{\mu}\|\nabla\boldsymbol{u}\|^2_{L_2(\Omega)} + (\bar{\lambda} + \bar{\mu})\|\nabla\cdot\boldsymbol{u}\|^2_{L_2(\Omega)} \\ &\le \frac{\delta_1 d_3}{2}\|\nabla\boldsymbol{u}\|^2_{L_2(\Omega)} + \frac{1}{2\delta_1}\|\boldsymbol{Q}\boldsymbol{u}\|^2_{L_2(\Omega)}, \quad \delta_1 > 0. \end{aligned}$$

Thus, choosing δ_1 so that $\delta_1 d_3 = \bar{\mu}$, we have
$\|\boldsymbol{Q}^{1/2}\boldsymbol{u}\|^2_{L_2(\Omega)} \le \frac{d_3}{\bar{\mu}}\|\boldsymbol{Q}\boldsymbol{u}\|^2_{L_2(\Omega)}$, which proves $(4.5)_1$.

To show $(4.5)_2$, let us note that by the Poincaré-Friedrichs inequality (1.20) and $(4.5)_1$ it follows immediately that

$$\text{(4.9)} \quad \frac{\bar{\mu}}{d_3}\|\boldsymbol{u}\|^2_{L_2(\Omega)} \le \bar{\mu}\|\nabla\boldsymbol{u}\|^2_{L_2(\Omega)} \le \|\boldsymbol{Q}^{1/2}\boldsymbol{u}\|^2_{L_2(\Omega)} \le \frac{d_3}{\bar{\mu}}\|\boldsymbol{Q}\boldsymbol{u}\|^2_{L_2(\Omega)},$$

that is $(4.5)_2$. This completes the proof. □

Proof of Lemma 4.1. Multiplying $(1.14)_1$ by $\boldsymbol{Q}\boldsymbol{u}(t)$ and integrating over Ω gives

$$\text{(4.10)} \qquad \begin{aligned} \frac{\nu}{2}\frac{d}{dt}\|\boldsymbol{Q}\boldsymbol{u}\|^2_{L_2(\Omega)} + \|\boldsymbol{Q}\boldsymbol{u}\|^2_{L_2(\Omega)} &= \int_\Omega \boldsymbol{u}_{tt}\cdot\boldsymbol{Q}\boldsymbol{u}dx \\ &- \int_\Omega (z'(\chi)\boldsymbol{B}\nabla\chi + \boldsymbol{b})\cdot\boldsymbol{Q}\boldsymbol{u}dx. \end{aligned}$$

With the help of the Hölder and Young inequalities, using $(2.11)_3$, the right-hand side of (4.10) is estimated by

$$\frac{3\delta_1}{2}\int_\Omega |\boldsymbol{Q}\boldsymbol{u}|^2dx+\frac{1}{2\delta_1}\int_\Omega |\boldsymbol{u}_{tt}|^2dx+\frac{a_5^2}{2\delta_1}\int_\Omega |\nabla\chi|^2dx+\frac{1}{2\delta_1}\int_\Omega |\boldsymbol{b}|^2dx, \quad \delta_1>0.$$

Hence, setting $\delta_1=\frac{1}{3}$, it follows that

$$\tag{4.11} \begin{aligned} &\frac{\nu}{2}\frac{d}{dt}\|\boldsymbol{Q}\boldsymbol{u}\|^2_{\boldsymbol{L}_2(\Omega)}+\frac{1}{2}\|\boldsymbol{Q}\boldsymbol{u}\|^2_{\boldsymbol{L}_2(\Omega)}\le\frac{3}{2}\|\boldsymbol{u}_{tt}\|^2_{\boldsymbol{L}_2(\Omega)}\\ &\quad+\frac{3}{2}a_5^2\|\nabla\chi\|^2_{\boldsymbol{L}_2(\Omega)}+\frac{3}{2}\|\boldsymbol{b}\|^2_{\boldsymbol{L}_2(\Omega)}. \end{aligned}$$

Now let us multiply $(1.14)_1$ by $\boldsymbol{Q}\boldsymbol{u}_t(t)$, integrate over Ω and by parts using $(1.14)_3$, to get

$$\tag{4.12} \begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\boldsymbol{Q}^{1/2}\boldsymbol{u}_t\|^2_{\boldsymbol{L}_2(\Omega)}+\|\boldsymbol{Q}\boldsymbol{u}\|^2_{\boldsymbol{L}_2(\Omega)})+\nu\|\boldsymbol{Q}\boldsymbol{u}_t\|^2_{\boldsymbol{L}_2(\Omega)}\\ &=-\int_\Omega (z'(\chi)\boldsymbol{B}\nabla\chi+\boldsymbol{b})\cdot\boldsymbol{Q}\boldsymbol{u}_t dx. \end{aligned}$$

Again, by the Hölder and Young inequalities, using $(2.11)_3$, the right-hand side of (4.12) is estimated by

$$\delta_2\int_\Omega |\boldsymbol{Q}\boldsymbol{u}_t|^2dx+\frac{a_5^2}{2\delta_2}\int_\Omega |\nabla\chi|^2dx+\frac{1}{2\delta_2}\int_\Omega |\boldsymbol{b}|^2dx, \quad \delta_2>0.$$

Hence, setting $\delta_2=\frac{\nu}{2}$ we conclude that

$$\tag{4.13} \begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\boldsymbol{Q}^{1/2}\boldsymbol{u}_t\|^2_{\boldsymbol{L}_2(\Omega)}+\|\boldsymbol{Q}\boldsymbol{u}\|^2_{\boldsymbol{L}_2(\Omega)})+\frac{\nu}{2}\|\boldsymbol{Q}\boldsymbol{u}_t\|^2_{\boldsymbol{L}_2(\Omega)}\\ &\le\frac{a_5^2}{\nu}\|\nabla\chi\|^2_{\boldsymbol{L}_2(\Omega)}+\frac{1}{\nu}\|\boldsymbol{b}\|^2_{\boldsymbol{L}_2(\Omega)}. \end{aligned}$$

Since, on account of Lemma 4.2,

$$\tag{4.14} \|\boldsymbol{Q}^{1/2}\boldsymbol{u}_t\|^2_{\boldsymbol{L}_2(\Omega)}\le\frac{d_3}{\bar{\mu}}\|\boldsymbol{Q}\boldsymbol{u}_t\|^2_{\boldsymbol{L}_2(\Omega)},$$

inequality (4.13) yields

$$\tag{4.15} \begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\boldsymbol{Q}^{1/2}\boldsymbol{u}_t\|^2_{\boldsymbol{L}_2(\Omega)}+\|\boldsymbol{Q}\boldsymbol{u}\|^2_{\boldsymbol{L}_2(\Omega)})+\frac{\nu}{4}\|\boldsymbol{Q}\boldsymbol{u}_t\|^2_{\boldsymbol{L}_2(\Omega)}\\ &\quad+\frac{\nu\bar{\mu}}{4d_3}\|\boldsymbol{Q}^{1/2}\boldsymbol{u}_t\|^2_{\boldsymbol{L}_2(\Omega)}\le\frac{a_5^2}{\nu}\|\nabla\chi\|^2_{\boldsymbol{L}_2(\Omega)}+\frac{1}{\nu}\|\boldsymbol{b}\|^2_{\boldsymbol{L}_2(\Omega)}. \end{aligned}$$

Now, let us differentiate $(1.14)_1$ with respect to t, multiply by $\boldsymbol{Q}\boldsymbol{u}_{tt}(t)$, integrate over Ω and by parts to get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\boldsymbol{Q}^{1/2}\boldsymbol{u}_{tt}\|^2_{L_2(\Omega)} + \|\boldsymbol{Q}\boldsymbol{u}_t\|^2_{L_2(\Omega)}) + \nu\|\boldsymbol{Q}\boldsymbol{u}_{tt}\|^2_{L_2(\Omega)} \\
&= -\int_\Omega [(z'(\chi)\boldsymbol{B}\nabla\chi)_{,t} + \boldsymbol{b}_t]\cdot \boldsymbol{Q}\boldsymbol{u}_{tt}dx.
\end{aligned}
\tag{4.16}
$$

Performing differentaition on the right-hand side of (4.16) and applying the Young inequality we find

$$
\begin{aligned}
&\left|\int_\Omega (z''(\chi)\chi_t\boldsymbol{B}\nabla\chi + z'(\chi)\boldsymbol{B}\nabla\chi_t + \boldsymbol{b}_t)\cdot\boldsymbol{Q}\boldsymbol{u}_{tt}dx\right| \le \frac{3\delta_3}{2}\|\boldsymbol{Q}\boldsymbol{u}_{tt}\|^2_{L_2(\Omega)} \\
&+\frac{1}{2\delta_3}\int_\Omega (|z''(\chi)\boldsymbol{B}|^2\chi_t^2|\nabla\chi|^2 + |z'(\chi)\boldsymbol{B}|^2|\nabla\chi_t|^2 + |\boldsymbol{b}_t|^2)dx, \quad \delta_3 > 0.
\end{aligned}
$$

Hence, setting $\delta_3 = \frac{\nu}{3}$ and using $(2.11)_3$, we conclude that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\boldsymbol{Q}^{1/2}\boldsymbol{u}_{tt}\|^2_{L_2(\Omega)} + \|\boldsymbol{Q}\boldsymbol{u}_t\|^2_{L_2(\Omega)}) + \frac{\nu}{2}\|\boldsymbol{Q}\boldsymbol{u}_{tt}\|^2_{L_2(\Omega)} \\
&\le \frac{3a_5^2}{2\nu}\int_\Omega (\chi_t^2|\nabla\chi|^2 + |\nabla\chi_t|^2)dx + \frac{3}{2\nu}\|\boldsymbol{b}_t\|^2_{L_2(\Omega)}.
\end{aligned}
\tag{4.17}
$$

Since $\boldsymbol{u}_{tt} = \boldsymbol{0}$ on S, by Lemma 4.2 we have

$$
\|\boldsymbol{Q}^{1/2}\boldsymbol{u}_{tt}\|^2_{L_2(\Omega)} \le \frac{d_3}{\bar{\mu}}\|\boldsymbol{Q}\boldsymbol{u}_{tt}\|^2_{L_2(\Omega)}, \quad \|\boldsymbol{u}_{tt}\|^2_{L_2(\Omega)} \le \frac{d_3^2}{\bar{\mu}^2}\|\boldsymbol{Q}\boldsymbol{u}_{tt}\|^2_{L_2(\Omega)}.
\tag{4.18}
$$

Using (4.18) we conclude from (4.17) that

$$
\begin{aligned}
&\frac{d}{dt}(\|\boldsymbol{Q}^{1/2}\boldsymbol{u}_{tt}\|^2_{L_2(\Omega)} + \|\boldsymbol{Q}\boldsymbol{u}_t\|^2_{L_2(\Omega)}) + \frac{\nu}{2}\|\boldsymbol{Q}\boldsymbol{u}_{tt}\|^2_{L_2(\Omega)} \\
&\quad + \frac{\nu\bar{\mu}^2}{4d_3^2}\|\boldsymbol{u}_{tt}\|^2_{L_2(\Omega)} + \frac{\nu\bar{\mu}}{4d_3}\|\boldsymbol{Q}^{1/2}\boldsymbol{u}_{tt}\|^2_{L_2(\Omega)} \\
&\le \frac{3a_5^2}{\nu}(\|\chi_t\nabla\chi\|^2_{L_2(\Omega)} + \|\nabla\chi_t\|^2_{L_2(\Omega)}) + \frac{3}{\nu}\|\boldsymbol{b}_t\|^2_{L_2(\Omega)}.
\end{aligned}
\tag{4.19}
$$

Multiplying (4.19) by $4\cdot\frac{4d_3^2}{\nu\bar{\mu}^2}$ and summing up with inequalities (4.11)

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and (4.15), each multiplied by 2, we arrive at

$$
\begin{aligned}
&\frac{d}{dt}\bigg[(\nu+1)\|\boldsymbol{Q}\boldsymbol{u}\|^2_{L_2(\Omega)} + \|\boldsymbol{Q}^{1/2}\boldsymbol{u}_{tt}\|^2_{L_2(\Omega)} \\
&\quad + \frac{16d_3^2}{\nu\bar{\mu}^2}(\|\boldsymbol{Q}\boldsymbol{u}_t\|^2_{L_2(\Omega)} + \|\boldsymbol{Q}^{1/2}\boldsymbol{u}_{tt}\|^2_{L_2(\Omega)})\bigg] \\
&\quad + \|\boldsymbol{Q}\boldsymbol{u}\|^2_{L_2(\Omega)} + \frac{\nu\bar{\mu}}{2d_3}\|\boldsymbol{Q}^{1/2}\boldsymbol{u}_t\|^2_{L_2(\Omega)} + \frac{\nu}{2}\|\boldsymbol{Q}\boldsymbol{u}_t\|^2_{L_2(\Omega)} \\
&\quad + \frac{4d_3}{\bar{\mu}}\|\boldsymbol{Q}^{1/2}\boldsymbol{u}_{tt}\|^2_{L_2(\Omega)} + \|\boldsymbol{u}_{tt}\|^2_{L_2(\Omega)} + \frac{8d_3^2}{\bar{\mu}^2}\|\boldsymbol{Q}\boldsymbol{u}_{tt}\|^2_{L_2(\Omega)} \\
&\le a_5^2\left(3+\frac{2}{\nu}\right)\|\nabla\chi\|^2_{L_2(\Omega)} + \frac{48a_5^2d_3^2}{\nu^2\bar{\mu}^2}(\|\chi_t\nabla\chi\|^2_{L_2(\Omega)} + \|\nabla\chi_t\|^2_{L_2(\Omega)}) \\
&\quad + \left(3+\frac{2}{\nu}\right)\|\boldsymbol{b}\|^2_{L_2(\Omega)} + \frac{48d_3^2}{\nu^2\bar{\mu}^2}\|\boldsymbol{b}_t\|^2_{L_2(\Omega)}.
\end{aligned} \tag{4.20}
$$

Hence, letting

$$
\beta_2 = \min\left\{\frac{1}{\nu+1}, \frac{\nu\bar{\mu}}{2d_3}, \frac{\nu^2\bar{\mu}^2}{32d_3^2}, \frac{\nu\bar{\mu}}{4d_3}\right\},
$$

and defining $H(t)$ according to (4.2), we conclude (4.4) what finishes the proof. □

4.2. Estimates for χ

Our goal now is to obtain estimates on χ which are appropriate to handle the right-hand side of inequality (4.4). To this end we consider system (1.15), (1.16) expressed in the form (1.17). We have

Lemma 4.3. *Assume that*

$$
\sup_{0\le t\le T}\|\boldsymbol{\varepsilon}(\boldsymbol{u}_t(t))\|_{L_2(\Omega)} < \infty. \tag{4.21}
$$

Then solutions χ of problem (1.17) satisfy the differential inequality

$$
\begin{aligned}
&\frac{d}{dt}(\|\chi(t)\|^2_{L_2(\Omega)} + c_2\|\Delta\chi(t)\|^2_{L_2(\Omega)}) + \|\chi_t(t)\|^2_{L_2(\Omega)} \\
&\quad + c_3(\|\chi_t(t)\|^2_{L_2(\Omega)} + \|\Delta\chi_t(t)\|^2_{L_2(\Omega)} \\
&\quad + \|\chi(t)\|^2_{H^2(\Omega)} + \|\Delta\chi(t)\|^2_{L_2(\Omega)}) \\
&\le c_4(\|\boldsymbol{\varepsilon}(\boldsymbol{u}_t(t))\|^2_{L_2(\Omega)} + \chi_m^2 + 1) \quad \textit{for} \;\; t\in(0,T),
\end{aligned} \tag{4.22}
$$

where $c_2 = c_2(c_1)$, $c_3 = c_3(c_1)$, $c_4 = c_4(c_1)$ denote positive constants which are monotone increasing as functions of constant c_1 from energy estimate (3.58). They are given by

$$c_2 = \gamma c_2', \quad c_3 = \min\left\{c_2', \frac{\gamma}{4}, \frac{\gamma}{2d_5}\right\}, \quad c_4 = \max\left\{c_3', c_5', \frac{1}{2d_5}\right\}$$

where

$$\begin{aligned}
c_2' &= d_7\left[\frac{80(9c_* + a_5^2 + a_6^2)d_5}{\gamma^2 c_*}\right]^3 c_1^3 + \frac{10}{\gamma}(a_5^2 + a_6^2 + 1),\\
c_3' &= \frac{10}{\gamma}(a_5^2 + a_6^2),\\
c_4' &= \frac{32}{\gamma^3} d_4^3(d_2+1)^3 c_1^3 + 2\left(\frac{d_2}{\gamma} + \frac{4a_5^2}{c_*}\right)c_1\\
&\quad + (4d_4^3\chi_m^6|\Omega|^2 + \chi_m^2 + 2a_6^2)|\Omega|,\\
c_5' &= \frac{2}{\gamma}c_4'(4c_2'^2 + 1).
\end{aligned}$$

Proof. Differentiating $(1.17)_1$ with respect to t, multiplying by $\chi_t(t)$, integrating over Ω and by parts using boundary conditions $(1.17)_{3,4}$, and applying the Young inequality we find

(4.23)

$$\frac{d}{dt}\|\chi_t\|^2_{L_2(\Omega)} + \gamma\|\Delta\chi_t\|^2_{L_2(\Omega)} \le \frac{1}{\gamma}\int\limits_\Omega [\psi'(\chi) + z'(\chi)(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}) + Dz(\chi) + E)]^2_{,t}dx.$$

After performing differentiation and using (2.7), $(2.11)_{3,4}$ the right-hand side of (4.23) is bounded by

$$\begin{aligned}
&\frac{1}{\gamma}\int\limits_\Omega [\psi''(\chi)\chi_t + z''(\chi)\chi_t(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}) + Dz(\chi) + E)\\
&\quad + z'(\chi)(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_t) + Dz'(\chi)\chi_t)]^2 dx\\
&\le \frac{5}{\gamma}[\psi''(\chi)^2\chi_t^2 + |z''(\chi)\boldsymbol{B}|^2|\boldsymbol{\varepsilon}(\boldsymbol{u})|^2\chi_t^2 + |z''(\chi)(Dz(\chi) + E)|^2\chi_t^2\\
&\quad + |z'(\chi)\boldsymbol{B}|^2|\boldsymbol{\varepsilon}(\boldsymbol{u}_t)|^2 + |Dz'(\chi)|^2\chi_t^2]dx\\
&\le \frac{5}{\gamma}\int\limits_\Omega [(3\chi^2 - 1)^2\chi_t^2 + a_5^2|\boldsymbol{\varepsilon}(\boldsymbol{u})|^2\chi_t^2 + 2a_6^2\chi_t^2 + a_5^2|\boldsymbol{\varepsilon}(\boldsymbol{u}_t)|^2]dx\\
&\le \frac{5}{\gamma}\int\limits_\Omega (9\chi^4 + 1)\chi_t^2 dx + \frac{10}{\gamma}(a_5^2 + a_6^2)\int\limits_\Omega (|\boldsymbol{\varepsilon}(\boldsymbol{u})|^2\chi_t^2 + \chi_t^2 + |\boldsymbol{\varepsilon}(\boldsymbol{u}_t)|^2)dx.
\end{aligned}$$

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Hence, recalling estimate (3.58),

$$
\begin{aligned}
&\frac{d}{dt}\|\chi_t\|^2_{L_2(\Omega)} + \gamma\|\Delta\chi_t\|^2_{L_2(\Omega)} \\
&\le \frac{45}{\gamma}\|\chi\|^4_{L_4(\Omega)}\|\chi_t\|^2_{L_\infty(\Omega)} + \frac{10}{\gamma}(a_5^2+a_6^2+1)\|\chi_t\|^2_{L_2(\Omega)} \\
&\quad + \frac{10}{\gamma}(a_5^2+a_6^2)\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|^2_{\boldsymbol{L}_2(\Omega)}\|\chi_t\|^2_{L_\infty(\Omega)} \\
&\quad + \frac{10}{\gamma}(a_5^2+a_6^2)\|\boldsymbol{\varepsilon}(\boldsymbol{u}_t)\|^2_{\boldsymbol{L}_2(\Omega)} \\
&\le 8\cdot\frac{45}{\gamma}c_1\|\chi_t\|^2_{L_\infty(\Omega)} + \frac{10}{\gamma}(a_5^2+a_6^2+1)\|\chi_t\|^2_{L_2(\Omega)} \\
&\quad + \frac{40}{\gamma c_*}(a_5^2+a_6^2)c_1\|\chi_t\|^2_{L_\infty(\Omega)} + \frac{10}{\gamma}(a_5^2+a_6^2)\|\boldsymbol{\varepsilon}(\boldsymbol{u}_t)\|^2_{\boldsymbol{L}_2(\Omega)}.
\end{aligned}
\tag{4.24}
$$

By the interpolation inequality (see (1.24)) we have

$$
\|\chi_t\|^2_{L_\infty(\Omega)} \le \delta_1^{1-\varkappa}\|\chi_t\|^2_{H^2(\Omega)} + d_7\delta_1^{-\varkappa}\|\chi_t\|^2_{L_2(\Omega)}, \tag{4.25}
$$

with $\varkappa = \frac{3}{4} < 1$, $\delta_1 > 0$, $d_7 > 0$.

Further, in view of the identity $\int_\Omega \chi_t dx = 0$ (see (3.1)) and the boundary condition $(1.17)_3$, by virtue of the elliptic inequality (1.22),

$$
\|\chi_t\|^2_{H^2(\Omega)} \le d_5\|\Delta\chi_t\|^2_{L_2(\Omega)}, \quad d_5 > 0. \tag{4.26}
$$

Using (4.25) and (4.26) in (4.24) and choosing δ_1 so that

$$
\frac{40}{\gamma}\left(9 + \frac{a_5^2+a_6^2}{c_*}\right)d_5 c_1 \delta_1^{1/4} = \frac{\gamma}{2},
$$

we arrive at the inequality

$$
\frac{d}{dt}\|\chi_t\|^2_{L_2(\Omega)} + \frac{\gamma}{2}\|\Delta\chi_t\|^2_{L_2(\Omega)} \le c_2'\|\chi_t\|^2_{L_2(\Omega)} + c_3'\|\boldsymbol{\varepsilon}(\boldsymbol{u}_t)\|^2_{\boldsymbol{L}_2(\Omega)}, \tag{4.27}
$$

with positive constants c_2', c_3' given by

$$
\begin{aligned}
c_2' &= d_7\left[\frac{80(9c_* + a_5^2 + a_6^2)d_5}{\gamma^2 c_*}\right]^3 c_1^3 + \frac{10}{\gamma}(a_5^2+a_6^2+1), \\
c_3' &= \frac{10}{\gamma}(a_5^2+a_6^2).
\end{aligned}
$$

Now, let us multiply $(1.17)_1$ by $\chi_t(t)$, integrate over Ω and by parts using boundary conditions $(1.17)_{3,4}$ to get

$$
\begin{aligned}
&\frac{\gamma}{2}\frac{d}{dt}\|\Delta\chi\|^2_{L_2(\Omega)} + \|\chi_t\|^2_{L_2(\Omega)} \\
&= \int_\Omega [\psi'(\chi) + z'(\chi)(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}) + Dz(\chi) + E)]\Delta\chi_t dx.
\end{aligned}
\tag{4.28}
$$

By the Young inequality the right-hand side of (4.28) is estimated by

$$\frac{\delta_2}{2}\|\Delta\chi_t\|^2_{L_2(\Omega)}+\frac{1}{\delta_2}\int_\Omega[|\psi'(\chi)|^2+|z'(\chi)(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u})+Dz(\chi)+E)|^2]dx, \tag{4.29}$$

where $\delta_2 > 0$. Using (2.7) and $(2.11)_{3,4}$ the last integral in the above inequality is bounded by

$$\begin{aligned}&\int_\Omega(\chi^6+\chi^2)dx+2\int_\Omega(a_5^2|\boldsymbol{\varepsilon}(\boldsymbol{u})|^2+a_6^2)dx\\&=\|\chi\|^6_{L_6(\Omega)}+\|\chi\|^2_{L_2(\Omega)}+2a_5^2\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|^2_{L_2(\Omega)}+2a_6^2|\Omega|\equiv I.\end{aligned} \tag{4.30}$$

Recalling estimates (3.23) and (3.58) we have

$$\|\chi\|^2_{L_2(\Omega)}\le d_2\|\nabla\chi\|^2_{L_2(\Omega)}+|\Omega|\chi_m^2\le\frac{2d_2}{\gamma}c_1+|\Omega|\chi_m^2,\quad d_2>0,$$

so that

$$\|\chi\|^2_{H^1(\Omega)}\le\frac{2}{\gamma}(d_2+1)c_1+|\Omega|\chi_m^2.$$

Moreover, by the Sobolev imbedding (1.21),

$$\begin{aligned}\|\chi\|^6_{L_6(\Omega)}&\le d_4^3\|\chi\|^6_{H^1(\Omega)}\le d_4^3\left[\frac{2}{\gamma}(d_2+1)c_1+|\Omega|\chi_m^2\right]^3\\&\le 4d_4^3\left[\frac{8}{\gamma^3}(d_2+1)^3c_1^3+|\Omega|^3\chi_m^6\right].\end{aligned}$$

Hence, we conclude the bound

$$I\le c_4' \tag{4.31}$$

with constant c_4' given by

$$c_4'=\frac{32}{\gamma^3}d_4^3(d_2+1)^3c_1^3+2\left(\frac{d_2}{\gamma}+\frac{4a_5^2}{c_*}\right)c_1+(4d_4^3\chi_m^6|\Omega|^2+\chi_m^2+2a_6^2)|\Omega|.$$

Combining estimates (4.29)–(4.31) in (4.28) we arrive at

$$\frac{\gamma}{2}\frac{d}{dt}\|\Delta\chi\|^2_{L_2(\Omega)}+\|\chi_t\|^2_{L_2(\Omega)}\le\frac{\delta_2}{2}\|\Delta\chi_t\|^2_{L_2(\Omega)}+\frac{1}{\delta_2}c_4'. \tag{4.32}$$

Let us multiply now (4.32) by $2c_2'$ and add by sides to (4.27) to get, after setting $\delta_2=\frac{\gamma}{4c_2'}$, the inequality

$$\begin{aligned}&\frac{d}{dt}(\|\chi_t\|^2_{L_2(\Omega)}+\gamma c_2'\|\Delta\chi\|^2_{L_2(\Omega)})+c_2'\|\chi_t\|^2_{L_2(\Omega)}+\frac{\gamma}{4}\|\Delta\chi_t\|^2_{L_2(\Omega)}\\&\le c_3'\|\varepsilon(\boldsymbol{u}_t)\|^2_{L_2(\Omega)}+\frac{8}{\gamma}c_2'^2c_4'.\end{aligned} \tag{4.33}$$

Finally, we multiply $(1.17)_1$ by $\chi(t)$, integrate over Ω and by parts using boundary conditions $(1.17)_{3,4}$ to get after applying the Young inequality

$$
(4.34)\qquad \begin{aligned} &\frac{d}{dt}\|\chi\|^2_{L_2(\Omega)} + \gamma\|\Delta\chi\|^2_{L_2(\Omega)} \\ &\le \frac{2}{\gamma}\int\limits_\Omega [\psi'(\chi)^2 + |z'(\chi)(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}) + Dz(\chi) + E)|^2]dx \le \frac{2}{\gamma}c_4', \end{aligned}
$$

where in the last line we used (4.30) and (4.31). Adding (4.33) and (4.34) by sides gives

$$
(4.35)\qquad \begin{aligned} &\frac{d}{dt}(\|\chi\|^2_{L_2(\Omega)} + \gamma c_2'\|\Delta\chi\|^2_{L_2(\Omega)} + \|\chi_t\|^2_{L_2(\Omega)}) \\ &\quad + c_2'\|\chi_t\|^2_{L_2(\Omega)} + \frac{\gamma}{4}\|\Delta\chi_t\|^2_{L_2(\Omega)} + \gamma\|\Delta\chi\|^2_{L_2(\Omega)} \\ &\le c_3'\|\boldsymbol{\varepsilon}(\boldsymbol{u}_t)\|^2_{L_2(\Omega)} + c_5' \end{aligned}
$$

where $c_5' = \frac{2}{\gamma}c_4'(4c_2'^2 + 1)$. Moreover, using the elliptic estimate (1.22) and property (3.2), we deduce from (4.35) that

$$
(4.36)\qquad \begin{aligned} &\frac{d}{dt}(\|\chi\|^2_{L_2(\Omega)} + \gamma c_2'\|\Delta\chi\|^2_{L_2(\Omega)} + \|\chi_t\|^2_{L_2(\Omega)}) \\ &+ c_2'\|\chi_t\|^2_{L_2(\Omega)} + \frac{\gamma}{4}\|\Delta\chi_t\|^2_{L_2(\Omega)} + \frac{\gamma}{2d_5}\|\chi\|^2_{H^2(\Omega)} + \frac{\gamma}{2}\|\Delta\chi\|^2_{L_2(\Omega)} \\ &\le c_3'\|\boldsymbol{\varepsilon}(\boldsymbol{u}_t)\|^2_{L_2(\Omega)} + \frac{1}{2d_5}\chi_m^2 + c_5'. \end{aligned}
$$

Hence, setting

$$
(4.37)\qquad c_2 = \gamma c_2', \quad c_3 = \min\left\{c_2', \frac{\gamma}{4}, \frac{\gamma}{2d_5}\right\}, \quad c_4 = \max\left\{c_3', c_5', \frac{1}{2d_5}\right\},
$$

we conclude inequality (4.22). This finishes the proof. □

4.3. Estimates for $\boldsymbol{u}$ and χ

Combining the estimates from Lemmas 4.1 and 4.3 we deduce

Lemma 4.4. *Assume that*

$$
\sup_{0\le t\le T}\|\boldsymbol{\varepsilon}(\boldsymbol{u}_t(t)\|_{L_2(\Omega)} < \infty, \qquad \sup_{0\le t\le T}(\|\boldsymbol{b}(t)\|_{L_2(\Omega)} + \|\boldsymbol{b}_t(t)\|_{L_2(\Omega)}) < \infty,
$$

Let $K : [0,\infty) \to [0,\infty)$ be the function defined by

$$
(4.38)\qquad K(t) = H(t) + c_5(\|\chi(t)\|^2_{L_2(\Omega)} + c_2\|\Delta\chi(t)\|^2_{L_2(\Omega)} + \|\chi_t(t)\|^2_{L_2(\Omega)})
$$

where the function $H(t)$ and constant c_2 are given in Lemmas 4.1 and 4.3, and $c_5 = c_5(c_1)$ is a positive constant, monotone increasing in c_1, given by

$$c_5 = \frac{96a_5^2 d_3^2 d_6}{\nu^2 \bar{\mu}^2 c_3}\left(\frac{2}{\gamma}c_1 + 1\right). \tag{4.39}$$

Then there exists positive constant

$$\beta_3 = \beta_3(c_1) = \min\left\{\beta_2,\ \frac{c_3}{2},\ \frac{c_3}{c_2}\right\}, \tag{4.40}$$

with β_2, c_2, c_3 defined in Lemmas 4.1, 4.3, such that solutions $(\boldsymbol{u}, \chi, \mu)$ of problem (1.14)–(1.16) satisfy the differential inequality

$$\begin{aligned}
&\frac{d}{dt}K(t) + \beta_3 K(t) + \|\boldsymbol{u}_{tt}(t)\|^2_{\boldsymbol{L}_2(\Omega)} + \frac{8d_3^2}{\bar{\mu}^2}\|\boldsymbol{Q}\boldsymbol{u}_{tt}(t)\|^2_{\boldsymbol{L}_2(\Omega)} \\
&\quad + \frac{c_3 c_5}{2}(\|\chi(t)\|^2_{H^2(\Omega)} + \|\Delta\chi_t(t)\|^2_{L_2(\Omega)}) \\
&\leq c_4 c_5(\|\boldsymbol{\varepsilon}(\boldsymbol{u}_t(t))\|^2_{\boldsymbol{L}_2(\Omega)} + \chi_m^2 + 1) \\
&\quad + \frac{2a_5^2}{\gamma}\left(3 + \frac{2}{\nu}\right)c_1 + \left(3 + \frac{2}{\nu}\right)\|\boldsymbol{b}(t)\|^2_{\boldsymbol{L}_2(\Omega)} \\
&\quad + \frac{48d_3^2}{\nu^2\bar{\mu}^2}\|\boldsymbol{b}_t(t)\|^2_{\boldsymbol{L}_2(\Omega)} \quad \textit{for} \quad t \in (0, T).
\end{aligned} \tag{4.41}$$

Proof. Applying estimate (3.58) to the right-hand side of inequality (4.4) we get

$$\begin{aligned}
&\frac{d}{dt}H(t) + \beta_2 H(t) + \|\boldsymbol{u}_{tt}\|^2_{\boldsymbol{L}_2(\Omega)} + \frac{8d_3^2}{\bar{\mu}^2}\|\boldsymbol{Q}\boldsymbol{u}_{tt}\|^2_{\boldsymbol{L}_2(\Omega)} \\
&\leq \frac{2a_5^2}{\gamma}\left(3 + \frac{2}{\nu}\right)c_1 + \frac{96a_5^2 d_3^2}{\nu^2\bar{\mu}^2\gamma}c_1\|\chi_t\|^2_{L_\infty(\Omega)} \\
&\quad + \frac{48a_5^2 d_3^2}{\nu^2\bar{\mu}^2}\|\nabla\chi_t\|^2_{\boldsymbol{L}_2(\Omega)} + \left(3 + \frac{2}{\nu}\right)\|\boldsymbol{b}\|^2_{\boldsymbol{L}_2(\Omega)} + \frac{48d_3^2}{\nu^2\bar{\mu}^2}\|\boldsymbol{b}_t\|^2_{\boldsymbol{L}_2(\Omega)}.
\end{aligned} \tag{4.42}$$

By virtue of imbedding (1.23), since $\int_\Omega \chi_t dx = 0$,

$$\|\chi_t\|^2_{L_\infty(\Omega)} + \|\nabla\chi_t\|^2_{\boldsymbol{L}2(\Omega)} \leq d_6\|\Delta\chi_t\|^2_{L_2(\Omega)}. \tag{4.43}$$

The use of (4.43) in (4.42) gives

$$\begin{aligned}
&\frac{d}{dt}H(t) + \beta_2 H(t) + \|\boldsymbol{u}_{tt}\|^2_{\boldsymbol{L}_2(\Omega)} + \frac{8d_3^2}{\bar{\mu}^2}\|\boldsymbol{Q}\boldsymbol{u}_{tt}\|^2_{\boldsymbol{L}_2(\Omega)} \\
&\leq \frac{2a_5^2}{\gamma}\left(3 + \frac{2}{\nu}\right)c_1 + \frac{48a_5^2 d_3^2 d_6}{\nu^2\bar{\mu}^2}\left(\frac{2}{\gamma}c_1 + 1\right)\|\Delta\chi_t\|^2_{L_2(\Omega)} \\
&\quad + \left(3 + \frac{2}{\nu}\right)\|\boldsymbol{b}\|^2_{\boldsymbol{L}_2(\Omega)} + \frac{48d_3^2}{\nu^2\bar{\mu}^2}\|\boldsymbol{b}_t\|^2_{\boldsymbol{L}_2(\Omega)}.
\end{aligned} \tag{4.44}$$

Now, multiplying inequality (4.22) by constant c_5 such that

$$\frac{48a_5^2d_3^2d_6}{\nu^2\bar{\mu}^2}\left(\frac{2}{\gamma}c_1+1\right)=\frac{1}{2}c_3c_5, \tag{4.45}$$

adding the result to (4.44), and denoting

$$K(t)=H(t)+c_5(\|\chi(t)\|^2_{L_2(\Omega)}+\|\chi_t(t)\|^2_{L_2(\Omega)}+c_2\|\Delta\chi(t)\|^2_{L_2(\Omega)}),$$

we arrive at

$$\begin{aligned}&\frac{d}{dt}K(t)+\beta_2H(t)+\|\boldsymbol{u}_{tt}\|^2_{\boldsymbol{L}_2(\Omega)}+\frac{8d_3^2}{\bar{\mu}^2}\|\boldsymbol{Q}\boldsymbol{u}_{tt}\|^2_{\boldsymbol{L}_2(\Omega)}\\&\quad+c_3c_5\left(\|\chi_t\|^2_{L_2(\Omega)}+\frac{1}{2}\|\Delta\chi_t\|^2_{L_2(\Omega)}+\|\chi\|^2_{H^2(\Omega)}+\|\Delta\chi\|^2_{L_2(\Omega)}\right)\\&\le c_4c_5(\|\varepsilon(\boldsymbol{u}_t)\|^2_{\boldsymbol{L}_2(\Omega)}+\chi_m^2+1)\\&\quad+\frac{2a_5^2}{\gamma}\left(3+\frac{2}{\nu}\right)c_1+\left(3+\frac{2}{\nu}\right)\|\boldsymbol{b}\|^2_{\boldsymbol{L}_2(\Omega)}+\frac{48d_3^2}{\nu^2\bar{\mu}^2}\|\boldsymbol{b}_t\|^2_{\boldsymbol{L}_2(\Omega)}.\end{aligned} \tag{4.46}$$

Let us note that

(4.47)

$$\begin{aligned}&\beta_2H(t)+c_3c_5\left(\|\chi_t\|^2_{L_2(\Omega)}+\frac{1}{2}\|\Delta\chi_t\|^2_{L_2(\Omega)}+\|\chi\|^2_{H^2(\Omega)}+\|\Delta\chi\|^2_{L_2(\Omega)}\right)\\&\ge\beta_2H(t)+c_3c_5\left(\frac{1}{2}\|\chi\|^2_{L_2(\Omega)}+\|\chi_t\|^2_{L_2(\Omega)}+c_2^{-1}c_2\|\Delta\chi\|^2_{L_2(\Omega)}\right)\\&\quad+\frac{c_3c_5}{2}(\|\chi\|^2_{H^2(\Omega)}+\|\Delta\chi_t\|^2_{L_2(\Omega)})\\&\ge\beta_3K(t)+\frac{c_3c_5}{2}(\|\chi\|^2_{H^2(\Omega)}+\|\Delta\chi_t\|^2_{L_2(\Omega)})\end{aligned}$$

where $\beta_3=\min\left\{\beta_2,\ \frac{c_3}{2},\ \frac{c_3}{c_2}\right\}$. On account of (4.47), inequality (4.46) leads to (4.41) which together with condition (4.45) proves the lemma. □

Finally, combining the results of Lemmas 4.4 and 3.3, we get

Lemma 4.5. *Let the assumptions of Lemma 4.4 be satisfied and $N:[0,\infty)\to[0,\infty)$ be the funciton defined by*

$$N(t)=K(t)+c_6G(t) \tag{4.48}$$

with $K(t)$ and $G(t)$ given respectively by (4.38) and (3.14), and positive constant $c_6=c_6(c_1)$, monotone increasing in c_1, such that

$$c_6=\frac{16}{\nu c_*d_1}c_4c_5. \tag{4.49}$$

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Then solutions $(\boldsymbol{u}, \chi, \mu)$ of problem (1.14)–(1.16) satisfy the differential inequality

$$\begin{aligned}
&\frac{d}{dt}N(t) + \beta_4 N(t) + c_6\frac{\nu c_* d_1}{16}\|\boldsymbol{u}_t(t)\|^2_{H^1(\Omega)} + \frac{c_6}{2}\|\nabla\mu(t)\|^2_{L_2(\Omega)} \\
&+ \|\boldsymbol{u}_{tt}(t)\|^2_{L_2(\Omega)} + \frac{8d_3^2}{\bar{\mu}^2}\|\boldsymbol{Q}\boldsymbol{u}_{tt}(t)\|^2_{L_2(\Omega)} + \frac{c_3c_5}{2}(\|\chi(t)\|^2_{H^2(\Omega)} \\
&+ \|\Delta\chi_t(t)\|^2_{L_2(\Omega)}) \leq \Lambda_3\|\boldsymbol{b}(t)\|^2_{L_2(\Omega)} + \Lambda_4\|\boldsymbol{b}_t(t)\|^2_{L_2(\Omega)} + \Lambda_5
\end{aligned} \tag{4.50}$$

for $t \in (0,T)$, where $\beta_4 = \beta_4(c_1)$, $\Lambda_3 = \Lambda_3(c_1)$, Λ_4, $\Lambda_5 = \Lambda_5(c_1)$ are positive constants given by:

$$\begin{aligned}
&\beta_4 = \min\{\beta_3, \beta_1\}, \qquad \Lambda_3 = 3 + \frac{2}{\nu} + c_6\Lambda_1, \qquad \Lambda_4 = \frac{48d_3^2}{\nu^2\bar{\mu}^2}, \\
&\Lambda_5 = c_4c_5(\chi_m^2 + 1) + \frac{2a_5^2}{\gamma}\left(3 + \frac{2}{\nu}\right)c_1 + c_6\Lambda_2,
\end{aligned} \tag{4.51}$$

with $\beta_3 = \beta_3(c_1)$, β_1, Λ_1, Λ_2 defined in (4.40), (3.16), (3.18).

Proof. Multiplying inequality (3.17) by constant $c_6 = c_6(c_1)$ satisfying condition (4.49), adding the result to (4.41) and denoting $N(t) = K(t) + c_6G(t)$, we arrive at

$$\begin{aligned}
&\frac{d}{dt}N(t) + \beta_1 c_6 G(t) + c_6\frac{\nu c_* d_1}{16}\|\boldsymbol{u}_t(t)\|^2_{H^1(\Omega)} + \frac{c_6}{2}\|\nabla\mu(t)\|^2_{L_2(\Omega)} \\
&+ \beta_3 K(t) + \|\boldsymbol{u}_{tt}(t)\|^2_{L_2(\Omega)} + \frac{8d_3^2}{\bar{\mu}^2}\|\boldsymbol{Q}\boldsymbol{u}_{tt}(t)\|^2_{L_2(\Omega)} \\
&+ \frac{c_3c_5}{2}(\|\chi(t)\|^2_{H^2(\Omega)} + \|\Delta\chi_t(t)\|^2_{L_2(\Omega)}) \\
&\leq c_6(\Lambda_1\|\boldsymbol{b}(t)\|^2_{L_2(\Omega)} + \Lambda_2) + c_4c_5(\chi_m^2 + 1) + \frac{2a_5^2}{\gamma}\left(3 + \frac{2}{\nu}\right)c_1 \\
&+ \left(3 + \frac{2}{\nu}\right)\|\boldsymbol{b}(t)\|^2_{L_2(\Omega)} + \frac{48d_3^2}{\nu^2\bar{\mu}^2}\|\boldsymbol{b}_t(t)\|^2_{L_2(\Omega)}.
\end{aligned} \tag{4.52}$$

Using the notation (4.51), in view of the inequality

$$\beta_1 c_6 G(t) + \beta_3 K(t) \geq \beta_4 N(t),$$

we conclude the assertion. □

Let us define the function $\tilde{N} : [0,\infty) \to [0,\infty)$ by

$$\tilde{N}(t) = \|\boldsymbol{u}_{tt}(t)\|^2_{H^2(\Omega)} + \|\chi_t(t)\|^2_{H^2(\Omega)}. \tag{4.53}$$

Then inequality (4.50) implies the following one in the concised form

$$(4.54)\qquad \begin{aligned}&\frac{d}{dt}N(t)+\beta_4N(t)+\beta_5\tilde{N}(t)\\&\le\Lambda_3\|\boldsymbol{b}(t)\|^2_{\boldsymbol{L}_2(\Omega)}+\Lambda_4\|\boldsymbol{b}_t(t)\|^2_{\boldsymbol{L}_2(\Omega)}+\Lambda_5\quad\text{for}\quad t\in(0,T),\end{aligned}$$

where

$$\beta_5=\beta_5(c_1)=\min\left\{\frac{8d_3^2c_Q^2}{\bar{\mu}^2},\ \frac{c_3c_5}{2d_5}\right\}.$$

Similarly as in Lemma 3.4, we deduce from (4.54) an absorbing estimate.

Lemma 4.6. *Let $N(t)$ be defined by (4.48), $N(0)<\infty$, $|\chi_m|<\infty$, and*

$$b_{10}=\sup_{0\le t\le T}\|\boldsymbol{b}(t)\|_{\boldsymbol{L}_2(\Omega)}<\infty,\qquad b_{20}=\sup_{0\le t\le T}\|\boldsymbol{b}_t(t)\|_{\boldsymbol{L}_2(\Omega)}<\infty.$$

Then the function $N(t)$ is Lipschitz continuous on $[0,T]$ and satisfies the following estimates:

$$(4.55)\qquad N(t)\le A_{20}(1-e^{-\beta_4t})+N(0)e^{-\beta_4t},$$

and

$$(4.56)\qquad N(t)+\beta_4\int\limits_0^t N(t')dt'+\beta_5\int\limits_0^t\tilde{N}(t')dt'\le\beta_4A_{20}t+N(0)$$

with positive constant

$$A_{20}=\frac{1}{\beta_4}(\Lambda_3b_{10}^2+\Lambda_4b_{20}^2+\Lambda_5),$$

and constants $\beta_4,\Lambda_3,\Lambda_4,\Lambda_5$ from Lemma 4.5.

Let us note that by the definitions of $K(t)$ and $H(t)$ (see (4.38), (4.2)) it follows from (4.48) that

(4.57)

$$\begin{aligned}N(t)&=(\nu+1)\|\boldsymbol{Q}\boldsymbol{u}(t)\|^2_{\boldsymbol{L}_2(\Omega)}+\|\boldsymbol{Q}^{1/2}\boldsymbol{u}_t(t)\|^2_{\boldsymbol{L}_2(\Omega)}\\&+\frac{16d_3^2}{\nu\bar{\mu}^2}(\|\boldsymbol{Q}\boldsymbol{u}_t(t)\|^2_{\boldsymbol{L}_2(\Omega)}+\|\boldsymbol{Q}^{1/2}\boldsymbol{u}_{tt}(t)\|^2_{\boldsymbol{L}_2(\Omega)})+c_5(\|\chi(t)\|^2_{L_2(\Omega)}\\&+\|\chi_t(t)\|^2_{L_2(\Omega)}+c_2\|\Delta\chi(t)\|^2_{L_2(\Omega)})+c_6G(t).\end{aligned}$$

Hence, on account of (3.57) and (1.22), (2.3), (2.5), (3.2), we have
(4.58)

$$
\begin{aligned}
N(t) \geq c_7(&\|\boldsymbol{u}(t)\|^2_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{u}(t)\|^2_{\boldsymbol{H}^2(\Omega)} + \|\boldsymbol{u}_t(t)\|^2_{\boldsymbol{L}_2(\Omega)} + \|\boldsymbol{u}_t(t)\|^2_{\boldsymbol{H}^1(\Omega)} \\
&+ \|\boldsymbol{u}_t(t)\|^2_{\boldsymbol{H}^2(\Omega)} + \|\boldsymbol{u}_{tt}(t)\|^2_{\boldsymbol{H}^1(\Omega)} + \|\chi(t)\|^2_{L_2(\Omega)} + \|\chi(t)\|^4_{L_4(\Omega)} \\
&+ \|\nabla\chi(t)\|^2_{\boldsymbol{L}_2(\Omega)} + \|\chi(t)\|^2_{H^2(\Omega)} + \|\chi_t(t)\|^2_{L_2(\Omega)}) - c_8' \\
\geq c_7(&\|\boldsymbol{u}(t)\|^2_{\boldsymbol{H}^2(\Omega)} + \|\boldsymbol{u}_t(t)\|^2_{\boldsymbol{H}^2(\Omega)} + \|\boldsymbol{u}_{tt}(t)\|^2_{\boldsymbol{H}^1(\Omega)} + \|\chi(t)\|^2_{H^2(\Omega)} \\
&+ \|\chi_t(t)\|^2_{L_2(\Omega)}) - c_8',
\end{aligned}
$$

with positive constants $c_7 = c_7(c_1)$ and $c_8' = c_8'(c_1)$ given by

$$
\begin{aligned}
c_7 &= \min\left\{\frac{c_* d_1}{4} c_6,\ \frac{c_6}{8},\ \frac{\gamma}{2} c_6,\ c_5,\ \frac{c_5 c_2}{d_5},\ (\nu+1)\underline{c}_Q^2,\ c_* d_1,\right. \\
&\qquad \left.\frac{16 d_3^2 \underline{c}_Q^2}{\nu\bar{\mu}^2},\ \frac{16 d_3^2 c_* d_1}{\nu\bar{\mu}^2}\right\}, \\
c_8' &= \frac{|\Omega|}{2}\left(c_* a_3^2 + \frac{1}{2}\right) c_6 + \frac{\chi_m^2}{d_5^2} c_5 c_2.
\end{aligned}
$$

In view of the bound (4.58) we deduce from (4.55) the following

Corollary 4.1. *Let (A1)–(A6) hold, $N(t)$ be defined by (4.48), $N(0) < \infty$, and $\boldsymbol{b}, \boldsymbol{b}_t \in L_\infty(0,T;\boldsymbol{L}_2(\Omega))$. Then*

$$
\begin{aligned}
&c_7(\|\boldsymbol{u}(t)\|^2_{\boldsymbol{H}^2(\Omega)} + \|\boldsymbol{u}_t(t)\|^2_{\boldsymbol{H}^2(\Omega)} + \|\boldsymbol{u}_{tt}(t)\|^2_{\boldsymbol{H}^1(\Omega)} + \|\chi(t)\|^2_{H^2(\Omega)} \\
&\quad + \|\chi_t(t)\|^2_{L_2(\Omega)}) \leq N(t) + c_8' \leq A_{20} + N(0) + c_8' \equiv c_8
\end{aligned} \tag{4.59}
$$

with constants A_{20} and c_7, c_8' defined in (4.56) and (4.58).

5. Proof of Theorem 2.1 (Global existence)

To prove global existence we consider problem (1.14)–(1.16) on time intervals $[kT,(k+1)T]$ where $k \in \mathbb{N}\cup\{0\}$ and $T > 0$ is an arbitrary finite number:

$$
\begin{aligned}
&\boldsymbol{u}_{tt} - \boldsymbol{Q}\boldsymbol{u} - \nu\boldsymbol{Q}\boldsymbol{u}_t = z'(\chi)\boldsymbol{B}\nabla\chi + \boldsymbol{b} && \text{in } \Omega\times(kT,(k+1)T), \\
&\boldsymbol{u}|_{t=kT} = \boldsymbol{u}(kT) \quad \boldsymbol{u}_t|_{t=kT} = \boldsymbol{u}_t(kT) && \text{in } \Omega, \\
&\boldsymbol{u} = 0 && \text{on } S\times(kT,(k+1)T),
\end{aligned} \tag{5.1}
$$

$$
\begin{aligned}
&\chi_t - \Delta\mu = 0 && \text{in } \Omega\times(kT,(k+1)T), \\
&\chi|_{t=kT} = \chi(kT) && \text{in } \Omega, \\
&\boldsymbol{n}\cdot\nabla\mu = 0 && \text{on } S\times(kT,(k+1)T),
\end{aligned} \tag{5.2}
$$

$$
(5.3) \qquad \begin{aligned} &\mu = -\gamma\Delta\chi + \psi'(\chi) \\ &\quad + z'(\chi)(\boldsymbol{B}\cdot\varepsilon(\boldsymbol{u}) + Dz(\chi) + E) && \text{in } \Omega\times(kT,(k+1)T), \\ &\boldsymbol{n}\cdot\nabla\chi = 0 && \text{on } S\times(kT,(k+1)T). \end{aligned}
$$

Let the functions $G(t)$, $N(t)$, $\tilde{N}(t) : [0,\infty) \to [0,\infty)$ be defined by (3.14), (4.48) and (4.53). Moreover, let us denote

$$
(5.4) \quad b_{1k} = \sup_{t\in[kT,(k+1)T]} \|\boldsymbol{b}(t)\|_{L_2(\Omega)}, \quad b_{2k} = \sup_{t\in[kT,(k+1)T]} \|\boldsymbol{b}_t(t)\|_{L_2(\Omega)}
$$

and

$$
b_1 = \sup_{k\in\mathbb{N}\cup\{0\}} b_{1k}, \quad b_2 = \sup_{k\in\mathbb{N}\cup\{0\}} b_{2k}.
$$

Repeating the estimates from Section 3 and 4 on the subsequent time intervals $[kT,(k+1)T]$, $k\in\mathbb{N}\cup\{0\}$ we conclude the following

Lemma 5.1. *Let $G(0) < \infty$, $b_1 < \infty$ and*

$$
A_1 = \frac{1}{\beta_1}(\Lambda_1 b_1^2 + \Lambda_2)
$$

with constants $\beta_1, \Lambda_1, \Lambda_2$ defined in Lemma 3.3. Then

$$
(5.5) \quad G(kT) \le A_1(1 - e^{-\beta_1 kT}) + G(0)e^{-\beta_1 kT} \le A_1 + G(0) \quad \textit{for} \quad k\in\mathbb{N}.
$$

Proof. Considering inequality (3.17) on time interval $[(l-1)T, lT]$, multiplying by $e^{\beta_1 t}$ and integrating from $t = (l-1)T$ to $t = lT$, we obtain

$$
(5.6) \qquad G(lT) \le A_1(1 - e^{-\beta_1 T}) + e^{-\beta_1 T} G((l-1)T).
$$

Iterating (5.6) with respect to l from 1 to k implies estimate (5.5). □

On account of Lemma 5.1, energy estimate (3.58) on time interval $[kT,(k+1)T]$ takes the form

$$
(5.7) \quad \begin{aligned} &\frac{1}{4}\|\boldsymbol{u}_t(t)\|^2_{L_2(\Omega)} + \frac{\gamma}{2}\|\nabla\chi(t)\|^2_{L_2(\Omega)} + \frac{1}{8}\|\chi(t)\|^4_{L_4(\Omega)} + \frac{c_*}{4}\|\varepsilon(\boldsymbol{u}(t))\|^2_{L_2(\Omega)} \\ &\le G(kT) + c_1' \le A_1 + G(0) + c_1' \equiv c_1 \quad \text{for } t\in[kT,(k+1)T]. \end{aligned}
$$

This shows that constant c_1 from Section 3, and consequently all other constants $c_i = c_i(c_1)$, $i = 2,\dots,8$ from Section 4 are independent of the time step $k\in\mathbb{N}\cup\{0\}$. Thus, according to (4.54), we have

$$
(5.8) \quad \begin{aligned} &\frac{d}{dt}N(t) + \beta_4 N(t) + \beta_5\tilde{N}(t) \\ &\le \Lambda_3\|\boldsymbol{b}(t)\|^2_{L_2(\Omega)} + \Lambda_4\|\boldsymbol{b}_t(t)\|^2_{L_2(\Omega)} + \Lambda_5 \quad \text{for } t\in(kT,(k+1)T), \end{aligned}
$$

where constants $\beta_4, \beta_5, \Lambda_3, \Lambda_4, \Lambda_5$ are independent of k, defined in (4.51) and (4.53).

Repeating Lemma 4.6 on time intervals $[kT,(k+1)T]$ we conclude

Lemma 5.2. *Let $N(0) < \infty$, $b_1, b_2 < \infty$, and*

$$A_2 = \frac{1}{\beta_4}(\Lambda_3 b_1^2 + \Lambda_4 b_2^2 + \Lambda_5).$$

Then

$$N(kT) \le A_2(1 - e^{-\beta_4 kT}) + N(0)e^{-\beta_4 kT} \le A_2 + N(0) \quad \textit{for} \quad k \in \mathbb{N}. \tag{5.9}$$

Proof. Considering inequality (5.8) on time interval $[(l-1)T, lT]$, multiplying by $e^{\beta_4 t}$ and integrating from $t = (l-1)T$ to $t = lT$ we obtain

$$N(lT) \le A_2(1 - e^{-\beta_4 T}) + e^{-\beta_4 T} N((l-1)T). \tag{5.10}$$

Iterating (5.10) with respect to l from 1 to k gives (5.9). □

For $k \in \mathbb{N} \cup \{0\}$ we introduce the spaces

$$\begin{aligned} \mathcal{N}(kT) = \{&(\boldsymbol{u}, \chi)|_{t=kT} : N(kT) < \infty, \\ &\text{and } \boldsymbol{u}(kT) = \boldsymbol{0}, \quad \boldsymbol{n} \cdot \nabla\chi(kT) = 0 \text{ on } S\}, \\ \mathcal{M}(kT, (k+1)T) = \Big\{&(\boldsymbol{u}, \chi) : \max_{t \in [kT,(k+1)T]} N(t) \\ &+ \int_{kT}^{(k+1)T} (N(t) + \tilde{N}(t))dt < \infty, \\ &\text{and } \boldsymbol{u}(t) = \boldsymbol{0}, \quad \boldsymbol{n} \cdot \nabla\chi(t) = 0 \text{ on } S,\ t \in [kT, (k+1)T]\Big\}. \end{aligned} \tag{5.11}$$

Let us note that by (4.58),

$$\begin{aligned} N(t) \ge c_7(&\|\boldsymbol{u}(t)\|^2_{\boldsymbol{H}^2(\Omega)} + \|\boldsymbol{u}_t(t)\|^2_{\boldsymbol{H}^2(\Omega)} + \|\boldsymbol{u}_{tt}(t)\|^2_{\boldsymbol{H}^1(\Omega)} \\ &+ \|\chi(t)\|^2_{H^2(\Omega)} + \|\chi_t(t)\|^2_{L_2(\Omega)}) - c_8' \quad \text{for} \quad t \in \mathbb{R}_+, \end{aligned} \tag{5.12}$$

with positive constants c_7 and c_8' independent of k. Moreover, according to (4.53),

$$\tilde{N}(t) = \|\boldsymbol{u}_{tt}(t)\|^2_{\boldsymbol{H}^2(\Omega)} + \|\chi_t(t)\|^2_{H^2(\Omega)}. \tag{5.13}$$

We have the following local existence result on each time interval $[kT, (k+1)T]$, $k \in \mathbb{N} \cup \{0\}$.

Lemma 5.3. *Assume that $(\boldsymbol{u},\chi)\in\mathcal{N}(kT)$, and $b_1,b_2<\infty$. Then there exists a local solution $(\boldsymbol{u},\chi,\mu)$ to problem (5.1)–(5.3) such that $(\boldsymbol{u},\chi)\in\mathcal{M}(kT,(k+1)T)$.*

Proof. Multiplying inequality (5.8) by $e^{\beta_4 t}$, integrating from $t=kT$ to $t\in[kT,(k+1)T]$, and using (5.9) we get

$$N(t)\le A_2(1-e^{-\beta_4 t})+N(kT)e^{-\beta_4 t}\le A_2+N(0)\quad\text{for}\ \ t\in[kT,(k+1)T].$$

Hence,

$$\max_{t\in[kT,(k+1)T]} N(t)\le A_2+N(0). \tag{5.14}$$

Moreover, integrating inequality (5.8) from $t=kT$ to $t\in[kT,(k+1)T]$ and using (5.9) gives

$$\begin{aligned}&N(t)+\int_{kT}^{t}(\beta_4 N(t')+\beta_5\tilde{N}(t'))dt'\\&\le T(\Lambda_3 b_1^2+\Lambda_4 b_2^2+\Lambda_5)+N(kT)\\&\le TA_2\beta_4+A_2+N(0)\quad\text{for}\ \ t\in[kT,(k+1)T].\end{aligned}\tag{5.15}$$

In view of estimates (5.14) and (5.15) the existence of a solution $(\boldsymbol{u},\chi)\in\mathcal{M}(kT,(k+1)T)$ can be concluded rigorously with the help of a Faedo-Galerkin method. □

The next lemma states the global existence.

Lemma 5.4. *Assume that $(\boldsymbol{u},\chi)|_{t=0}\in\mathcal{N}(0)$ and $b_1,b_2<\infty$. Then there exists a global solution $(\boldsymbol{u},\chi,\mu)$ to problem (5.1)–(5.3) such that $(\boldsymbol{u},\chi)\in\bigcup_{k\in\mathbb{N}\cup\{0\}}\mathcal{M}(kT,(k+1)T)$. Moreover, the following uniform (in k) estimates hold true:*

$$\begin{aligned}&\sup_{k\in\mathbb{N}\cup\{0\}}\ \max_{t\in[kT,(k+1)T]} N(t)\le A_2+N(0),\\&\sup_{k\in\mathbb{N}\cup\{0\}}\int_{kT}^{(k+1)T}(\beta_4 N(t')+\beta_5\tilde{N}(t'))dt'\le TA_2\beta_4+A_2+N(0),\end{aligned}\tag{5.16}$$

with constants A_2,β_4 independent of k, given in (5.8) and (5.9).

Proof. By virtue of the uniform in k estimate (5.9), Lemma 5.3 can be successively repeated on time intervals $[kT,(k+1)T]$, $k\in\mathbb{N}\cup\{0\}$, to entail the global existence. Estimates (5.16) are direct consequences of the bounds (5.14) and (5.15). □

In view of (5.12), (5.13) it is seen that the global solution constructed in Lemma 5.4 satisfies $(2.12)_{1,2}$, (2.13) and (2.14). The mean value property in $(2.12)_3$ results from (3.2). The statement $\mu \in C([0,\infty); H^2_N(\Omega))$ in $(2.12)_3$ follows from the elliptic regulartiy (1.22) since $\Delta\mu = \chi_t \in C([0,\infty); L_2(\Omega))$, and by (3.8),

$$\begin{aligned}\Big|\int\limits_\Omega \mu dx\Big| &= \Big|\int\limits_\Omega [\psi'(\chi) + z'(\chi)(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}) + Dz(\chi) + E)]dx\Big| \\ &\le \|\chi\|^3_{L_3(\Omega)} + |\chi_m||\Omega| + a_5|\Omega|^{1/2}\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{\boldsymbol{L}_2(\Omega)} + a_6|\Omega| \\ &\le c(c_0) \quad \text{for} \quad t\in[0,\infty).\end{aligned}$$

Clearly, energy estimate (3.8), holding true for $t\in[0,\infty)$, implies $(2.12)_4$. Furthermore, the bound (2.15) is a direct consequence of (3.8). Estimates (2.16) and (2.17) follow immediately from (5.16) on account of (5.12) and (5.13). This completes the proof of Theorem 2.1. □

6. Proof of Theorem 2.2 (Absorbing estimate)

According to Lemma 5.1 the following estimate holds true for all $t\in\mathbb{R}_+$:

$$G(t) \le A_1(1-e^{-\beta_1 t}) + G(0)e^{-\beta_1 t} \tag{6.1}$$

where

$$A_1 = \frac{1}{\beta_1}(\Lambda_1 b_1^2 + \Lambda_2) > 0, \quad b_1 = \sup_{t\in\mathbb{R}_+}\|\boldsymbol{b}(t)\|_{\boldsymbol{L}_2(\Omega)},$$

and $\beta_1, \Lambda_1, \Lambda_2$ are positive constants dependent only on absolute data, defined in Lemma 3.3. Thus

$$\limsup_{t\to\infty} G(t) < A_1. \tag{6.2}$$

From inequality (6.1) we deduce that for any positive number $G(0)$ and any positive number A_1' satisfying $A_1' > A_1$, there exists time moment $t_1 = t_1(G(0), A_1')$, given by

$$t_1 = \frac{1}{\beta_1}\log\frac{G(0)}{A_1' - A_1},$$

such that $G(t) < A_1'$ for all $t \ge t_1$. In view of the bound (3.57) this proves the following

Lemma 6.1. *Let $G(t)$ given by (3.14), $G(0) < \infty$, and $\boldsymbol{b} \in L_\infty(\mathbb{R}_+; \boldsymbol{L}_2(\Omega))$. Then for any positive number A_1' satisfying $A_1' > A_1$ with A_1 defined in (6.1), there exists a time moment*

$$t_1 = t_1(G(0), A_1') = \frac{1}{\beta_1} \log \frac{G(0)}{A_1' - A_1}, \tag{6.3}$$

such that

$$\begin{aligned} &\frac{1}{4}\|\boldsymbol{u}_t(t)\|^2_{\boldsymbol{L}_2(\Omega)} + \frac{\gamma}{2}\|\nabla\chi(t)\|^2_{\boldsymbol{L}_2(\Omega)} + \frac{1}{8}\|\chi(t)\|^4_{L_4(\Omega)} \\ &\quad + \frac{c_*}{4}\|\varepsilon(\boldsymbol{u}(t))\|^2_{\boldsymbol{L}_2(\Omega)} \le G(t) + c_1' \\ &< A_1' + c_1' \equiv c_{1a} \quad \textit{for all } t \ge t_1, \end{aligned} \tag{6.4}$$

where $c_1' = \frac{|\Omega|}{2}\left(c_* a_3^2 + \frac{1}{2}\right)$.

Constant c_{1a} in estimate (6.4) is independent of the initial condition $G(0)$. Consequently, for $t \ge t_1$ all estimates from Section 4 and 5 hold true with constant c_1 replaced by c_{1a}. Let $c_{ia} = c_i(c_{1a})$, $i = 2, \dots, 8$, $\beta_{4a} = \beta_4(c_{1a})$, $\beta_{5a} = \beta_5(c_{1a})$, $\Lambda_{3a} = \Lambda_3(c_{1a})$, $\Lambda_{4a} = \Lambda_4$, $\Lambda_{5a} = \Lambda_5(c_{1a})$ denote the corresponding constants independent of $G(0)$.

On account of Lemma 5.2 the following estimate is satisfied for all $t \ge t_1$:

$$N(t) \le A_{2a}(1 - e^{-\beta_{4a} t}) + N(0)e^{-\beta_{4a} t} \tag{6.5}$$

where β_{4a} and A_{2a} are positive constants independent of $N(0)$:

$$\begin{aligned} &\beta_{4a} = \beta_4(c_{1a}) \text{ defined in (4.51)}, \\ &A_{2a} = \frac{1}{\beta_{4a}}(\Lambda_{3a} b_1^2 + \Lambda_{4a} b_2^2 + \Lambda_{5a}), \end{aligned} \tag{6.6}$$

with

$$b_1 = \sup_{t\in\mathbb{R}_+} \|\boldsymbol{b}(t)\|_{\boldsymbol{L}_2(\Omega)}, \quad b_2 = \sup_{t\in\mathbb{R}_+} \|\boldsymbol{b}_t(t)\|_{\boldsymbol{L}_2(\Omega)}.$$

Thus,

$$\limsup_{t\to\infty} N(t) < A_{2a}. \tag{6.7}$$

From inequality (6.5) it follows that for any positive number $N(0)$ and any positive number A_2' satisfying $A_2' > A_{2a}$, there exists time moment

$$t_2 = \frac{1}{\beta_{4a}} \log \frac{N(0)}{A_2' - A_{2a}},$$

such that

$$N(t) < A_2' \quad \text{for all} \quad t \geq t_* = \max\{t_1, t_2\}. \tag{6.8}$$

Hence, taking into account that by (4.58),

$$\begin{aligned} N(t) \geq c_{7a}(&\|\boldsymbol{u}(t)\|^2_{\boldsymbol{H}^2(\Omega)} + \|\boldsymbol{u}_t(t)\|^2_{\boldsymbol{H}^2(\Omega)} + \|\boldsymbol{u}_{tt}(t)\|^2_{\boldsymbol{H}^1(\Omega)} \\ &+ \|\chi(t)\|^2_{H^2(\Omega)} + \|\chi_t(t)\|^2_{L_2(\Omega)}) - c_{8a}' \quad \text{for} \quad t \geq t_1, \end{aligned} \tag{6.9}$$

we deduce the following

Lemma 6.2. *Let $N(t)$ be given by (4.48), $N(0) < \infty$ and $\boldsymbol{b} \in$ $\in W^1_\infty(0,\infty; \boldsymbol{L}_2(\Omega))$. Moreover, let the numbers t_1 and c_{1a} be defined in Lemma 6.1. Then for any positive number A_2' satisfying $A_2' > A_{2a}$, with A_{2a} defined by (6.6), there exists time moment $t_2 = t_2(N(0), A_2')$, given by*

$$t_2 = \frac{1}{\beta_{4a}} \log \frac{N(0)}{A_2' - A_{2a}}, \tag{6.10}$$

such that

$$\begin{aligned} &c_{7a}(\|\boldsymbol{u}(t)\|^2_{\boldsymbol{H}^2(\Omega)} + \boldsymbol{u}_t(t)\|^2_{\boldsymbol{H}^2(\Omega)} + \|\boldsymbol{u}_{tt}(t)\|_{\boldsymbol{H}^1(\Omega)} + \|\chi(t)\|^2_{H^2(\Omega)} \\ &\quad + \chi_t(t)\|^2_{L_2(\Omega)}) \\ &\leq N(t) + c_{8a}' < A_2' + c_{8a}' \quad \textit{for all} \quad t \geq t_* = \max\{t_1, t_2\}, \end{aligned} \tag{6.11}$$

where constants c_{7a}, c_{8a}', independent of $N(0)$, are defined in (4.58) with constant c_{1a} in place of c_1.

The above lemma completes the proof of Theorem 2.2. □

7. Proof of Theorem 2.3 (Uniqueness)

Let $(\boldsymbol{u}_1, \chi_1, \mu_1)$ and $(\boldsymbol{u}_2, \chi_2, \mu_2)$ be two solutions of problem (1.14)–(1.16) corresponding to the same data. Subtracting the corresponding equations and denoting

$$\boldsymbol{U} = \boldsymbol{u}_1 - \boldsymbol{u}_2, \quad H = \chi_1 - \chi_2, \quad Y = \mu_1 - \mu_2,$$

we obtain the following system for $(\boldsymbol{U}, H, Y)$:

(7.1)

$$\begin{aligned} &\boldsymbol{U}_{tt} - \boldsymbol{Q}\boldsymbol{U} - \nu\boldsymbol{Q}\boldsymbol{U}_t = (z'(\chi_1) - z'(\chi_2))\boldsymbol{B}\nabla\chi_1 + z'(\chi_2)\boldsymbol{B}\nabla H && \text{in } \Omega^T, \\ &\boldsymbol{U}|_{t=0} = \boldsymbol{0}, \quad \boldsymbol{U}_t|_{t=0} = \boldsymbol{0} && \text{in } \Omega, \\ &\boldsymbol{U} = \boldsymbol{0} && \text{on } S^T, \end{aligned}$$

$$
(7.2)\qquad \begin{aligned} &H_t - \Delta Y = 0 && \text{in } \Omega^T, \\ &H|_{t=0} = 0 && \text{in } \Omega, \\ &\boldsymbol{n}\cdot\nabla Y = 0 && \text{on } S^T, \end{aligned}
$$

(7.3)

$$
\begin{aligned} Y &= -\gamma\Delta H + \psi'(\chi_1) - \psi'(\chi_2) + (z'(\chi_1) - z'(\chi_2))(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_1) \\ &\quad + Dz(\chi_1) + E) + z'(\chi_2)[\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{U}) + D(z(\chi_1) - z(\chi_2))] \quad \text{in } \Omega^T, \\ \boldsymbol{n}\cdot\nabla H &= 0 \quad \text{on } S^T. \end{aligned}
$$

Multiplying (7.1) by $\boldsymbol{QU}_t(t)$, integrating over Ω and by parts, using boundary condition $(7.1)_3$ we get

$$
(7.4)\qquad \begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big[\int_\Omega |\boldsymbol{Q}^{1/2}\boldsymbol{U}_t|^2dx + \int_\Omega |\boldsymbol{QU}|^2dx\Big] + \nu\int_\Omega |\boldsymbol{QU}_t|^2dx \\ &\le \Big|\int_\Omega (z''(\chi_*)H\boldsymbol{B}\nabla\chi_1 + z'(\chi_2)\boldsymbol{B}\nabla H)\cdot\boldsymbol{QU}_t dx\Big|, \end{aligned}
$$

where $\chi_* \in (\chi_1, \chi_2)$. Hence, by the Young inequality and the boundedness of $z'(\cdot)$, $z''(\cdot)$, it follows that

$$
(7.5)\qquad \begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\boldsymbol{Q}^{1/2}\boldsymbol{U}_t\|^2_{\boldsymbol{L}_2(\Omega)} + \|\boldsymbol{QU}\|^2_{\boldsymbol{L}_2(\Omega)}) + \frac{\nu}{2}\|\boldsymbol{QU}_t\|^2_{\boldsymbol{L}_2(\Omega)} \\ &\le c\int_\Omega (H^2|\nabla\chi_1|^2 + |\nabla H|^2)dx \\ &\le c(\|H\|^2_{L_3(\Omega)}\|\nabla\chi_1\|^2_{\boldsymbol{L}_6(\Omega)} + \|\nabla H\|^2_{\boldsymbol{L}_2(\Omega)}) \\ &\le c(\|H\|^2_{L_3(\Omega)} + \|\nabla H\|^2_{\boldsymbol{L}_2(\Omega)}) \quad \text{for } t\in(0,T), \end{aligned}
$$

where we used the fact that by virtue of (2.13), $\|\nabla\chi_1\|_{L_\infty(0,T;\boldsymbol{L}_6(\Omega))} \le c$. Now, let us multiply $(7.2)_1$ by $\Delta H(t)$, integrate over Ω and by parts using boundary condition $(7.2)_3$, to get after substituting $(7.3)_1$

(7.6)

$$
\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla H|^2dx + \gamma\int_\Omega |\nabla\Delta H|^2dx \\ &= \int_\Omega \nabla[\psi'(\chi_1) - \psi'(\chi_2) + (z'(\chi_1) - z'(\chi_2))(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_1) + Dz(\chi_1) + E) \\ &\quad + z'(\chi_2)(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{U}) + D(z(\chi_1) - z(\chi_2)))]\cdot\nabla\Delta H dx. \end{aligned}
$$

In view of the estimate $\|\chi_i\|_{L_\infty(\Omega^T)} \le c$, $i = 1,2$, and assumption (2.21) on $z(\cdot)$, we have

$$
|\nabla[(\psi'(\chi_1) - \psi'(\chi_2)]| \le c|\nabla H|(|\nabla\chi_1| + |\nabla\chi_2| + 1),
$$

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$$
\begin{aligned}
&|\nabla[(z'(\chi_1)-z'(\chi_2))(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_1)+Dz(\chi_1)+E)]| \\
&\le c(|\nabla H|+|H|\,|\nabla\chi_2|)(|\boldsymbol{\varepsilon}(\boldsymbol{u}_1)|+1) \\
&\quad + c|H|(|\nabla^2 u_1|+|\nabla\chi_1|),
\end{aligned}
$$

$$
\begin{aligned}
&|\nabla[z'(\chi_2)(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{U})+D(z(\chi_1)-z(\chi_2)))] \\
&\le c|\nabla\chi_2|(|\boldsymbol{\varepsilon}(\boldsymbol{U})|+|H|)+c(|\nabla^2\boldsymbol{U}|+|H|\,|\nabla\chi_1|+|\nabla H|).
\end{aligned}
$$

Hence, by the Young inequality, we conclude from (7.6) that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla H\|^2_{L_2(\Omega)}+\frac{\gamma}{2}\|\nabla\Delta H\|^2_{L_2(\Omega)} \\
&\le c\int_\Omega [|\nabla H|^2+H^2(|\nabla\chi_1|^2+|\nabla\chi_2|^2)+|\nabla H|^2|\boldsymbol{\varepsilon}(\boldsymbol{u}_1)|^2 \\
&\quad + H^2|\nabla\chi_2|^2|\boldsymbol{\varepsilon}(\boldsymbol{u}_1)|^2+H^2|\nabla^2\boldsymbol{u}_1|^2 \\
&\quad + |\nabla\chi_2|^2|\boldsymbol{\varepsilon}(\boldsymbol{U})|^2+|\nabla^2\boldsymbol{U}|^2]dx.
\end{aligned}
\tag{7.7}
$$

We estimate the subsequent terms on the right-hand side of (7.7). On account of the bound $\|\nabla\chi_i\|_{L_\infty(0,T;L_6(\Omega))}\le c$,

$$
\begin{aligned}
&\int_\Omega H^2(|\nabla\chi_1|^2+|\nabla\chi_2|^2)dx \le \|H\|^2_{L_3(\Omega)}(\|\nabla\chi_1\|^2_{L_6(\Omega)}+\|\nabla\chi_2\|^2_{L_6(\Omega)}) \\
&\le c\|H\|^2_{L_3(\Omega)} \le \delta_1\|\nabla^3 H\|^2_{L_2(\Omega)}+c(1/\delta_1)\|H\|^2_{L_2(\Omega)}, \quad \delta_1>0,
\end{aligned}
\tag{7.8}
$$

where in the last line we applied the interpolation inequality. Similarly,

$$
\begin{aligned}
&\int_\Omega |\nabla H|^2|\boldsymbol{\varepsilon}(\boldsymbol{u}_1)|^2dx \le \|\nabla H\|^2_{L_3(\Omega)}\|\boldsymbol{\varepsilon}(\boldsymbol{u}_1)\|^2_{L_6(\Omega)} \\
&\le c\|\nabla H\|^2_{L_3(\Omega)} \le \delta_2\|\nabla^3 H\|^2_{L_2(\Omega)}+c(1/\delta_2)\|\nabla H\|^2_{L_2(\Omega)}, \quad \delta_2>0,
\end{aligned}
\tag{7.9}
$$

and

$$
\begin{aligned}
&\int_\Omega H^2|\nabla\chi_2|^2|\boldsymbol{\varepsilon}(\boldsymbol{u}_1)|^2dx \le \|H\|^2_{L_6(\Omega)}\|\nabla\chi_2\|^2_{L_6(\Omega)}\|\boldsymbol{\varepsilon}(\boldsymbol{u}_1)\|^2_{L_6(\Omega)} \\
&\le c\|H\|^2_{L_6(\Omega)} \le \delta_3\|\nabla^3 H\|^2_{L_2(\Omega)}+c(1/\delta_3)\|H\|^2_{L_2(\Omega)}, \quad \delta_3>0.
\end{aligned}
\tag{7.10}
$$

Next, recalling the bound $\|\boldsymbol{u}_1\|_{L_\infty(0,T;\boldsymbol{H}^2(\Omega))}\le c$, and applying the interpolation inequality, we have

$$
\begin{aligned}
&\int_\Omega H^2|\nabla^2\boldsymbol{u}_1|^2dx \le \|H\|^2_{L_\infty(\Omega)}\|\nabla^2\boldsymbol{u}_1\|^2_{L_2(\Omega)} \\
&\le c\|H\|^2_{L_\infty(\Omega)} \le \delta_4\|\nabla^3 H\|^2_{L_2(\Omega)}+c(1/\delta_4)\|H\|^2_{L_2(\Omega)}, \quad \delta_4>0.
\end{aligned}
\tag{7.11}
$$

Finally,

$$(7.12)\qquad \int\limits_{\Omega} |\nabla \chi_2|^2 |\boldsymbol{\varepsilon}(\boldsymbol{U})|^2 dx \le \|\nabla \chi_2\|^2_{L_6(\Omega)} \|\boldsymbol{\varepsilon}(\boldsymbol{U})\|^2_{L_3(\Omega)} \le c\|\boldsymbol{\varepsilon}(\boldsymbol{U})\|^2_{L_3(\Omega)}.$$

Let us note that since $\int\limits_{\Omega} H dx = 0$ and $\boldsymbol{n} \cdot \nabla H = 0$ on S, the Poincaré inequality and the elliptic regularity theory yield

$$(7.13)\qquad \|H\|^2_{L_2(\Omega)} \le c\|\nabla H\|^2_{L_2(\Omega)}, \quad \|\nabla^3 H\|^2_{L_2(\Omega)} \le c\|\nabla \Delta H\|^2_{L_2(\Omega)}.$$

Combining estimates (7.8)–(7.12) in (7.7), using (7.13) and choosing constants δ_i, $i = 1, 2, 3, 4$ appropriately, we arrive at

$$(7.14)\qquad \begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla H\|^2_{L_2(\Omega)} + \frac{\gamma}{4}\|\nabla \Delta H\|^2_{L_2(\Omega)} \\ &\le c(\|\nabla H\|^2_{L_2(\Omega)} + \|H\|^2_{L_2(\Omega)} + \|\boldsymbol{\varepsilon}(\boldsymbol{U})\|^2_{L_3(\Omega)} + \|\nabla^2 \boldsymbol{U}\|^2_{L_2(\Omega)}) \\ &\le c(\|\nabla H\|^2_{L_2(\Omega)} + \|\boldsymbol{Q}\boldsymbol{U}\|^2_{L_2(\Omega)}). \end{aligned}$$

Summing up inequalities (7.14) and (7.5) leads to

$$(7.15)\qquad \begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\nabla H\|^2_{L_2(\Omega)} + \|\boldsymbol{Q}^{1/2}\boldsymbol{U}_t\|^2_{L_2(\Omega)} + \|\boldsymbol{Q}\boldsymbol{U}\|^2_{L_2(\Omega)}) \\ &\quad + \frac{\gamma}{4}\|\nabla \Delta H\|^2_{L_2(\Omega)} + \frac{\nu}{2}\|\boldsymbol{Q}\boldsymbol{U}_t\|^2_{L_2(\Omega)} \\ &\le c(\|H\|^2_{L_3(\Omega)} + \|\nabla H\|^2_{L_2(\Omega)} + \|\boldsymbol{Q}\boldsymbol{U}\|^2_{L_2(\Omega)}). \end{aligned}$$

Hence, applying again the interpolation inequality

$$\|H\|^2_{L_3(\Omega)} \le \delta_5\|\nabla^3 H\|_{L_2(\Omega)} + c(1/\delta_5)\|H\|^2_{L_2(\Omega)}, \quad \delta_5 > 0,$$

and using (7.13), we conclude finally that

$$(7.16)\qquad \begin{aligned} &\frac{d}{dt}(\|\nabla H\|^2_{L_2(\Omega)} + \|\boldsymbol{Q}^{1/2}\boldsymbol{U}_t\|^2_{L_2(\Omega)} + \|\boldsymbol{Q}\boldsymbol{U}\|^2_{L_2(\Omega)}) \\ &\quad + \|\nabla \Delta H\|^2_{L_2(\Omega)} + \|\boldsymbol{Q}\boldsymbol{U}_t\|^2_{L_2(\Omega)} \\ &\le c(\|\nabla H\|^2_{L_2(\Omega)} + \|\boldsymbol{Q}\boldsymbol{U}\|^2_{L_2(\Omega)}). \end{aligned}$$

Thus, denoting

$$D(t) = \|\nabla H(t)\|^2_{L_2(\Omega)} + \|\boldsymbol{Q}^{1/2}\boldsymbol{U}_t(t)\|^2_{L_2(\Omega)} + \|\boldsymbol{Q}\boldsymbol{U}(t)\|^2_{L_2(\Omega)},$$

we find the inequality

$$\frac{d}{dt}D(t) \le cD(t) \quad \text{for} \quad t \in (0, T),$$

which implies that $D(t) \le D(0)e^{ct}$. Hence, since $D(0) = 0$,

$$c(\|H(t)\|^2_{H^1(\Omega)} + \|\boldsymbol{U}_t(t)\|^2_{H^1(\Omega)} + \|\boldsymbol{U}(t)\|^2_{\boldsymbol{H}^2(\Omega)}) \le D(t) = 0 \quad \text{for} \quad t \in [0, T],$$

that is, $\boldsymbol{U} = \boldsymbol{0}$ and $H = 0$ in Ω^T. Besides, from $(7.3)_1$ it follows immediately that $Y = 0$ in Ω^T. This finishes the proof. □

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