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VARIABLE LOADING AND IMPOSED DISPLACEMENTS IN THE SHAKEDOWN THEORY

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Some of the results presented in this dissertation were obtained under supervision of late

Professor Jan Andrzej König

1937–1990

This dissertation is dedicated to his memory.

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Variables in continuum description

V	volume of the body
S_T	boundary of the body with surface tractions
S_D	boundary of the body with imposed displacements
S_U	boundary of the body with vanishing displacements
$\mathbf{f}$	body forces specified in the volume V
$\mathbf{t}$	surface tractions applied to the boundary S_T
$\mathbf{t}^{eq}$	elastically equivalent surface tractions on S_D
$\mathbf{u}^D$	imposed displacements on S_D
$\mathbf{u}^{res}$	residual displacements produced by plastic strains
$\beta_{(l)}$	parameter of the $l-th$ independent load system
$\beta^o_{(l)}$	constant part of the $l-th$ load parameter
$\beta^s_{(l)}$	portion of the $l-th$ load parameter, symmetrically variable with respect to the constant part $\beta^o_{(l)}$
$a_{(l)}$, $b_{(l)}$	lower and upper limits of the $l-th$ load parameter
$f(.)$	yield function
Ω	domain of variation of load parameters $\beta_{(l)}(t)$
$\mathcal{B}^k$	vertex of load polyhedron Ω
μ^{sh}	shakedown load multiplier
λ^a	inverse statical multiplier
λ^{sh}	inverse shakedown multiplier
ϵ_{ij}	total strain tensor
ϵ^e_{ij}	elastic part of the total strain
ϵ^p_{ij}	plastic part of the total strain
ϵ^{res}_{ij}	kinematically admissible residual strains produced by plastic strains
ϵ^E_{ij}	strains in a purely elastic reference body
σ_{ij}	total stress tensor
σ^E_{ij}	stress in a purely elastic reference body
ρ_{ij}	residual stress
σ^{EM}_{ij}	elastic stress in the reference body under external forces $\mathbf{f}$, $\mathbf{t}$
σ^{ED}_{ij}	elastic stress in the reference body under imposed displacements $\mathbf{u}^D$

Variables in structural description

A	bar axis or middle surface in the case of plates (shells)
H	area of the cross-section or thickness of the plate (shell)
ξ	position of the cross-section described in curvilinear coordinate system related to the bar axis or to the middle surface of the plate (shell)
$F^L = 0$	yield locus for the cross-section
$F^E = 0$	elastic locus for the cross-section
$F^R = 0$	incremental collapse (rattcheting) locus for the cross-section
k_r	vector of plastic moduli for generalized variables
$\mathbf{Q}$	generalized stress vector
$\mathbf{q}$	generalized strain vector
$\mathbf{w}$	displacements of the bar axis or of the middle surface
$\mathbf{r}$	generalized displacement vector (displacements and rotations of the bar axis or of the middle surface)
$\mathbf{R}$	generalized external forces
$\mathbf{p}$	surface tractions
$\mathbf{L}$	linear differential operator of equilibrium equations
$\mathbf{L}^*$	linear differential operator of kinematical equations
$\mathbf{N}$	linear diferential operator of statical boundary conditions
$\mathbf{M}$	linear differential operators of kinematical boundary conditions
$\eta^a = \bar{F}$	inverse static multiplier for generalized variable approach
η^{inc}	inverse shakedown multiplier with respect to incremental collapse
η^{alt}	inverse shakedown multiplier with respect to alternating plasticity
$\xi^{(e_\phi)}$	the $\phi - th$ integration point of the $e - th$ finite element
$\varphi^{(e)}$	shape function for displacements
$\psi^{(e_\phi)}$	shape functions for plastic strains
$\bar{\mathbf{r}}^{(e)}$	generalized displacement vector defined in nodal points of the finite element
$\bar{\mathbf{r}}$	global vector of generalized displacements
$\mathbf{K}$	stiffness matrix for the cross-section (denoted also by $\mathbf{E}$), later the stiffness matrix of the structure
$\mathbf{Z}$	influence matrix relating residual generalized stress vector $\mathbf{Q}^{res}$ and plastic generalized strain vector $\mathbf{q}^p$

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VARIABLE LOADING AND IMPOSED DISPLACEMENTS IN THE SHAKEDOWN THEORY*

Summary

The paper deals with a theoretical and a numerical analysis of the action of variable combined loads (imposed displacements and external forces) on elastic-plastic bodies (structures). Three fundamental parts of the paper can be distinguished

1. In the first of them an influence of the combined loads onto continuum is studied. Various cases of a body behaviour undergoing imposed displacements were considered. A notion of a load equivalent to to the combined one was introduced. Some theorems concerning bounds for a safety factor of elastic-plastic systems were proved.

2. Second part is devoted to an alternative formulation of the shakedown problem, which appeared in the literature recently. It permits to determine a shakedown load multiplier for a nonlinear yield condition in an efficient way. A theorem concerning the equivalence of the classical continuum approach to the proposed one was proved. An extension to a generalized variable description was proposed. A unified approach was presented in order to determine the shakedown load multiplier for the two inadaptation plastic mechanisms: incremental collapse and alternating plasticity.

3. The third part concerns the numerical analysis. An algorithm for solution of the alternatively formulated shakedown problem was proposed. The procedure is illustrated with examples of plane grids subjected to the variable repeated imposed displacements and the external loads.

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1 Introduction

1.1 Historical outline

A collapse phenomenon, treated as a result of an occurrence of plastic strains in the structure under variable repeated loads, was first investigated by Grüning [27] in 1926. Redundant beam systems with I-cross-section were examined by Bleich [2], who proved for them a shakedown theorem in 1932. A general formulation of the shakedown problem for continuum was presented by Melan [53] in 1938r. It may be interesting to remark that it has been done before a rigorous formulation of limit analysis theorems.

It was in 1956 that Koiter [34] noticed that the limit analysis theorems are particular case of the shakedown theorems. He proved also a kinematical theorem for continuum. In 1960 he summarized the knowledge concerning the elastic-plastic bodies [35] and formulated the shakedown and the limit analysis theorems in the form, which is used nowadays. Thus, the first period of the research concerning the influence of variable repeated loads on the elastic-plastic bodies was closed. The statical Melan's and the kinematical Koiter's theorems were acknowledged to be the classical ones.

A generalization of the classical theorems to other types of loads, more complex material models and nonlinear geometrical effects were subject of an intensive study in the next period. Prager [68] in 1956 and, independently of him, Rozenblum [73] in 1957, extended the statical Melan's theorem to the case of variable repeated thermal effects. The influence of the temperature on the variation of elastic moduli was investigated by Kõnig [37] in 1969.

It was also Rosenblum [74], who in 1965 generalized the kinematical Koiter's theorem to the case of the thermal loading taking into account a dependence of plastic moduli upon the temperature. Independently, the extension was also given by De Donato [13] in 1970.

It appears that there was no problem with application of the statical Melan's shakedown theorem, but the kinematical Koiter's theorem had no practical meaning during a long period of time. It was because of difficulties related to integration of unknown quantities over a time and a volume of the body. It was Gokhfeld [20] in 1966 and Sawczuk [75] in 1967, who showed how to integrate the Koiter's inequality over the time using linearized yield conditions. Thus, a new way was opened to employ the kinematical approach to the theory of plates and shells. Two inadaptation plastic mechanisms, i.e. an incremental collapse and an alternating plasticity were also clearly defined. The former is characterized by a progressive accumulation of the plastic strain increments over a cycle, whereas the latter is connected with the vanishing plastic strain increments over the whole cycle in each considered point of the body. A strict mathematical proof for determination of the shakedown load, which protects the structure against the collapse by the alternating plasticity was given by König [41] in 1979.

Both the statical Melan's and the kinematical Koiter's theorems were proved for continuum. In the contrary to the limit analysis theory it is impossible to transform the shakedown theorems directly from continuum to the case of generalized variables. It was König [36], who presented in 1966 the shakedown theory in the terms of generalized variables. He introduced some subsequent elastic loci on a level of a cross-section, fully described by a residual stresses inside the cross-section. At the end of sixties Sawczuk [76] and Gokhfeld together with Cherniavsky [21] presented new theoretical results and proposed methods for calculation of plates and shells.

Further investigations by Gokhfeld and Cherniavsky [22] concerned new methods for solutions of the shakedown problem, based on the statical Melan's theorem. A notion of a fictitive yield surface was introduced by the authors in 1972, in order to reduce the shakedown problem to the case of the limit analysis for a body with some plastic heterogeneity. A detailed description of this method and of the kinematical approach (analytical methods) can be found together with some engineering applications in the monography [24] by Gokhfeld and Cherniavsky published in 1980.

A simple model of a linear kinematical hardening (so called the Prager model) was already incorporated in the proof of the theorem by Melan [54] in 1938. This idea was examined in detail by Neal [59] in 1950, König [38] in 1971, Maier [50] in 1973r., and Ponter [67] in 1975. The fundamental weakness of this model was related to the assumption of unlimited hardening, what practically excluded from considerations the incremental collapse mode of inadaptation. An extension of Melan's theorem to the case of the hardening model with internal parameters, first introduced by Halphen and Nguyen [28] in 1975, was done by Mandel [52] in 1976. Recently, Weichert and Gross-Weege [88] in 1988 used the Mandel's formulation with a two-surfaces hardening model. Next, a generalization to nonlinear hardening (the material behaviour is stable in the sense of the Drucker's postulate), using a discrete model of the structure, was presented by Maier and Novati [51] in 1990. A different approach based on examination of subsequent shakedown surfaces depending on an initially chosen process of deformation was proposed by Życzkowski [93] in 1988.

The above description concerned only quasi-statical loads. An extension of Melan's theorem to the case of dynamic loads can be found in the following papers: Ceradini [9] in 1969., Hwa-Shan-Ho [30] in 1972, Corradi and Maier [10] in 1973. At the beginning of seventies a theorem concerning the steady cyclic state for the structure under dynamic loads was proved by Mróz [58]. Viscous effects and kinematical hardening material model were taken into account. It was shown, that the shakedown problem can be treated as a particular case of the steady cyclic state with plastic strains vanishing during the loading cycle.

The statical Melan's and the kinematical Koiter's theorems are dual with respect to each other. The former uses statical quantities represented by the stress state in the the body, whereas the latter deals with a history of kinematically admissible plastic strains, which forms a plastic mechanism over the loading cycle. Unfortunately, in both the cases, calculations of structural displacements are already excluded on the level of the formulation. It follows, among other things from the fact, that the shakedown theorems are valid for any arbitrary load path included in prescribed limits. Because of the arbitrary load history (not specified in many practical cases) it is impossible to determine final displacements of the body in a unique way. Instead of this, it is reasonable to estimate an upper bound to the displacements. It was the subject of work for many researchers like Brzeziński and König [6], [7], Vitello [84], Ponter et all. [66] at the beginning of seventies. Their investigations were limited only to some beam and frame systems with sandwich cross-sections i.e. when plastic strains were assumed to be localized in plastic hinges. It was Dorosz [16], who took into account a spreading of the plastic zone along the beam axis in order to estimate the maximal displacements.

Besides the material nonlinearity some nonlinear geometrical effects were also studied. Nevertheless, no general approach has been proposed until now accounting for an arbitrary of geometrical nonlinearity under variable loads. Up to now, only the second order effects were incorporated into the analysis. It was Maier [50], who as the first published a paper on this subject in 1973. He proved, that the shakedown theorems are also valid if an additional term depending on the displacements of the structure is incorporated into equilibrium equations. This term may describe the influence of axial forces on flexure of beams. Proofs of the theorems concerned discretized structures and linearized yield conditions.

Another approach to the problem was presented by Nquyen, Gary and Baylac [62] in 1982 and by Nguyen [63]. Using Maier's formulation of the problem authors pointed out a destabilization effect of the shakedown process. It is caused by the accumulation of the plastic strains, which influences changes in geometry and may induce a local or a global instability.

It was Weichert [87] who in 1986 derived the shakedown theorem for continuum in the case of nonlinear geometrical effects. However, to determine the shakedown multiplier one should know a supposed process of the deformation, which makes the theorem rather difficult in a direct application. Further considerations were limited only to the particular case of the second order geometrical effects. Variable repeated loads with the amplitude, which do not affect the changes of geometry, were superimposed on a initial fully nonlinear deformation process of the body undergoing a known load history. The problem was described in Lagrange coordinates referred to the undeformed configuration. A detailed description of such deformation and its influence on the shakedown theorems were presented by Gross-Weege [25] in 1990. The results of Maier, Weichert and Gross-Weege were extended by Pycko and König [69] in 1991, who studied occurrence of the steady cyclic state. The problem was described in the updated Lagrange coordinate system, therefore any arbitrary initial deformation process could be considered. It was shown (as Mróz [58] did in the linear case) that the shakedown problem can be considered as a particular case of the limit steady cycle. Depending on the choice of the residual stress representation the Maier's [50] or Gross-Weege's [25] formulation was obtained.

Another approach, allowing for investigation of any arbitrary state of deformations was presented by Siemaszko and König [80] in 1985 as an extension of papers by Gokhfeld and Cherniavsky [23], and König [42]. This was based on a subsequent application of the linear geometrical model, assuming small elastic displacements. The changes of the geometry were induced by progressive plastic deformations corresponding to the most dangerous plastic mechanism in a given step of analysis. In spite of the simplicity of the method the authors obtained very interesting results. Further papers include an analysis of any arbitrary material hardening [81], and the influence of geometry changes on an optimal structure under variable loading [82].

A more complete literature review of the shakedown theory can be found in the following monographs (Sawczuk, Janas, König [77], Gokhfeld, Cherniavsky [24], König [45], Zarka et al. [89]).

1.2 Aim and scope

The classical statical and kinematical shakedown theorems were proved for the loads consisted of external forces. Next, they were extended to the case of the thermal loading with thermal dependent plastic moduli. Up to now, the subject of variable repeated imposed displacements was not considered in detail in the literature. The notion of variable repeated imposed displacements is understood as displacements fields specified on the boundary S_D. They can be described by finite number of parameters, independently varying within arbitrary chosen limits. In some papers (i.e. Weichert and Gross-Weege [88]) imposed displacements were taken into account in the general formulation of the boundary value problem but they were neglected in the further analysis. On the other hand the action of this type of the load may have an essential influence on safety of important engineering systems like: off-shore structures, floating bridges or compensators of pipelines. Some important conclusions concerning a serviceability of the structures could be drawn from the analysis of the problem mentioned above. Namely, for the safety of structure subjected to external forces the maximum possible stiffness may be recommended, whereas the opposite situation takes place in the case of imposed displacements.

The state of the theoretical investigations presented above and the technological importance of the problem motivated the author of this dissertation to undertake research concerning the influence of the combined load onto the structure. This load is defined as the variable repeated systems of the imposed displacements and the external forces, both described by a finite number of independent parameters.

In this contribution the influence of the imposed displacements onto the structure and the shakedown theorems concerning the combined loads described above were investigated. Two separate stress states resulting from external forces and imposed displacements, respectively, were distinguished. Some general properties of the problem described in this way were pointed out. For a qualitative presentation of the theoretical reasoning and some important resulting effects, some examples of simple plane frames were given. Main results were published in the papers of König and Pycko [47], Pycko and König [70].

The second part of the dissertation was devoted to some numerical aspects. Until now, the shakedown problems were solved by using the mathematical programming methods. Unfortunately, because of numerous constraints (a linearized or a nonlinear form of the yield condition) these methods don't seem to be efficient in the case of complex structures with many components of the generalized stress state. Therefore, seeking of more effective methods of the solution is needed.

To follow this requirements, a relatively new formulation of a min-max problem was used. First, it was proposed by Zwoliński and Bielawski [90] in order to develop a numerical program for a continuum. Unfortunately, no equivalence between the above formulation and the shakedown problems was pointed out. Because of the attractiveness of the mentioned approach (lack of constraints) the second part of this contribution was devoted to a detailed study of the problem. The main attention was focused on an extension of the formulation from the level of a continuum to the level of a structure (description in the terms of generalized variables). Some appropriate theorems allowing for a unified treatment of both inadaptation mechanisms were proved (Pycko and Mróz [71]). A method of solution of the min-max problem, verified on some examples of plane grids, was proposed.

This contribution attempts to treat the shakedown problem of the structure under the variable loads and imposed displacements in a comprehensive manner. Some theoretical reasoning presented here have their own simple physical interpretations. They help us to understand the behaviour of structures under the combined loads. The numerical method presented in the contribution appears to be very efficient for the analysis of the bar structures and is very promising in the case of plate and shell structures. It seems to become a convenient tool in the engineering practice.

There are some limitations imposed onto the consideration made in the dissertation. Namely the model assumes:

1. the elastic-perfectly plastic model of the material;

2. the associated plastic flow rule;

3. the small displacements theory;

4. a quasi-static character of loads.

An extension of the considerations given in this contribution to more complex material models, thermal loading and second order geometrical effects will be a subject of a future research of the author.

2 Fundamental relations

2.1 Scope of the chapter

Fundamental relations needed for description of the shakedown problem are presented. To clarify notation and description some basic notions are quoted. That concerns first of all the elastic-perfectly plastic material model, load variation (independently varying systems of external forces and imposed displacements), equilibrium equations and boundary conditions for the problem considered.

2.2 Material model

The elastic-perfectly plastic material is considered. Such a model implies existence of a non-negative scalar function of stress components

$$F(\sigma_{ij}, k) \leq 0, \tag{2.1}$$

which may be represented (especially in the case of homogeneous function of stresses) in the form

$$f(\sigma_{ij}) - k \leq 0, \tag{2.2}$$

where
f yield function,
k plastic modulus,
σ_{ij} stress components.

The inequality $F(\sigma_{ij}, k) < 0$ defines an elastic domain in the stress space whereas the condition

$$F(\sigma_{ij}, k) = 0 \tag{2.3}$$

describes a yield surface in this space.

The yield function $f(\sigma_{ij})$ is assumed here to be homogeneous of the first degree

$$f(\alpha\sigma_{ij}) = |\alpha| f(\sigma_{ij}), \tag{2.4}$$

what, among other, assumes its symmetry with respect to the coordinate origin in the case $\alpha = -1$

$$f(-\sigma_{ij}) = f(\sigma_{ij}). \tag{2.5}$$

Because of the assumption of small displacements, additivity of elastic $\dot{\epsilon}^e_{ij}$ and plastic $\dot{\epsilon}^p_{ij}$ strain rates implies additivity of the strains themselves

$$\epsilon_{ij} = \epsilon^e_{ij} + \epsilon^p_{ij}. \tag{2.6}$$

From the Drucker's postulate [18] of the material stability it follows

$$F(\sigma_{ij}) = 0, \qquad F(\sigma^s_{ij}) \leq 0, \qquad (\sigma_{ij} - \sigma^s_{ij})\dot{\epsilon}^p_{ij} \geq 0, \tag{2.7}$$

what means that:

1. the yield surface is convex;

2. the plastic strain rate tensor is orthogonal to the yield surface $F = 0$ ("the associated plastic flow rule")

$$\dot{\epsilon}^p_{ij} = \dot{\lambda}\frac{\partial F}{\partial \sigma_{ij}}, \qquad \dot{\lambda} \geq 0, \qquad F(\sigma_{ij}) \leq 0, \qquad \dot{\lambda}F = 0. \tag{2.8}$$

Active and passive plastic processes, as well as (elastic) unloading from the yield surface are described by the inequalities

$$F = 0, \qquad \dot{\lambda} \geq 0, \qquad \dot{F}(\sigma_{ij}) \leq 0, \qquad \dot{\lambda}\dot{F} = 0. \tag{2.9}$$

The elastic part of the strains is determined by the Hooke's law

$$\epsilon^e_{ij} = E^{-1}_{ijkl}\sigma_{kl}, \tag{2.10}$$

where E_{ijkl} is the fourth-order tensor of elastic moduli, symmetric with respect to the indices i, j and k, l.

2.3 Variable loads

Let a body occupying a volume V, bounded by a surface S, undergoes the following variable actions:

1. body forces $\mathbf{f}(\mathbf{x}, t)$ defined in V and surface tractions $\mathbf{t}(\mathbf{x}, t)$ acting on the surface S_T,

2. imposed displacements $\mathbf{u}^D(\mathbf{x}, t)$ specified on the surface S_D.

On the remaining part of surface $S_U = S \backslash (S_D \cup S_T)$ displacements disappear $\mathbf{u} = 0$.

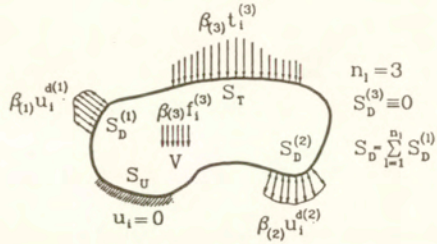
Without any loss in generality of the theoretical considerátions, the above set consisting of $l = 1$ to n_l independently varying load systems may be described by a finite number of loading parameters $\beta_{(l)}$ in the following way (Kōnig [36])

$$f_i = \sum_{l=1}^{n_l} \beta_{(l)}(t) f_i^{(l)}(\mathbf{x}), \qquad t_i = \sum_{l=1}^{n_l} \beta_{(l)}(t) t_i^{(l)}(\mathbf{x}), \qquad u_i^D = \sum_{l=1}^{n_l} \beta_{(l)}(t) u_i^{d\,(l)}(\mathbf{x}), \tag{2.11}$$

$$a_{(l)} \leq \beta_{(l)}(t) \leq b_{(l)}, \qquad l = 1, \ldots, n_l, \tag{2.12}$$

where

$f_i^{(l)}, t_i^{(l)}, u_i^{D(l)}$	components of a time-independent nominal load for the $l - th$ load system,
$\beta_{(l)}(t)$	a parameter describing the time-dependence of the $l - th$ load system,
$a_{(l)}, b_{(l)}$	lower and upper limits imposed on the parameters $\beta_{(l)}$ for the $l - th$ independent set of load.



$$f_i(x,t) = \beta_{(3)}(t) f_i^{(3)}(x)$$
$$t_i(x,t) = \beta_{(3)}(t)\, t_i^{(3)}(x)$$
$$u_i^D(x,t) = \sum_{l=1}^{2} \beta_{(l)}(t) u_i^{d(l)}(x)$$

Figure 2.1: Body undergoing three independent load systems.

Values introduced in (2.11) and (2.12) are illustrated in Fig. 2.1 in the case of 3 independent loads:

1. imposed displacements **u** on $S_D^{(1)}$

$$a_{(1)} u_i^{d(1)}(\mathbf{x}) \le u_i(\mathbf{x},t) \le b_{(1)} u_i^{d(1)}(\mathbf{x}), \qquad \mathbf{x} \in S_D^{(1)}; \tag{2.13}$$

2. imposed displacements **u** on $S_D^{(2)}$

$$a_{(2)} u_i^{d(2)}(\mathbf{x}) \le u_i(\mathbf{x},t) \le b_{(2)} u_i^{d(2)}(\mathbf{x}), \qquad \mathbf{x} \in S_D^{(2)}; \tag{2.14}$$

3. external forces **f** in V, and **t** on S_T

$$\begin{aligned} a_{(3)} f_i^{(3)}(\mathbf{x}) \le f_i(\mathbf{x},t) \le b_{(3)} f_i^{(3)}(\mathbf{x}), \qquad & \mathbf{x} \in V, \\ a_{(3)} t_i^{(3)}(\mathbf{x}) \le t_i(\mathbf{x},t) \le b_{(3)} t_i^{(3)}(\mathbf{x}), \qquad & \mathbf{x} \in S_T. \end{aligned} \tag{2.15}$$

The loads (2.13)–(2.15) may be described by the corresponding values (2.11)–(2.12). The limits of the variation and the corresponding nominal loads are given in the Table 1.

Table 1. Identification of the parameters for 3 loading systems.

l	$a_{(l)}$	$b_{(l)}$	$f_i^{(l)}(\mathbf{x})$ $\mathbf{x} \in V$	$t_i^{(l)}(\mathbf{x})$ $\mathbf{x} \in S_T$	$u_i^{d(l)}(\mathbf{x})$ $\mathbf{x} \in S_D^{(1)}$	$u_i^{d(l)}(\mathbf{x})$ $\mathbf{x} \in S_D^{(2)}$
1	$a_{(1)}$	$b_{(1)}$	0	0	$u_i^{d(1)}$	0
2	$a_{(2)}$	$b_{(2)}$	0	0	0	$u^{d(2)}$
3	$a_{(3)}$	$b_{(3)}$	$f_i^{(3)}$	$t_i^{(3)}$	0	0

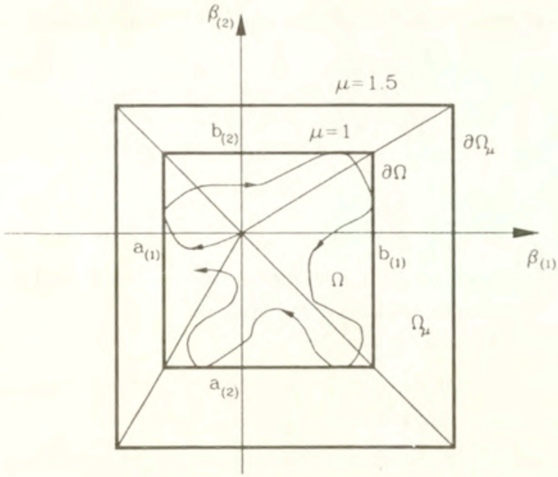


Figure 2.2: Domains of variation of loading parameters.

Let Ω_μ defines a domain for variation of the loading parameters $\mu\beta_{(l)}(t)$, $\partial\Omega_\mu$ its boundary, and μ a load intensity multiplier transforming the nominal domain Ω into Ω_μ (Fig. 2.2)

$$\beta_{(l)}(t) \in \Omega, \qquad \Omega_\mu = \mu\Omega, \qquad \mu\beta_{(l)}(t) \in \Omega_\mu. \tag{2.16}$$

The structure response depends upon the size of the loading domain Ω_μ. Depending on the value of the load multiplier μ it may occur:

$0 \le \mu \le \mu^e$	the elastic response (μ^e limit elastic multiplier);
$\mu^e < \mu \le \mu^{sh}$	the shakedown (adaptation) of the structure, i.e. the elastic response after a transitory phase of plastic deformations (μ^{sh} shakedown load multiplier);
$\mu^{sh} \le \mu \le \mu^l$	a progressive accumulation of plastic deformations leading to the collapse due either to appearence of incremental mechanism or due to alternating plasticity or to a combination of the both mechanisms or finally to unconstrained plastic flow (μ^l the lowest multiplier for the most unfavorable combination of independent load systems).

The case of certain load parameters $\beta_{(k)}$=const ($a_{(k)} = b_{(k)}$) or more generally, when the load domain does not include the coordinate origin ($a_{(k)}\ b_{(k)} > 0$) should be treated separately. In the first of these cases only the variable part of the load should be multiplied by μ, whereas in the second case multiplying by μ concerns only the upper bound, e.g. $a_{(k)} \le \beta_{(k)} \le \mu b_{(k)}$. In numerical procedures such an approach requires an iterative corrections of the nominal loads.

2.4 Equilibrium equations and boundary conditions

The internal equilibrium, the statical and the kinematical boundary conditions are described by:

$$\begin{aligned} &\sigma_{ij,j} + f_i = 0 && \text{in } V, \\ &\sigma_{ij} n_j = t_i && \text{on } S_T, \end{aligned} \tag{2.17}$$

$$\begin{aligned} &\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) && \text{in } V, \\ &u_i = u_i^D && \text{on } S_D, \\ &u_i = 0 && \text{on } S_U. \end{aligned} \tag{2.18}$$

The above system of equations, together with the constitutive law defined by relations (2.2)–(2.10) present a complete formulation of the problem.
Admissible states are defined as below:

Statically admissible stress state σ_{ij}^s should satisfy the equilibrium equations $(2.17)_1$ and the statical boundary conditions $(2.17)_2$.

Kinematically admissible strain state ϵ_{ij}^k should satisfy the geometrical relations $(2.18)_1$ and the kinematical boundary conditions $(2.18)_{2,3}$.

For any arbitrary statically and kinematically admissible stress and strain states the principle of virtual work holds

$$\int_V \sigma_{ij}^s \epsilon_{ij}^k dV = \int_V f_i u_i^k dV + \int_{S_T} t_i u_i^k dS_T + \int_{S_D} \sigma_{ij}^s n_j u_i^D dS_D, \tag{2.19}$$

or

$$\int_V \sigma_{ij}^s \delta\epsilon_{ij}^k dV = \int_V f_i \delta u_i^k dV + \int_{S_T} t_i \delta u_i^k dS_T, \tag{2.20}$$

where δu is a virtual displacement satisfying the following constraints

$$\delta u = 0 \qquad \text{on} \quad S_D \cup S_U. \tag{2.21}$$

Let us introduce, following Koiter [35], a decomposition of the stress and strain states (the superscript "E" is related to a perfectly elastic reference body)

$$\sigma_{ij} = \sigma_{ij}^E + \rho_{ij}, \tag{2.22}$$

$$\epsilon_{ij} = \epsilon_{ij}^E + \epsilon_{ij}^{res}, \tag{2.23}$$

$$\epsilon_{ij}^E = E_{ijkl}^{-1} \sigma_{kl}^E, \tag{2.24}$$

$$\epsilon_{ij}^{res} = E_{ijkl}^{-1} \rho_{kl} + \epsilon_{ij}^p = \frac{1}{2}(u_{i,j}^{res} + u_{j,i}^{res}) \tag{2.25}$$

where

σ_{ij}^E	is the stress state in the perfectly elastic reference body undergoing the external agents **f**, **t**, $\mathbf{u}^D$;
ρ_{ij}	stands for residual stresses (induced by a plastic deformation) being in the equilibrium with the vanishing external loads;
ϵ_{ij}^{res}	kinematically consistent (resulting from a field of residual displacements $\mathbf{u}^{res}$) residual strains produced by the plastic deformation.

The Koiter's decomposition [35] quoted above concerned external forces only. Since we deal here also with the imposed displacements, it is necessary to specify clearly the boundary conditions for each type of the independent external actions. Due to the linearity of the equilibrium equations and (2.11), a subsequent decomposition of the stress state can be proposed

$$\sigma_{ij}(\mathbf{x},t) = \sigma_{ij}^{EM}(\mathbf{x},t) + \sigma_{ij}^{ED}(\mathbf{x},t) + \rho_{ij}(\mathbf{x},t) =$$
$$\sum_{l=1}^{r} \beta_{(l)}(t)(\sigma_{ij}^{Em(l)}(\mathbf{x}) + \sigma_{ij}^{Ed(l)}(\mathbf{x})) + \rho_{ij}(\mathbf{x},t), \tag{2.26}$$

where

σ_{ij}^{EM}, σ_{ij}^{ED}	define stresses occurring in the perfectly elastic reference body under the action of external forces $\mathbf{f}$, $\mathbf{t}$, and imposed displacements $\mathbf{u}^D$, respectively,
$\sigma_{ij}^{Em(l)}$, $\sigma_{ij}^{Ed(l)}$	are stresses obtained for $l = 1,\ldots,n_l$ independently varying systems of loads and imposed displacements (2.11).

The decomposition of the stress state σ_{ij}^E into $\sigma_{ij}^{Em(l)}$, and $\sigma_{ij}^{Ed(l)}$ may be done in different ways, depending upon the choice of boundary conditions on S_D and S_U. From a numerical point of view as well as because of the simplicity of the proofs concerning some theorems and for some general conclusions it is convenient to describe the corresponding boundary value problems in the following way:

$$\sigma_{ij,j}^{Em(l)} + f_i^{(l)}(\mathbf{x}) = 0, \qquad \sigma_{ij,j}^{Ed(l)} = 0, \qquad \text{in } V \tag{2.27}$$

$$\sigma_{ij}^{Em(l)} n_j = t_i^{(l)}(\mathbf{x}), \qquad \sigma_{ij}^{Ed(l)} n_j = 0, \qquad \text{on } S_T; \tag{2.28}$$

$$\sigma_{ij}^{Em(l)} = E_{ijkl}\epsilon_{kl}^{Em(l)} \qquad \sigma_{ij}^{Ed(l)} = E_{ijkl}\epsilon_{kl}^{Ed(l)} \tag{2.29}$$

$$\epsilon_{ij}^{Em(l)} = \frac{1}{2}(u_{i,j}^{Em(l)} + u_{j,i}^{Em(l)}) \qquad \epsilon_{ij}^{Ed(l)} = \frac{1}{2}(u_{i,j}^{Ed(l)} + u_{j,i}^{Ed(l)}) \tag{2.30}$$

$$u_i^{Em(l)} = 0, \qquad u_i^{Ed(l)} = u_i^{d\,(l)}(\mathbf{x}) \qquad \text{on } S_D^{(l)}; \tag{2.31}$$

$$u_i^{Em(l)} = 0, \qquad u_i^{Ed(l)} = 0, \qquad \text{on } S_D \cup S_U \setminus S_D^{(l)}; \tag{2.32}$$

where ρ_{ij} are residual stresses determined by the following equations

$$\rho_{ij,j} = 0, \qquad \text{in } V \tag{2.33}$$

$$\rho_{ij} n_j = 0, \qquad \text{on } S_T; \tag{2.34}$$

$$\rho_{ij}^{Em(l)} = E_{ijkl}\epsilon_{kl}^{res} \tag{2.35}$$

$$\epsilon_{ij}^{res} + \epsilon_{ij}^{p} = \frac{1}{2}(u_{i,j}^{res} + u_{j,i}^{res}) \tag{2.36}$$

$$u_i^{res} = 0 \qquad \text{on } S_D; \tag{2.37}$$

$$u_i^{res} = 0 \qquad \text{on } S_U; \tag{2.38}$$

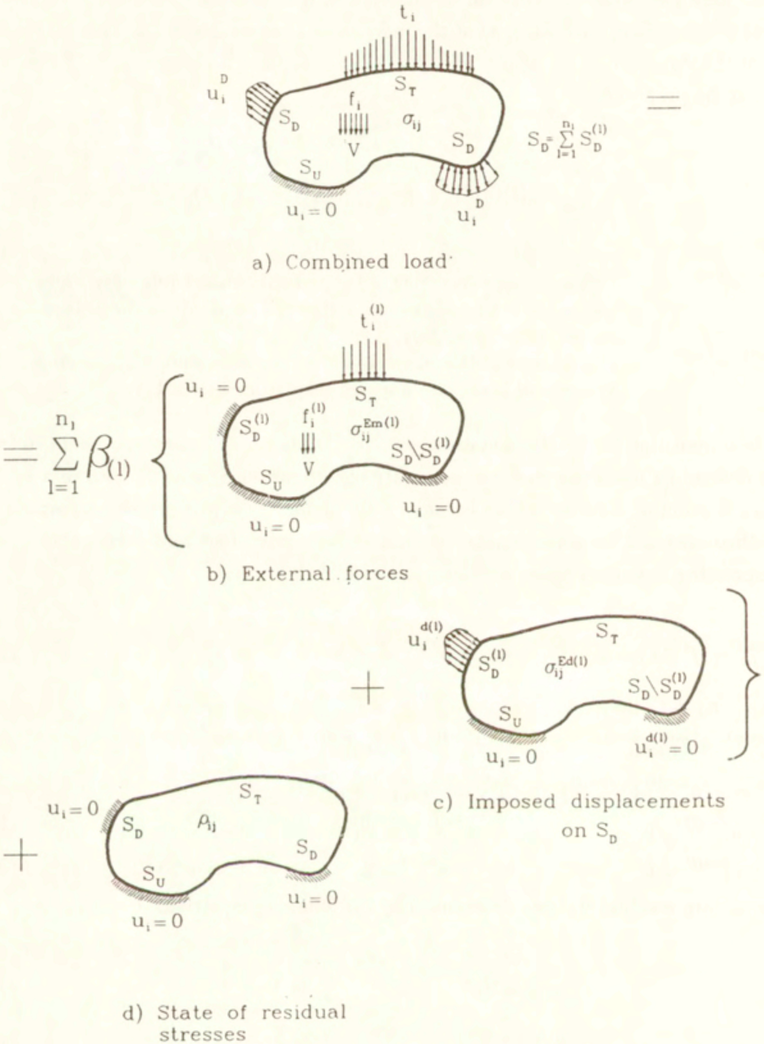


Figure 2.3: Superposition of stress states (b), (c), (d) for a composed load (a)

It is worthwhile to mention that in the boundary value problems concerning the external loads and the residual stresses, the displacements on the surface of the body S_D disappear. The imposed displacements are specified by $\mathbf{u}^{d(l)}$ on $S_D^{(l)}$ (for the $l-th$ independent system) and disappear on the remaining surface $S_D \backslash S_D^{(l)}$. The superposition of the states introduced above is consistent with the global equations (2.17), (2.18).

The stresses resulting from Eqs.(2.26)-(2.38) are schematically shown in Fiq. (2.3).

Similarly to the above, the displacements can be presented also as a sum of the corresponding components

$$u_i = \sum_{l=1}^{r} \beta_{(l)}(u_i^{Em(l)} + u_i^{Ed(l)}) + u_i^{res}. \tag{2.39}$$

The above notation is convenient for a proper description of the shakedown phenomena as well as for an investigation of differences in the structural response to various types of loads. In the case when the whole elastic-plastic process of deformation is concerned the corresponding equations should be presented in the incremental form (c.f. Życzkowski [92]).

3 Generalized shakedown theorems

3.1 Scope of the chapter

Statical and kinematical shakedown theorems generalized to the case of imposed displacements are presented. Definitions for different kind of inadaptation modes and corresponding safety criteria known in the literature were quoted. The problem of the load determination, which ensures the shakedown of the body under the specified variable repeated systems of external forces and imposed displacements was formulated by means of classical mathematical programming.

3.2 Principal assumptions

Shakedown theorems discussed later were proved using the assumption mentioned in the Section 1.2, here they are given in details:

1. The elastic-perfectly plastic model of material is considered. Experiments show [17], that shakedown may occur if a length of plastic plateau on the uni-dimensional diagram $\sigma - \epsilon$ is about 6-7 times greater than the limit value of the elastic strain.

2. The associated plastic flow rule is valid (the normality of the plastic strain rate tensor to the yield surface and the convexity of this surface with respect to their arguments).

3. The small displacement theory is assumed and stability conditions are satisfied during the plastic process of deformations

Moreover:

1. The volume of the considered body must be finite, otherwise proofs regarding the corresponding theorems provoke some objections, similarly like in the case of limit analysis

2. The elastic stress state have to be finite in each point of the body.

3. Effects of the time scale is not taken into account.

4. The quasi-static loads and the kinematical displacements depend on the finite number of independent load parameters $\beta_{(l)}$, whereas the limits variation are described by (2.12).

3.3 Generalized shakedown theorems

Theorem 3.1 Generalized statical theorem (König, Pycko [47])

If there exists a safety load factor $\mu^{st} > 1$, and a statically admissible time-independent residual stress field $\rho_{ij}(x)$

$$\rho_{ij,j}(\mathbf{x}) = 0 \quad \text{in } V, \qquad \rho_{ij}(\mathbf{x})n_j = 0 \quad \text{on } S_T, \tag{3.1}$$

such that for any combination of mechanical loads $\mathbf{f}$, $\mathbf{t}$ *and imposed displacements* $\mathbf{u}^D$ *possible to happen during the live-time of the body, the following condition holds true*

$$f\left(\mu^{st}\left(\sigma_{ij}^{EM}(\mathbf{x},t) + \sigma_{ij}^{ED}(\mathbf{x},t)\right) + \rho_{ij}(\mathbf{x})\right) \leq k, \tag{3.2}$$

then the given body shakes down to the prescribed load program.

Theorem 3.2 Generalized kinematical theorem (König, Pycko [47])

If there exists for a certain time interval $< t_1,\ t_2 >$

- *a history of loads* $\mathbf{f}$, $\mathbf{t}$, $\mathbf{u}^D$ *included inside a domain Ω, and connected with it an elastic stress state σ_{ij}^{ED}*
- *any history of the plastic strain rate field $\dot{\epsilon}_{ij}(\mathbf{x},t)$, resulting during the time interval $< t_1,\ t_2 >$ in a kinematically admissible strain increment $\Delta\epsilon_{ij}$*

$$\begin{aligned} \Delta\epsilon_{ij}(\mathbf{x}) = \int_{t_1}^{t_2} \dot{\epsilon}_{ij}^{p}(\mathbf{x},t)dt = \tfrac{1}{2}(\Delta u_{i,j}^{res} + \Delta u_{j,i}^{res}) \qquad & \text{in } V, \\ \Delta u_i^{res} = 0 \qquad & \text{on } S_U \cup S_D, \end{aligned} \tag{3.3}$$

such that the following inequality holds

$$\int_{t_1}^{t_2}\int_V (\sigma_{ij}^{EM} + \sigma_{ij}^{ED})\dot{\epsilon}_{ij}^{p} dV dt > \int_{t_1}^{t_2}\int_V D(\dot{\epsilon}_{ij}^{p}) dV dt, \tag{3.4}$$

then the body may not shake down to the specified load program $\mathbf{f}$, $\mathbf{t}$, $\mathbf{u}^D$. *The symbol $D(\dot{\epsilon}_{ij}^{p})$ denotes a rate of plastic dissipation, uniquely defined by $\dot{\epsilon}_{ij}^{p}$.*

Theorems, quoted on the basis of the paper by König, Pycko [47], are direct extension of Melan's [54] and Koiter's [35] theorems to the case of imposed displacements. The influence of the imposed displacements appears as the additional stress term σ_{ij}^{ED}, obtained from the solution of the boundary value problem $(2.27)_2$–$(2.32)_2$. The proofs of the theorems mentioned above proceed in an analogous manner to the classical ones, what was pointed out in the paper of König and Pycko [47]. This is the reason why they are omitted here.

For simplicity of the physical interpretation, the integral occurred on the left hand side of the inequality (3.4) can be expressed by means of the external agencies. Using the principle of the virtual work (2.19) and the boundary conditions (2.34), $(2.31)_1$, $(2.32)_1$ we arrive at

$$\int_V \sigma_{ij}^{EM} E_{ijkl}^{-1} \dot{\rho}_{kl} dV = \int_V \epsilon_{ij}^{EM} \dot{\rho}_{ij} dV = 0, \tag{3.5}$$

where $\dot{\rho}_{ij}$ defines a rate of residual stress field related with a history of plastic strains $\dot{\epsilon}_{ij}^{p}$.

Similarly, due to the kinematical admissibility of the rate field of residual strains (2.36)–(2.38)

$$\int_V \sigma_{ij}^{ED} (\dot{\epsilon}_{ij}^{p} E_{ijkl}^{-1} \dot{\rho}_{kl}) dV = 0, \tag{3.6}$$

and after using Eqs. $(2.31)_2$, $(2.32)_2$, (2.34) one obtains

$$\int_V \sigma_{ij}^{ED} \dot{\epsilon}_{ij}^{p} dV = -\int_V \sigma_{ij}^{ED} E_{ijkl}^{-1} \dot{\rho}_{kl} dV = -\int_{S_D} \dot{\rho}_{ij} n_j u_i^D dV. \tag{3.7}$$

Finally, taking into account the relations mentioned above one can write

$$\int_{t_1}^{t_2} \int_V (\sigma_{ij}^{EM} + \sigma_{ij}^{ED}) \dot{\epsilon}_{ij}^{p} dV dt =$$

$$\int_{t_1}^{t_2} \left(\int_V \sigma_{ij}^{EM} (\dot{\epsilon}_{ij}^{p} + E_{ijkl}^{-1} \dot{\rho}_{ij}) dV - \int_{S_D} \dot{\rho}_{ij} n_j u_i^D dV \right) dt = \tag{3.8}$$

$$\int_{t_1}^{t_2} \left(\int_V b_i(\mathbf{x},t) \dot{u}_i^{res}(\mathbf{x},t) dV + \int_{S_T} t_i(\mathbf{x},t) \dot{u}_i^{res}(\mathbf{x},t) dS - \int_{S_D} u_i^D(\mathbf{x},t) \dot{\rho}_{ij}(\mathbf{x},t) n_j dS \right) dt.$$

where $\dot{\mathbf{u}}^{res}$ is a state of residual displacements rates induced by the plastic strain increments $\dot{\epsilon}_{ij}^{p}$ (see (2.36)–(2.38)). Taking into account (3.3) one arrives at

$$\Delta u_i^{res} = \int_{t_1}^{t_2} \dot{u}_i^{res} dt. \tag{3.9}$$

It is worthwhile to mention that the first and the second integral on the r.h.s. of Eq. (3.8) has a very similar form to a strain energy, whereas the third one to the complementary energy. The difference in the notation also reflects different action of the external forces $\mathbf{f}$, $\mathbf{t}$ and the imposed displacements $\mathbf{u}^D$ upon the the structure.

The Theorem 3.2 states about the inadaptation of the structures to given loads (unsafety loads) if one can find such a history of plastic strain rates, which satisfies the inequality (3.4). Making the statement opposite, i.e. considering all possible load paths in the load space Ω and all history of kinematically admissible plastic strains possible to happen and changing the sign in the inequality (3.4) we arrive, similarly like in the classical case, to the safe formulation of the shakedown problem.

Theorem 3.3 Generalized kinematical theorem (safe formulation)

The body will shake down to a given load program if there exists a load multiplier $\mu^k > 1$ such that for all load paths $\beta_{(l)}(t)$ and for all kinematically admissible strain rate fields (3.3), (3.4) the following inequality holds

$$\mu^k \int_{t_1}^{t_2} \int_V (\sigma_{ij}^{EM} + \sigma_{ij}^{ED}) \dot{\epsilon}_{ij}^p dV dt =$$

$$\mu^k \int_{t_1}^{t_2} \left(\int_V b_i(\mathbf{x},t) \dot{u}_i^{res}(\mathbf{x},t) dV + \int_{S_T} t_i(\mathbf{x},t) \dot{u}_i^{res}(\mathbf{x},t) dS - \int_{S_D} u_i^D(\mathbf{x},t) \dot{\rho}_{ij}(\mathbf{x},t) n_j dS \right) dt \tag{3.10}$$

$$\leq \int_{t_1}^{t_2} \int_V D(\dot{\epsilon}_{ij}^p) dV dt.$$

The first equality is satisfied on the basis of (3.8). Let us try to interpret the theorem mentioned above. Namely, the subsequent integrands occurring in the second integral of Eq. (3.10) are composed of a scalar product of the body forces $\mathbf{f}$ and the surface tractions $\mathbf{t}$ on the rates of the residual displacements $\dot{\mathbf{u}}^{res}$, respectively and of a scalar product of the imposed displacements $\mathbf{u}^D$ on the rates of the residual surface tractions $\mathbf{t}^{res} = \rho\mathbf{n}$. All these integrands are integrated over the time. Rates of residual stresses and of displacements result from the assumed history of the plastic strains. Thus, the analyzed integral can be identified as a "work" done by the combined loads $\mathbf{b}$, $\mathbf{t}, \mathbf{u}^D$ on the corresponding residual quantities. Moreover, the integral occurring on the r.h.s. of the inequality (3.10) denotes the plastic dissipation of the body. Summarizing our considerations, one can say that unless the "work" done by the combined loads is less than the ability of storing the plastic work in the body resulting from all kinematically admissible plastic strains rates, the shakedown will not occur.

3.4 Criteria for different types of inadaptation

If the load multiplier exceeds the value of the shakedown multiplier then the body may be destroyed due to:

- incremental collapse;
- alternating plasticity;
- a mixed kind of mechanism i.e. the alternating plasticity and the incremental collapse simultaneously;

- an instantaneous collapse mechanism (the mechanism of unlimited plastic flow).

Some generalization of the shakedown theorems to the case of the imposed displacements were presented in the previous section. They were proved in an analogous manner to their classical equivalents. Taking into account this similarity one can directly extend some suitable definitions and criteria against the incremental collapse and the alternating plasticity (see. König [45]) to the case of imposed displacements.

Definition 3.1 *A perfect incremental collapse process (over a certain time interval* $< t_1, t_2 >$ *is the process of plastic deformation* ϵ_{ij}^{inc}*, in which a kinematically admissible plastic strain increment* $\Delta\epsilon_{ij}^{p} = \epsilon_{ij}^{p}(\mathbf{x}, t_2) - \epsilon_{ij}^{p}(\mathbf{x}, t_1)$ *is attained in a proportional and monotonic way*

$$\Delta\epsilon_{ij}^{p} = \frac{1}{2}(\Delta u_{i,j}^{res} + \Delta u_{j,i}^{res}), \tag{3.11}$$

$$\Delta u_i^{res} = 0, \qquad \text{on } S_D \cup S_U; \tag{3.12}$$

$$\epsilon_{ij}^{inc}(x, t) = \Lambda(\mathbf{x}, t)\Delta\epsilon_{ij}(\mathbf{x}), \tag{3.13}$$

$$\dot{\Lambda}(\mathbf{x}, t) \geq 0, \qquad \Lambda(\mathbf{x}, t_1) = 0, \qquad \Lambda(\mathbf{x}, t_2) = 1, \tag{3.14}$$

$$\Delta\epsilon_{ij}^{inc} = \epsilon_{ij}^{inc}(\mathbf{x}, t_2) - \epsilon_{ij}^{inc}(\mathbf{x}, t_1). \tag{3.15}$$

Since Λ is a scalar quantity, then all components of the plastic strain rate increase proportionally to each another in a specified point $\mathbf{x} \in V$ in the process of the incremental collapse. After the whole cycle $\Delta t = t_2 - t_1$ the increment of the total plastic strain $\Delta\epsilon_{ij}^{inc}$ is equal to the increment of total strains $\Delta\epsilon_{ij}^{p}$. Physically, the process of the incremental collapse may lead to a deterioration of the structure or to significant change of the geometry, what can cause unserviceability of the structure.

Definition 3.2 *Alternating plasticity process is any process of plastic deformation* $\epsilon_{ij}^{alt}(\mathbf{x}, t)$*, within a certain time interval* $< t_1, t_2 >$ *such that the total increment of the plastic strain* $\Delta\epsilon_{ij}^{alt}$ *in each point of the body* $\mathbf{x}$ *over this period is zero*

$$\Delta\epsilon_{ij}^{alt}(\mathbf{x}) = \int_{t_1}^{t_2} \dot{\epsilon}_{ij}^{alt}(\mathbf{x}, t)dt = 0. \tag{3.16}$$

As a consequence of this phenomenon the low cyclic fatigue of the material may occur.

According to the definition of the perfect process of incremental collapse and of alternating plasticity, the total plastic strain in the time interval $< t_1, t_2 >$ can be written (König [45]) as follows

$$\epsilon_{ij}(\mathbf{x}, t) = \epsilon_{ij}^{inc}(\mathbf{x}, t) + \epsilon_{ij}^{alt}(\mathbf{x}, t). \tag{3.17}$$

Separate safety criteria with respect to the incremental collapse and the alternating plasticity can be derived from the safety formulation of the kinematical theorem (Theorem (3.3)). If they are both satisfied the structure is not exposed to the collapse also in the case of a mixed mechanism (König [45]).

The respective criteria can be proved in an analogous manner to their classical equivalents (see Sawczuk [75], König [43]).

Theorem 3.4 Safety criteria against incremental collapse

There will be no incremental collapse if there exists a load multiplier $\mu^k > 1$ and if for all possible kinematically admissible plastic strain increments $\Delta\epsilon_{ij}^{inc}$ satisfying the condition

$$\Delta\epsilon_{ij}^{inc} = \frac{1}{2}(\Delta u_{i,j} + \Delta u_{j,i}), \qquad \Delta u_i = 0 \quad \text{on } S_D \cup S_U, \tag{3.18}$$

the following inequality holds true

$$\mu^k \int_V \sum_{l=1}^{n_l} \alpha_k(\mathbf{x}) J(\mathbf{x}) dV \le \int_V D(\Delta\epsilon_{ij}^{inc})(\mathbf{x}) dV, \tag{3.19}$$

where

$$J_{(l)} = \left(\sigma_{ij}^{Em(l)}(\mathbf{x}) + \sigma_{ij}^{Ed(l)}(\mathbf{x})\right) \Delta\epsilon_{ij}^{inc}(\mathbf{x}), \tag{3.20}$$

$$\alpha_{(l)}(\mathbf{x}) = \begin{cases} b_{(l)} & \text{if} \quad J_{(l)} \ge 0, \\ a_{(l)} & \text{if} \quad J_{(l)} < 0. \end{cases} \tag{3.21}$$

The function $D(.)$ (homogeneous of degree one) denotes the plastic dissipation, which is uniquely determined by the yield condition $f(.)$ (2.2) and the plastic strain tensor. Other quantities are defined by Eqs. (2.11) and (2.26).

Theorem 3.5 Safety criterion with respect to alternating plasticity

There is no alternating plasticity, if there exists a load multiplier $\mu^k > 1$, and time independent field $\psi_{ij}(\mathbf{x})$, such that for each point $\mathbf{x} \in V$ and for any time instant $t \ge 0$ the following inequality holds:

$$f\left(\mu^k(\sigma_{ij}^{EM}(\mathbf{x},t) + \sigma_{ij}^{ED}(\mathbf{x},t)) + \psi_{ij}(\mathbf{x})\right) \le k. \tag{3.22}$$

The quantity $\psi_{ij}(\mathbf{x})$ has not to satisfy any other condition.

Let us pay our attention to an apparent similarity between the inequalities (3.2) and (3.22). The difference between the statical Theorem 3.1 and Theorem 3.5 is due to the fact that the residual stress tensor $\rho_{ij}(\mathbf{x})$ contrarily to the tensor $\psi_{ij}(\mathbf{x})$, has to satisfy additional constraints i.e. equilibrium equations and statical boundary conditions. In the second case lack of constraints concerning the function $\psi_{ij}(\mathbf{x})$ causes that in the case of alternating plasticity criterion only separate points in the body should be analyzed.

3.5 Formulation of the shakedown problem in the form of mathematical programming

The statical theorem and the safety criteria against the incremental collapse and the alternating plasticity concern the problem of adaptation of the body to any load program included inside the domain Ω_μ. A question arises, what is the largest load domain $\Omega_{\mu^{sh}}$ to which the body is still able to shake down. To answer the question it is necessary to solve a certain optimization problem taking into account a well known fact that the statical theorem, in the contrary to the kinematical one gives a lower bound load multiplier. Taking advantage from this statement we arrive to the following mathematical programming problems:

1. the statical formulation

$$\mu^{sh} = \max \mu^{st}, \tag{3.23}$$

$$f\left(\mu^{st}\left(\sigma_{ij}^{EM}(\mathbf{x},t) + \sigma_{ij}^{ED}(\mathbf{x},t)\right) + \rho_{ij}(\mathbf{x})\right) \le k, \tag{3.24}$$

$$\rho_{ij,j} = 0, \qquad \text{in } V; \tag{3.25}$$

$$\rho_{ij}n_j = 0, \qquad \text{on } S_T; \tag{3.26}$$

2. the kinematical approach

$$\mu^{sh} = \min(\mu^{inc}, \mu^{alt}); \tag{3.27}$$

   (a) the safety multiplier with respect to the incremental collapse can be obtained from the following optimization problem

$$\mu^{inc} = \min_{\Delta\epsilon_{ij}^{inc}} \mu^k, \tag{3.28}$$

$$\mu^k = \int_V D(\Delta\epsilon_{ij}^{inc}), \tag{3.29}$$

   under subsidiary conditions

$$1 = \sum_{l=1}^{n_l} \int_V \alpha_{(l)}(\mathbf{x}) J_{(l)}(\mathbf{x}) dV, \tag{3.30}$$

   where

$$J_{(l)} = \left(\sigma_{ij}^{Em(l)}(\mathbf{x}) + \sigma_{ij}^{Ed(l)}(\mathbf{x})\right) \Delta\epsilon_{ij}^{inc}(\mathbf{x}), \tag{3.31}$$

$$\alpha_{(l)}(\mathbf{x}) = \begin{cases} b_{(l)} & \text{if } J_{(l)} \ge 0 \\ a_{(l)} & \text{if } J_{(l)} < 0 \end{cases} \tag{3.32}$$

$$\Delta\epsilon_{ij}^{inc} = \frac{1}{2}(\Delta u_{i,j} + \Delta u_{j,i}) \qquad \Delta u_i = 0, \qquad \text{on } S_U \cup S_D; \tag{3.33}$$

(b) the safety multiplier with respect to the alternating plasticity results from the problem

$$\mu^{alt} = \max \mu^k, \tag{3.34}$$

$$f\left(\mu^k(\sigma_{ij}^{EM}(\mathbf{x},t) + \sigma_{ij}^{ED}(\mathbf{x},t)) + \psi_{ij}(\mathbf{x})\right) \le k. \tag{3.35}$$

The decomposition of the kinematical theorem into two safety criteria against the incremental collapse and the alternating plasticity is necessary in the case of nonlinear yield condition, because of some difficulties in integration over the time of the inequality of (3.10). However, in the case of linearized yield condition the integration is possible and the load shakedown multiplier can be determined from one generalized optimization problem (Maier [48]).

3.6 Conclusions

The shakedown theorems extended to the case of variable imposed displacements were presented on the basis of the paper by König, and Pycko [47]. In some other papers i.e. Weichert and Gross-Wege [88] the imposed displacements were specified on the boundary S_U within prescribed limits. However, the general formulation was then restricted to the particular case with vanishing displacements on S_U.

Due to the similarity of proofs concerning the generalized classical theorems, it was possible nearly a direct transfer of some classical definitions and of the safety criteria with respect to the incremental collapse and the alternating plasticity to the case of imposed displacements.

The formulation presented in this chapter uses decomposition of the total elastic stress state σ_{ij}^E into stresses σ_{ij}^{EM} produced by the external forces and σ_{ij}^{ED} induced by the imposed displacements (2.27)–(2.32). Such decomposition appears to be very convenient for discovering some interesting properties concerning the type of loading, which is given in details in the following chapter. Now, let us draw our attention to a certain difference in response of the bodies to the action of external forces and imposed displacements. It is seen during the analysis of the expression (3.8) being the l.h.s. integral of the inequality (3.4) in the generalized kinematical theorem

$$\int_{t_1}^{t_2}\int_V (\sigma_{ij}^{EM} + \sigma_{ij}^{ED})\dot{\epsilon}_{ij}^{p} dV dt = \int_{t_1}^{t_2}\left(\int_V f_i(\mathbf{x},t)\dot{u}_i^{res}(\mathbf{x},t)dV\right.$$

$$\left. + \int_{S_T} t_i(\mathbf{x},t)\dot{u}_i^{res}(\mathbf{x},t)dS - \int_{S_D} u_i^D(\mathbf{x},t)\dot{\rho}_{ij}(\mathbf{x},t)n_j dS\right) dt. \tag{3.36}$$

Namely, the two first integrals on the r.h.s. of the equality (3.36) are very similar to the expression for the strain energy, whereas the third one resembles a form of the complementary energy. Of course, the rates of the residual displacements $\dot{u}_{ij}^{res}$ and the stresses

$\dot{\rho}_{ij}$ result here from the history of the kinematically admissible plastic strains rates $\dot{\epsilon}^p_{ij}$ (see (2.33)–(2.38) and Theorem 3.2).

4 Imposed displacements

4.1 Scope of the chapter

Some general conclusions are discussed on the basis of the theorems presented in the Chapter 3 and particular cases of the combined loads are studied. For this purpose three types of loading are considered, namely:

1. variable imposed displacements,
2. constant external forces and variable imposed displacements,
3. variable external forces and imposed displacements.

Some statements concerning the imposed displacements are formulated and proved for particular cases of loads. A notion of the load equivalent to the given combined load is introduced. It permits to compare the shakedown multipliers corresponding to different types of loading. Some simple examples clarifying main ideas of the proposed statements are given.

4.2 Influence of imposed displacements on behaviour of structure

Corollary 4.1 *In the case of imposed displacements alone* $\mathbf{u}^D(\mathbf{x}, t)$ *(external loads are absent* $\mathbf{f} = \mathbf{t} = \mathbf{0}$*) the shakedown multiplier* μ *is uniquely determined from the alternating plasticity criterion.*

Although an incremental collapse mechanism cannot develop under one system of imposed displacements, some doubts may appear in the case of several independent displacements systems.

Proof:
Let us decompose the load parameters $\beta_{(l)}$ (2.12) for $l = 1, \ldots, n_l$ into constant parts $\beta^o_{(l)}$ being averages of the load limits $a_{(l)}, b_{(l)}$ and parameters $\beta^s_{(l)}(t)$ symmetrically varying with respect to $\beta^o_{(l)}$

$$\beta_{(l)}(t) = \beta^o_{(l)} + \beta^s_{(l)}(t), \tag{4.1}$$

$$\beta^o_{(l)} = \frac{1}{2}(a_{(l)} + b_{(l)}), \qquad -\frac{b_{(l)} - a_{(l)}}{2} \leq \beta^s_{(l)}(t) \leq \frac{b_{(l)} - a_{(l)}}{2}. \tag{4.2}$$

Let us take an advantage from the safe formulation of the kinematical theorem (Theorem 3.3). Taking into account Eqs. (4.1) and (4.2) the left hand side of Eq. (3.10) takes the following form

$$\int_{t_1}^{t_2} \sum_{l=1}^{n_l} \beta_{(l)}(t) \int_V \sigma_{ij}^{Ed(l)}(\mathbf{x}) \dot{\epsilon}^p_{ij}(\mathbf{x}, t) dV dt = \int_{t_1}^{t_2} \sum_{l=1}^{n_l} \beta^s_{(l)}(t) \int_V \sigma_{ij}^{Ed(l)}(\mathbf{x}) \dot{\epsilon}^p_{ij}(\mathbf{x}, t) dV dt. \tag{4.3}$$

because the following integral containing the parameters $\beta^o_{(l)}$ vanishes

$$\int\limits_{t_1}^{t_2}\sum_{l=1}^{n_l}\beta^o_{(l)}\int\limits_V \sigma_{ij}^{Ed(l)}(\mathrm{x})\dot{\epsilon}^p_{ij}(\mathrm{x},t)dVdt=\sum_{l=1}^{n_l}\beta^o_{(l)}\int\limits_V \sigma_{ij}^{Ed(l)}(\mathrm{x})\Delta\epsilon_{ij}(\mathrm{x})dV=0, \tag{4.4}$$

The above results from the integration of the plastic strain rates $\dot{\epsilon}^p_{ij}$ over the time and applying the virtual work principle (2.19) with the corresponding boundary conditions on S_T $(2.28)_2$ and (3.3), (3.4).

The equality (4.3) denotes equivalence (with respect to the shakedown multiplier) of the following two systems of imposed displacements:

1. the real domain of displacement variations

$$u_i^D=\sum_{l=1}^{n_l}\beta_{(l)}(t)u_i^{d(l)}(\mathrm{x}), \qquad a_{(l)}\le\beta_{(l)}(t)\le b_{(l)}. \tag{4.5}$$

2. a symmetrical part of the real domain

$$u_i^{Ds}=\sum_{l=1}^{n_l}\beta^s_{(l)}(t)u_i^{d(l)}(\mathrm{x}), \qquad -\frac{b_{(l)}-a_{(l)}}{2}\le\beta^s_{(l)}(t)\le\frac{b_{(l)}-a_{(l)}}{2}. \tag{4.6}$$

Let us consider the symmetrical part of the displacement domain (4.6) and a certain critical point $\hat{\mathrm{x}}$ in the body for which the value of the yield function $f(\sigma_{ij}(\hat{\mathrm{x}},t))$ at a time instant $\hat{t}$ attains its maximum over the time and over the volume of the body

$$f(\sigma_{ij}^{ED}(\hat{\mathrm{x}},\hat{t}))\ge f(\sigma_{ij}^{ED}(\mathrm{x},t)), \qquad \mathrm{x}\in V, \qquad t>0. \tag{4.7}$$

Taking into account the symmetry of the displacement domain (4.6) and the symmetry of the yield function with respect to zero (2.5) it is easy to see that there exists such a time instant $\check{t}$ for which

$$\sigma_{ij}^{ED}(\hat{\mathrm{x}},\check{t})=-\sigma_{ij}^{ED}(\hat{\mathrm{x}},\hat{t}) \qquad \text{and} \qquad f(\sigma_{ij}^{ED}(\hat{\mathrm{x}},\check{t}))=f(\sigma_{ij}^{ED}(\hat{\mathrm{x}},\hat{t})). \tag{4.8}$$

So, the stresses at the critical point of the body $\hat{\mathrm{x}}$ and at the time instants $\hat{t}$, and $\check{t}$ reach the surface of the yield function on the opposite sides with respect to zero. For such symmetrical part of loading, exist no different from zero residual stresses $\rho_{ij}=0$, which may increase the shakedown with larger load multiplier. This conclusion results from the statical shakedown theorem (3.2). Therefore only alternating plasticity mechanism may develop at one or at several domains of measure zero in the body. Due to the equivalence of the shakedown load multipliers for real domain of displacement variation and for symmetric portion of this domain the following Corollary 4.1 is satisfied.

Corollary 4.2 *If a time-independent system of external forces is applied to a body under variable imposed displacements, which produce alternating plasticity, the mode of inadaptation may change into an incremental collapse one.*

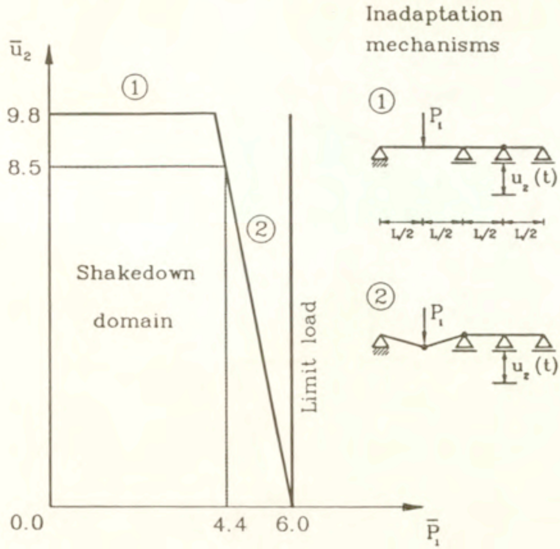


Figure 4.1: A shakedown loads envelope and inadaptation mechanisms for a constant force P_1 and a variable support displacement u_2 .

The above corollary can be illustrated by the following example.

Example 4.1 Let us consider a beam on three supports (Fig. 4.1). The ideal I-cross-section of the beam is characterized by its plastic moment M_o and a second inertia moment J. The cross-section remains elastic if the bending moment does not exceed the plastic moment

$$|M| < M_o. \tag{4.9}$$

If the equality $|M| = M_o$ is satisfied at the specified cross-section then a plastic hinge may develop and residual state of bending moments appear in the beam.

The beam is subjected to two independent load systems:

1. a constant external force P_1=const;
2. an imposed variable displacement u_2, $0 \leq u_2(t) \leq u$.

It is seen in Fig. 4.1 that even if the load $\bar{P}_1 > 4.4$ ($\bar{P} = PL/M_o$, L-beam length) is much smaller than the limit load $\bar{P}_L = 6$, the incremental collapse mode occurs as the result of variable in time support displacement u_2 imposed.

Corollary 4.3 *If we apply any imposed displacements to a body under given external loads, producing incremental collapse mode, the mechanism may change into another incremental collapse one.*

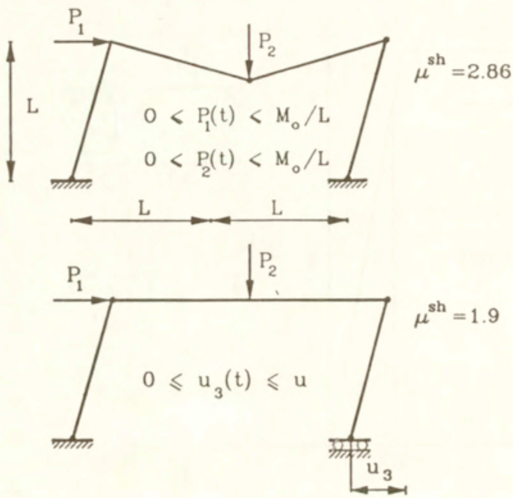


Figure 4.2: Change of incremental collapse mode as a result of applying imposed displacements u_3.

This property is illustrated by the example presented below.

Example 4.2 A clamped portal plane frame (Fig. 4.2) is subjected to two types of loads

1. external forces P_1, P_2

$$0 \leq P_1(t) \leq M_o/L,$$

$$0 \leq P_2(t) \leq M_o/L;$$

2. combined loads

$$0 \leq P_1(t) \leq M_o/L,$$

$$0 \leq P_2(t) \leq M_o/L,$$

$$0 \leq u_3(t) \leq u,$$

where u denotes such a value of a specified displacement, which generates the same distribution of elastic bending moments as in the case of action of support force $P_u = 0.8M_o/L$.

It is clearly seen from the figure that different incremental collapse modes develop for two systems of loads. So the property mentioned above is satisfied, what proves the corollary.

Of course, alternating plasticity may also occur in the case of increasing intensity of imposed displacement variation.

Corollary 4.4 *Any loading in the form of imposed displacements, which satisfies the following relations*

$$u_i^D = \sum_{l=1}^{n_l} \bar{\beta}_{(l)}(t) u_i^{d\,(l)}(\mathbf{x}), \qquad \bar{\beta}_{(l)}(t) = \bar{\beta}^o_{(l)} + \beta^s_{(l)}(t), \tag{4.10}$$

$$\bar{\beta}^o_{(l)} \text{ any constant value}, \qquad a^s_{(l)} \le \beta^s_{(l)} \le b^s_{(l)}, \qquad b^s_{(l)} - a^s_{(l)} = b_{(l)} - a_{(l)}, \tag{4.11}$$

is equivalent with respect to the shakedown load multiplier to the real domain of displacement variation (4.5).

The above follows from the fact that the value of the load parameter $\bar{\beta}^o_{(l)}$ does not influence the vanishing integral (4.4).

Conclusion 4.1 *In the case of imposed displacements the shakedown load multiplier depends only on the difference between the load limits $b_{(l)} - a_{(l)}$ (4.11)$_2$.*

Very similar statement was pointed out by Mróz [58], who considered uniqueness of a steady stress state in the body under cyclic loading. He proved that any constant set of displacements imposed on the boundary S_D does not affect the steady cyclic state of the body. The proof was performed on the basis of a material model with a kinematical hardening or (and) with a viscousity what strongly influences the uniqueness condition of the solution. But in the case of perfect plasticity with a piecewise linear yield condition some difficulties appears in proving the uniqueness of the steady cycle state. That was analyzed in the paper by Pycko and König [69]. On the contrary to the Conclusion 4.1, the statement formulated by Mróz [58], cannot be extended directly to the elastic-perfectly plastic model of material with the piecewise linear yield condition.

The inequality (3.10) of the Theorem 3.3 can be presented, following Conclusion 4.1, in the equivalent form

$$\mu^k \int_{t_1}^{t_2} \left[\sum_{l=1}^{n_l} \left(\beta_{(l)}(t) \int_V \sigma_{ij}^{Em(l)}(\mathbf{x}) \dot{\epsilon}_{ij}^p(\mathbf{x},t) dV + \beta^s_{(l)}(t) \int_V \sigma_{ij}^{Ed(l)}(\mathbf{x}) \dot{\epsilon}_{ij}^p(\mathbf{x},t) dV \right) \right] dt$$

$$\le \int_{t_1}^{t_2} \int_V D(\dot{\epsilon}_{ij}^p dV dt. \tag{4.12}$$

$$a^s_{(l)} \le \beta^s_{(l)} \le b^s_{(l)}, \qquad b^s_{(l)} - a^s_{(l)} = b_{(l)} - a_{(l)}, \tag{4.13}$$

where instead of the load parameters $\beta_{(l)}$ concerning the imposed displacements the quantities $\beta^s_{(l)}$ (4.10) were introduced describing the symmetric variable portion of the real load domain for displacement variation.

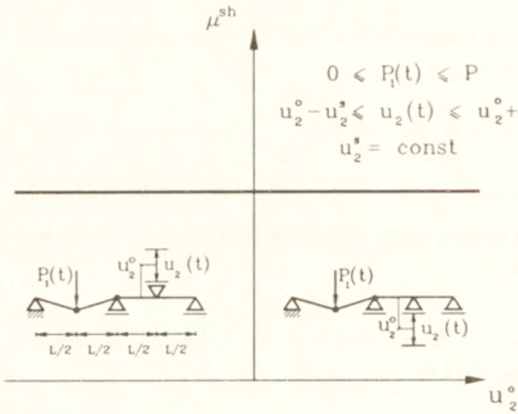


Figure 4.3: The case of imposed displacements with a constant amplitude.

Example 4.3 A two span plane beam is subjected to two systems of independent loading (Fig. 4.3)

1. an external force P_1, which varies within the limits $0 \leq P_1(t) \leq P$;

2. an imposed displacement $u_2(t)$, $u_2^o - u_2^s \leq u_2(t) \leq u_2^o + u_2^s$ with u_2^s=const, and u_2^o treated as a free parameter.

According to the Conclusion 4.1, the shakedown load multiplier (Fig. 4.3) does not depend on the displacement u_2^o.

4.3 Equivalent loads

Let us define elastically equivalent surface tractions $\mathbf{t}^{eq}$ applied on the boundary S_D, which ensure the same elastic stress field as in the case of the imposed displacements $\mathbf{u}^D$ acting alone

$$t_i^{eq} = \sigma_{ij}^{ED} n_j \quad \text{on} \quad S_D, \qquad \sigma_{ij}^{eq}(\mathbf{t}^{eq}) = \sigma_{ij}^{ED}(\mathbf{u}^D) \quad \text{in } V.$$

The elastic stress field σ_{ij}^{eq} results from the action of the surface tractions $\mathbf{t}^{eq}$ and has to satisfy the equilibrium equation and the boundary conditions

$$\begin{aligned} &\sigma^{eq}_{ij,j} = \sigma^{ED}_{ij,j} = 0 && \text{in } V;\\ &\sigma^{eq}_{ij} n_j = \sigma^{ED}_{ij} n_j = 0 && \text{on } S_T;\\ &\sigma^{eq}_{ij} n_j = \sigma^{ED}_{ij} n_j = t^{eq}_i && \text{on } S_D;\\ &u^{eq}_i = u^{ED}_i = 0. && \text{on } S_U. \end{aligned} \tag{4.14}$$

It is worthy to mention, that although the elastic stress state induced by the imposed displacements $\mathbf{u}^D$ is the same as in the case of the external forces $\mathbf{t}^{eq}$ the boundary conditions on the same surface S_D are changed from the kinematical $\mathbf{u} = \mathbf{u}^D$ to the statical ones $\mathbf{t}^{eq} = \boldsymbol{\sigma}\mathbf{n}$.

The surface tractions $\mathbf{t}^{eq}$, together with the external forces $\mathbf{f}$, $\mathbf{t}$ are defined as a loads equivalent to the combined loads (i.e. the external forces $\mathbf{f}$, $\mathbf{t}$ and the imposed displacements $\mathbf{u}^D$). The equivalent loads produce a stress state σ^{EQ}_{ij} in the perfectly elastic reference body, which is a solution of the following boundary value problem:

$$\begin{aligned} &\sigma^{EQ}_{ij,j} + f_i = 0 && \text{in } V;\\ &\sigma^{EQ}_{ij} n_j = t_i && \text{on } S_T;\\ &\sigma^{EQ}_{ij} n_j = t^{eq}_i && \text{on } S_D;\\ &\sigma^{EQ}_{ij} = E_{ijkl}\epsilon^{EQ}_{kl} && \text{in } V;\\ &\epsilon^{EQ}_{ij} = \tfrac{1}{2}(u^{EQ}_{i,j} + u^{EQ}_{j,i}) && \text{in } V;\\ &u^{EQ}_i = 0 && \text{on } S_U. \end{aligned} \tag{4.15}$$

This state differs from the elastic one, produced by the combined loads (2.27)-(2.32), by the boundary conditions on the surface S_D. The elastic stress state resulting from the equivalent loads can be expressed, by the stresses generated by the combined loads (see (2.27)-(2.32)) in the following form

$$\sigma^{EQ}_{ij} = \sigma^{EM}_{ij} + \sigma^{ED}_{ij} + \hat{\sigma}^{EM}_{ij}, \qquad \sigma^{ED}_{ij} = \sigma^{eq}_{ij}, \tag{4.16}$$

Stresses $\hat{\sigma}^{EM}_{ij}$ are due to the action of the body forces $\mathbf{f}$ and the surface tractions $\mathbf{t}$ and to the change of the kinematical boundary conditions $\mathbf{u} = \mathbf{u}^D$ into the statical ones $\boldsymbol{\sigma}\mathbf{n} = \mathbf{t}^{eq}$ on the boundary S_D

$$\begin{aligned}
&\hat{\sigma}^{EM}_{ij,j} = 0 && \text{in } V;\\
&\hat{\sigma}^{EM}_{ij} n_j = 0 && \text{on } S_T;\\
&\hat{\sigma}^{EM}_{ij} n_j = t^M_i(b_i, t_i) = -\sigma^{EM}_{ij} n_j && \text{on } S_D;\\
&\hat{u}^{EM}_i = 0 && \text{on } S_U.
\end{aligned} \tag{4.17}$$

Considering $l = 1, \ldots, n_l$ independent systems of imposed displacements $\mathbf{u}^{d(l)}$ we obtain the same number of the elastically equivalent systems of the surface tractions $\mathbf{t}^{eq(l)}$. For the combined loads presented on Fig. 2.3, some suitable equivalent loads can be determined (Fig. 4.4). The surface tractions $\mathbf{t}^{eq(l)}$ applied to the boundary S_D, (Fig. 4.4c), generate the same elastic stress state as in the case of the imposed displacements $\mathbf{u}^{d(l)}$ on $S_D^{(l)}$ (Fig. 2.3c). Comparison of the Figures 2.3 and 4.4 shows that the elastic stress states produced by the combined loads and by the equivalent loads are different.

The equivalent loads are characterized by the systems of variable external forces applied to the body. The problem of the shakedown of the body to such loading is classical. Denoting the equivalent loads by body forces $\mathbf{f}^{EQ} = \mathbf{f}$ and surface tractions $\mathbf{t}^{EQ}$ $(\mathbf{t},\ \mathbf{t}^{eq})$ acting on the boundary S_T^{EQ} (S_T and S_D), we arrive at the formulation of the kinematical theorem (Koiter [35]):

Theorem 4.1 *The body will shake down to the given loading program* $\mathbf{f}^{EQ}$, $\mathbf{t}^{EQ}$, *if there exists a load multiplier* $\mu^k > 1$ *and if for all possible plastic strain rates resulting in a kinematically admissible strain increment during a time interval* $< t_1,\ t_2 >$

$$\Delta\epsilon_{ij}(\mathbf{x}) = \int_{t_1}^{t_2} \dot{\epsilon}^p_{ij}(\mathbf{x}, t)dt = \frac{1}{2}(\Delta u^{res}_{i,j} + \Delta u^{res}_{j,i}), \quad \text{in } V; \qquad \Delta u^{res}_i = 0 \quad \text{on } S_U, \tag{4.18}$$

the following inequality holds

$$\mu^k \int_{t_1}^{t_2}\int_V \sigma^{EQ}_{ij}\dot{\epsilon}^p_{ij} dV dt = \mu^k \int_{t_1}^{t_2}\left(\int_V f_i(\mathbf{x},t)\dot{u}^{res}_i(\mathbf{x},t)dV + \int_{S_T^{EQ}} t^{EQ}_i(\mathbf{x},t)\dot{u}^{res}_i(\mathbf{x},t)dS\right)dt \leq$$

$$\leq \int_{t_1}^{t_2}\int_V D(\dot{\epsilon}^p_{ij})dV dt, \tag{4.19}$$

where

$$\Delta u^{res}_i = \int_{t_1}^{t_2} \dot{u}^{res}_i(\mathbf{x}, t)dt.$$

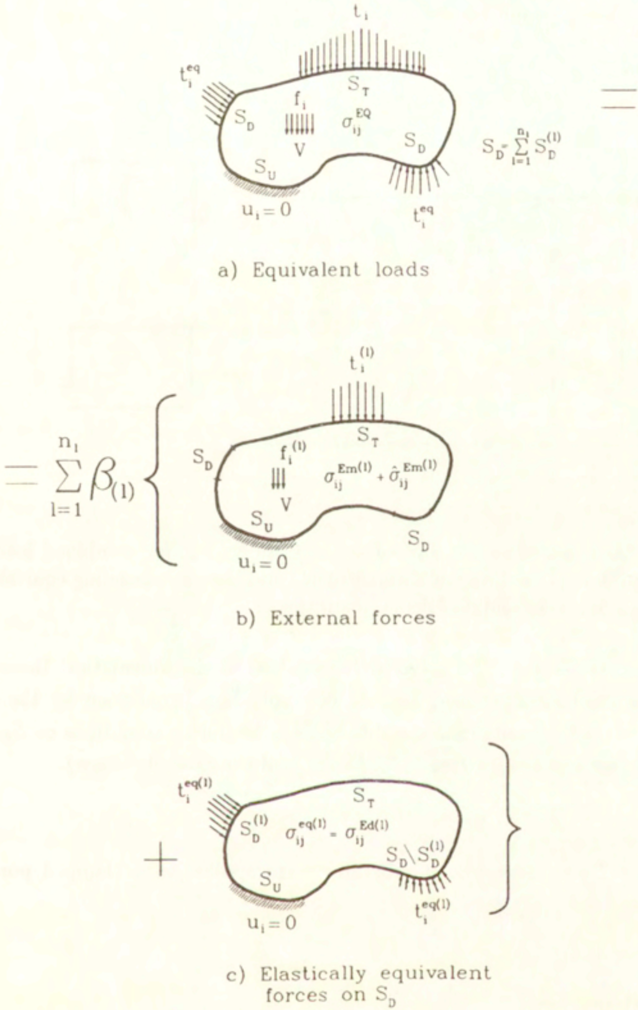


Figure 4.4: Superposition of elastic stress states (b), (c) for equivalent loads (a).

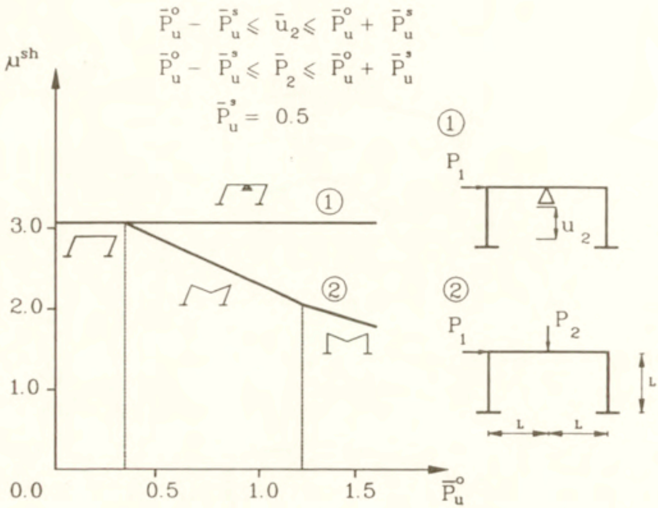


Figure 4.5: Comparison of the shakedown multipliers μ_{sh} for combined loads with a constant amplitude of the imposed displacements and the corresponding equivalent loads. $\bar{P}_u = PL/M_o$ - denotes nondimensional quantities.

The difference between the classical formulation of the kinematical theorem (Theorem 4.1) for the equivalent load and the corresponding formulation for the combined loads (Theorem 3.3), results from the difference in boundary conditions on S_D specified for those two cases of loading (see (3.3), (4.18), and the example below).

Example 4.4 Two different load programs were applied to a clamped portal frame (Fig. 4.5)

1. combined loads:

   (a) external force P_1

   $$0 \leq P_1(t) \leq M_o/L;$$

   (b) imposed displacement u_2

   $$u_2^o - u_2^s \leq u_2(t) \leq u_2^o + u_2^s,$$

   where u_2^s =const, u_2^o is considered as an arbitrary parameter;

2. the equivalent loads

$$0 \le P_1(t) \le M_o/L,$$

$$P_u^o - P_u^s \le P_2(t) \le P_u^o + P_u^s,$$

where P_u^o is any arbitrary parameter depending on u_2^o, and $P_u^s = 0.5\, M_o/L$.

Displacements u_2^o, u_2^s are related to P_u^o, P_u^s, respectively in such a way that bending moments produced by the imposed displacements and by the corresponding support reactions are the same in the purely elastic body. We are looking for the shakedown load multipliers for both types of loads. In the case of the combined loads the multiplier does not depend on u_2^o but for the equivalent loads it strongly depends on the quantity P_2^o.

4.4 Constant external forces

Theorem 4.2 *If combined loads consist of variable repeated systems of independent imposed displacements and of constante external forces, then the shakedown load multiplier is greater than or equal to that one corresponding to the equivalent loads.*

Proof: Our consideration is based on the safe formulation for the kinematical theorems concerning the combined loads (3.10) and the equivalent loads (4.18), (4.19), respectively. For particular types of loads the following relations hold true

1. for the combined loads $\mathbf{f}$, $\mathbf{t}$, $\mathbf{u}^D$

$$\mu^{k1} \int_{t_1}^{t_2} \sum_{l=1}^{n_l} \beta_{(l)}(t) \int_V (\sigma_{ij}^{Em(l)} + \sigma_{ij}^{Ed(l)}) \dot{\epsilon}_{ij}^{p1} dV dt \le \int_{t_1}^{t_2} \int_V D(\dot{\epsilon}_{ij}^{p1}) dV dt, \tag{4.20}$$

$$\Delta\epsilon_{ij}^1(\mathbf{x}) = \int_{t_1}^{t_2} \dot{\epsilon}_{ij}^{p1}(\mathbf{x},t)dt = \frac{1}{2}(\Delta u_{i,j}^{\mathrm{res1}} + \Delta u_{j,i}^{\mathrm{res1}}) \quad \text{in } V, \quad \Delta u_i^{\mathrm{res1}} = 0 \quad \text{on } S_U \cup S_D; \tag{4.21}$$

2. for the equivalent loads $\mathbf{f}$, $\mathbf{t}$, $\mathbf{t}^{eq}$

$$\mu^{k2} \int_{t_1}^{t_2} \sum_{l=1}^{n_l} \beta_{(l)}(t) \int_V (\sigma_{ij}^{Em(l)} + \sigma_{ij}^{Ed(l)} + \hat{\sigma}_{ij}^{Em(l)}) \dot{\epsilon}_{ij}^{p2} dV dt \le \int_{t_1}^{t_2} \int_V D(\dot{\epsilon}_{ij}^{p2}) dV dt, \tag{4.22}$$

$$\Delta\epsilon_{ij}^2(\mathbf{x}) = \int_{t_1}^{t_2} \dot{\epsilon}_{ij}^{p2}(\mathbf{x},t)dt = \frac{1}{2}(\Delta u_{i,j}^{\mathrm{res2}} + \Delta u_{j,i}^{\mathrm{res2}}) \quad \text{in } V, \qquad \Delta u_i^{\mathrm{res2}} = 0 \quad \text{on } S_U. \tag{4.23}$$

Let the most stringent collapse mechanism for the first case be either an incremental or an alternating mechanism described by a history of plastic strains $\dot{\epsilon}_{ij}^{p1} = \dot{\epsilon}_{ij}^{p}(\mathbf{x}, t)$. The maximal multiplier μ^{k1} satisfying the inequality (4.20) for all possible kinematically admissible fields of the plastic strain rates (4.21) is called the shakedown multiplier for this load.

Let us consider now the second loading program and the plastic strain mechanism corresponding to the first considered case. The stresses $\beta_{(l)}\hat{\sigma}_{ij}^{Em(l)}$ produced by the system of external forces are time-independent and therefore the plastic strain rates can be integrated over a time. Using the virtual work principle with the boundary conditions (4.17) and $(4.17)_4$ we conclude that the following integral vanishes

$$\int_{t_1}^{t_2} \sum_{l=1}^{n_l} \beta_{(l)}^{o} \int_V \hat{\sigma}_{ij}^{Em(l)}(\mathbf{x})\dot{\epsilon}_{ij}^{p}(\mathbf{x}, t)dV dt =$$

$$\sum_{l=1}^{n_l} \beta_{(l)}^{o} \int_V \hat{\sigma}_{ij}^{Em(l)}(\mathbf{x})\Delta\epsilon_{ij}(\mathbf{x})dV = 0. \tag{4.24}$$

In this way the inequality (4.22) is reduced to the inequality (4.20). Thus, we obtain the same kinematical multipliers for the most stringent collapse mechanism in the two cases considered

$$\mu^{k1} = \mu^{k2}, \qquad \dot{\epsilon}_{ij}^{p2} = \dot{\epsilon}_{ij}^{p1} = \dot{\epsilon}_{ij}^{p}(\mathbf{x}, t). \tag{4.25}$$

Comparing the boundary conditions for the displacements (4.21) and (4.23) it is clearly seen that each incremental or alternating plasticity mechanism generated by the combined loads "1" becomes also plastic mechanism for the equivalent loads "2". A reciprocal relation does not hold true. It means that it may exist such a plastic mechanism for the equivalent loads, for which the kinematical multiplier μ^{k2} is smaller than that μ^{k1} obtained for the combined loads and the associated plastic mechanism. Thus, the theorem is proved.

Example 4.5 A plane one-storey portal frame like that in the Example 4.4 is considered, subjected to two loading programs (Fig. 4.6):

1. combined loads

$$P_1 = M_o/L, \qquad 0 \leq u_2(t) \leq u;$$

2. equivalent loads

$$P_1 = M_o/L, \qquad 0 \leq P_2(t) \leq P_u.$$

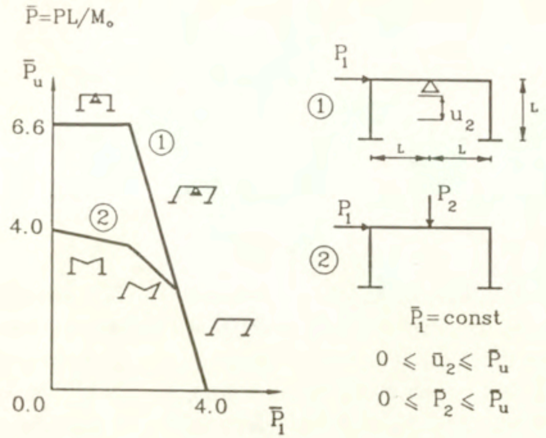


Figure 4.6: Shakedown load envelopes-the combined loads (constant external force P_1 and variable support displacement u_2) are safer than the equivalent loads.

Similarly to the Example 4.4, the external force described by P_u is treated as an elastic support reaction induced by the imposed displacement u. It is easy to seen that, according to the Theorem 4.2 the shakedown load envelope is much larger for the first considered case than for the second loading program.

Theorem 4.2 can be used to estimate the shakedown multiplier for the combined loads $\mathbf{f}$, $\mathbf{t}$, $\mathbf{u}^D$ from below. For this purpose the suitable equivalent loads $\mathbf{f}$, $\mathbf{t}$, $\mathbf{t}^{eq}$ should be introduced which consists of independent external forces only.

4.5 Variable imposed displacements and external forces

Corollary 4.5 *Theorem 4.2 cannot be extended to the case of simultaneous action of variable repeated imposed displacements and external forces.*

Proof: Let us assume such a generalization possible. This would mean that the shakedown multiplier for the variable combined loads $\mathbf{f}$, $\mathbf{t}$, $\mathbf{u}^D$ (2.11) is greater than or equal to the multiplier resulting from the equivalent loads $\mathbf{f}$, $\mathbf{t}$, $\mathbf{t}^{eq}$ (4.15).

If this statement were valid, it would be also satisfied for the particular case when the displacements imposed on the boundary S_D are time-independent

$$u_i(\mathbf{x}, t) = u_i^D(\mathbf{x}) \qquad \text{on} \qquad S_D, \tag{4.26}$$

It leads to the following equilibrium equations and the boundary conditions:

combined loads	equivalent loads		
$\sigma_{ij,j} + f_i(\mathbf{x},t) = 0,$	$\sigma_{ij,j}^{EQ} + f_i(\mathbf{x},t) = 0$	in V;	
$\sigma_{ij}n_j = t_i(\mathbf{x},t),$	$\sigma_{ij}^{EQ}n_j = t_i(\mathbf{x},t)$	on S_T;	(4.27)
$u_i = u_i^D(\mathbf{x})$	$\sigma_{ij}^{EQ}n_j = t_i^{eq}(u_i^D(\mathbf{x})),$	on S_D;	
$u_i = 0,$	$u_i^{EQ} = 0$	on S_U.	

In the boundary value problem concerning the combined loads, constant displacements $\mathbf{u}^D$ are applied to the boundary S_D. The total elastic stresses are decomposed into the stresses σ_{ij}^{EM} resulting from the external forces $\mathbf{f}$, $\mathbf{t}$, and the stresses σ_{ij}^{ED} produced by the time-independent system of the imposed displacements $\mathbf{u}^D$. According to the equations $(2.27)_1$–$(2.32)_1$ the stresses σ_{ij}^{EM} are determined with displacements vanishing on the boundary S_D, which can be treated as supports. The supports may have a strong influence on occurrence of a stress concentration generated by the external forces $\mathbf{f}$, $\mathbf{t}$. Because these external forces vary in time, then for a sufficiently large coefficient of the stress concentration (resulting from a large variation of the stress amplitude) there exists a possibility of the inadaptation by a local alternating plasticity (Theorem 3.5).

In the case of the equivalent loads the statical (not kinematical) boundary conditions are specified on S_D in the form of the surface tractions $\mathbf{t}^{eq}$. Thus, the similar effect of the stress concentration obtained from the variable repeated external forces (as it was in the case of the combined loads) cannot occur, because of the absence of supports on S_D. Therefore, there exists the possibility of the inadaptation by alternating plasticity for the combined loads, whereas for the equivalent loads this effect will not occur (would contradict the assumption).

Example 4.6 (Fig. 4.7) The case of the inadaptation of the body by the alternating plasticity described above as a result of the unfavorable influence of supports producing a local stress concentration can first of all occur in the shells. However, as a possibly simple illustration of the Corollary 4.5, let us consider a two span beam (Fig. 4.7). Elastic stiffness and plastic moduli are suitably chosen to permit us modelling the expected effect.

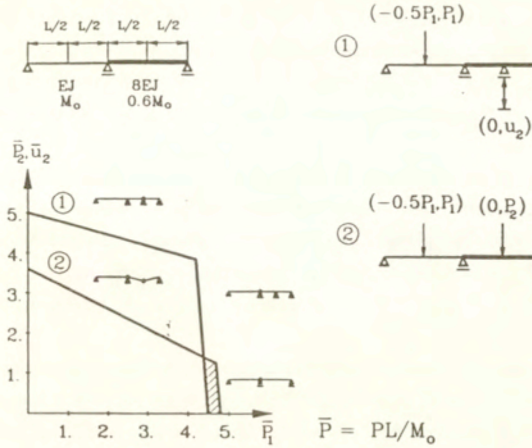


Figure 4.7: Shakedown load envelopes-the combined loads (the variable external force P_1 and support displacement u_2) are less safe than the equivalent loads in the case of inadaptation by alternating plasticity.

The beam is subjected to two loading programs:

1. the combined loads

$$-0.5M_o/L \leq P_1 \leq M_o/L, \qquad 0 \leq u_2(t) \leq u;$$

2. the equivalent loads

$$-0.5M_o/L \leq P_1 \leq M_o/L, \qquad 0 \leq P_2(t) \leq P_u$$

where the external force P_u and the displacement u produce the same elastic bending moments.

It follows from the Fig. 4.7 that the shakedown domain for combined loads can be either larger or smaller than the domain corresponding to the equivalent loads.

4.6 Conclusions

In the case when only imposed displacements acts upon the body the inadaptation is always due to the alternating plasticity, according to the Corollary 4.1. This statement seems to be obvious for a one set of imposed displacements but in the case of several independent systems of loads acting simultaneously there is no assurance that a progressive accumulation of plastic strains, leading to the incremental collapse, cannot occur.

It was pointed out that although incremental collapse mechanism cannot develop under imposed displacements alone, but in presence of additional time-independent external forces an incremental collapse mechanism may occur (Corollary 4.2, Example 4.1).

Superimposing variable displacements to variable loads producing the incremental collapse mode may change its mechanism into another of the same mode (Corollary 4.3, Example 4.2).

A direct conclusion from the Corollary 4.1 is independence of the shakedown load multiplier from constant in time displacements on the boundary S_D (Conclusion 4.1).

A notion of the loads equivalent to the combined loads was introduced. It was pointed out (Theorem 4.2), that the combined loads consisted of a time-independent set of external forces and of variable imposed displacements are safer than the corresponding equivalent loads.

Let us notice that the above theorem is simply a generalization of the well-known limit analysis theorem. It states that adding support cannot decrease the carrying capacity of the body. To prove that, let us assume, that the imposed displacements vanish on the boundary S_D, i.e. this boundary is supported. The equilibrium equations and the boundary conditions for the combined and the corresponding equivalent loads are as follows:

$$\sigma_{ij,j} + f_i = 0, \qquad \sigma_{ij,j}^{EQ} + f_i = 0, \qquad \text{in } V; \tag{4.28}$$

$$\sigma_{ij} n_j = t_i, \qquad \sigma_{ij}^{EQ} n_j = t_i, \qquad \text{on } S_T; \tag{4.29}$$

$$u_i = 0, \qquad \sigma_{ij}^{EQ} n_j = 0, \qquad \text{on } S_D; \tag{4.30}$$

$$u_i = 0, \qquad u_i = 0, \qquad \text{on } S_U, \tag{4.31}$$

Theorem 4.2 is valid both for the variable and the constant in time imposed displacements. Because the shakedown theorems are extensions of the limit analysis theorems, so we can state that the combined loads (the external forces $\mathbf{f}$, $\mathbf{t}$ and the additional support $\mathbf{u} = 0$ on S_D) are safer than the equivalent loads (the same external forces $\mathbf{f}$, $\mathbf{t}$ without support on S_D). It is just the contents of the limit analysis theorem.

If the combined loads consist of variable external forces and the variable imposed displacements, the conclusion concerning a higher safety level for this loads with respect to the equivalent ones is no more valid (Corollary 4.5). Namely, in the case of improper supports on the boundary S_D, the combined loads may generate a stress concentration, violating the alternating plasticity condition. Example 4.6 shows that in some cases the combined loads may be more dangerous for the body than the equivalent loads.

Main results presented in this chapter were published in the paper by Pycko and König [70].

5 Alternative shakedown theorems

5.1 Scope of the chapter

Theorems alternative to the classical ones concerning determination of the shakedown multiplier are proved. Instead of the previous approach, based on the determination of constrained extremum, a new formulation is presented in the form of a min-max problem.

A correctness of the considered problem for continuum is pointed out. The formulation and proofs of the alternative theorems with respect to the classical ones concerning inadaptation by incremental collapse and by alternating plasticity are presented in terms of the generalized variable description. This approach leads to a unified treatment of the both inadaptation mechanisms.

5.2 Min-max problem for continuum

The classical approach to the determination of the shakedown load multiplier is based on the solution of a mathematical programming problem with constraints. In the statical case, the solution of the problem requires calculation of the maximal load multiplier (3.23), for which the yield condition, expressed by the sum of elastic stresses multiplied by a scalar μ and of residual stresses (3.24), is satisfied everywhere. Depending on the piece-wise linearity or nonlinearity of the yield condition we deal either with linear or nonlinear mathematical programming, respectively. The complete formulation is based upon the generalized statical shakedown theorem (Theorem 3.1).

The alternative formulation of the problem of determination of the shakedown load multiplier with respect to the classical formulation is considered in this chapter. It permits us to elaborate qualitatively new numerical procedures, which can be competitive with the classical mathematical programming methods, especially in the case of complex stress state.

For this purpose, instead of enlarging the initial load domain Ω (2.16) proportionally to the multiplier μ, we keep it all time constant $\mu = 1$ and introduce a certain scalar multiplier λ responsible for a change of the plastic modulus. It means that in our analysis loads are considered to vary arbitrary only within the following limits

$$f_i = \sum_{l=1}^{n_l} \beta_{(l)}(t) f_i^{(l)}(\mathrm{x}), \qquad t_i = \sum_{l=1}^{n_l} \beta_{(l)}(t) t_i^{(l)}(\mathrm{x}), \qquad u_i^D = \sum_{l=1}^{n_l} \beta_{(l)}(t) u_i^{d\,(l)}(\mathrm{x}), \tag{5.1}$$

$$a_{(l)} \leq \beta_{(l)}(t) \leq b_{(l)}, \qquad l = 1, \ldots, n_l, \tag{5.2}$$

The intensity of the plastic deformation is now controlled by the multiplier of the plastic modulus λ. Its magnitude allow us to answer the question whether the shakedown takes place or not.

Let us introduce the following definition:

Definition 5.1 *For a given statically admissible time-independent residual stress field, the maximal multiplier for plastic modulus determined from all multipliers λ satisfying the following equality*

$$f\left(\sigma_{ij}^{E}(\mathbf{x},t)+\rho_{ij}(\mathbf{x})\right)=\lambda k(\mathbf{x}),\qquad \sigma_{ij}^{E}=\sigma_{ij}^{EM}+\sigma_{ij}^{ED},\tag{5.3}$$

for each point $\mathbf{x}\in V$, and a time instant $t>0$, is defined as an inverse static multiplier λ^{a}

$$\lambda^{a}(\rho_{ij})=\max_{x,t}\lambda(\mathbf{x},t,\rho_{ij})=\max_{x,t}f\left(\sigma_{ij}^{E}(\mathbf{x},t)+\rho_{ij}(\mathbf{x})\right)/k(\mathbf{x}).\tag{5.4}$$

Using the above definition the following theorem can be proved:

Theorem 5.1 *If from all statically admissible time-independent residual stress fields ρ_{ij}, is chosen that one, which minimizes the inverse static multiplier λ^{a}*

$$\lambda^{sh}=\min_{\rho_{ij}}\lambda^{a}(\rho_{ij}),\tag{5.5}$$

then the shakedown load is obtained as follows

$$f_{i}^{sh}=\frac{f_{i}}{\lambda^{sh}},\qquad t_{i}^{sh}=\frac{t_{i}}{\lambda^{sh}},\qquad u_{i}^{sh}=\frac{u_{i}^{D}}{\lambda^{sh}}.\tag{5.6}$$

The multiplier λ^{sh} is called the inverse shakedown multiplier.

Proof - follows directly from the Theorem 3.1.

Let the shakedown inverse multiplier λ^{sh}, corresponds to a time-independent residual stress field $\bar{\rho}_{ij}(\mathbf{x})$. Thus, there exists an inverse statical multiplier which satisfies the following equation

$$\lambda^{sh}=\bar{\lambda}^{a}(\bar{\rho}_{ij})\tag{5.7}$$

According to the Definition (5.1) the following inequality holds true

$$f\left(\sigma_{ij}^{E}(\mathbf{x},t)+\bar{\rho}_{ij}(\mathbf{x})\right)\leq\bar{\lambda}^{a}k(\mathbf{x}),\qquad \sigma_{ij}^{E}=\sigma_{ij}^{EM}+\sigma_{ij}^{ED},\tag{5.8}$$

at any point $\mathbf{x}\in V$ and at any time instant $t>0$.

Let us define a multiplier $\bar{\mu}^{sh}$ in the following form

$$\bar{\mu}^{sh}=1/\bar{\lambda}^{a}.\tag{5.9}$$

Multiplying the inequality (5.8) by $\bar{\mu}^{sh}$ and taking into account the homogeneity of the degree one of the yield function we arrive at the following relation

$$f\left(\bar{\mu}^{sh}\sigma_{ij}^{E}(\mathbf{x},t)+\bar{\mu}^{sh}\bar{\rho}_{ij}(\mathbf{x})\right)\leq k(\mathbf{x}).\tag{5.10}$$

The above inequality is fulfilled for the time-independent residual stress field $\bar{\mu}^{sh}\bar{\rho}_{ij}(\mathbf{x})$ at any point $\mathbf{x} \in V$ and instant $t > 0$. According to the generalized statical theorem (Theorem 3.1) the body shakes down to the load described in the following form

$$\bar{f}_i = \bar{\mu} f_i, \qquad \bar{t}_i = \bar{\mu} t_i, \qquad \bar{u}_i^D = \bar{\mu} u_i^D, \qquad \bar{\mu} < \bar{\mu}^{sh}. \tag{5.11}$$

This load produces the stress state σ_{ij}^E in the reference purely elastic body

$$\sigma_{ij}^E = \sigma_{ij}^{EM} + \sigma_{ij}^{ED} \tag{5.12}$$

$$\bar{\mu}\sigma_{ij,j}^{EM}(\mathbf{x},t) + \bar{f}_i = 0, \qquad \bar{\mu}\sigma_{ij,j}^{ED}(\mathbf{x},t) = 0, \qquad \text{in } V; \tag{5.13}$$

$$\bar{\mu}\sigma_{ij}^{EM}(\mathbf{x},t)n_j = \bar{t}_i, \qquad \bar{\mu}\sigma_{ij}^{ED}(\mathbf{x},t)n_j = 0, \qquad \text{on } S_T; \tag{5.14}$$

$$\bar{\mu}u_i^{EM}(\mathbf{x},t) = 0, \qquad \bar{\mu}u_i^{ED}(\mathbf{x},t) = \bar{u}_i^D, \qquad \text{on } S_D; \tag{5.15}$$

$$\bar{\mu}u_i^{EM}(\mathbf{x},t) = 0, \qquad \bar{\mu}u_i^{ED}(\mathbf{x},t) = 0, \qquad \text{on } S_U; \tag{5.16}$$

Similarly to the previous analysis, let us consider any other time-independent residual stress field $\hat{\rho}_{ij}(\mathbf{x})$ and the corresponding inverse statical multiplier $\hat{\lambda}^a$. According to the relation (5.5) we arrive at

$$\lambda^{sh} = \bar{\lambda}^a \leq \hat{\lambda}^a. \tag{5.17}$$

Denoting by $\hat{\mu}^{st} = 1/\hat{\lambda}^a$, we obtain the inequality

$$f\left(\hat{\mu}^{st}\sigma_{ij}^E(\mathbf{x},t) + \hat{\mu}^{st}\hat{\rho}_{ij}(\mathbf{x})\right) \leq k(\mathbf{x}), \tag{5.18}$$

which is valid at each point $\mathbf{x} \in V$ and instant $t > 0$. In this case the shakedown takes place for the load $\hat{\mu} f_i$, $\hat{\mu} t_i$, $\hat{\mu} u_i^D$. The following inequality is satisfied with the mentioned multipliers

$$\hat{\mu} < \hat{\mu}^{st}. \tag{5.19}$$

In view of Eqs. (5.17) and (5.9) the following holds true

$$(\lambda^{sh})^{-1} = (\bar{\lambda}^a)^{-1} \geq (\hat{\lambda}^a)^{-1}. \tag{5.20}$$

Hence, the loading $(\lambda^{sh})^{-1} f_i$, $(\lambda^{sh})^{-1} t_i$, $(\lambda^{sh})^{-1} u_i^D$, is the maximum shakedown load. The residual stress field resulting from the shakedown loads can be determined as follows

$$\rho_{ij}(\mathbf{x}) = \frac{\bar{\rho}_{ij}(\mathbf{x})}{\lambda^{sh}}. \tag{5.21}$$

Let us notice that the inverse shakedown multiplier (5.5) occurring in the Theorem 5.1 can be determined from the solution of a min-max problem on the basis of the Definition 5.1

$$\lambda^{sh} = \min_{\rho_{ij}} \max_{x,t} f\left(\sigma_{ij}^E(\mathbf{x},t) + \rho_{ij}(\mathbf{x})\right)/k(\mathbf{x}), \quad \sigma_{ij}^E = \sigma_{ij}^{EM} + \sigma_{ij}^{ED}. \tag{5.22}$$

We proceed here with the maximization over the space and time of the yield function divided by the plastic modulus and with the minimization of this expression over all possible statically admissible residual stress fields.

The min-max formulation was first used by Hill [29] who introduced the concept of extremal fields in the limit analysis theory. Next, Zwoliński and Bielawski [90] formulated the min-max problem in order to obtain a numerical procedure capable for solving the problem of an optimal selection of residual stresses for the limit and the shakedown analysis. In the present contribution we complete this formulation by a proof of the equivalence to the classical formulation. An extension to the case of generalized variables will be also presented.

Let us notice that no inequality constraints occur in the optimization problem (5.22). Therefore, it seems to be more advantageous to use this formulation for numerical applications than the classical mathematical programming methods.

5.3 Generalized variable approach

Let us first introduce a notion of a cross-section $\boldsymbol{\xi}$ for plate and shell structures. It is understood as the intersection of the plate or the shell by a plane perpendicular to a given unit vector. This vector is situated on the middle surface at the point with coordinates $\boldsymbol{\xi}$ in a curvilinear system spanned on the middle surface of the plate (shell).

Generalized variables, i.e., generalized stresses Q_r (stress resultants in the specified cross-section $\boldsymbol{\xi}$) and corresponding generalized strains q_r, $r = 1, \ldots, n_r$ are used in the analysis of bar and surface structures. The above selection is of course not unique but a relationship between statical and kinematical variables has to satisfy the independence of the strain energy increment from its form of description (see, e.g., Życzkowski [92])

$$\delta W = \int_A Q_r(\boldsymbol{\xi})\delta q_r(\boldsymbol{\xi})dA = \int_V \sigma_{ij}(\mathbf{x})\delta\epsilon_{ij}(\mathbf{x})d\mathbf{x} = \int_A\int_H \sigma_{ij}(\boldsymbol{\xi},\mathbf{z})\delta\epsilon_{ij}(\boldsymbol{\xi},\mathbf{z})dH\,dA. \qquad (5.23)$$

The corresponding variables denote:

dA	an element of length of the bar axis or an element of area of the middle surface of the plate (shell);
dH	an element of the area of the bar or an element of the thickness of the plate (shell);
A	denotes the length of the bar axis or the area of the middle surface of the plate (shell);
H	is the area of the bar cross-section or the thickness of the plate (shell).

The equality (5.23) together with a kinematical hypothesis described by a linear operator (König [45]) in the form

$$\epsilon_{ij}(\boldsymbol{\xi},\mathbf{z}) = D^r_{ij}(\mathbf{z})q_r(\boldsymbol{\xi}), \qquad \boldsymbol{\xi} \in A, \quad \mathbf{z} \in H. \qquad (5.24)$$

ensure the uniqueness of transformation from continuum variables σ_{ij}, ϵ_{ij} to the generalized variables Q_r, q_r. Taking into account Eqs. (5.23) i (5.24) we arrive at a definition of generalized stresses:

$$Q_r(\sigma_{ij}) = \int_H \sigma_{ij}(\boldsymbol{\xi}, \mathbf{z}) D^r_{ij}(\mathbf{z}) dH, \qquad \mathbf{z} \in H, \tag{5.25}$$

which is a linear function of the components of the stress state. In the further analysis the stresses Q_s, for which the corresponding generalized strains $q_s = 0$ are equal to zero, according to the kinematical hypothesis (5.24), are not treated as generalized stresses.

The constitutive relations for generalized variables can be expressed in the elastic range as follows

$$Q_r = K_{rs} q_s, \tag{5.26}$$

where

$$K_{rs} = \int_H D^r_{ij}(\mathbf{z}) E_{ijkl} D^s_{kl}(\mathbf{z}) dH, \tag{5.27}$$

is a symmetrical positive definite stiffness matrix of the cross-section $\boldsymbol{\xi}$, which depends on the operator D^r_{ij} (5.24) and on the tensor of elastic moduli E_{ijkl} (2.10).

According to (5.24), a linear dependence between the elastic stress state σ^E_{ij} and the generalized stress state Q^E_r takes place at the specified cross-section $\boldsymbol{\xi}$

$$\sigma^E_{ij}(\boldsymbol{\xi}, \mathbf{z}) = h^r_{ij}(\boldsymbol{\xi}, \mathbf{z}) Q^E_r(\boldsymbol{\xi}), \tag{5.28}$$

where

$$h^r_{ij}(\boldsymbol{\xi}, \mathbf{z}) = E_{ijkl} D^m_{kl}(\mathbf{z}) K^{-1}_{mr}(\boldsymbol{\xi}). \tag{5.29}$$

In general, no unique dependence exists between the stress state σ_{ij} and the generalized stress state Q_r for elastic plastic structures. For each cross-section $\boldsymbol{\xi}$ can exist a non-zero stress field $S_{ij}(\boldsymbol{\xi}, \mathbf{z})$, $\mathbf{z} \in H$ for which the resultant generalized stresses are equal to zero

$$Q_r(S_{ij}(\boldsymbol{\xi}, \mathbf{z})) \equiv 0. \tag{5.30}$$

Stresses S_{ij} are called pseudoresidual ones (see König [45]). They differ from the stresses which correspond to the self-equilibrated (on the level of the whole structure) states of generalized stresses $Q^{res}_r \neq 0$.

Thus, the residual stresses can be finally decomposed into two parts:

$$\rho_{ij}(\boldsymbol{\xi}, \mathbf{z}) = \rho^{res}_{ij}(\boldsymbol{\xi}, \mathbf{z}) + S_{ij}(\boldsymbol{\xi}, \mathbf{z}). \tag{5.31}$$

The first of them corresponds to the resultant generalized stress, is different from zero in the cross-section $\boldsymbol{\xi}$

$$Q_r(\rho^{res}_{ij}) = Q^{res}_r \neq 0, \tag{5.32}$$

whereas the second one gives

$$Q_r(S_{ij}) \equiv 0. \tag{5.33}$$

In this manner the total stress state in the body (2.22) can be expressed in the following form

$$\sigma_{ij}(\boldsymbol{\xi},\mathbf{z}) = \sigma_{ij}^{EM} + \sigma_{ij}^{ED} + \rho_{ij} = h_{ij}^{r}(\boldsymbol{\xi},\mathbf{z})Q_r(\boldsymbol{\xi}) + S_{ij}(\boldsymbol{\xi},\mathbf{z}),$$
$$Q_r = Q_r^{EM} + Q_r^{ED} + Q_r^{res}, \tag{5.34}$$

where Q_r^{EM}, Q_r^{ED} are vectors of generalized stresses obtained for the purely elastic reference structure subjected to external forces and imposed displacements, respectively. Q_r^{res} denotes a vector of difference between the total generalized stress vector and the elastic stress vector.

Equilibrium equations and boundary conditions take the following form for the structure described in the generalized variables

$$\mathcal{L}_{ir}Q_r(\boldsymbol{\xi},\ t)+p_i(\boldsymbol{\xi},\ t) = 0, \qquad \text{on } A; \tag{5.35}$$

$$\mathcal{N}_{mr}Q_r(\boldsymbol{\xi},\ t) = R_m(\boldsymbol{\xi},\ t), \qquad \text{on } S_T; \tag{5.36}$$

$$\mathcal{M}_{mi}w_i(\boldsymbol{\xi},\ t) = r_m^D(\boldsymbol{\xi},\ t), \qquad \text{on } S_D; \tag{5.37}$$

$$\mathcal{M}_{mi}w_i(\boldsymbol{\xi},\ t) = 0, \qquad \text{on } S_U; \tag{5.38}$$

$$r = 1,\ldots,n_r, \qquad i = 1,\ldots,n_i, \qquad m = 1,\ldots,n_m,$$

where $\mathcal{L}_{ir}$ denote linear differential equilibrium operators, p_i–are external loads integrated over the cross-section, w_i–are displacements of the bar axis or of the middle surface. R_m, r_m denote the generalized forces (forces and moments) and generalized displacements (displacements and rotations of the bar axis or the middle surface), respectively, described on the boundary S of the structure. $\mathcal{N}_{mr}$, $\mathcal{M}_{mi}$ denote linear operators, which transform the generalized stresses and displacements into generalized forces and generalized displacements, respectively (see Section 6.2).

Loads can vary arbitrarly within the prescribed limits (2.11)

$$\begin{aligned} p_i(\boldsymbol{\xi},\ t) &= \sum_{l=1}^{n_l}\beta_{(l)}(t)p_i^{(l)}(\boldsymbol{\xi}),\\ R_m(\boldsymbol{\xi},\ t) &= \sum_{l=1}^{n_l}\beta_{(l)}(t)R_m^{(l)}(\boldsymbol{\xi}), \quad a_{(l)} \le \beta_{(l)}(t) \le b_{(l)}.\\ r_m^D(\boldsymbol{\xi},\ t) &= \sum_{l=1}^{n_l}\beta_{(l)}(t)r_m^{d(l)}(\boldsymbol{\xi}), \end{aligned} \tag{5.39}$$

Let us denote by $F^L = 0$ the yield surface described for a specified cross-section $\boldsymbol{\xi}$ by means of the stress components Q_r. Statically admissible generalized stresses have to satisfy the following relation (k_r denotes a vector of plastic moduli) (Fig. 5.1a)

$$F^L(Q_r,\ k_r) \le 0. \tag{5.40}$$

The function F^L can be considered as an arbitrary nonhomogeneous function of the variables Q_r. It does not depend on the loading path (e.g. Janas [31]). The equality in the expression (5.40) denotes a fully plastified cross-section with except of a domain of measure zero. In other words we can say that at each point $\mathbf{z} \in H$ of the cross-section $\boldsymbol{\xi} \in A$, the stresses satisfy the yield condition for a continuum

$$f\left(\sigma_{ij}(\boldsymbol{\xi}, \mathbf{z})\right) = k, \qquad \boldsymbol{\xi} \in A, \quad \mathbf{z} \in H. \tag{5.41}$$

In the same space of generalized stresses we introduce an elastic surface (surface of the first yielding of the cross-section), which is described by the equation $F^E = 0$ (Fig. 5.1a).

The interior of this surface

$$F^E(Q_r,\ k_r) \leq 0, \tag{5.42}$$

becomes the domain of elastic behaviour of the structure without initial stresses.

The knowledge of the yield surface for a given cross-section $\boldsymbol{\xi}$, is not sufficient to the exact determination of the shakedown load multiplier.

For multi-parameter loads inadaptation of the cross-section may occur by incremental mechanism or alternating plasticity, before the generalized stresses reach the yield surface.

Statical and kinematical shakedown theorems for generalized variables were formulated by Kõnig [36]. The kinematical theorem appeared to be a direct generalization of the theorem for a continuum, but the statical one was significantly modified. The latter is based on the concept of the elastic locus defined in the space of generalized stresses for the given cross-section. Both the magnitude and the shape of this surface depends on the pseudo-residual stresses S_{ij}. The generalized residual stresses Q_r^{res} are responsible for a rigid translation of this surface in the space of generalized stresses.

It was Sawicki [79] who noticed that after reducing the shakedown problem for plane grids to the mathematical programming problem with constraints, a separation of constraints responsible for the incremental collapse and the alternating plasticity is possible. The former constraint is equivalent to an envelope of elastic surfaces depending on the pseudoresidual stresses S_{ij}. This envelope is called ratchetting (incremental collapse) surface for the cross-section $\boldsymbol{\xi}$ of the structure. The method of determination of such surfaces for a circular cross-section subjected to bending and torque moments was proposed by Sawicki [78]. In the present dissertation we suggest another method of finding this surface.

If a domain $\mathcal{D}^{NR}$ described in the space of generalized stresses for a specified cross-section $\boldsymbol{\xi}$ satisfies the following relation

$$Q_r \in \mathcal{D}^{NR}, \tag{5.43}$$

$$\underset{\mathbf{z} \in H_\xi}{\exists} \quad \underset{Q_r \in \mathcal{D}^{NR}}{\forall} \quad \underset{S_{ij}(\boldsymbol{\xi}, \mathbf{z})}{\exists} \qquad f\left(h_{ij}^r(\boldsymbol{\xi}, \mathbf{z}) Q_r(\boldsymbol{\xi}) + S_{ij}(\boldsymbol{\xi}, \mathbf{z})\right) < k, \tag{5.44}$$

then this domain will be safe against the inadaptation by incremental collapse. The expression (5.44) states, that the mechanism of incremental collapse will develop for the cross-section $\xi \in A$ unless there exists a finite (it can be arbitrarly small) domain H_e, which remains elastic for any history of generalized stresses varying within $\mathcal{D}^{NR}$. It means that the cross-section is always partially plastic. In the domain $\mathbf{z} \in H_e$ behaviour of the cross-section is perfectly elastic for any variation of stresses Q_r, whereas the remaining portion of the cross-section $H \backslash H_e$ undergoes alternating plastic strains. The existing elastic domain H_e makes impossible to occur the incremental mechanism for a given cross-section.

The closure of the domain $\mathcal{D}^{NR}$ is defined as a surface of ratchetting (incremental collapse)

$$F^R(Q_r,\ k_r) = 0, \qquad F^R = \partial\mathcal{D}^{NR}. \tag{5.45}$$

The specification of the incremental collapse (ratchetting) domain for a beam of rectangular cross-section subjected to cyclic bending moment and a fixed axial force was discussed by Mróz [58], Kōnig [44], Cegielski [8]. In the Fig. 5.1a limit loci (yield, ratchetting, and elastic loci) are presented for the rectangular cross-section. In the part b) of this figure some statically admissible stress distributions are shown inside the cross-section corresponding to the yield and ratchetting (incremental collapse) loci. We shall briefly discuss the way of specification of the surface $F^R = 0$, following the above definition.

The expression (5.44) introduces the notion of a finite elastic domain H_e. The closure of the domain $\mathcal{D}^{NR}$ (5.45) causes that the elastic domain H_e, which has to occur for any variation of the generalized stresses, is of zero measure. This domain is reduced to the middle axis of the rectangular cross-section, as it was shown in Fig. 5.1b (two lower diagrams). Next, we specify some statically admissible states of stresses, which result from the bending moment and the fixed axial force varying within the limits $(-M, +M)$, and $(-N, +N)$, respectively. The resulting state of statically admissible stresses produced by the loads varying within prescribed limits should not violate the yield condition for continuum at any point $\mathbf{z} \in H$ of the cross-section. Moreover, for any ratio of variation limits M/N the elastic zero-measure domain H_e with measure equal to zero, has to remain always at the same layer of the cross-section, for any combination of generalized stresses. Two distributions, schematically presented at the bottom of Fig. 5.1b, satisfy the requirements mentioned above. The left and the right diagrams determine the curve $F^R = 0$ (curves 1 and 2 in the Fig. 5.1a, respectively). The elastic domain of zero measure is located at the middle axis of the cross-section for any variation of generalized forces.

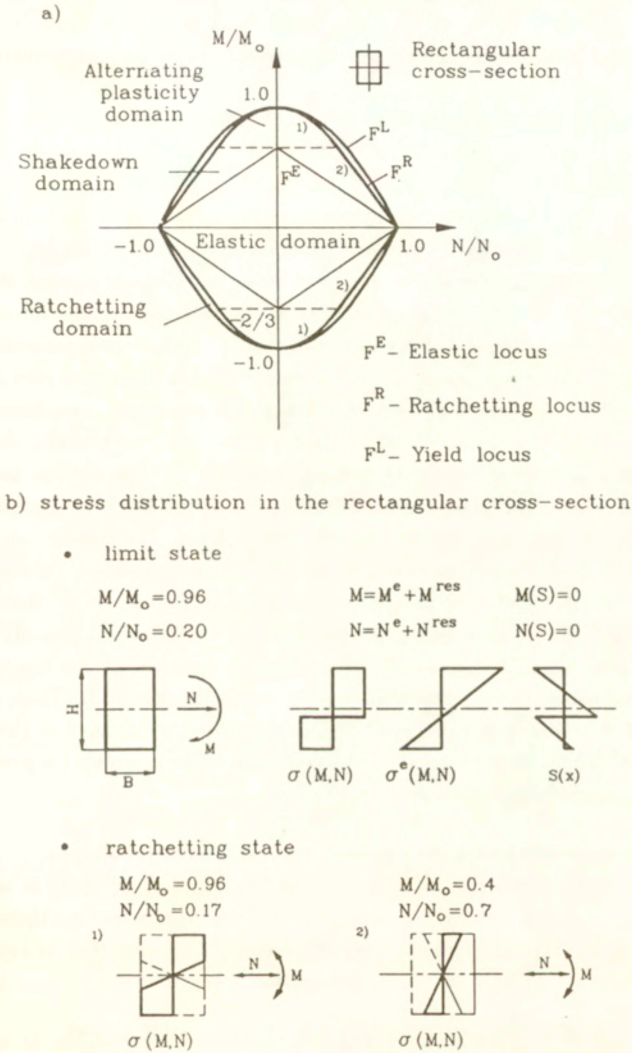


Figure 5.1: a) Yield, ratchetting and elastic loci, respectively for the rectangular cross-section. b) Distributions of statically admissible stresses within the cross-section corresponding to the yield and the ratchetting loci.

The method outlined above is based on a suitable specification of statically admissible states. This is the reason why it can be relatively simpler, in some cases, than the method proposed by Sawicki [78], [79] requiring determination of the envelope of subsequent elastic surfaces.

It appears that, for the rectangular cross-section, the ratchetting (incremental collapse) locus F^R is very close to the yield locus F^L (Fig. 5.1a). In the case of a sandwich cross-section these loci coincide.

It was Kõnig [45], who on the basis of considerations of the elastic loci, made a separation of the statical shakedown theorem in generalized variables (Kõnig [36]) into two criteria protecting the structure against the incremental collapse and against the alternating plasticity. The former uses the yield locus, what gives exact results for sandwich cross-sections. For other shapes of the cross-section inadaptation of the cross-section may appear before the yield locus is attained. So, determination of the load multiplier concerning the incremental collapse is evaluated in this way. On the other hand, introduction of ratchetting (incremental collapse) locus does not ensure the exact result. It follows from the way of determination of the ratchetting locus $F^R = 0$. Namely, the domain of variation of the generalized stresses depends on the location of the cross-section in the structure and on the domain of variation of the external loads. The domain can be very small e.q. in the case of small variations of loads superimposed onto a constant load. The ratchetting surface will tend then toward the yield surface $F^L = 0$. Because the ratchetting surface $F^R = 0$ given in Fig. 5.1 was obtained for the largest possibly domain of variation of generalized stresses in the cross-section, it becomes a lower bound of the real surface obtained for the external loads varying within the domain Ω. Thus, the theorems formulated by Kõnig [45] (pages 67/68), may be directly extended to the case of imposed displacements, when formulation of incremental collapse problem is precised by introducing the ratchetting surface $F^R = 0$.

Theorem 5.2 The safety criterion against the incremental collapse.

A given structure subjected to variable loads $p_i(\boldsymbol{\xi},t)$, $R_m(\boldsymbol{\xi},t)$, $r^D_m(\boldsymbol{\xi},t)$, is safe with respect to the incremental collapse (Definition 3.1), if there exists a load multiplier $\mu > 1$, and a time-independent residual generalized stress vector $Q^{res}_r(\boldsymbol{\xi})$ such that for any instant $t > 0$ and for each point $\boldsymbol{\xi} \in A$ the following condition holds true

$$F\left(\mu Q^E_r(\boldsymbol{\xi},t) + Q^{res}_r(\boldsymbol{\xi}),\ k_r(\boldsymbol{\xi})\right) \leq 0, \qquad Q^E_r = Q^{EM}_r + Q^{ED}_r, \tag{5.46}$$

The surface $F = 0$ denotes either the yield ,F^L, or ratchetting locus F^R.

Moreover, the maximal multiplier μ satisfying the inequality (5.46) is

1. *the lower bound for the shakedown multiplier if $F = F^R$;*

2. *the upper bound for the shakedown multiplier if $F = F^L$.*

Theorem 5.3 The safety criterion against the alternating plasticity.

A given structure subjected to variable loads $p_i(\boldsymbol{\xi}, t)$, $R_m(\boldsymbol{\xi}, t)$, $r_m^D(\boldsymbol{\xi}, t)$ is safe with respect to the alternating plasticity (Definition 3.2), if for each cross-section $\boldsymbol{\xi} \in A$ there exists a load multiplier $\mu > 1$ and such a time-independent pseudoresidual stress field $S_{ij}(\boldsymbol{\xi}, \mathbf{z})$, for which the following condition is satisfied

$$
\begin{gathered}
f\left(\mu h_{ij}^r(\boldsymbol{\xi}, \mathbf{z}) Q_r^E(\boldsymbol{\xi}, t) + S_{ij}(\boldsymbol{\xi}, \mathbf{z})\right) \leq k, \qquad \boldsymbol{\xi} \in A, \quad \mathbf{z} \in H, \\
Q_r^E = Q_r^{EM} + Q_r^{ED},
\end{gathered}
\tag{5.47}
$$

for any time instant $t > 0$, and for each point $(\boldsymbol{\xi},\ \mathbf{z})$.

Let us notice that the Theorem 5.2 was formulated in generalized variables only for the given cross-section $\boldsymbol{\xi}$. On the other hand, the Theorem 5.3 requires a solution of the problem on the level of a point of the body $\mathbf{x} = (\boldsymbol{\xi}, \mathbf{z})$, because of the presence of the pseudoresidual stresses S_{ij}. So, a certain inconsistency arises concerning the description, which leads to the need of different algorithms for both inadaptation modes.

5.4 Incremental collapse mechanism

Let us turn back to the alternative formulation of the statical shakedown theorem (Section 5.2). Instead of seeking the maximal load multiplier we introduce a multiplier for the plastic modulus η.

Definition 5.2 *For a given statically admissible time-independent generalized residual stresses Q_r^{res}, the maximal multiplier of plastic moduli obtained from all possible multipliers η satisfying the following equality*

$$
F\left(\frac{Q_r^E(\boldsymbol{\xi},\ t) + Q_r^{res}(\boldsymbol{\xi})}{\eta},\ k_r(\boldsymbol{\xi})\right) = 0, \qquad Q_r^E = Q_r^{EM} + Q_r^{ED},
\tag{5.48}
$$

for each point $\boldsymbol{\xi} \in A$, and any time $t > 0$ is defined as an inverse statical multiplier

$$
\eta^a(Q_r^{res}) = \max_{\boldsymbol{\xi}, t} \eta(\boldsymbol{\xi}, t, Q_r^{res}, k_r).
\tag{5.49}
$$

$F = 0$ denotes one of the limit loci F^L or F^R (see Theorem 5.2).

After the solution of the equation (5.48) with respect to η we obtain

$$
\eta(\boldsymbol{\xi}, t, Q_r^{res}, k_r) = \bar{F}\left(Q_r^E(\boldsymbol{\xi}, t) + Q_r^{res}(\boldsymbol{\xi}), k_r(\boldsymbol{\xi})\right),
\tag{5.50}
$$

Employing the multiplier η introduced above enable us (contrarily to the use of λ) to deal with arbitrarily yield or ratchetting loci nonhomogeneous with respect to the components of generalized stress vector Q_r.

The inverse statical multiplier can be also expressed by the function $\bar{F}$ obtained from the solution of the equation (5.48) with respect to η

$$\eta^a(Q_r^{res}) = \max_{\boldsymbol{\xi},t} \bar{F}\left(Q_r^E(\boldsymbol{\xi},t) + Q_r^{res}(\boldsymbol{\xi}), k_r(\boldsymbol{\xi})\right). \tag{5.51}$$

For such a description in terms of generalized variables the alternative criterion against the incremental collapse takes the following form:

Theorem 5.4 *If from all statically admissible time-independent residual stress fields $Q_r^{res}(\boldsymbol{\xi})$ is selected that one, which minimizes the inverse statical multiplier η^a*

$$\eta^{inc} = \min_{Q_r^{res}} \eta^a(Q_r^{res}) = \min_{Q_r^{res}} \max_{\boldsymbol{\xi},t} \bar{F}\left(Q_r^E(\boldsymbol{\xi},t) + Q_r^{res}(\boldsymbol{\xi}), k_r(\boldsymbol{\xi})\right), \qquad Q_r^E = Q_r^{EM} + Q_r^{ED}, \tag{5.52}$$

then the shakedown load protecting the structure against the incremental collapse is obtained as follows

$$p_i^{inc} = \frac{p_i}{\eta^{inc}}, \qquad R_m^{inc} = \frac{R_m}{\eta^{inc}}, \qquad r_m^{inc} = \frac{r_m^D}{\eta^{inc}}. \tag{5.53}$$

The multiplier η^{inc} is called inverse shakedown multiplier with respect to the incremental collapse.

Proof: Let the statical shakedown multiplier corresponds to a time-independent generalized residual stress field $\bar{Q}_r^{res}(\boldsymbol{\xi})$. According to the Definition 5.2 there is

$$\eta^{inc} = \bar{\eta}^a(\bar{Q}_r^{res}) \geq \bar{\eta}(\boldsymbol{\xi}, t, \bar{Q}_r^{res}, k_r), \tag{5.54}$$

and

$$\frac{1}{\bar{\eta}^a} \leq \frac{1}{\bar{\eta}}. \tag{5.55}$$

It follows from Eq. (5.48) that for a given $\bar{Q}_r^{res}(\boldsymbol{\xi})$ the following inequality is satisfied

$$F\left(\frac{Q_r^E(\boldsymbol{\xi},\ t) + \bar{Q}_r^{res}(\boldsymbol{\xi})}{\bar{\eta}^a}, k_r(\boldsymbol{\xi})\right) \leq 0, \tag{5.56}$$

for any point $\boldsymbol{\xi} \in A$ and time $t > 0$.

According to the Theorem 5.2, the structure shakes down with respect to the incremental collapse to any load program described by

$$\bar{p}_i = \bar{\mu} p_i, \qquad \bar{R}_m = \bar{\mu} R_m, \qquad \bar{r}_m^D = \bar{\mu} r_m^D, \qquad \bar{\mu} < \tfrac{1}{\bar{\eta}^a}. \tag{5.57}$$

Let assume another arbitrary residual stress field $\hat{Q}_r^{res}(\boldsymbol{\xi})$ and a corresponding inverse multiplier $\hat{\eta}^a(\hat{Q}_r^{res})$. Similarly to the previous considerations, the structure shakes down to any load

$$\hat{p}_i = \hat{\mu} p_i, \qquad \hat{R}_m = \hat{\mu} R_m, \qquad \hat{r}_m^D = \hat{\mu} r_m^D, \qquad \hat{\mu} < \tfrac{1}{\hat{\eta}^a}. \tag{5.58}$$

In view of Eq. (5.52) there is

$$\eta^{inc} = \bar{\eta}^{a} \leq \hat{\eta}^{a}, \tag{5.59}$$

and

$$\bar{\mu}^{inc} = \frac{1}{\bar{\eta}^{inc}} \geq \hat{\mu}^{a} = \frac{1}{\hat{\eta}^{a}}, \tag{5.60}$$

so the theorem is proved.

5.5 Alternating plasticity

Let us return to the continuum formulation of the shakedown problem and consider the load in the form of Eq. (2.11). Let us decompose the load parameter $\beta_{(l)}(t)$ into a time-independent part $\beta^{o}_{(l)}$ which is an average of the limits $a_{(l)}$, $b_{(l)}$ and a time-dependent part $\beta^{s}_{(l)}(t)$ symmetrically variable with respect to the mean value $\beta^{o}_{(l)}$

$$\beta_{(l)}(t) = \beta^{o}_{(l)} + \beta^{s}_{(l}(t), \qquad \beta^{o}_{(l)} = \frac{a_{(l)} + b_{(l)}}{2}, \qquad |\beta^{s}_{(l)}(t)| \leq \frac{b_{(l)} - a_{(l)}}{2}. \tag{5.61}$$

The following theorem can be proved:

Theorem 5.5 *The fixed load described by parameter $\beta^{o}_{(l)}$ cannot affect the alternating plasticity condition, which is dependent only on the symmetrically varying loads described by $\beta^{s}_{(l)}(t)$.*

Proof: Since in view of the Definition 3.2 concerning the alternating plasticity the total plastic strain increments is equal to zero over a cycle

$$\Delta\bar{\epsilon}^{p}_{ij}(\mathbf{x}) = \int_{t_1}^{t_2} \dot{\epsilon}_{ij}(\mathbf{x},t)dt = 0, \qquad \mathbf{x} \in V, \tag{5.62}$$

then for the fixed portion of load parameter $\beta^{o}_{(l)}$ the following integral vanishes

$$\int_{t_1}^{t_2} \sum_{l=1}^{n_l} \beta^{o}_{(l)} \int_V (\sigma_{ij}^{Em(l)}(\mathbf{x}) + \sigma_{ij}^{Ed(l)}(\mathbf{x}))\dot{\bar{\epsilon}}^{p}_{ij}(\mathbf{x},t)dV\ dt =$$

$$= \sum_{l=1}^{n_l} \beta^{o}_{(l)} \int_V (\sigma_{ij}^{Em(l)}(\mathbf{x}) + \sigma_{ij}^{Ed(l)}(\mathbf{x}))dV \int_{t_1}^{t_2} \dot{\bar{\epsilon}}^{p}_{ij}(\mathbf{x},t)dt = 0. \tag{5.63}$$

Thus, the inequality expressing the generalized kinematical theorem (Theorem 3.3) can be reduced according to the Eq. (5.61) and (5.63) to the form

$$\int_{t_1}^{t_2} \sum_{l=1}^{n_l} \beta_{(l)}(t) \int_V (\sigma_{ij}^{E(l)}(\mathbf{x}) + \sigma_{ij}^{Ed(l)}(\mathbf{x}))\dot{\epsilon}_{ij}^p(\mathbf{x},t)dV\ dt =$$

$$\int_{t_1}^{t_2} \sum_{l=1}^{n_l} \beta_{(l)}^s(t) \int_V (\sigma_{ij}^{E(l)}(\mathbf{x}) + \sigma_{ij}^{Ed(l)}(\mathbf{x}))\dot{\epsilon}_{ij}^p(\mathbf{x},t)dV\ dt \leq \tag{5.64}$$

$$\leq \int_{t_1}^{t_2} \int_V D(\dot{\epsilon}_{ij}^p(\mathbf{x},t)dV\ dt.$$

It results that the constant load part described by the parameter $\beta_{(l)}^o$ does not enter into the generalized Koiter's inequality (5.64), so the theorem is proved.

A slightly different formulation of the above theorem can be found in the monography of Kõnig [45]. It was presented as a conclusion drawn from the criterion of the alternating plasticity for even yield conditions. But the proof of this criterion is more complex that that one presented above. So, it seems to be useful to show another way of proving the theorem for deeper understanding of the analysis carried out.

Let us turn back to the formulation in terms of generalized variables and consider, according to the Theorem 5.5, only the symmetrically variable part of loads specified by the parameter $\beta_{(l)}^s(t)$

$$\mu p_i^s(\boldsymbol{\xi},\ t) = \mu \sum_{l=1}^{n_l} \beta_{(l)}^s(t) p_i^{(l)}(\boldsymbol{\xi}), \qquad \mu R_m^s(\boldsymbol{\xi},\ t) = \mu \sum_{l=1}^{n_l} \beta_{(l)}^s(t) R_m^{(l)}(\boldsymbol{\xi}), \tag{5.65}$$

$$\mu r_m^{Ds}(\boldsymbol{\xi},\ t) = \mu \sum_{l=1}^{n_l} \beta_{(l)}^s(t) r_m^{d(l)}(\boldsymbol{\xi}), \qquad -\frac{b_{(l)} - a_{(l)}}{2} \leq \beta_{(l)}^s \leq \frac{b_{(l)} - a_{(l)}}{2}, \tag{5.66}$$

which is equivalent with respect to the shakedown multiplier against the alternating plasticity to the total load. For the symmetrical loads there exist such instants of time $\hat{t}_1,\ \hat{t}_2$ for which at every point $\boldsymbol{\xi}$ the corresponding vectors of elastic generalized stresses are equal each other

$$Q_r^{Es}(\boldsymbol{\xi},\hat{t}_1) = -Q_r^{Es}(\boldsymbol{\xi},\hat{t}_2), \quad Q_r^{Es} = Q_r^{EMs} + Q_r^{EDs}, \tag{5.67}$$

Moreover, at every point of the cross-section $z \in H$ the stresses satisfy the yield condition

$$f\left(\mu h_{ij}^r(\boldsymbol{\xi},z) Q_r^{Es}(\boldsymbol{\xi},\hat{t}_1)\right) = f\left(-\mu h_{ij}^r(\boldsymbol{\xi},z) Q_r^{Es}(\boldsymbol{\xi},\hat{t}_2)\right), \tag{5.68}$$

where $Q_r^{Es} = Q_r^{EMs} + Q_r^{EDs}$ denote generalized stresses for the purely elastic reference structure under symmetrically variable loads $p_i^s,\ R_m^s,\ r_m^{Ds}$. In view of the statical shakedown Theorem 3.1 for such loads there is no time-independent residual stress field different from zero $\rho_{ij}(\mathbf{x}) \neq 0$, which would improve the shakedown condition. Thus, the Theorem 5.3 is reduced to the following one

Theorem 5.6 *The shakedown multiplier μ^{alt} protecting the structure against alternating plasticity is obtained from the following optimization problem (see (3.34), (3.35))*

$$\mu^{alt} = \max \mu,$$

$$f\left(\mu h^{r}_{ij}(\mathbf{x})^{r} Q^{Es}_{r}(\boldsymbol{\xi}, t)\right) \leq k, \qquad Q^{Es}_{r} = Q^{EMs}_{r} + Q^{EDs}_{r} \tag{5.69}$$

where $\mathbf{x}$ *denote a point of the cross-section* $(\boldsymbol{\xi}, \mathbf{z})$ *of the structure.*

Due to the fact, that the stresses Q^{Es}_{r} depend on the symmetrical loads p^{s}_{i}, R^{s}_{m} r^{Ds}_{m} and because of the Eq. (5.68), the load multiplier μ for which the first yielding occurs at any point $\mathbf{x} \in V$ in the structure is also the shakedown multiplier with respect to the alternating plasticity.

It follows from the above considerations that for typical bar and surface structures, which obey the linear kinematical hypothesis it is sufficient to check only the intensity of stresses in external fibers of the cross-sections in order to determine the shakedown load multiplier with respect to the alternating plasticity.

When introducing a notion of the inverse multiplier η^{E} for the elastic locus (5.42) (Definition 5.2)

$$F^{E}\left(\frac{Q^{Es}_{r}}{\eta^{E}}, k_{r}\right) = 0, \qquad \eta^{E} = \bar{F}^{E}\left(Q^{Es}_{r}, k_{r}\right), \qquad Q^{Es}_{r} = Q^{EMs}_{r} + Q^{EDs}_{r} \tag{5.70}$$

the optimization problem presented in the Theorem 5.6 becomes equivalent to the following one

$$1/\mu^{alt} = \eta^{alt} = \max_{\boldsymbol{\xi}, t} \bar{F}^{E}\left(Q^{Es}_{r}(\boldsymbol{\xi}, t), k_{r}(\boldsymbol{\xi})\right). \tag{5.71}$$

Let us notice, that in the contrary to the Theorem 5.6 the formulation (5.71) is described exclusively by generalized variables and without any constraints.

5.6 Shakedown multiplier

To determine the shakedown multiplier by a generalized variable approach it is necessary to find:

1. the inverse shakedown multiplier η^{inc} protecting the structure against incremental collapse (elastic-plastic analysis)

$$\eta^{inc} = \min_{Q^{res}_{r}} \eta^{a}(Q^{res}_{r}) = \min_{Q^{res}_{r}} \max_{\boldsymbol{\xi}, t} \bar{F}\left(Q^{E}_{r}(\boldsymbol{\xi}, t) + Q^{res}_{r}(\boldsymbol{\xi}), k_{r}(\boldsymbol{\xi})\right), \tag{5.72}$$

$$Q^{E}_{r} = Q^{EM}_{r} + Q^{ED}_{r}, \qquad \mu^{inc} = (\eta^{inc})^{-1}.$$

2. the inverse shakedown multiplier η^{alt} with respect to the alternating plasticity (purely elastic analysis)

$$\eta^{alt} = \max_{\boldsymbol{\xi}, t} \bar{F}^{E}\left(Q^{Es}_{r}(\boldsymbol{\xi}, t), k_{r}(\boldsymbol{\xi})\right), \tag{5.73}$$

$$Q_r^{Es} = Q_r^{EMs} + Q_r^{EDs}, \qquad \mu^{alt} = (\eta^{alt})^{-1}$$

with Q_r^{Es} denoting elastic stress resulting from the symmetrical portion of the loads p_i^s, R_m^s r_m^{Ds} (5.65), (5.66).

3. the inverse shakedown multiplier η^{sh} (the maximal one), which will be responsible for the inadaptation mechanism of the structure

$$\eta^{sh} = \sup(\ \eta^{inc},\ \eta^{alt}). \tag{5.74}$$

The shakedown loads is denoted by

$$\begin{aligned} p_i^{sh} &= \mu^{sh} p_i = p_i/\eta^{sh}, \qquad R_m^{sh} = \mu^{sh} R_m = R_m/\eta^{sh}, \\ r_m^{Dsh} &= \mu^{sh} r_m^D = r_m^D/\eta^{sh}, \qquad \mu^{sh} = 1/\eta^{sh} \end{aligned} \tag{5.75}$$

It is worthy to mention that determination of the shakedown multiplier η^{alt} with respect to the alternating plasticity becomes partial result of the shakedown problem with respect to the incremental collapse. The presented approach enables us to give a unified description of the shakedown analysis in terms of generalized variables only. Moreover, it permits to elaborate a common algorithm solution for both inadaptation modes.

5.7 Examples

For clarity of our considerations let us assume that imposed displacements vanish.

Example 5.1 Determination of the inverse multiplier η with respect to the incremental collapse for a rectangular cross-section.

Let us consider a rectangular cross-section of a plane beam. Generalized stresses (bending moment M and an axial force N) satisfy the equation of the yield surface $F^L = 0$ written as follows

$$F^L(N, M, N_o, M_o) = \left(\frac{N}{N_o}\right)^2 + \left|\frac{M}{M_o}\right| - 1 = 0, \tag{5.76}$$

where N_o M_o denote the axial force and the bending moment at yielding, respectively, when acting sparately. Let us remember that, in view of the Theorem 5.2, the analysis performed for the yield locus $F = F^L$ leads to an upper bound of the shakedown load concerning incremental collapse. However, the obtained results will be satisfactory enough because of the meaningless difference between the yield and the ratchetting surfaces (Fig. 5.1).

Using Definition 5.2 we determine the multiplier η from the following equality

$$F^L(N, M, N_o, M_o) = \left(\frac{N}{\eta N_o}\right)^2 + \left|\frac{M}{\eta M_o}\right| - 1 = 0. \tag{5.77}$$

Finally we arrive at

$$\eta = \bar{F}(N,\ M,\ N_o,\ M_o) = \frac{1}{2}\left(\left|\frac{M}{M_o}\right| + \sqrt{\left(\frac{M}{M_o}\right)^2 + 4\left(\frac{N}{N_o}\right)^2}\right), \qquad \text{for} \qquad \eta > 0. \quad (5.78)$$

The quantity η obtained in such a way describes the stress intensity for a given cross-section. The carrying capacity of the cross-section is characterized by $\eta = 1$.

The inverse shakedown multiplier with respect to the incremental collapse is obtained from the solution of the following min-max problem (5.72)

$$\eta^{inc} = \min_{\theta} \max_{\boldsymbol{\xi},t} \frac{1}{2}\left(\left| \frac{M^E(\boldsymbol{\xi},t) + M^{res}(\boldsymbol{\xi},\theta)}{M_o} \right| + \right.$$

$$\left. + \sqrt{\left(\frac{M^E(\boldsymbol{\xi},t) + M^{res}(\boldsymbol{\xi},\theta)}{M_o}\right)^2 + 4\left(\frac{N^E(\boldsymbol{\xi},t) + N^{res}(\boldsymbol{\xi},\theta)}{N_o}\right)^2} \right), \qquad (5.79)$$

where the following quantities denote

$M^E, M^{res}, N^E, N^{res}$	the elastic and residual bending moments and axial forces, respectively;
$\boldsymbol{\xi}$	location of the specified cross-section in the structure
θ	free parameters responsible for the determination of the residual generalized stresses in the frame structure.

A solution of the problem (5.79) for a simply plane frame will be presented in the next example.

Example 5.2 Plane frame subjected to two variable loads.

Let us consider a plane frame with a rectangular cross-section with dimensions $H \times 1$ (Fig. 5.2) subjected to variable external forces: horizontal P_1 and vertical P_2 with the loading program described in the Table 1.

Table 1. Load program.

Load program	Forces	
	P_1	P_2
I	P	0
II	P	P
III	0	P
IV	0	0

We are looking for the shakedown load multipliers with respect to the incremental collapse and the alternating plasticity. To solve the first problem, let us analyze the behaviour of the multiplier η described by the expression (5.78), namely

$$\eta(\boldsymbol{\xi}_i, t_j, X^{res}) = \frac{1}{2}\left(\left| \frac{M^E(\boldsymbol{\xi}_i, t_j) + M^{res}(\boldsymbol{\xi}_i, X^{res})}{M_o} \right| + \right.$$

$$+\sqrt{\left(\frac{M^E(\xi_i,t_j)+M^{res}(\xi_i,X^{res})}{M_o}\right)^2+4\left(\frac{N^E(\xi_i,t_j)+N^{res}(\xi_i,X^{res})}{N_o}\right)^2}\ \Bigg), \qquad (5.80)$$

where X^{res} is a vertical redundant force applied to the upper support. Each curve η presented at the diagram $\eta - X^{res}$ (Fig. 5.2) is determined for one set of quantities (ξ_i, t_j), where $\xi_i = 1, 2, 3, 4$ represent nodal points of the frame and t_j describes time instants I, II, III, IV corresponding to the particular points of the loading program. This program presented at the graph $P_1 - P_2$ is described by a polyhedron with vertices I, II, III, IV (Fig. 5.2).

In order to find the shakedown multiplier it is sufficient (König [39]) to consider only the time instants which correspond to the vertices of the load polyhedron. In this manner we can avoid a difficult analysis concerning any arbitrary load program included inside the polyhedron. The Table 2 presents an elastic state of generalized stresses for particular sets of vertices of the load polyhedron and nodal points as well as the distribution of the residual generalized stresses at nodal points of the frame.

Table 2. Elastic and residual generalized stresses.

Load program	M^E/PL at the node			N^E/P at the node	
	1	2, 3	4	1, 2	3, 4
I	13/64	–3/32	–3/64	–3/32	–19/32
II	10/64	–6/32	10/64	–22/32	–22/32
III	–3/64	–3/32	13/64	–19/32	–3/32
	M^{res}			N^{res}	
	$0.5X^{res}l$	$X^{res}l$	$0.5X^{res}l$	X^{res}	X^{res}

The solution of the min-max problem corresponds to the point A on the Fig. 5.2d, and is obtained as an intersection of curves representing $\eta(2, II, X^{res})$ and $\eta(1, I, X^{res}) \equiv \eta(4, III, X^{res})$ plotted in the $\eta - X^{res}$ plane.

Taking into account the corresponding plastic moduli for the rectangular cross-section

$$M_o = \sigma_o H^2/4, \qquad N_o = \sigma_o H, \qquad M_o/N_o = H/4, \qquad (5.81)$$

we obtain, for the ratio of thickness to span $H/L = 0.2$, the following results

$$X^{res} = -0.006487, \qquad \eta^{inc} = 0.200007PL/M_o, \qquad \mu^{inc}P = \frac{P}{\eta^{inc}} = 4.9998M_o/L. \qquad (5.82)$$

For comparison, in the case of a sandwich cross-section $F^L = F^E$ and the yield locus neglecting the axial forces (Fig. 5.2c, the case equivalent to the ratio H/L=0) we obtain the solution which ensures the shakedown for the frame

$$X^{res} = -0.010417, \qquad \eta^{inc} = 0.197917PL/M_o, \qquad \mu^{inc}P = \frac{P}{\eta^{inc}} = 5.0526M_o/L. \qquad (5.83)$$

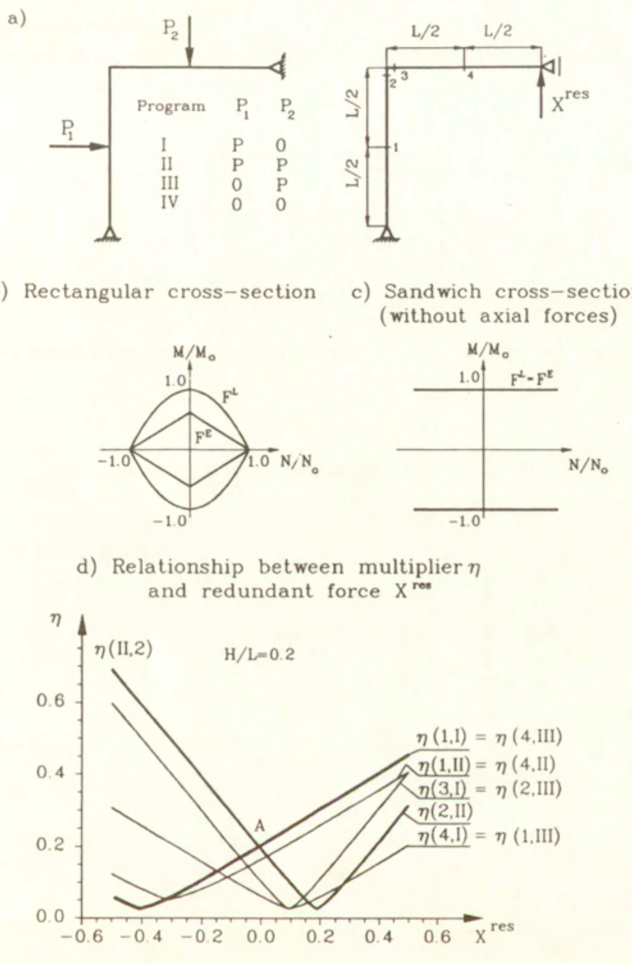


Figure 5.2: a) Plane frame under variable loads b)the yield locus for the rectangular cross-section, c)the yield locus without axial force for a sandwich cross-section d) relation between multiplier η and the redundant force X^{res}.

Comparing the Eqs. (5.82) and (5.83) it is clearly seen, that the results obtained for different yield loci are very close to each other.

To obtain the shakedown load multiplier with respect to the alternating plasticity it is necessary to consider only the symmetrical part of the variable loads described by the Eq. (5.61). The load programs for this particular case are listed in the Table 3

Table 3. Programs for the symmetrical loads.

Load program	Forces P_1	Forces P_2
I	P/2	-P/2
II	P/2	P/2
III	-P/2	P/2
IV	-P/2	-P/2

The respective elastic distributions of the bending moments and axial forces are shown in the Table 4.

Table 4. Elastic bending moments and axial forces.

Load	M^E/PL at the node			N^E/P at the node	
program	1	2, 3	4	1, 2	3, 4
I	8/64	0	−8/64	8/32	−8/32
II	5/64	−3/32	5/64	−11/32	−11/32
III	−8/64	0	8/64	−8/32	8/32
IV	−5/64	3/32	−5/64	11/32	11/32

The elastic locus for the rectangular cross-section (Fig.5.1) and corresponding multiplier η^E (5.70) are described as follows

$$F^E = \left|\frac{N}{N_o}\right| + \frac{3}{2}\left|\frac{M}{M_o}\right| - 1 = 0, \qquad \eta^E = \frac{Pl}{M_o}\left(\left|\frac{N}{P}\right|\frac{H}{L} + \frac{3}{2}\left|\frac{M}{PL}\right|\right). \tag{5.84}$$

The following optimization problem (5.73) should be solved

$$\eta^{alt} = \max_{\boldsymbol{\xi}_i, t_j} \frac{PL}{M_o}\left(\left|\frac{N^{Es}(\boldsymbol{\xi}_i, t_j)}{P}\right|\frac{H}{L} + \frac{3}{2}\left|\frac{M^{Es}(\boldsymbol{\xi}_i, t_j)}{PL}\right|\right) \tag{5.85}$$

The maximum value of η^{alt} is obtained for the following sets of $(\boldsymbol{\xi}_i, t_j)$

(1,I), (1,III), (4,I), (4,III).

For $H/L = 0.2$ the shakedown load multiplier with respect to the alternating plasticity is equal to

$$\eta^{alt} = \frac{Pl}{M_o}\left(\left|\frac{N}{P}\right|\frac{H}{L} + \frac{3}{2}\left|\frac{M}{PL}\right|\right)_{(\boldsymbol{\xi}_i=1, t_j=I)} = 0.2375\frac{PL}{M_o}. \tag{5.86}$$

In view of (5.74) the inverse shakedown multiplier is equal to the largest of the multipliers η^{inc} and η^{alt}. Thus, in the case of the plane frame shown on the Fig. (5.2), the shakedown loads is as follows

$$\mu^{sh} P = \frac{P}{\eta^{alt}} = 4.21052 M_o/L \tag{5.87}$$

The alternating plasticity is then responsible for the inadaptation of the frame.

Let us compare the results with those (5.83) obtained for the sandwich cross-section and the yield locus $F^L = F^E$ independent of the axial force. In the latter case the shakedown multiplier is determined by the incremental collapse mechanism and is equal to $\mu^{sh} P = 5.0526 M_o/L$. In the case of rectangular cross-section, the shakedown multiplier is about 16% smaller than that for the sandwich cross-section. Such a difference results from the change of the incremental collapse mechanism of inadaptation by the alternating one.

5.8 Conclusions

Min-max formulation was first used by Zwoliński and Bielawski [90] for a continuum in order to derive a numerical procedure allowing to solve the shakedown problem for a nonlinear yield condition. Correctness of such formulation was pointed out in the proof of Theorem 5.1. The only constraint concerning the equivalence of this theorem to the classical one requires a homogeneity of the yield function of degree one.

The generalized variable description introduces some modifications to the min-max problem. Previous methods of determination of the shakedown load were based on two criteria with respect to the incremental collapse and the alternating plasticity. The former (see König [45]) was fully reduced to the generalized variable description. This criterion was completed in the thesis by the introduction of the ratchetting (incremental collapse) locus $F^R = 0$ for the given cross-section. The latter (Theorem 5.3) combines some properties of descriptions for a continuum (on the level of a point) with generalized variable approach (on the level of a cross-section). This formulation causes the lack of consistency and make shakedown analysis more difficult.

The min-max problem with respect to the incremental collapse was formulated using the definition of inverse shakedown multiplier (Definition 5.2). Such an approach makes possible considerations concerning any nonlinearity and any nonhomogeneuity with respect to the generalized stress components for the limit locus F (being the yield or the ratchetting loci for the given cross-section). The above analysis could be automatically extended to continuum if an inverse multiplier η (Definition 5.2) is introduced to the yield condition instead of the definition of the multiplier λ.

A slightly different situation concerns the case of alternating plasticity. Independence of the alternating inadaptation mechanism from the time-independent load, represented by multiplier $\beta^{o}_{(l)}$ (5.61) (being an arithmetic average of loads variation limits), was pointed out in the Theorem 5.5. Formulation of the above problem exclusively in terms of generalized variables was possible after introducing the elastic locus F^E (5.42). Because of the elimination of the pseudoresidual stresses, the problem was reduced to a purely elastic analysis. In this way the alternating plasticity condition written in the form of the maximum of the function $\bar{F}^E$ (5.73) can be treated as the first step in the analysis of the min-max problem, described for the incremental collapse. So, both inadaptation mechanisms expressed in terms of generalized variables can be analyzed in an unified way.

6 Solution algorithm for the min-max problem

6.1 Scope of the chapter

The most common in use methods for determination of the shakedown multiplier were based on the mathematical programming. A theoretical background concerning the formulation of the shakedown problem as a dual problem of linear programming was laid by Maier [48]. The description of the problem by this author for a linearized form of the yield condition became a certain standard for the shakedown analysis. Nevertheless, this approach is connected with some inconveniences. Namely, too exact approximation (linearization) of the yield condition considerably increases the number of constraints in the formulation of the mathematical programming problem. It is an obstacle in the effective solution of the problem, especially for space structures with more than two stress components involved in the yield condition. The linear programming methods were employed to plane frames Ĉyras [12], Maier [50], Corradi and Poggi [11], grids Grundy and Spencer [26] and slabs Nguyen Dang Hung and Palgen [60], Weichert and Gross-weege [88].

Application of a nonlinear programming methods is connected with increased difficulties, concerning complexity of some numerical procedures. A relatively large range of RAM memory and connected with this high computer qualities are required for solution of the considered problem. The first paper devoted to this subject concerning the adaptation of a slab was published by Belytchko [1] in 1972, and later by Nguyen Dang Hung and Palgen [61]. Recently, some new papers appeared, that took into account limitation number of variables and constraints on each level of iterative procedure (Genna [19], Stein et al. [83], Orkisz and Pazdanowski [64], Orkisz et al. [65]).

Some classical elastic-plastic incremental step-by-step procedures can also be applied (see König and Kleiber [40], Borkowski and Kleiber [4], Waszczyszyn and Pabisek [86]), but these methods are very time consuming from the numerical point of view.

A new way for further numerical investigations was opened by a formulation of the shakedown as a min-max problem. First attempts to develop the numerical solution were undertaken by Zwoliński and Bielawski [90].

In this chapter a numerical algorithm concerning the solution of the shakedown problem based on the inverse theorems formulated in terms of the generalized variables was proposed. Thus, the shakedown analysis is reduced to the solution of a min-max problem in two phases. In the first of them the shakedown load multiplier with respect to the incremental collapse is determined. In the second step the corresponding multiplier with respect to the alternating plasticity is looked for.

The equivalence of the alternating plasticity conditions to a certain elastic problem concerning a part of the external load symmetrically variable with respect to the mean load value, was shown out in the Chapter 5. The equivalent problem enables us to determine the load multiplier in the first iterative step and furnishes a very powerful method for bounding from above the shakedown load multiplier. Determination of the load multiplier with respect to the incremental collapse is much more complex. To solve it a finite element method is employed in order to find the elastic generalized stress state and relations between generalized residual stress and plastic strain states. These relations can be derived for arbitrary types of structures described by suitable chosen sets of partial derivative equations. Next, a scheme of the numerical algorithm which solve the min-max problem is proposed. A particular attention is focused on the problem of determination of an optimal direction improving the solution of min-max at a given step of iteration.

6.2 Boundary value problem

Engineering structures: bar, plate and shell structures can be described by means of a unified set of the equilibrium equations and the boundary conditions with suitable chosen differential operators for a considered kind of the structure (see Washizu [85]).

For the generalized variable description the following set of equations is used (the corresponding equations are presented in index and matrix notations):

Equilibrium equations

$$\begin{gathered} \mathcal{L}_{ir}Q_r + p_i = 0, \\ \mathbf{LQ} + \mathbf{p} = 0, \qquad \text{on } A, \\ i = 1,\ldots,n_i, \quad \mathrm{r}{=}1,\ldots,n_r, \end{gathered} \tag{6.1}$$

where $\mathcal{L}_{ir}$ denote linear differential operators of the equilibrium equations, Q_r are (like previously) components of the generalized stress vector for a specified cross-section ξ of the structure, p_i stand for external load integrals over a cross-section ξ. Indices i, r vary from 1 to n_i, n_r, respectively.

Stress boundary conditions

$$\begin{gathered} \mathcal{N}_{mr}Q_r = R_m, \\ \mathbf{NQ} = \mathbf{R}, \qquad \text{on } S_T, \\ m = 1,\ldots,n_m, \quad r = 1,\ldots,n_r, \end{gathered} \tag{6.2}$$

where $\mathcal{N}_{mr}$ are linear differentiation operators described on the boundary S_T, and R_m denote the prescribed loads in the form of membrane and shear forces and moments applied to the boundary S_T.

Elastic law

$$
\begin{gathered}
Q_r = E_{rs} q_s^e, \\
\mathbf{Q} = \mathbf{E}\mathbf{q}, \qquad \text{on } A, \\
r = 1, \ldots, n_r, \quad s = 1, \ldots, n_r,
\end{gathered}
\tag{6.3}
$$

where E_{rs} is an elastic stiffness of a given cross-section (notation K_{rs} was used in the Chapter 5.3 see Eq. (5.27)) , q_s^e describes a generalized elastic strain state consisted of the elongation and curvatures of the beam axis or the middle surface of the plate or shell.

Geometrical relations

$$
\begin{gathered}
q_s = q_s^e + q_s^p = \mathcal{L}_{si}^* w_i, \\
\mathbf{q} = \mathbf{q}^e + \mathbf{q}^p = \mathbf{L}^* \mathbf{w},
\end{gathered}
\qquad \text{on } A,
\tag{6.4}
$$

where q_s^p denote generalized plastic (permanent) strains, whereas $\mathcal{L}_{si}^*$ are linear differential operators conjugated by means of the virtual work principle with operators $\mathcal{L}_{mr}$, while w_i denote a displacement vector of the beam axis or the middle surface of the plate or shell.

Displacement boundary conditions

$$
\begin{gathered}
\mathcal{M}_{mi} w_i = r_m^D, \\
\mathbf{M}\mathbf{w} = \mathbf{r}^D,
\end{gathered}
\qquad \text{on } S_D,
\tag{6.5}
$$

$$
\begin{gathered}
\mathcal{M}_{mi} w_i = 0, \\
\mathbf{M}\mathbf{w} = \mathbf{0},
\end{gathered}
\qquad \text{on } S_U,
\tag{6.6}
$$

$\mathcal{M}_{mi}$ are linear differential operators described on the boundary $S_T \cup S_D \cup S_U$, whereas $\mathbf{r}^D$ denote the vector of the generalized displacements imposed on the boundary S_D in the form of displacements and rotations of the beam axis or the middle surface.

Vectors R_m and r_m are conjugated by the fact that their scalar product describes the work done by the external forces R_m on the generalized displacements r_m.

Let us draw our attention to the fact, that in the equation (6.4) the total generalized strain vector is decomposed into the purely elastic and plastic parts. Instead of the latter we can consider any permanent strains. They may appear in the structure as a result of some technological processes, thermal effects or as a result of some other reasons. Generally speaking for any shape of the cross-section the permanent (plastic) strains may appear also in a portion of the cross-section. Relations (6.4) cannot describe such a phenomenon.

The assumption made in the equation (6.4) concerning the additivity of the elastic and plastic strains does not affect the accuracy of the solution of the shakedown problem. It follows from the fact that the shakedown analysis is based on two theorems (Chapter 5) which protect the structure against the incremental collapse and the alternating plasticity.

In the Theorem 5.4 pseudoresidual stresses S_{ij} do not appear in an evident way. They were used to determinate the ratchetting (incremental collapse) locus F^R on the level of the cross-section of the structure and next they are not considered in the process of the solution of the shakedown problem. The second type of the generalized residual stresses Q_r^{res} corresponds to a linear distribution of stresses within the cross-section of the beam or along the thickness of the plate or shell. The field $Q_r^{res}(\xi)$ can be determined in the same way like for elastically equivalent sandwich structures with certain plastic (permanent) distortions. So, the assumption concerning additivity of elastic and plastic strain states described by the relation (6.4) is proper one for our analysis.

Taking into account Theorem 5.6 referred to the alternating plasticity the structure can be analyzed as a purely elastic one subjected to the symmetrically portion of loads, without any need of investigation of the residual stresses.

Summarizing, the solution of the shakedown problem consists from:

1. determination of the elastic, F^E, and yield ,F^L, or ratchetting (incremental collapse), F^R, loci for a given cross-section taking into account a distribution of of psedoresidual stresses S_{ij} inside cross-section;

2. the substitution of the given cross-section by the elastically equivalent sandwich cross-section. Determination of the elastic solution and relation concerning residual stresses Q_r^{res} versus generalized plastic strains (6.4) for such a new structure;

3. applying of the shakedown theorems protecting the structure against:

   (a) the incremental collapse (Theorem 5.4). In the analysis the yield, F^L, or ratchetting, F^R, loci and relation between generalized residual stresses Q^{res} and plastic strains are taken into account;

   (b) the alternating plasticity (5.71), where the elastic locus F^E is employed to the calculations.

The formulation of the shakedown problem on the basis of the theorems proved in the previous chapter allows us to restrict our considerations only to the sandwich structure with limit loci determined for the real cross-section. As a result of such analysis a considerable reduction of the number of variables is made in the problem.

Previous methods concerning the problem of determination of the shakedown multiplier for shell or plate structures (Morelle and Fonder [56]) took into account a distribution of the pseudoresidual stresses along the cross-section. It was connected with a division of the cross-section into some layers and it led to the additional increasement of the number of variables, which were responsible for the description of the pseuderesidual stress state

S_{ij} in these layers. In this case the size of the problem increases considerably and the mathematical programming methods are very costly.

The virtual work principle for the generalized variables approach can be obtained in two manners. The first consists of derivation of the principle on the basis of partial differential equations (6.1)–(6.6). Having the form of differential operators for a bar, plate or shell problem a suitable integral expression should be considered. As a result of the integration by parts (see Washizu [85]) we obtain relation, which can be written in the same general form for all considered problems. The second way relies upon the application of the virtual work principle directly to the continuum. Next, a passage from the continuum to the generalized variable approach is performed. As a result we obtain

$$\begin{aligned} \int_A Q_r \delta q_r dA &= \int_A p_i \delta w_i dA + \int_{S_T} R_m \delta r_m dS, \\ \int_A \mathbf{Q}^T \delta \mathbf{q} dA &= \int_A \mathbf{p}^T \delta \mathbf{w} dA + \int_{S_T} \mathbf{R}^T \delta \mathbf{r} dS, \end{aligned} \tag{6.7}$$

with the following constraints imposed on the virtual strains and displacements

$$\begin{aligned} \delta q_r &= \delta\left(\mathcal{L}^*_{ri} w_i\right), \\ \delta \mathbf{q} &= \delta\left(\mathbf{L}^* \mathbf{w}\right), \end{aligned} \qquad \text{on } A, \tag{6.8}$$

and

$$\begin{aligned} \delta r_m &= \delta\left(\mathcal{M}_{mi} w_i\right) = 0, \\ \delta \mathbf{r} &= \delta\left(\mathbf{M}\mathbf{w}\right) = 0, \end{aligned} \qquad \text{on} \qquad S_D \cup S_U. \tag{6.9}$$

Similarly, like in the case of continuum the l.h.s. of Eq.(6.7) expresses the work of the generalized internal forces on virtual generalized strains whereas the r.h.s. denotes the work done by the external forces on the virtual displacements and rotations of the axis or the middle surface.

Taking into account the constitutive law (6.3) and the additivity conditions for elastic and plastic generalized strains (6.4) the virtual work principle can be transformed into a stationary condition of the strain energy for the elastic body with plastic distortions

$$\delta\, \mathcal{J}_q[\mathbf{w}] = 0,$$

$$\begin{aligned} \mathcal{J}_q[\mathbf{w}] = \tfrac{1}{2} \int_A & E_{rs} q_r(\mathbf{w}) q_s(\mathbf{w}) dA \\ & - \int_A p_i w_i dA - \int_{S_T} R_m r_m(\mathbf{w}) dS \\ & \qquad - \int_A E_{rs} q_r(\mathbf{w}) q^p_s(\xi) dA, \\ \mathcal{J}_q[\mathbf{w}] = \tfrac{1}{2} \int_A & (\mathbf{q}(\mathbf{w}))^T \mathbf{E} \mathbf{q}(\mathbf{w}) dA \\ & - \int_A \mathbf{p}^T \mathbf{w} dA - \int_{S_T} \mathbf{R}^T \mathbf{r}(\mathbf{w}) dS \\ & \qquad - \int_A (\mathbf{q}(\mathbf{w}))^T \mathbf{E} \mathbf{q}^p dA. \end{aligned} \tag{6.10}$$

6.3 Finite element discretization

Before solving the shakedown problem the generalized stresses for the purely elastic body and the relations for the residual stresses should be determined. For this reason the structure is descretized using displacement version of finite elements.

The shakedown theorem concerning incremental collapse (Theorem 5.4) assumes a statically admissible residual stress field Q_r^{res}. Usually in order to obtain such a field by means of standard methods, a functional of a complementary energy must be considered. An approximation of stress state is assumed by means of a suitable chosen shape functions satisfying identically the internal equilibrium equations inside the element and the boundary conditions. Minimization of the functional of the complementary energy gives the solution for both elastic and residual stresses.

In practice the displacement version of the finite elements is more often used. It is based on the minimization of the functional of the strain energy under the condition of a kinematically admissible distribution of displacements. The stress state is obtained as a result of differentiation of displacements what inevitably introduces an error which depends on the accuracy of the approximation of the displacement field.

The solution obtained under the condition of a suitably high-order approximation of the displacement fields or sufficiently large number of the finite elements should lead to better results. So, the error resulting from the solution of the stress state may be estimated on the basis of elastic analysis.

The second type of errors occur as a result of the approximation of continuum by means of discrete systems with a finite number of degrees of freedom. It is strongly related with a possibility of a violation of the yield condition in other points than the integration points, in which the yield criterion is checked.

The errors mentioned above are connected with the approximation of continuum by a discrete model. Their total contribution depends on the manner of the finite element discretization and on the number of degree of freedom inside the considered element. It should be underlined that these errors are standard and always arise during the finite element discretization of continuous systems.

Because of popularity and universality of the displacements method this version of the finite element formulation will be applied in the present contribution. Of course, every other approach resulting from, e.g., hybrid elements is also applicable. The only thing we must know concerns the amount of errors, which can be committed during the discretization of the structures.

The displacement version of the finite element with initial plastic strains relies on the approximation of both generalized displacement and plastic strain vectors by means of the corresponding nodal parameters inside the finite element.

In the case of beam, plate and shell the nodal parameters will be connected with generalized displacements r_i which consist of displacements and rotations of the beam axis or of the middle surface described on the boundary or inside the finite element. The approximation of the displacement state can be written as follows:

$$\begin{aligned} w_i^{(e)} &= \varphi_{i\xi}^{(e)}\ \bar{r}_\xi^{(e)}, \\ \mathbf{w}^{(e)} &= \boldsymbol{\varphi}^{(e)}\ \bar{\mathbf{r}}^{(e)}, \qquad w\ A^e \\ i &= 1,\ldots,n_i, \\ \xi &= 1,\ldots,n_\xi, \end{aligned} \tag{6.11}$$

where

$\varphi_{i\xi}^{(e)}$ are shape functions for generalized displacements of the beam axis or the middle surface. These functions disappear outside the finite element domain A^e;

$\bar{r}_\xi^{(e)}$ denote nodal parameters of the displacement state (the displacements and the rotations of the beam axis or the middle surface) described for a local coordinate system connected with the considered finite element.

The initial plastic (permanent) strains can be treated as a particular kind of the external loading acting upon the structure. They can be arbitrary distributed inside the structure. Because in the discrete structure all quantities are expressed by means of nodal parameters so, the approximation of the plastic (permanent) strains have to be performed. We assume, that in the given finite element there exist n_ϕ integration points. In each of them n_r plastic strain components $\bar{q}_r^{(e_\phi)p}$ written in a local coordinate systems can be distinguished. The subscript ϕ denotes the number of integration points (numeration from 1 to n_ϕ) for the given finite element. Plastic strains can be approximated in each point of the finite element by means of a suitably chosen shape functions $\psi_{rs}^{(e_\phi)}$ in the following way:

$$\begin{aligned} q_r^{(e)p} &= \sum_\phi \psi_{rs}^{(e_\phi)}\ \bar{q}_s^{(e_\phi)p}, \\ \mathbf{q}^{(e)p} &= \sum_\phi \boldsymbol{\psi}^{(e_\phi)}\ \bar{\mathbf{q}}^{(e_\phi)p}, \\ r &= 1,\ldots,n_r, \\ \phi &= 1,\ldots,n_\phi. \end{aligned} \tag{6.12}$$

Summation $\sum_\phi$ is performed over all integration points ϕ of the given finite element e. If we assume shape function $\boldsymbol{\psi}^{(e_\phi)}$ as a Dirac's function, then we will obtain, in the case of frame structures, a model with localized plastic strains (see Borkowski [5]).

The plastic (permanent) strains do not usually form a kinematically admissible field. Therefore, the shape functions can be different from the corresponding ones described by the total strain distribution inside the finite element

$$\begin{aligned} \sum_\phi \psi_{rs}^{(e_\phi)} \bar{q}_s^{(e_\phi)p} &\neq \mathcal{L}_{si}^{*} \psi_{i\xi}^{(e)} \bar{r}_\xi^{(e)}, \\ \sum_\phi \boldsymbol{\psi}^{(e_\phi)} \bar{\mathbf{q}}^{(e_\phi)p} &\neq \mathbf{L}^{*} \boldsymbol{\psi}^{(e)} \bar{\mathbf{r}}^{(e)}. \end{aligned} \tag{6.13}$$

The element nodal parameters expressed in the local coordinate system connected with the finite element can be represented by a global vector, composed of all nodal parameters referred to one common (global) coordinate system.

In the case of the element vectors of generalized displacements and plastic strains we obtain

$$\begin{aligned} \bar{r}_\xi^{(e)} &= T_{\xi\alpha}^{(e)}\, \bar{r}_\alpha, & \bar{q}_r^{(e_\phi)p} &= H_{r\rho}^{(e_\phi)}\, \bar{q}_\rho^{p}, \\ \bar{\mathbf{r}}^{(e)} &= \mathbf{T}^{(e)}\, \bar{\mathbf{r}}, & \bar{\mathbf{q}}^{(e_\phi)p} &= \mathbf{H}^{(e_\phi)}\, \bar{\mathbf{q}}^{p}, \\ \alpha &= 1,\ldots,n_\alpha, & \rho &= 1,\ldots,n_\rho, \end{aligned} \tag{6.14}$$

where the matrix $T_{\xi\alpha}^{(e)}$ transform nodal displacements of a given element expressed in the local coordinate system in the global vector referred to the global coordinate system. This matrix consists of the product of orthogonal rotation matrix with Boole'an (zero-unity) matrix. In the case of generalized plastic strain vector it is convenient to assemble the vectors of the element parameters to a global vector without previous rotation to the global coordinate system. Matrix $H_{rs}^{(e_\phi)}$ becomes in this way the Boole'an matrix with the following properties

$$\begin{aligned} & e_\phi = e_\phi, & & e_\phi \neq e_\varphi, \\ & H_{r\sigma}^{(e_\phi)}\, H_{s\sigma}^{(e_\phi)} = \delta_{rs}, \quad \text{and} & & H_{r\sigma}^{(e_\phi)}\, H_{s\sigma}^{(e_\varphi)} = 0, \\ & \mathbf{H}^{(e_\phi)}\, (\mathbf{H}^{(e_\phi)})^T = \mathbf{I}, & & \mathbf{H}^{(e_\phi)}\, (\mathbf{H}^{(e_\varphi)})^T = \mathbf{0}. \end{aligned} \tag{6.15}$$

Details concerning the finite element method can be found in any book devoted to this subject. A large bibliography can be found in the book of Kleiber [33].

Having in mind a clarification of further description let us introduce the following variation ranges for indices accompanying the vectors **q**, **r**:

indices on the level of cross-section of the structure

$$\begin{array}{ccc} w_i^{(e)} & r_m^{(e)} & q_r^{(e)} \\ i,j,k,l = 1,\ldots,n_i, & m,n,o,p = 1,\ldots,n_m, & r,s,t,u = 1,\ldots,n_r, \end{array} \tag{6.16}$$

indices for element nodal parameters

$$\begin{array}{cc} \bar{r}_\xi^{(e)} & \bar{q}_r^{(e_\phi)p} \\ \xi,\ \zeta,\ \eta,\ \varepsilon = 1,\ldots,n_\xi, & r,\ s,\ t,\ u = 1,\ldots,n_r, \\ & \phi,\ \varphi,\ \chi,\ \psi = 1,\ldots,n_\phi, \end{array} \tag{6.17}$$

indices for global vectors

$$\begin{array}{cc} \bar{r}_\alpha & \bar{q}_\rho^{p} \\ \alpha,\ \beta,\ \gamma,\ \delta = 1,\ldots,n_\alpha, & \rho,\ \varrho,\ \sigma,\ \varsigma = 1,\ldots,n_\rho. \end{array} \tag{6.18}$$

6.4 Generalized stress state

After the finite element discretization of the structure, the approximated quantities obtained in the previous sections are introduced into the functional of the strain energy with plastic distortions. The functional is expressed by the global vectors of generalized displacements $\bar{\mathbf{r}}$ and plastic strains $\bar{\mathbf{q}}^p$

$$\begin{aligned} \mathcal{J}_q[\bar{\mathbf{r}}] &= \bar{r}_\alpha K_{\alpha\beta}\bar{r}_\beta - \bar{r}_\alpha \bar{R}_\alpha - \bar{r}_\alpha U_{\alpha\rho}\bar{q}^p_\rho, \\ \mathcal{J}_q[\bar{\mathbf{r}}] &= \bar{\mathbf{r}}^T\mathbf{K}\bar{\mathbf{r}} - \bar{\mathbf{r}}^T\bar{\mathbf{R}} - \bar{\mathbf{r}}^T\mathbf{U}\bar{\mathbf{q}}^p. \end{aligned} \tag{6.19}$$

Corresponding matrices denote:

1. Structural stiffness matrix

$$\begin{aligned} K_{\alpha\beta} &= \sum_{e=1}^{n_e} \int_{A^e} T^{(e)}_{\xi\alpha} \mathcal{L}^*_{ri}\varphi^{(e)}_{i\xi} E_{rs} \mathcal{L}^*_{sj}\varphi^{(e)}_{j\zeta}\; T^{(e)}_{\zeta\beta} dA, \\ \mathbf{K} &= \sum_{e=1}^{n_e} \int_{A^e} (\mathbf{T}^{(e)})^T (\mathbf{L}^*\varphi^{(e)})^T \mathbf{E}\mathbf{L}^*\varphi^{(e)}\; \mathbf{T}^{(e)} dA. \end{aligned} \tag{6.20}$$

2. Vector of global forces

$$\begin{aligned} \bar{R}_\alpha &= \sum_{e=1}^{n_e} \int_{A^e} p_i\varphi^{(e)}_{i\xi} T^{(e)}_{\xi\alpha} dA + \sum_{e=1}^{n_e} \int_{S^e_T} R_m \mathcal{M}_{mi}\varphi^{(e)}_{i\xi} T^{(e)}_{\xi\alpha} dS, \\ \bar{\mathbf{R}} &= \sum_{e=1}^{n_e} \int_{A^e} \bar{\mathbf{p}}\varphi^{(e)}\mathbf{T}^{(e)} dA + \sum_{e=1}^{n_e} \int_{S^e_T} \mathbf{R}\mathbf{M}\varphi^{(e)}\mathbf{T}^{(e)} dS. \end{aligned} \tag{6.21}$$

3. Matrix dependent on the plastic (permanent) stress distribution

$$\begin{aligned} U_{\alpha\varrho} &= \sum_{e=1}^{n_e} \int_{A^e} T^{(e)}_{\xi\alpha} \mathcal{L}^*_{ri}\varphi^{(e)}_{i\xi} E_{rs} \sum_{\phi} \psi^{(e\phi)}_{st} H^{(e\phi)}_{t,\varrho} dA, \\ \mathbf{U} &= \sum_{e=1}^{n_e} \int_{A^e} (\mathbf{T}^{(e)})^T (\mathbf{L}^*\varphi^{(e)})^T \mathbf{E} \sum_{\phi} \psi^{(e\phi)} \mathbf{H}^{(e\phi)} dA. \end{aligned} \tag{6.22}$$

The stationarity conditions for the functional of the strain energy with the plastic distortions correspond to vanishing of the first variation of $\mathcal{J}_q$

$$\begin{aligned} \delta\mathcal{J}_q &= \tfrac{\partial \mathcal{J}_q}{\partial \bar{r}_\alpha}\delta\bar{r}_\alpha = 0, \\ \delta\mathcal{J}_q &= \tfrac{\partial \mathcal{J}_q}{\partial \bar{\mathbf{r}}}\delta\bar{\mathbf{r}} = 0. \end{aligned} \tag{6.23}$$

A free choice of the generalized displacement variation $\delta\bar{r}_\alpha$ lead us to a fulfillment of the following linear algebraical system of equations

$$\begin{aligned} K_{\alpha\beta}\bar{r}_\beta &= \bar{R}_\alpha + U_{\alpha\varrho}\bar{q}^p_\varrho, \\ \mathbf{K}\bar{\mathbf{r}} &= \bar{\mathbf{R}} + \mathbf{U}\bar{\mathbf{q}}^p. \end{aligned} \tag{6.24}$$

The above system of equations should be solved in a way to incorporate the imposed displacements $\bar{\mathbf{r}}^D$ applied on the boundary S_D. It can be done in various manners commonly used in papers of FEM (see, e.g., Dhatt and Touzot [14]). Let the following solution of the equilibrium equations (6.24)

$$\begin{aligned} \bar{r}_\beta &= K^{-1}_{\alpha\beta}(\bar{R}_\alpha + U_{\alpha\rho}\bar{q}^p_\rho), \\ \bar{\mathbf{r}} &= \mathbf{K}^{-1}(\bar{\mathbf{R}} + \mathbf{U}\bar{\mathbf{q}}^p). \end{aligned} \tag{6.25}$$

be understood as the solution satisfying kinematical boundary conditions $\mathbf{r}^D$ on S_D. So, having the distribution of displacements field we can easily determine the elastic strain field in any the cross-section $\boldsymbol{\xi}$ of the finite element using the equation (6.4). Namely,

$$\begin{aligned} q_s{}^{(e)e} &= \mathcal{L}^*_{sj}\varphi^{(e)}_{j\zeta}T^{(e)}_{\zeta\beta}\bar{r}_\beta - \sum_\phi \psi^{(e_\phi)}_{st}H^{(e_\phi)}_{t\varrho}\bar{q}^p_\varrho, \\ \mathbf{q}^{(e)e} &= \mathcal{L}^*\varphi^{(e)}\mathbf{T}^{(e)}\bar{\mathbf{w}} - \sum_\phi \psi^{(e_\phi)}\mathbf{H}^{(e_\phi)}\bar{\mathbf{q}}^p. \end{aligned} \tag{6.26}$$

The elastic strain vector reduced to the integration points can be expressed as follows

$$\begin{aligned} \bar{q}_s^{(e_\phi)e} &= \mathcal{L}^*_{sj}\bar{\varphi}^{(e_\phi)}_{j\zeta}T^{(e)}_{\zeta\beta}\bar{r}_\beta - H^{(e_\phi)}_{s\varrho}\bar{q}^p_\varrho, \\ \bar{\mathbf{q}}^{(e_\phi)e} &= \mathbf{L}^*\bar{\varphi}^{(e_\phi)}\mathbf{T}^{(e)}\bar{\mathbf{w}} - \mathbf{H}^{(e_\phi)}\bar{\mathbf{q}}^p, \end{aligned} \tag{6.27}$$

where

$\mathcal{L}^*_{sj}\bar{\varphi}^{(e_\phi)}_{j\zeta}$ denotes the value of differential operator imposed on the shape function of displacements $\varphi^{(e)}_{i\xi}$ at the integration point ξ_ϕ of the finite element;

e_ϕ is a number of the integration points in the given cross-section ξ_ϕ of the finite element.

The stress state is determinated in the integration points of the structure from the Hooke's law (6.3). Expressing the global stress vector by the components of the element stress vectors described at the integration points ξ_ϕ of the finite element

$$\begin{aligned} \bar{Q}_\rho &= \sum_{e=1}^{n_e}\sum_\phi H^{(e_\phi)}_{r\rho}\bar{Q}^{(e_\phi)}_r, \\ \bar{\mathbf{Q}} &= \sum_{e=1}^{n_e}\sum_\phi (\mathbf{H}^{(e_\phi)})^T\bar{\mathbf{Q}}^{(e_\phi)}, \end{aligned} \tag{6.28}$$

we obtain elastic solution in stresses

$$\begin{aligned} \bar{Q}_\rho &= \sum_{e=1}^{n_e}\sum_\phi \left(H^{(e_\phi)}_{r\rho}E_{rs}\mathcal{L}^*_{rj}\varphi^{(e_\phi)}_{j\zeta}T^{(e)}_{\zeta\beta}\bar{r}_\beta - H^{(e_\phi)}_{r\rho}E_{rs}H^{(e_\phi)}_{s\varrho}\bar{q}^p_\varrho\right), \\ \bar{\mathbf{Q}} &= \sum_{e=1}^{n_e}\sum_\phi \left((\mathbf{H}^{(e_\phi)})^T\mathbf{E}\mathbf{L}^*\varphi^{(e_\phi)}\mathbf{T}^{(e)}\bar{\mathbf{r}} - (\mathbf{H}^{(e_\phi)})^T\mathbf{E}\mathbf{H}^{(e_\phi)}\bar{\mathbf{q}}^p\right). \end{aligned} \tag{6.29}$$

Let us decompose the total stress state described by (6.29) into

1. the stress state for perfectly elastic structure. It results from the applied external loads in the form of surface tractions p_i and boundary forces R_m and imposed displacements r^D_m on S_D (absence of plastic strains)

$$\begin{aligned} \bar{Q}^E_\rho &= \sum_{e=1}^{n_e}\sum_\phi H^{(e_\phi)}_{r\rho}E_{rs}\mathcal{L}^*_{sj}\varphi^{(e_\phi)}_{j\zeta}T^{(e)}_{\zeta\beta}K^{-1}_{\alpha\beta}\bar{R}_\alpha, \\ \bar{\mathbf{Q}}^E &= \sum_{e=1}^{n_e}\sum_\phi (\mathbf{H}^{(e_\phi)})^T\mathbf{E}\mathbf{L}^*\varphi^{(e_\phi)}\mathbf{T}^{(e)}\mathbf{K}^{-1}\bar{\mathbf{R}}; \end{aligned} \tag{6.30}$$

2. the stress state for the elastic structure with initial plastic strains and vanishing external loads

$$\begin{aligned}\bar{Q}^{res}_{\rho} &= \sum_{e=1}^{n_e}\sum_{\phi}\left(H^{(e_\phi)}_{r\rho}E_{rs}\mathcal{L}^{*}_{rj}\varphi^{(e_\phi)}_{j\zeta}T^{(e)}_{\zeta\beta}K^{-1}_{\alpha\beta}U_{\alpha\varrho}\bar{q}^{p}_{\varrho} - H^{(e_\phi)}_{r\rho}E_{rs}H^{(e_\phi)}_{s\varrho}\bar{q}^{p}_{\varrho}\right),\\ \bar{\mathbf{Q}}^{res} &= \sum_{e=1}^{n_e}\sum_{\phi}\left((\mathbf{H}^{(e_\phi)})^{T}\mathbf{E}\mathbf{L}^{*}\boldsymbol{\varphi}^{(e_\phi)}\mathbf{T}^{(e)}\mathbf{K}^{-1}\mathbf{U}\bar{\mathbf{q}}^{p} - (\mathbf{H}^{(e_\phi)})^{T}\mathbf{E}\mathbf{H}^{(e_\phi)}\bar{\mathbf{q}}^{p}\right).\end{aligned} \tag{6.31}$$

The global vector of residual stresses can be expressed by means of the plastic strain vector through the influence matrix $Z_{\rho\varrho}$ as follows

$$\begin{aligned}\bar{Q}^{res}_{\rho} &= Z_{\rho\varrho}\bar{q}^{p}_{\varrho},\\ \bar{\mathbf{Q}}^{res} &= \mathbf{Z}\bar{\mathbf{q}}^{p},\end{aligned} \tag{6.32}$$

where $Z_{\rho\varrho}$ is, in general, a nonsymmetrical matrix

$$\begin{aligned}Z_{\rho\varrho} = {} & \left(\sum_{e=1}^{n_e}\sum_{\phi}H^{(e_\phi)}_{r\rho}E_{rs}\mathcal{L}^{*}_{rj}\varphi^{(e_\phi)}_{j\zeta}T^{(e)}_{\zeta\beta}\right)K^{-1}_{\alpha\beta}\\ & \times\left(\sum_{e=1}^{n_e}\int_{A^e}T^{(e)}_{\xi\alpha}\mathcal{L}^{*}_{ri}\varphi^{(e)}_{i\xi}E_{rs}\sum_{\phi}\psi^{(e_\phi)}_{st}H^{(e_\phi)}_{t,\varrho}dA\right)\\ & -\sum_{e=1}^{n_e}\sum_{\phi}H^{(e_\phi)}_{r\rho}E_{rs}H^{(e_\phi)}_{s\varrho},\\ \mathbf{Z} = {} & \left(\sum_{e=1}^{n_e}\sum_{\phi}(\mathbf{H}^{(e_\phi)})^{T}\mathbf{E}\mathbf{L}^{*}\boldsymbol{\varphi}^{(e_\phi)}\mathbf{T}^{(e)}\right)\mathbf{K}^{-1}\\ & \times\left(\sum_{e=1}^{n_e}\int_{A^e}(\mathbf{T}^{(e)})^{T}(\mathbf{L}^{*}\boldsymbol{\varphi}^{(e)})^{T}\mathbf{E}\sum_{\phi}\boldsymbol{\psi}^{(e_\phi)}\mathbf{H}^{(e_\phi)}dA\right)\\ & -\sum_{e=1}^{n_e}\sum_{\phi}(\mathbf{H}^{(e_\phi)})^{T}\mathbf{E}\mathbf{H}^{(e_\phi)}.\end{aligned} \tag{6.33}$$

The symmetry of the influence matrix $Z_{\varrho\rho}$ is disturbed by the appearance of the shape function $\psi^{(e_\phi)}$ for plastic (permanent) strains and by the integration of the expression including these shape functions over the middle surface. Such a situation results from the way of discretization of the structure by finite elements. The assumption concerning the occurrence of the plastic strains at any integration point is equivalent to a distribution of the plastic strains in the finite element according to the shape function $\psi^{(e_\phi)}_{rs}$. So, the residual stresses described at the integration points of the finite element are determined by the plastic strain distribution inside this element but not by the nodal parameters alone. It is the main reason of the lack of the symmetry of the matrix $Z_{\rho\varrho}$. For bar structures the shape function for generalized plastic strains can be represented by the Dirac's function, describing localized plastic strains at the end of the bar. It permits for integration the expression containing the shape functions $\psi^{(e_\phi)}_{rs}$ in the formula (6.33) over the element and for symmetrization of the matrix $Z_{\rho\varrho}$. Of course, in this case instead of the generalized plastic strains $\bar{q}^{p}_{\varrho}$ some appropriate integral quantities corresponding to discontinuities of the generalized displacements at the nodal points should be introduced (see Borkowski [5]).

6.5 A particular case of symmetrization of the matrix Z

The generalized stress vector for the purely elastic reference structure and the influence matrix $Z_{\varrho\rho}$ (6.76) are used in numerical procedure. They are calculated in the initial phase of the program and are stored in disk files. Thus, it is recommended that the matrix $Z_{\rho\varrho}$ should be stored in the file of a possible small size.

Let us investigate transformations of variables for surface structures insuring the symmetry of the matrix $Z_{\rho\varrho}$ (what takes place for the bar structures).

For this purpose, let us assume such a form of the displacement shape function that gives the total strains constant at all over the finite element

$$\begin{aligned} q_s^{(e)} &= q_s^{(e)e} + q_s^{(e)p} = \mathcal{L}^*_{sj}\varphi_{j\zeta}^{(e)}\bar{w}_\zeta^{(e)} = \text{const}, \\ \mathbf{q}^{(e)} &= \mathbf{q}^{(e)e} + \mathbf{q}^{(e)p} = \mathbf{L}^*\varphi^{(e)}\bar{\mathbf{w}}^{(e)} = \text{const}, \end{aligned} \qquad \text{on } A^e, \tag{6.34}$$

Therefore, the generalized elastic and plastic strain distribution is also constant inside the finite element. These conditions are satisfied, e.g, for plate by the 6-th nodes triangular Morley element [57] (nonconsistant element), and for shell structures by extended version of this element described by Mołdach [55]. To determine the generalized plastic (permanent) strain state it is sufficient to consider only one nodal point ($\phi = 1$) in the element. In this case the shape functions $\psi_{rs}^{(e_1)}$ become the unity matrix

$$\begin{aligned} q_s^{(e)p} &= \sum_\phi \psi_{st}^{(e_\phi)}\bar{q}_t^{(e_\phi)p}, \\ \mathbf{q}^{(e)p} &= \sum_\phi \boldsymbol{\psi}^{(e_\phi)}\bar{\mathbf{q}}^{(e_\phi)p}, \end{aligned} \qquad \Longrightarrow \qquad \phi = 1, \qquad \begin{aligned} \psi_{st}^{(e_1)} &= \psi_{st}^{(e)} = \delta_{st}, \\ \boldsymbol{\psi}^{(e_1)} &= \boldsymbol{\psi}^{(e)} = \mathbf{I}. \end{aligned} \tag{6.35}$$

For the element with constant strains also the stress state has to be constant. Thus, we can introduce a new variable $\hat{Q}_r^{(e)}$ of the generalized stress state as a product of the stress vector at the nodal point and area of the finite element

$$\begin{aligned} \hat{Q}_r^{(e)} &= \bar{Q}_r^{(e)}A^e, \qquad & \hat{Q}_\rho &= \sum_{e=1}^{n_e} H_{r\rho}^{(e)}\bar{Q}_r^{(e)}A^e, \\ \hat{\mathbf{Q}}^{(e)} &= \bar{\mathbf{Q}}^{(e)}A^e, \qquad & \hat{\mathbf{Q}} &= \sum_{e=1}^{n_e}(\mathbf{H}^{(e)})^T\bar{\mathbf{Q}}^{(e)}A^e. \end{aligned} \tag{6.36}$$

Taking into account the relations (6.34)–(6.36) we obtain relations between global residual stress $\hat{Q}_\rho^{res}$ and plastic strain $\bar{q}_\varrho^p$ vectors

$$\begin{aligned} \hat{Q}_\rho^{res} &= \hat{Z}_{\rho\varrho}\bar{q}_\varrho^p, \\ \hat{\mathbf{Q}}^{res} &= \hat{\mathbf{Z}}\bar{\mathbf{q}}^p, \end{aligned} \tag{6.37}$$

with a symmetric influence matrix $\hat{Z}_{\varrho\rho}$

$$\begin{aligned} \hat{Z}_{\rho\varrho} &= \hat{U}_{\beta\rho}\hat{K}_{\alpha\beta}^{-1}\hat{U}_{\alpha\varrho} - \hat{E}_{\rho\varrho}, \\ \hat{\mathbf{Z}} &= \hat{\mathbf{U}}^T\hat{\mathbf{K}}^{-1}\hat{\mathbf{U}} - \hat{\mathbf{E}}, \end{aligned} \tag{6.38}$$

The corresponding matrices denote

$$\hat{U}_{\alpha\varrho} = \sum_{e=1}^{n_e} T^{(e)}_{\xi\alpha}\mathcal{L}^{*}_{ri}\varphi^{e}_{i\xi}E_{rs}H^{(e)}_{s\varrho}A^e,$$
$$\hat{\mathbf{U}} = \sum_{e=1}^{n_e}(\mathbf{T}^{(e)})^T(\mathbf{L}^{*}\varphi^{(e)})^T\mathbf{E}\mathbf{H}^{(e)}A^e, \tag{6.39}$$

and

$$\hat{E}_{\rho\varrho} = \sum_{e=1}^{n_e} H^{(e)}_{r\rho}E_{rs}H^{(e)}_{s\varrho}A^e,$$
$$\hat{\mathbf{E}} = \sum_{e=1}^{n_e}(\mathbf{H}^{(e)})^T\mathbf{E}\mathbf{H}^{(e)}A^e. \tag{6.40}$$

Introduction of the new variables of generalized stresses $\hat{Q}^{(e)}_r$ (6.36) is convenient because it does not change the yield, F^L, or ratchetting, F^R, loci if they are expressed in terms of nondimensional variables Q_r/k_r (k_r denote vector of plastic moduli for the corresponding components of the generalized stresses)

$$F\left(\frac{\hat{Q}^{(e)}_r}{\hat{k}^{(e)}_r}\right) \le 0, \qquad \hat{Q}^{(e)}_r = \bar{Q}^{(e)}_r A^e, \qquad \hat{k}^{(e)}_r = k^{(e)}_r A^e. \tag{6.41}$$

Let us remark, that the global vectors of elastic stresses $\hat{Q}_\rho$ and strains $\bar{q}^e_\varrho$ are related by the matrix $\hat{E}$ (6.40)

$$\hat{Q}_\rho = \hat{E}_{\rho\varrho}\bar{q}^e_\varrho,$$
$$\hat{\mathbf{Q}} = \hat{\mathbf{E}}\bar{\mathbf{q}}^e. \tag{6.42}$$

In a similar way we can specify a compatibility matrix $\hat{C}$ as a relation between the global vectors of the total generalized strains and displacements of the middle surface of the structure

$$\bar{q}_\varsigma = \hat{C}_{\varsigma\alpha}\bar{r}_\alpha, \qquad \hat{C}_{\varsigma\alpha} = \sum_{e=1}^{n_e} H^{(e)}_{r\varsigma}\mathcal{L}^{*}_{ri}\varphi^{(e)}_{i\xi}T^{(e)}_{\xi\alpha},$$
$$\bar{\mathbf{q}} = \hat{\mathbf{C}}\bar{\mathbf{r}}, \qquad \hat{C} = \sum_{e=1}^{n_e}(\mathbf{H}^{(e)})^T\mathbf{L}^{*}\varphi^{(e)}\mathbf{T}^{(e)}. \tag{6.43}$$

On the basis of the property of the Boole'an matrix $H^{(e)}_{r\varsigma}$ (6.15) it can be shown that the product of the transposed consistent matrix $\hat{C}_{\varsigma\alpha}$ and the elasticity matrix $\hat{E}_{\varsigma\varrho}$ gives the matrix $\hat{U}_{\alpha\varrho}$ (6.39), what is seen below

$$\hat{C}_{\varsigma\alpha}\hat{E}_{\varsigma\varrho} = \left(\sum_{\dot{e}=1}^{n_e} T^{(\dot{e})}_{\xi\alpha}\mathcal{L}^{*}_{ri}\varphi^{(\dot{e})}_{i\xi}H^{(\dot{e}_1)}_{r\varsigma}\right)\left(\sum_{\dot{e}_1}^{n_e} H^{(\dot{e}_1)}_{t\varsigma}E_{ts}H^{(\dot{e}_1)}_{s\varrho}A^e\right) =$$
$$\hat{\mathbf{C}}^T\hat{\mathbf{E}} = \left(\sum_{\dot{e}=1}^{n_e}(\mathbf{T}^{(\dot{e})})^T(\mathbf{L}^{*}\varphi^{(\dot{e})})^T\mathbf{H}^{(e)}\right)\left(\sum_{\dot{e}=1}^{n_e}(\mathbf{H}^{(\dot{e}_1)})^T\mathbf{E}\mathbf{H}^{(\dot{e}_1)}A^e\right) =$$
$$\sum_{\dot{e}=1}^{n_e}\left(T^{(\dot{e})}_{\xi\alpha}\mathcal{L}^{*}_{ri}\varphi^{(\dot{e})}_{i\xi}H^{(\dot{e}_1)}_{r\varsigma}\sum_{\dot{e}_1=1}^{n_e}H^{(\dot{e}_1)}_{t\varsigma}E_{ts}H^{(\dot{e}_1)}_{s\varrho}A^e\right) =$$
$$\sum_{\dot{e}=1}^{n_e}\left((\mathbf{T}^{(\dot{e})})^T(\mathbf{L}^{*}\varphi^{(\dot{e})})^T\mathbf{H}^{(e)}\sum_{\dot{e}=1}^{n_e}(\mathbf{H}^{(\dot{e}_1)})^T\mathbf{E}\mathbf{H}^{(\dot{e}_1)}A^e\right) = \tag{6.44}$$
$$\sum_{\dot{e}=1}^{n_e}T^{(\dot{e})}_{\xi\alpha}\mathcal{L}^{*}_{ri}\varphi^{(\dot{e})}_{i\xi}E_{rs}H^{(\dot{e}_1)}_{s\varrho}A^e = \hat{U}_{\alpha\varrho}$$
$$\sum_{\dot{e}=1}^{n_e}(\mathbf{T}^{(\dot{e})})^T(\mathbf{L}^{*}\varphi^{(\dot{e})})^T\mathbf{E}\mathbf{H}^{(\dot{e}_1)}A^e = \hat{\mathbf{U}}$$

Summarizing our considerations, we can see that the assumption concerning constant distribution of the generalized strain vector in the finite element leads to the symmetrical form of the influence matrix $\hat{Z}_{\rho\varrho}$ which is expressed by the consistent matrix $\hat{C}_{\varsigma\alpha}$ and the elasticity matrix $\hat{E}_{\varsigma\varrho}$

$$\begin{gathered}\hat{Z}_{\rho\varrho} = \hat{E}_{\sigma\rho}\hat{C}_{\sigma\beta}\hat{K}^{-1}_{\alpha\beta}\hat{C}_{\varsigma\alpha}\hat{E}_{\varsigma\varrho} - \hat{E}_{\rho\varrho},\\ \hat{\mathbf{Z}} = \hat{\mathbf{E}}\hat{\mathbf{C}}\hat{\mathbf{K}}^{-1}\hat{\mathbf{C}}^T\hat{\mathbf{E}} - \hat{\mathbf{E}},\end{gathered} \tag{6.45}$$

The stiffness matrix $\hat{K}_{\alpha\beta}$ has the following form

$$\begin{gathered}\hat{K}_{\alpha\beta} = \hat{C}_{\sigma\alpha}\hat{E}_{\sigma\varsigma}\hat{C}_{\varsigma\beta},\\ \hat{\mathbf{K}} = \hat{\mathbf{C}}^T\hat{\mathbf{E}}\hat{\mathbf{C}}.\end{gathered} \tag{6.46}$$

Let us notice that the elasticity law (6.3) and the geometrical equations (6.4) for the surface structure were reduced to the matrix equations (6.42) and (6.43), which are satisfied for the structure discretized by finite elements with constant stresses.

The virtual work principle (6.7) can be presented in discrete variables in the following way

$$\begin{gathered}\bar{Q}_\rho\delta\bar{q}_\rho = \bar{R}_\alpha\delta\bar{r}_\alpha, \qquad \delta\bar{r}_\alpha = 0,\\ \bar{\mathbf{Q}}^T\delta\bar{\mathbf{q}} = \bar{\mathbf{R}}^T\delta\bar{\mathbf{r}}, \qquad \delta\bar{\mathbf{r}} = \mathbf{0},\end{gathered} \qquad \text{on } S_D \cup S_U, \tag{6.47}$$

Taking into account the kinematical relations (6.43) we arrive at a matrix form of the equilibrium equations

$$\begin{gathered}\hat{C}_{\rho\alpha}\hat{Q}_\alpha = \bar{R}_\alpha,\\ \hat{\mathbf{C}}^T\hat{\mathbf{Q}} = \bar{\mathbf{R}}.\end{gathered} \tag{6.48}$$

where the global force vector $\bar{R}_\alpha$ (6.21) is applied to the structure in nodal points of finite elements. These nodal points cannot belong to the boundary $S_U \cup S_D$ because of vanishing of variation $\delta\bar{r}_\alpha$ there.

Introducing the element with the constant distribution of the stress state we can pass from the differential equation system for the surface structure (6.1)–(6.7) to the equivalent matrix description

equilibrium equations

$$\begin{gathered}\hat{C}_{\rho\alpha}\hat{Q}_\alpha = \bar{R}_\alpha,\\ \hat{\mathbf{C}}^T\hat{\mathbf{Q}} = \bar{\mathbf{R}},\end{gathered} \qquad \text{on } A_T; \tag{6.49}$$

elasticity law

$$\begin{gathered}\hat{Q}_\rho = \hat{E}_{\rho\varrho}\bar{q}^e_\varrho,\\ \hat{\mathbf{Q}} = \hat{\mathbf{E}}\bar{\mathbf{q}}^e,\end{gathered} \qquad \text{on } A; \tag{6.50}$$

kinematical relations

$$\begin{gathered}\bar{q}_\varsigma = \hat{C}_{\varsigma\alpha}\bar{r}_\alpha,\\ \bar{\mathbf{q}} = \hat{\mathbf{C}}\bar{\mathbf{r}},\end{gathered} \qquad \text{on } A, \tag{6.51}$$

with the strain additivity

$$\begin{gathered}\bar{q}_\varsigma = \bar{q}^e_\varsigma + \bar{q}^p_\varsigma,\\ \bar{\mathbf{q}} = \bar{\mathbf{q}}^e + \bar{\mathbf{q}}^p,\end{gathered} \qquad \text{on } A; \tag{6.52}$$

kinematical boundary conditions

$$\begin{aligned} \bar{r}_\alpha &= r_\alpha^D, \\ \bar{\mathbf{r}} &= \mathbf{r}^D, \end{aligned} \qquad \text{on } S_D; \tag{6.53}$$

$$\begin{aligned} \bar{r}_\alpha &= 0, \\ \bar{\mathbf{r}} &= \mathbf{0}, \end{aligned} \qquad \text{on } S_U. \tag{6.54}$$

Corresponding matrices can be expressed as follows:

consistent matrix

$$\begin{aligned} \hat{C}_{\varsigma\alpha} &= \sum_{e=1}^{n_e} H_{r\varsigma}^{(e)} \mathcal{L}_{ri}^{*} \varphi_{i\xi}^{(e)} T_{\xi\alpha}^{(e)}, \\ \hat{C} &= \sum_{e=1}^{n_e} (\mathbf{H}^{(e)})^T \mathbf{L}^{*} \varphi^{(e)} \mathbf{T}^{(e)}; \end{aligned} \tag{6.55}$$

elasticity matrix

$$\begin{aligned} \hat{E}_{\rho\varrho} &= \sum_{e=1}^{n_e} H_{r\rho}^{(e)} E_{rs} H_{s\varrho}^{(e)} A^e, \\ \hat{\mathbf{E}} &= \sum_{e=1}^{n_e} (\mathbf{H}^{(e)})^T \mathbf{E} \mathbf{H}^{(e)} A^e; \end{aligned} \tag{6.56}$$

vector of global forces

$$\begin{aligned} \bar{R}_\alpha &= \sum_{e=1}^{n_e} \int_{A^e} p_i \varphi_{i\xi}^{(e)} T_{\xi\alpha}^{(e)} dA + \sum_{e=1}^{n_e} \int_{S_T^e} R_m \mathcal{M}_{mi} \varphi_{i\xi}^{(e)} T_{\xi\alpha}^{(e)} dS, \\ \bar{\mathbf{R}} &= \sum_{e=1}^{n_e} \int_{A^e} \bar{\mathbf{p}} \varphi^{(e)} \mathbf{T}^{(e)} dA + \sum_{e=1}^{n_e} \int_{S_T^e} \mathbf{R} \mathbf{M} \varphi^{(e)} \mathbf{T}^{(e)} dS. \end{aligned} \tag{6.57}$$

Within the framework of this description we obtain

1. the elastic state of the generalized stresses produced by external forces and imposed displacements on the boundary S_D

$$\begin{aligned} \hat{Q}_\rho^E &= \hat{E}_{\rho\sigma} \hat{C}_{\sigma\alpha} \hat{K}_{\alpha\beta}^{-1} \bar{R}_\alpha, \\ \hat{\mathbf{Q}}^E &= \hat{\mathbf{E}} \hat{\mathbf{C}} \hat{\mathbf{K}}^{-1} \bar{\mathbf{R}}; \end{aligned} \tag{6.58}$$

2. the state of generalized residual stresses produced by a state of plastic (permanent) strains

$$\begin{aligned} \hat{Q}_\rho^{res} &= \hat{Z}_{\rho\varrho} \bar{q}_\varrho^p, \\ \hat{\mathbf{Q}}^{res} &= \hat{\mathbf{Z}} \bar{\mathbf{q}}^p; \end{aligned} \tag{6.59}$$

where the influence matrix $\hat{Z}_{\rho\varrho}$ and the stiffness matrix $\hat{K}_{\alpha\beta}$ are symmetric

$$\begin{aligned} \hat{Z}_{\rho\varrho} &= \hat{E}_{\sigma\rho} \hat{C}_{\sigma\beta} \hat{K}_{\alpha\beta}^{-1} \hat{C}_{\varsigma\alpha} \hat{E}_{\varsigma\varrho} - \hat{E}_{\rho\varrho}, \\ \hat{\mathbf{Z}} &= \hat{\mathbf{E}} \hat{\mathbf{C}} \hat{\mathbf{K}}^{-1} \hat{\mathbf{C}}^T \hat{\mathbf{E}} - \hat{\mathbf{E}}, \end{aligned} \tag{6.60}$$

$$\begin{aligned} \hat{K}_{\alpha\beta} &= \hat{C}_{\sigma\alpha} \hat{E}_{\sigma\varsigma} \hat{C}_{\varsigma\beta}, \\ \hat{\mathbf{K}} &= \hat{\mathbf{C}}^T \hat{\mathbf{E}} \hat{\mathbf{C}}. \end{aligned} \tag{6.61}$$

It is worthy to underline, that an analogous matrix description of the structure was derived by Maier [49]. It is also valid for arbitrary bar structures. Moreover, this description can be easily implemented in the numerical code.

The passage from differential equations describing continuum to the matrix notation was given for bar structures in the monography of Borkowski [5]. For frame structures plastic hinges were introduced in order to describe generalized plastic strains. That corresponds to the substitution of the shape function $\psi_{rs}^{(e_\phi)}$ by the Dirac's function in the general relation (6.12), which approximates the plastic strains in the finite element. Although a physical interpretation of expressions (6.49)- (6.61) is a little different than that given by Borkowski [5], the form of the mathematical description is exactly the same. So, a discrete plate or shell structure approximated by the constant distribution of strains over the finite element can be analyzed in the same manner as a bar structure.

The relations derived above concerning the global vectors of elastic generalized stress and residual stress for any type of the structures will be applied in a numerical code, to solve the shakedown problem using a min-max formulation.

6.6 Time discretization

Following König [39] some theorems will be quoted in order to simplify operation dealing with the variable loads. Let the domain Ω defined by Eq.(2.16) be a polyhedron with vertices B^k, $k = 1, \ldots, n_k$ (Fig. 2.2) in the space of load parameters $\beta_{(l)}$. From the convexity of the yield condition as well as from the linearity of the equilibrium equations it follows:

1. *If a structure shakes down in any loading path contained within the boundary $\partial\Omega$ then it shakes also down in any loading path contained within the Ω.*

2. *If a structure shakes down in a cyclic loading path containing all the vertices $B^1, \ldots, B^{n_k}$ of the hyper-polyhedron domain Ω, then the structure shakes also down in any loading path contained within domain Ω.*

Theorems mentioned above were a basis for elaboration of a method concerning determination of the shakedown load multiplier by means of an incremental step-by-step FEM procedure for elastic-plastic structures (König and Kleiber [40]), Borkowski and Kleiber [4], Kleiber and König [32]. Due to error accumulation and not-too-good convergence for complex loading paths, these methods are rarely used in the shakedown analysis.

It follows from the second theorem that, instead of considering maximization of the function $\bar{F}$ over a time in min-max problem (5.72), we can maximize this function over all vertices $\mathcal{B}^k$, $k = 1, \ldots, n_k$ of load polyhedron $\partial\Omega$. This property will be used in the derivation of the numerical algorithm.

6.7 Numerical algorithm

Determination of the shakedown load multiplier for the structure analyzed in terms of the generalized variables requires a solution of two basic problems. The first, min-max formulation (5.72) yields the shakedown load multiplier, which protects the structure against a possibility of the occurrence of an incremental collapse mode. Solution of the second, i.e. maximization of the elastic inverse multiplier η^E (5.73) excludes inadaptation mechanism by alternating plasticity. Let us notice, that the numerical algorithm, which solves the second problem, is only a part of a more general solution algorithm concerning the min-max problem, which will be discussed in details.

Finally, after the finite element and time discretization of the structure we arrive at the following formulation of the min-max problem

$$\eta^{inc} = \min_{\bar{q}^p_\rho} \max_{\boldsymbol{\xi}^{(e_\phi)}, \mathcal{B}^k} \bar{F}\left(\bar{Q}^E_r(\boldsymbol{\xi}^{(e_\phi)}, \mathcal{B}^k) + \bar{Q}^{res}_r(\boldsymbol{\xi}^{(e_\phi)}, \bar{q}^p_\rho), k_r(\boldsymbol{\xi}^{(e_\phi)})\right),$$

$$\bar{Q}^E_r = \bar{Q}^{EM}_r + \bar{Q}^{ED}_r, \tag{6.62}$$

$$e = 1, \ldots, n_e, \quad \phi = 1, \ldots, n_\phi, \quad \rho = 1, \ldots, n_\rho, \quad k = 1, \ldots, n_k$$

where

$\boldsymbol{\xi}^{(e_\phi)}$	denotes the ϕ-th integration point on e-th finite element;
$\bar{q}^p_\rho$	is the global vector of the plastic (permanent) strains defined at integration points of the finite elements;
$\bar{Q}^E_r(\boldsymbol{\xi}^{(e_\phi)}, \mathcal{B}^k)$	denotes the total state of the generalized stresses obtained for the purely elastic structure, which is composed of the generalized stress states $\bar{Q}^{EM}_r$ and $\bar{Q}^{EM}_r$ produced by external forces and imposed displacements. It is described at a specified integration point $\boldsymbol{\xi}^{(e_\phi)}$ of the finite element and is determined for the k-th vertex $\mathcal{B}^k$ of the load polyhedron Ω;
$\bar{Q}^{res}_r(\boldsymbol{\xi}^{(e_\phi)}, \bar{q}^p_\rho)$	is the generalized residual stress state described at the specified integration point $\boldsymbol{\xi}^{(e_\phi)}$ induced by the generalized plastic (permanent) strain $\bar{q}^p_\rho$;
$k_r(\boldsymbol{\xi}^{(e_\phi)})$	is the vector of plastic moduli at the specified cross-section. $\boldsymbol{\xi}^{(e_\phi)}$

The solution of the min-max problem consists of the subsequent solutions of three basic subproblems:

1. determination of the most stressed integration points at which the function $\bar{F}$ attains its maximal values:

2. finding out an optimal direction for minimization of maximal values of the function $\bar{F}$;

3. calculation of the length of optimal slope vector obtained from the previous step.

If the subproblems mentioned above are solved at a given iterative step, then depending on the accuracy criterion for obtained results we can stop our analysis or continue the iteration process after updating the data.

Without any loss of generality we proceed directly with the analysis of the I-th iterative step. At the beginning both elastic $\bar{Q}_r^E = \bar{Q}_r^{EM} + \bar{Q}_r^{ED}$, and residual $\bar{Q}_r^{res}$, generalized stresses are given from the previous step of iteration. For every integration point $\xi^{(e_\phi)}$ and for every vertex B^k of the load polyhedron we are looking for a value of the function η_I (the first subproblem)

$$\eta = \bar{F}\left(\bar{Q}_r^E(\xi^{(e_\phi)},\, B^k) + \bar{Q}_r^{res}(\xi^{e_\phi},\, \bar{q}_\rho^p), k_r(\xi^{(e_\phi)})\right), \tag{6.63}$$

$$e = 1,\ldots,n_e, \quad \phi = 1,\ldots,n_\phi, \qquad k = 1,\ldots,n_k$$

For further analysis we choose only these couples $(\xi^v,\, B^v)$, $v = 1,\ldots,n_v$ for which η_I attains the maximal value on the current step of iteration

$$(\xi^v, B^v):$$

$$\eta_I(\xi^v, B^v, \bar{q}_\rho^p, K) = \eta_I^{max} = \max_{\xi^{(e_\phi)}, B^k} \bar{F}\left(\bar{Q}_r^E(\xi^{(e_\phi)},\, B^k) + \bar{Q}_r^{res}(\xi^{e_\phi},\, \bar{q}_\rho^p), k_r(\xi^{(e_\phi)})\right)_I, \tag{6.64}$$

$$v = 1,\ldots,n_v$$

The above considerations can be clarified using a very simple example below.

Example 6.1 . A one-storey portal frame is subjected to two parameter variable loading (Fig.6.1): horizontal and vertical forces P. Vertices of the load rectangle are given in Table 5.

Table 5. Vertices of the load rectangle.

Load vertices	Forces	
	P_1	P_2
I	P	0
II	P	P
III	0	P
IV	0	0

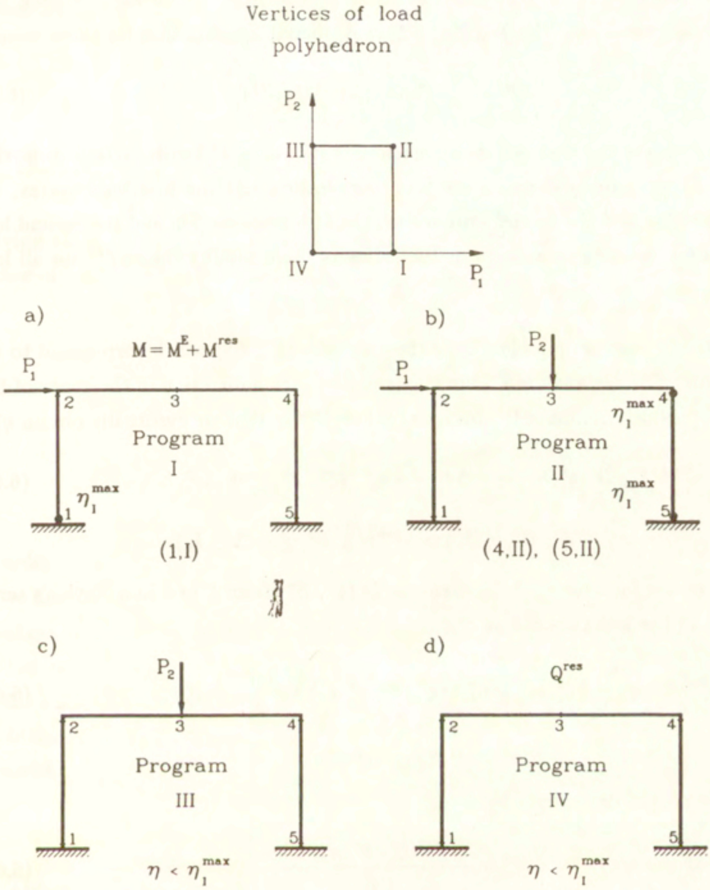


Figure 6.1: Determination of the couples $(\boldsymbol{\xi}^v, \mathcal{B}^v)$ representing the maximal value of the function $\bar{F}$.

Assume a certain state of residual stresses in the frame. If values of η are calculated according to the expression (6.63) for each cross-section from 1 to 5 and for each load vertex starting from $\mathcal{B}^I$ to $\mathcal{B}^{IV}$ (see Fig. $6.1_{a,b,c,d}$), then it appears that for three couples

$$(1, \mathcal{B}^I), \qquad (4, \mathcal{B}^{II}), \qquad (5, \mathcal{B}^{II}) \tag{6.65}$$

the values of η_I are the maximal and therefore, equal to η_I^{max} (with certain numerical accuracy). These couples denote: the first cross-section and the first load vertex, the forth cross-section and the second load vertex, the fifth cross-section and the second load vertex. In the remaining cross-sections the values of η are smaller than η_I^{max} for all load vertices

In the second basic subproblem only these couples ($\boldsymbol{\xi}^v$, $\mathcal{B}^v$) which correspond to the maximal value η_I^{max} are analyzed. Our aim is to find such a direction in the space of free parameters $\bar{q}^p_\rho$, which minimize the maximal value η_I^{max} so that we eventually obtain η_I^{min}

$$\eta_I^{max}(\boldsymbol{\xi}^v,\ \mathcal{B}^v, \bar{q}^p_\rho, k_r) \longrightarrow \Delta\bar{q}^p_\rho \longrightarrow \eta_I^{min}(\boldsymbol{\xi}^v,\ \mathcal{B}^v, \bar{q}^p_\rho + \Delta\bar{q}^p_\rho, k_r), \tag{6.66}$$

$$\eta_I^{max}(\boldsymbol{\xi}^v,\ \mathcal{B}^v, \bar{q}^p_\rho, k_r) > \eta_I^{min}(\boldsymbol{\xi}^v,\ \mathcal{B}^v, \bar{q}^p_\rho + \Delta\bar{q}^p_\rho, k_r).$$

Let us expand the function η_I^{min} for each couple ($\boldsymbol{\xi}^v$, $\mathcal{B}^v$) from 1 to v in a Taylor's series with respect to the free parameters $\bar{q}^p_\rho$

$$\eta_I^{min}(\boldsymbol{\xi}^v,\ \mathcal{B}^v, \bar{q}^p_\rho + \Delta\bar{q}^p_\rho, k_r) = \eta_I^{max}(\boldsymbol{\xi}^v,\ \mathcal{B}^v, \bar{q}^p_\rho, k_r) + \frac{\partial \eta_I^{max}}{\partial \bar{q}^p_\rho}\Delta\bar{q}^p_\rho +, \dots, +, \tag{6.67}$$

$$v = 1, \dots, n_v.$$

Denoting the partial derivatives by

$$g^v_\rho = \frac{\partial \eta_I^{max}(\boldsymbol{\xi}^v,\ \mathcal{B}^v, \bar{q}^p_\rho, k_r)}{\partial \bar{q}^p_\rho}, \tag{6.68}$$

$$v = 1, \dots, n_v, \qquad \rho = 1, \dots, n_\rho, \qquad n_v < n_\rho,$$

and neglecting in Eq. (6.67) high-order terms we arrive to the following set of n_v equations with n_ρ unknowns $n_v < n_\rho$

$$g^v_1 \Delta\bar{q}1^p +, \dots, + g^v_{n_\rho}\Delta\bar{q}_{n_\rho} = \eta_I^{min} - \eta_I^{max}, \qquad v = 1, \dots, n_v. \tag{6.69}$$

The set of equations can be presented in a graphical form as follows

$$\begin{bmatrix} \mathrm{A} & \mathrm{B} \\ \mathrm{C} & \mathrm{D} \end{bmatrix} \begin{bmatrix} \Delta\bar{\mathrm{q}}1 \\ \Delta\bar{\mathrm{q}}2 \end{bmatrix} = (\eta_I^{min} - \eta_I^{max}) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{matrix} (1 \times n_v - 1) \\ (1 \times n_\rho - n_v + 1) \end{matrix}$$

(6.70)

where the corresponding matrices denote

$$\Delta\bar{\mathbf{q}}\mathbf{1} = \lfloor \Delta\bar{q}_1^p, \ldots \Delta\bar{q}_{n_v-1}^p \rfloor^T \qquad \Delta\bar{\mathbf{q}}\mathbf{2} = \lfloor \Delta\bar{q}_{n_v}^p, \ldots \Delta\bar{q}_{n_\rho}^p \rfloor^T$$
$$\mathbf{A} = \lfloor g_1^1 \cdots g_{n_v-1}^1 \rfloor \qquad \mathbf{B} = \lfloor g_{n_v}^1 \cdots g_{n_\rho}^1 \rfloor \tag{6.71}$$

$$\mathbf{C}= \begin{bmatrix} g_1^2 & \cdots & g_{n_v-1}^2 \\ \cdots & \cdots & \cdots \\ g_1^{n_v} & \cdots & g_{n_v-1}^{n_v} \end{bmatrix}_{(n_v-1\times n_v-1)} \qquad \mathbf{D}= \begin{bmatrix} g_{n_v}^2 & \cdots & g_{n_\rho}^2 \\ \cdots & \cdots & \cdots \\ g_{n_v}^{n_v} & \cdots & g_{n_\rho}^{n_v} \end{bmatrix}_{(n_v-1\times n_\rho-n_v+1)} \tag{6.72}$$

Solving the second set of equations (6.70) with the square matrix $\mathbf{C}$ with respect to the vector of variables $\Delta\bar{\mathbf{q}}\mathbf{1}$ we obtain

$$\Delta\bar{\mathbf{q}}\mathbf{1} = \mathbf{C}^{-1}[(\eta_I^{min} - \eta_I^{max})\{\mathbf{1}\}_{(1\times n_v-1)} - \mathbf{D}\Delta\bar{\mathbf{q}}\mathbf{2}]. \tag{6.73}$$

After substitution $\Delta\bar{\mathbf{q}}\mathbf{1}$ to the first part of the set of equations we obtain

$$0 < \Delta\eta_I = \eta_I^{max} - \eta_I^{min} = -\mathbf{G}\,\Delta\bar{\mathbf{q}}\mathbf{2}, \tag{6.74}$$

$$\mathbf{G} = \frac{\mathbf{B} - \mathbf{A}\mathbf{C}^{-1}\mathbf{D}}{1 - \mathbf{A}\mathbf{C}^{-1}\{\mathbf{1}\}_{(1\times n_v-1)}}. \tag{6.75}$$

In order to find the optimal direction of minimization of η_I^{max} in the space of parameters $\bar{q}_\rho^p$ we are looking for the best orientation of the vector $\Delta\bar{\mathbf{q}}\mathbf{2}$ under the assumption of its constant length.

Let us notice that the expression (6.74) is a scalar product of two vectors $\mathbf{G}$ and $\Delta\bar{\mathbf{q}}\mathbf{2}$. The minimal value of $\eta_I^{min} = \eta_I^{max} - \Delta\eta_I$, what means the maximal value of $\Delta\eta_I$ is obtained if the direction of the vector $\Delta\bar{\mathbf{q}}\mathbf{2}$ is the same as the vector $\mathbf{G}$, but opposite oriented

$$\Delta\bar{\mathbf{q}}\mathbf{2} = -\mathbf{G}^T. \tag{6.76}$$

The vector $\Delta\bar{\mathbf{q}}\mathbf{1}$ can be determined according to the formula (6.73)

$$\Delta\bar{\mathbf{q}}\mathbf{1} = \mathbf{C}^{-1}[\mathbf{D}\mathbf{G}^T - \mathbf{G}\mathbf{G}^T\{\mathbf{1}\}_{(1\times n_v-1)}]. \tag{6.77}$$

Finally, after determination of the optimal direction $\Delta\bar{\mathbf{q}}$ responsible for the minimization of the function max $\bar{F}$

$$(\Delta\bar{\mathbf{q}}^p)^T = \lfloor \Delta\bar{\mathbf{q}}\mathbf{1}^T, \Delta\bar{\mathbf{q}}\mathbf{2}^T \rfloor, \tag{6.78}$$

we can reduce the min-max problem (6.62) to the minimization of the one-dimensional problem with respect to the length of the vector $\Delta\bar{\mathbf{q}}^p$ (the third subproblem)

$$\eta_I^{min} = \min_\alpha \eta_I^{max}(\alpha), \tag{6.79}$$

$$\eta_I^{max}(\alpha) = \max_{\boldsymbol{\xi}^{(e_\phi)}, \mathcal{B}^k} \bar{F}\left(\bar{Q}_r^E(\boldsymbol{\xi}^{(e_\phi)}, \mathcal{B}^k) + \bar{Q}_r^{res}(\boldsymbol{\xi}^{(e_\phi)},\ \bar{q}_\rho^p + \alpha * \Delta\bar{q}_\rho^p), k_r(\boldsymbol{\xi}^{(e_\phi)})\right). \tag{6.80}$$

This problem (6.79) can be easily solved by means of standard optimization methods.

After calculation of η_I^{min} we update the state of the generalized residual stresses and we check the accuracy criterion written symbolically in the following form

$$\mathcal{H}(\eta) < \epsilon, \tag{6.81}$$

where ϵ is any arbitrary quantity greater than zero.

The simplest criterion $\mathcal{H}$ can be expressed as follows

$$\mathcal{H}(\eta) = \left|\frac{\eta_{I+1}^{min} - \eta_I^{min}}{\eta_I^{min}}\right| < \epsilon. \tag{6.82}$$

It indicates that in the subsequent two steps of iteration the minimal values of η^{min} are located very close to each other.

The min-max formulation was first used by Zwoliński and Bielawski (as was mentioned before) [90], [91] in order to develop a numerical code for determination of the shakedown load multiplier. Both cited papers were presented at conferences and were published only in the form of summaries. For this reason numerical algorithms were not described in details. So, it is rather difficult to compare the authors' methods concerning determination of the direction which improve the solution with the method described above.

Comparison of the results obtained in the Chapter 6 with these obtained in [90], [91] is based on a discussion held with the first author of the mentioned papers. Here are shown some fundamental differences (data given within the parenthesis concerns the papers [90], [91]):

1. Formulation in the terms of generalized variables with decomposition of the problem into the incremental and the alternating mode of inadaptation. (Formulation for continuum without any distinguishing of the inadaptation modes);

2. Possibility of implementation of the higher-order finite elements to approximate the residual stress state. For the constant strain element the method for symmetrization of the influence matrix **Z** was proposed. (The constant strain elements CST, the nonsymmetrical influence matrix);

3. Formulation of the min-max problem permits to implement any nonlinear and non-homogeneous yield locus, what is necessary in the generalized variable description. (The procedure was developed for Huber-von Mises yield condition, the formulation is allowed only for a homogeneous function of order one with respect to the stress components what was proved in the Theorem 5.1);

4. The method for determination of the direction, which improve the solution at the given step of iteration (6.76)–(6.78) is proposed. Unfortunately, due to the lack of detailed description in [90] the comparison is not possible.

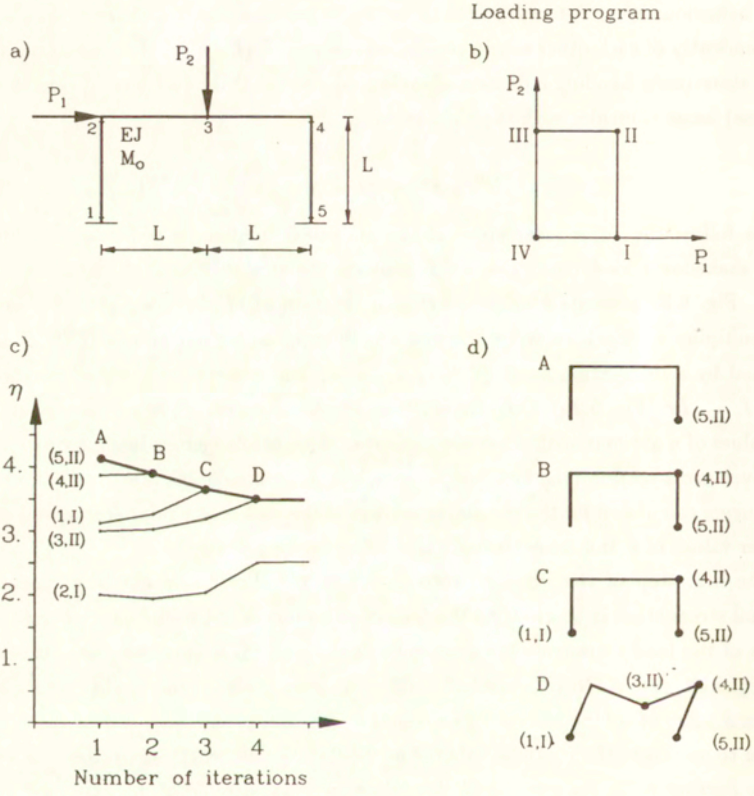
Example 6.2 In order to illustrate the numerical procedure described above let us analyze behaviour of one-storey portal frame under variable loading. Forces P_1, P_2 vary independently of each other within prescribed limits 0, P (Fig. 6.2). For one dimensional stress state (only bending moments are taken into account) the ratchetting (incremental collapse) locus coincides with the yield locus

$$F^R = F^L = |M| - M_o = 0 \tag{6.83}$$

Let us follow the subsequent steps of the numerical solution in order to find the inverse shakedown load multiplier which protects the structure against incremental collapse. Fig. 6.2c presents a relation between the number of iterations and the value of the multiplier η. Each curve on the graph is determined for one couple $(\xi^{(e_\phi)}, B^k)$ represented by a integration point $\xi^{(e_\phi)}$, $e_\phi = 1, \ldots, 5$ and a vertex of the load polyhedron $B^k = I, \ldots, IV$ (Fig. 6.2b). Only those curves are shown, for which at a chosen point $\xi^{(e_\phi)}$ the values of η are maximal. They correspond to the most dangerous loading program for the given cross-section (the first fundamental subproblem mentioned in the Section 6.7). The curves calculated for the remaining vertices of the load rectangular correspond to the smaller values of η and do not affect the solution of the problem.

The first step of the iteration corresponds to the elastic solution (the generalized residual stress state is absent). At the fifth cross-section of the frame and for the second vertex of the load rectangule the maximal value of η (6.63) is obtained, what is seen on the Fig. 6.2d, case A. Next, a residual bending moment state is generated to decrease the maximal value of η (the second fundamental subproblem). The intensity of this state is chosen to minimize the maximal value of η (the third fundamental subproblem according to the Section 6.7). As a result of this process the same value of η is attained at the second iteration step for the 4-th and the 5-th cross-sections and the second vertex of the load rectangule. In the each subsequent iteration the maximal value of η decreases as a result of generation of the proper increment of the residual bending moment state. So, in the third iteration a new couple composed of the first cross-section and the first load vertex is added to the previous existing, most stressed points. After the fourth iteration we arrive at the final solution. The maximal value of η was obtained at four cross-sections. The first cross-section correspond to the first case of loading whereas the third, the fourth and the fifth cross-sections are connected with the second case of loading. From the kinematical point of view it means that during the whole cycle of loading an incremental

$$F = \left| \frac{M}{M_o} \right| - 1 = 0$$



$$\mu^{sh} = 1/\eta^{sh} = 2.8571 M_o/(PL)$$

Figure 6.2: Relation between the inverse multiplier η and the number of iterations required for the determination of the shakedown load for the portal frame.

plastic mechanism in the form of a beam-sway mechanism developed (Fig. 6.2d case D). The exact solution was obtained at the fourth iteration for the case of variable loading consisted of four load vertices.

Obviously, it should be underlined that so fast solution of the problem was possible due to a very simple frame geometry and because of the one-dimensional yield condition dependent only upon the bending moment.

6.8 Numerical algorithm–scheme

A flow chart of the solution algorithm of the min-max problem proposed in the previous section is presented below. It concerns the method for determination of the shakedown load multiplier, which protects the structure against the incremental collapse. The problem dealing with alternating plasticity becomes a part of the more general min-max formulation, as was mentioned in the Section 5.6 The algorithm starts from zero iterative step:

1. I=0
2. Determination of the generalized elastic stress state $\bar{Q}_r^E = \bar{Q}_r^{EM} + \bar{Q}_r^{ED}$ (6.30) for subsequent vertices B^k of the load polyhedron.
3. Determination the influence matrix $Z_{\rho\varrho}$, which connects the vector of generalized residual stresses $\bar{Q}_r^{res}$ with the vector of free parameters $\bar{q}_\varrho^p$ (6.33)
$$\bar{Q}_\rho^{res} = Z_{\rho\varrho}\bar{q}_\varrho^p. \tag{6.84}$$
4. I=I+1
5. Determination of $v = 1, \ldots, n_v$ integration points $\boldsymbol{\xi}^v$ and the corresponding vertices of the load polyhedron B^v for which the inverse multiplier η attain the maximal value
$$(\boldsymbol{\xi}^v, B^v):$$
$$\eta_I(\boldsymbol{\xi}^v, B^v, \bar{q}_\rho^p, k_r) = \eta_I^{max} = \max_{\boldsymbol{\xi}^{(e\phi)}, B^k} \bar{F}\left(\bar{Q}_r^E(\boldsymbol{\xi}^{(e\phi)}, B^k) + \bar{Q}_r^{res}(\boldsymbol{\xi}^{e\phi}, \bar{q}_\rho^p), k_r\right)_I, \tag{6.85}$$
$$e = 1, \ldots, n_e, \qquad v = 1, \ldots, n_v,$$
$$\phi = 1, \ldots, n_\phi, \qquad k = 1, \ldots, n_k.$$
6. Calculation of the gradients g_ρ^v,
$$g_\rho^v = \frac{\partial \eta_I^{max}(\boldsymbol{\xi}^v, B^v, \bar{q}_\rho^p, k_r)}{\partial \bar{q}_\rho^p}, \tag{6.86}$$
$$v = 1, \ldots, n_v, \qquad \rho = 1, \ldots, n_\rho, \qquad n_v < n_\rho,$$
and specification of the matrices **A**, **B**, **C**, **D** according to (6.71), (6.72).

7. Determination of the optimal direction $\Delta\bar{q}^p_\rho$ which decrease the function η_I^{max}

$$(\Delta\bar{\mathbf{q}}^p)^T = \lfloor \Delta\bar{\mathbf{q}}\mathbf{1}^T, \Delta\bar{\mathbf{q}}\mathbf{2}^T \rfloor, \tag{6.87}$$

where

$$\Delta\bar{\mathbf{q}}\mathbf{1} = \mathbf{C}^{-1}[\mathbf{D}\mathbf{G}^T - \mathbf{G}\mathbf{G}^T\{\mathbf{1}\}_{(1\times n_v-1)}], \tag{6.88}$$

$$\Delta\bar{\mathbf{q}}\mathbf{2} = -\mathbf{G}^T, \tag{6.89}$$

$$\mathbf{G} = \frac{\mathbf{B} - \mathbf{A}\mathbf{C}^{-1}\mathbf{D}}{1 - \mathbf{A}\mathbf{C}^{-1}\{\mathbf{1}\}_{(1\times n_v-1)}}. \tag{6.90}$$

8. Solution of the one-dimensional problem in order to obtain the length of the vector of the optimal direction

$$\eta_I^{min} = \min_\alpha \eta_I^{max}(\alpha), \tag{6.91}$$

where

$$\eta_I^{max}(\alpha) = \max_{\boldsymbol{\xi}^{(e_\phi)},\mathcal{B}^k} \bar{F}\left(\bar{Q}_r^E(\boldsymbol{\xi}^{(e_\phi)}, \mathcal{B}^k) + \bar{Q}_r^{res}(\boldsymbol{\xi}^{(e_\phi)},\ \bar{q}^p_\rho + \alpha \times \Delta\bar{q}^p_\rho), k_r(\boldsymbol{\xi}^{(e_\phi)})\right). \tag{6.92}$$

9. Checking the accuracy criterion (6.82)

$$\mathcal{H}(\eta) < \epsilon. \tag{6.93}$$

10. In the case of violation of the accuracy criterion return to the point 4.

11. In the other case, calculate the shakedown load as follows

$$\begin{gathered} P_i^{sh}(\boldsymbol{\xi}, t) = P_i(\boldsymbol{\xi}, t)/\eta_I^{min}, \quad R_m^{sh}(\boldsymbol{\xi}, t) = R_m(\boldsymbol{\xi}, t)/\eta_I^{min}, \\ r_m^{sh}(\boldsymbol{\xi}, t) = r_m^D(\boldsymbol{\xi}, t)/\eta_I^{min}. \end{gathered} \tag{6.94}$$

Some properties of the numerical procedure are listed below:

1. The multidimensional min-max problem is considered as the solution of two standard problems with a good computer implementation:

   (a) Solution of a linear set of the algebraic equations with a full nonsymmetrical matrix of coefficients. This technique is used twice during determination of the vector $\mathbf{G}$ (6.90): for the first time in the numerator where an expression in the form of $\mathbf{C}^{-1}\mathbf{D}$ appears and for the second time in the denominator for calculation of $\mathbf{C}^{-1}\{\mathbf{1}\}_{1\times n_v-1}$. These two cases can be solved together in the following manner:

$$\mathbf{X} = \mathbf{C}^{-1}\bar{\mathbf{D}}, \tag{6.95}$$

   where

$$\bar{\mathbf{D}} = \lfloor \mathbf{D}, \{\mathbf{1}\}_{(1\times n_v-1)} \rfloor; \tag{6.96}$$

(b) Solution of the minimization of the one-dimensional problem (6.92);

2. Early iterations are very fast due to the small number of the integration points n_v, corresponding to the maximal value of η^{max}. It leads to a substantial reduction in size of the matrix **C** and accelerates the solution of the system of algebraical equations. The early steps of iterations contribute very significantly to the solution of the shakedown load multiplier;

3. A high efficiency of the solution is obtained for structure with local collapse. Because of small number of the nodal points with the maximal values η_I^{max} the order of the matrix **C** is also very small. It strongly influences the rapidity of the solution;

4. The min-max formulation for incremental collapse problem is based on the statical theorem. It means that each step of the iteration delivers a load multiplier which bounds from below the exact shakedown load multiplier. It is a fundamental property of the numerical program from engineering point of view, because according to the point (2) the largest contribution to the solution have the first iterative steps. The last iterations introduce only a meaningless correction to the solution and are very time consuming from the numerical point of view;

5. The procedure is also very convenient for solution of the problem concerning alternating plasticity. It is the case when purely elastic analysis is performed. The results are obtained in the first iterative step and estimation of the shakedown load multiplier from above becomes very fast. If this mode is a dominant the obtained result is the exact one;

6. In the case of the structure with only one stress component dominant (e.g., in the case of plane frames or grids, axial forces and torsion are of the secondary importance, respectively) it is possible to reduce the time of the solution. The calculation process should be divided into two phases. In the first we find an approximate solutions for one-dimensional yield condition (as it was in the Example 6.2). In the second phase we start from the approximate solution to obtain the final result for the exact yield condition;

7. During the determination of an envelope of the shakedown loads, it is preferable to calculate the next point of this envelope starting the solution process from the residual stress state obtained from the preceding point of the envelope. Such a starting point reduces considerably the time of the solution.

8. Analysis of large structures is possible using simple and well elaborated fundamental numerical procedures with standart input-output computer operations.

Some shortages of this method are listed below:

1. Determination of the influence matrix $\mathbf{Z}$ which relates the generalized residual stresses and generalized plastic strains is costly from numerical point of view.

2. The set of the algebraical equations with the full nonsymmetrical matrix $\mathbf{C}_{(n_v-1\times n_v-1)}$ (6.72) has to be solved at each step of the iteration;

3. The last iteration step are very time consuming.

In connection with the above it seems to be reasonable to follow further investigations concerning the convergence of the proposed method especially with an optimal selection of the direction vector improving the solution. A qualitative comparison with the known methods based on the linear mathematical programming is also needed.

6.9 Conclusions

Proposed numerical algorithm concerning the solution of min-max problem in the shakedown theory is based on gradient methods. It reduces the solution time through limitation the analysis to the most stressed cross-sections of the structures. It is convenient for numerical application because of taking advantage from procedures of finite element method. It seems also to be acceptable for calculations of large structures because of its high-speed solutions at the first iteration steps.

The proposition of the numerical algorithm seems to be especially useful for space frames with four stress components (axial force, torsion, bi-axial bending). It is possible to incorporate any yield condition corresponding to a specified cross-sectional properties without necessity of linearization of this condition. It is known (see Domaszewski [15]) that linearization of multidimensional yield loci leads to tremendous increase in the number of constraints for linear programming problems. Moreover, in the case of frames a symmetric influence matrix $\mathbf{Z}$ is obtained, which relates the residual stresses with plastic distortions. It considerably reduces the volume needed for storage of the matrix in the memory of the computer.

7 Application to bar structures

7.1 Space bar element

The previous chapter concerned general considerations on the discretization of the structure by means of the finite element method and on the numerical procedure. The obtained results can be used to calculate arbitrary bar, plate and shell structures. However, in order to verify the numerical procedure it seems to be reasonable to consider a possibly simple structure. It should be also representative for all considered types of structures.

According to the conclusions from the Section 6.5, both the structures discretized by means of the finite elements with a constant distribution of the stress and bar structures can be analyzed using the same matrix description introduced by Maier [49]. In the present chapter we extend directly the plane bar element proposed by Borkowski [3], [5] to its space version. A symmetrical form of the influence matrix **Z** for this element can be obtained from the general considerations. Namely, taking into account the matrix **Z** (6.33), with coefficients, which are integrals of the generalized plastic strains, the shape function $\psi_{rs}^{(e_\phi)}$ should be introduced in the form of Dirac's function (see final remarks in section 6.4).

The bar element has 12 degree of freedom, six in each nodal point. They consist of displacement components of the beam axis $\mathbf{w}^i$, $\mathbf{w}^j$ and rotations φ^i, φ^j, respectively for the left, "i", and the right, "j", cross-section, respectively:

$$\bar{\mathbf{r}}^{(e)} = \{w_x^i,\ w_y^i,\ w_z^i,\ \varphi_x^i\ ,\varphi_y^i,\ \varphi_z^i,\ w_x^j,\ w_y^j,\ w_z^j,\ \varphi_x^j,\ \varphi_y^j,\ \varphi_z^j\}^T. \tag{7.1}$$

Loads are considered as external forces and couples applied only at nodal points and are described by components corresponding to the conjugated degrees of freedom, in a local coordinate system

$$\bar{\mathbf{R}}^{(e)} = \{P_x^i,\ P_y^i,\ P_z^i,\ R_x^i,\ R_y^i,\ R_z^i,\ P_x^j,\ P_y^j,\ P_z^j,\ R_x^j,\ R_y^j,\ R_z^j\}^T. \tag{7.2}$$

Because the bar model assumes the loads applied only at the external nodal points of the element, the axial force, transverse forces and the torsional moment have to be constant along the bar element. Considering the transverse forces as internal reactions expressed by corresponding bending moments, the internal generalized stress state in the bar can be described by six components : the axial force N_x^j and the torque M_x^j in the second node of the element and two bending moments M_y, M_z in the first, "i", and the second, "j", cross-section, respectively

$$\bar{\mathbf{Q}}^{(e)} = \{N_x^j,\ M_x^j,\ M_y^i,\ M_z^i,\ M_y^j,\ M_z^j\}^T. \tag{7.3}$$

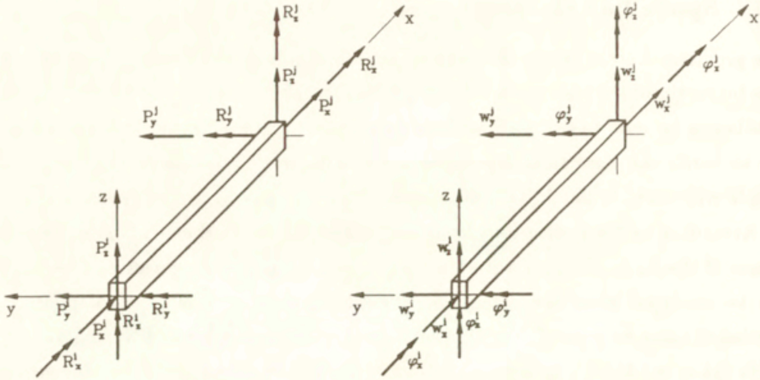


Figure 7.1: Space bar element

The generalized stress state is related to the state of generalized strains by a total axial elongation $\delta_x = \delta_x^i - \delta_x^j$, a total torque rotation $\theta_x = \theta_x^i - \theta_x^j$, and bending rotations θ_y i θ_z in the plane xz and xy, respectively

$$\bar{\mathbf{q}}^{(e)} = \{\delta_x,\ \theta_x,\ \theta_y^i,\ \theta_z^i,\ \theta_y^j,\ \theta_z^j\}^T. \tag{7.4}$$

The matrix of geometrical consistency $\bar{\mathbf{C}}^{(e)}$, which is responsible for the geometrical relations

$$\bar{\mathbf{q}}^{(e)} = \bar{\mathbf{C}}^{(e)}\bar{\mathbf{w}}^{(e)}, \tag{7.5}$$

can be expressed in a local coordinate system related to the bar in the following way (the vertical line separates the matrix components corresponding to the left and the right cross-section):

$$\bar{\mathbf{C}}^{(e)} = \left[\begin{array}{cccccc|cccccc}
-1 & & & & & & 1 & & & & & \\
 & & & -1 & & & & & & 1 & & \\
 & & \frac{-1}{L} & & 1 & & & & \frac{1}{L} & & & \\
 & \frac{1}{L} & & & & 1 & & \frac{-1}{L} & & & & \\
 & & \frac{-1}{L} & & & & & & \frac{1}{L} & & 1 & \\
 & \frac{1}{L} & & & & & & \frac{-1}{L} & & & & 1
\end{array}\right] \tag{7.6}$$

As it was stressed at the beginning, plastic distortions can occur only at the ends of the element, so the interior of the bar element remains elastic. The elastic law for the whole element can be written in the following way

$$\bar{\mathbf{Q}}^{(e)} = \bar{\mathbf{E}}^{(e)}\bar{\mathbf{q}}^{(e)e}, \tag{7.7}$$

where an the elasticity matrix depends both on the material constants (Young modulus E) and geometry of the bar (L-length of the bar, A-area of the cross-section, J-inertia moment)

$$\bar{\mathbf{E}}^{(e)} = \begin{bmatrix} \frac{EA}{L} & & & & & \\ & \frac{GJ_o}{L} & & & & \\ & & \frac{4EJ_y}{L} & & \frac{2EJ_y}{L} & \\ & & & \frac{4EJ_z}{L} & & \frac{2EJ_z}{L} \\ & & \frac{2EJ_y}{L} & & \frac{4EJ_y}{L} & \\ & & & \frac{2EJ_z}{L} & & \frac{4EJ_z}{L} \end{bmatrix}. \tag{7.8}$$

The total generalized strains can be decomposed into the elastic and the plastic parts

$$\bar{\mathbf{q}}^{(e)} = \bar{\mathbf{q}}^{(e)e} + \bar{\mathbf{q}}^{(e)p}. \tag{7.9}$$

Transformation of the respective local vectors and matrices to the global form is carried out by means of the general rules concerning the matrices $H^{(e)}$ and $\mathbf{T}^{(e)}$ defined by the formula (6.14). As a result, we obtain the following relations

equilibrium equations

$$\bar{\mathbf{C}}^T\bar{\mathbf{Q}} = \bar{\mathbf{R}}, \qquad \text{on } A_T; \tag{7.10}$$

elasticity law

$$\bar{\mathbf{Q}} = \bar{\mathbf{E}}\bar{\mathbf{q}}^e, \qquad \text{on } A; \tag{7.11}$$

kinematical relations

$$\bar{\mathbf{q}} = \bar{\mathbf{C}}\bar{\mathbf{r}}, \qquad \text{on } A; \tag{7.12}$$

where

$$\bar{\mathbf{q}} = \bar{\mathbf{q}}^e + \bar{\mathbf{q}}^p, \qquad \text{on } A; \tag{7.13}$$

kinematical boundary conditions

$$\bar{\mathbf{r}} = \mathbf{r}^D, \qquad \text{on } S_D; \tag{7.14}$$

$$\bar{\mathbf{r}} = \mathbf{0}, \qquad \text{on } S_U. \tag{7.15}$$

The following matrices denote:

consistency matrix

$$\bar{\mathbf{C}} = \sum_{e=1}^{n_e}(\mathbf{H}^{(e)})^T\bar{\mathbf{C}}^{(e)}\mathbf{T}^{(e)}; \tag{7.16}$$

elasticity matrix

$$\bar{\mathbf{E}} = \sum_{e=1}^{n_e} (\mathbf{H}^{(e)})^T \mathbf{E} \mathbf{H}^{(e)}; \tag{7.17}$$

Using the description introduced above we obtain:

1. the elastic state of generalized stresses produced by the external forces and the kinematical boundary conditions on S_D

$$\bar{\mathbf{Q}}^E = \bar{\mathbf{E}}\bar{\mathbf{C}}\bar{\mathbf{K}}^{-1}\bar{\mathbf{R}}; \tag{7.18}$$

2. the generalized state of residual stresses depending on the generalized state of the plastic strains

$$\bar{\mathbf{Q}}^{res} = \bar{\mathbf{Z}}\bar{\mathbf{q}}^p; \tag{7.19}$$

where matrix $\bar{\mathbf{Z}}$ and $\bar{\mathbf{K}}$ are symmetrical matrices.

$$\bar{\mathbf{Z}} = \bar{\mathbf{E}}\bar{\mathbf{C}}\bar{\mathbf{K}}^{-1}\bar{\mathbf{C}}^T\bar{\mathbf{E}} - \bar{\mathbf{E}}, \tag{7.20}$$

$$\bar{\mathbf{K}} = \bar{\mathbf{C}}^T\bar{\mathbf{E}}\bar{\mathbf{C}}. \tag{7.21}$$

It is clearly seen that the expressions (7.10)–(7.20) obtained for the bar structure and the expressions (6.49)–(6.61) describing any surface structure discretized by means of the finite elements with a constant distribution of the strain are almost identical. So, the analysis of both types may be performed by the same numerical algorithms.

7.2 Example–choice of the structure

For the clarity of the presentation of theorems and procedures let us restrict to some particular types of plane grids. Due to

1. two-dimensional generalized stress state in the cross-section (bending and torsional moments);

2. space character of the external loads (out of the plane of the structure);

the grids become good enough class of the bar structures in order to verify the proposed algorithm.

Let us consider a circular cross-section of the bar element. This type of the cross-section is chosen because of the knowledge of both elastic and limit locus as well as the ratchetting (incremental collapse) locus (Sawicki [79]).

A special attention was also paid to this type of the structure because of the possibility of modeling engineering systems, subjected to imposed displacements, which are an object of particular interest for us. That concerns floating bridges with transverse supports undergoing random excitations by displacements with an amplitude determined by sea waves.

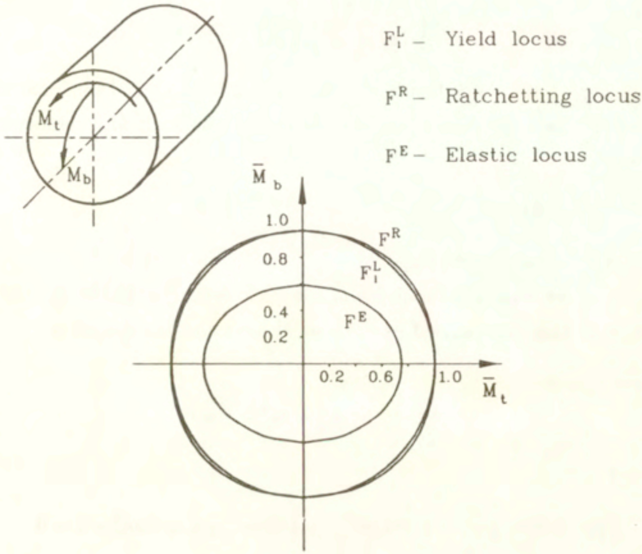


Figure 7.2: Limit loci for circular cross-section.

7.3 Limit loci

Two generalized stress components can be distinguished for a vertically loaded plane grid with a circular cross-section:

1. M_b - bending moment;

2. M_t - torsional moment.

The bending moment is responsible for an occurrence of normal stresses σ in the cross-section, whereas shear stresses τ are generated by the torsional moment (in the cylindrical coordinate system). The bar is made from the material obeying the Huber-von Mises yield condition

$$\sigma^2 + 3\tau^2 \leq \sigma_o^2, \tag{7.22}$$

where σ_o stands for yield stress in uniaxial tension.

Limit loci for the circular cross-section can be described as follows:

1. Elastic locus F^E (Sawicki [79])

$$F^E = \left(\frac{m_b}{3\pi/16}\right)^2 + \left(\frac{m_t}{3/4}\right)^2 - 1 = 0, \tag{7.23}$$

where $m_b = \frac{M_b}{M_{ob}}$, $m_t = \frac{M_t}{M_{ot}}$ denote nondimensional bending and torsional moments, respectively. The fully plastic bending, M_{ob}, and torsional, M_{ot}, moments are described by

$$M_{ob} = \frac{4}{3}\sigma_o R^3, \qquad M_{ot} = \frac{2\pi}{3\sqrt{3}}\sigma_o R^3. \tag{7.24}$$

2. Yield locus F^L (Życzkowski [92]).
Lower and upper bound estimations of the yield locus are known for the circular cross-section. They are derived from classical limit analysis theorems

   (a) lower bound

$$F_l^L = m_b^2 + m_t^2 - 1 = 0; \tag{7.25}$$

   (b) upper bound

$$F_u^L = 0.0958 m_b^4 + 0.9042 m_b^2 + 0.1155 m_t^4 + 0.8845 m_t^2 - 1 = 0. \tag{7.26}$$

3. Ratchetting (incremental collapse) locus F^R, derived by Sawicki [79] can be written in the following way

$$\left(\frac{16}{3\pi}m_b - \alpha\right)^2 + \left(\frac{4}{3}m_t \mp \frac{1}{3}\sqrt{1 - \left(\frac{\pi}{16/3 - \pi}\right)^2 \alpha^2}\right)^2 = 1, \tag{7.27}$$

where term α is of a very complex form (Sawicki [79]) and is dependent on the bending m_b and the torsional m_t moments.

The ratchetting (incremental collapse) locus is located between the lower and the upper bound of the yield locus, what follows from the analysis carried out by Sawicki [79]. Moreover, the bopunds of the yield locus are very close to each other. Therefore, we can assume without any loss of accuracy that the ratchetting and the yield loci can be treated as the same locus (Fig. 7.2)

$$F^R = F^L = F_l^L = m_b^2 + m_t^2 - 1 = 0. \tag{7.28}$$

7.4 Example of floating bridge–description of the scheme

Let us consider a rectangular plane grid shown on the Fig. 7.3. It consists of four spans in the 'x' direction and of six spans in the 'y' direction. The grid is simply supported with additional torque supports where torsional rotations vanish. Obviously, horizontal constraints imposed on some supports have to protect the structures against a rigid body motion in this plane.

Axial forces disappear in such a structure under external loads perpendicular to the grid plane. The above example can be regarded as a numerical model of a floating grid bridge. Nodal forces $1, 2, 3$, and $29, 30, 31$ of the grid (Fig. 7.3) rest on permanent supports, whereas nodal points $4, 9, 14, 19, 24$ and $8, 13, 18, 23, 28$ are located on floating pontoons. Influence of the waves, which front is parallel to the length of the bridge, can be modeled by imposed displacements with the amplitude equal to the height of the wave.

Variable loading in the form of external forces P_1, P_2, and P_3, P_4 applied along the spans 1–29 and 3–31, respectively consist of a dead load P and of variable portion of loads which attempts to model movable vehicles. Those loads are transmitted to the nodal points by means of a pavement structure, which is not an object of our interest.

1. the first set of external forces

$$P \leq P_1 \leq 2P, \qquad P \leq P_2 \leq 3P; \tag{7.29}$$

2. the second set of external forces

$$P \leq P_3 \leq 2P, \qquad P \leq P_4 \leq 3P; \tag{7.30}$$

3. the third set of imposed displacements

$$-u \leq u_1 \leq +u; \tag{7.31}$$

4. the fourth set of imposed displacements

$$-u \leq u_2 \leq +u. \tag{7.32}$$

The limits of variation of the external forces and the imposed displacements are expressed by means of a force P and a displacement u. The independent systems of loads $(P_1,\ P_2)$, $(P_3,\ P_4)$, u_1, u_2, vary arbitrarly within the polyhedron with sixteen vertices (Table 5), in the four dimensional space of loads.

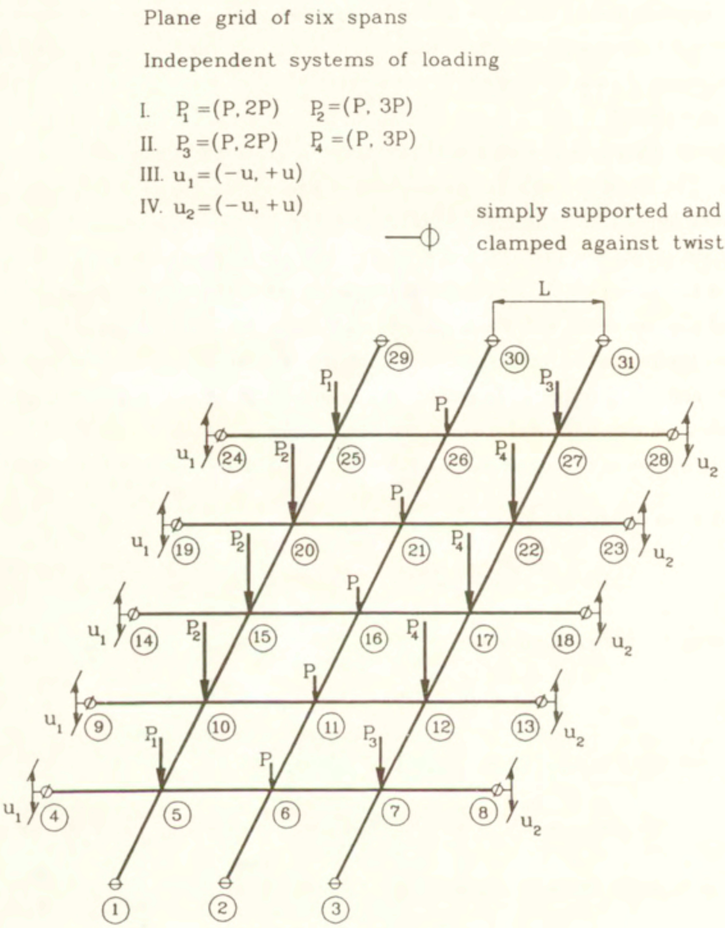


Figure 7.3: Plane grid under independent systems of external forces and imposed displacements.

Table 5. Vertices of the load polyhedron.

Number of vertex of load polyhedron	Independent sets of loading					
	I		II		III	IV
	P_1	P_2	P_3	P_4	u_1	u_2
1	P	P	P	P	-u	-u
2	2P	3P	P	P	-u	-u
3	P	P	2P	3P	-u	-u
4	2P	3P	2P	3P	-u	-u
5	P	P	P	P	+u	-u
6	2P	3P	P	P	+u	-u
7	P	P	2P	3P	+u	-u
8	2P	3P	2P	3P	+u	-u
9	P	P	P	P	-u	+u
10	2P	3P	P	P	-u	+u
11	P	P	2P	3P	-u	+u
12	2P	3P	2P	3P	-u	+u
13	P	P	P	P	+u	+u
14	2P	3P	P	P	+u	+u
15	P	P	2P	3P	+u	+u
16	2P	3P	2P	3P	+u	+u

The above sufficiently complex set of loading described by the four independently varying load parameters permits us to verify effectiveness of the proposed procedure.

The grid is divided into 36–bar elements. Each of them includes two nodes (at the beginning and at the end of the element) in which concentrated plastic strains (discountity of bending and torque rotations) may occur.

7.5 Verification of the numerical procedure

At the beginning, let us analyze a simplified model of loading without imposed displacements ($\mathbf{u}^D = 0$). It consists of the dead load P and two independently varying sets of external forces I, and II (Table 6).

Table 6. Simplified model of loading.

Number of vertex of the load polyhedron	Independent systems of external forces			
	I		II	
	P_1	P_2	P_3	P_4
1	P	P	P	P
2	2P	3P	P	P
3	P	P	2P	3P
4	2P	3P	2P	3P

Let us consider the problem of the limit analysis (Fig. 7.4) with loading described by the 4–th vertex of the load polyhedron (Table 6). The collapse load multiplier $(\mu^L)_n$ obtained from the numerical code is equal to

$$(\mu^L)_n = \frac{1}{\eta^L} = 0.44271\frac{M_{ob}}{PL}, \qquad \text{for} \quad \begin{array}{l} P_1 = P_3 = 2P, \\ P_2 = P_4 = 3P. \end{array} \tag{7.33}$$

It exists a simple method of verification of the obtained result. Namely, let us notice that the min-max problem (5.72) is based on the shakedown incremental collapse criterion (Theorem 5.2). In the case of proportional loading this theorem can be treated as a particular case of the limit analysis theorem. Thus, a generalized stress state obtained from the numerical analysis and multiplied by the scalar $(\mu^L)_n = \frac{1}{\eta^L}$ may be considered as a statically admissible stress state. Because it was determined as an optimal state, it should correspond to the real stress state (of course, with a certain numerical accuracy) at least at the most stressed nodal points. Considering the yield locus, the ratio between the plastic rotations rates $\dot{\theta}_t/\dot{\theta}_b$ can be determined at these points and the plastic collapse mechanism may be specified. Then, the collapse load multiplier can be calculated on the basis of kinematical theorem of limit analysis, which is an upper bound for the solution. In the Fig. 7.4 the kinematical mechanism is presented, which is associated with the statical solution. The kinematical load multiplier is equal to

$$(\mu^L)_k = \frac{1}{\eta^L} = 0.44275\frac{M_{ob}}{PL}. \tag{7.34}$$

A relative error calculated for the load multiplier $(\mu^L)_n$ obtained from the numerical solution versus its analytical equivalent $(\mu^L)_k$ is very small

$$\epsilon = \frac{(\mu^L)_k - (\mu^L)_n}{(\mu^L)_k}\, 100\% = 0.009\%. \tag{7.35}$$

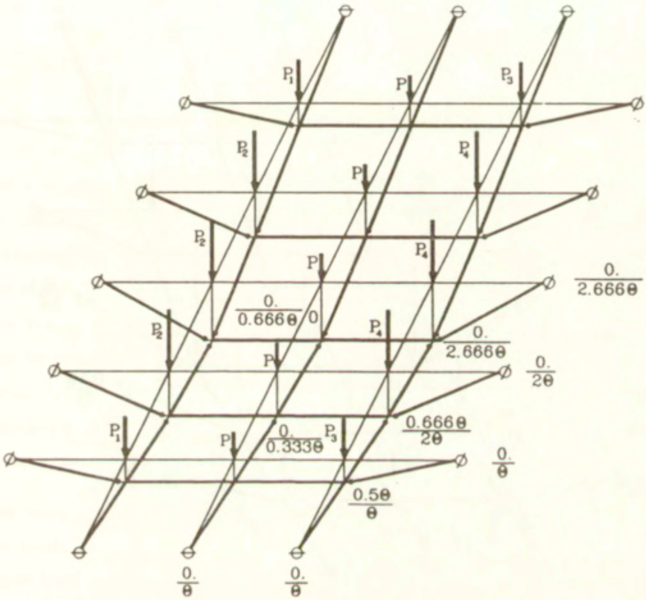
$$F = \left(\frac{M_b}{M_{ob}}\right)^2 + \left(\frac{M_t}{M_{ot}}\right)^2 - 1 = 0$$

$P_1 = P_3 = 2P$ Load–carrying capacity

$P_2 = P_4 = 3P$

$$(\mu^L)_k = 0.44275 \frac{M_{ob}}{PL} \qquad \varepsilon = 0.009\%$$

$$(\mu^L)_n = 0.44271 \frac{M_{ob}}{PL}$$



$P_1 = P_3 = (P, 2P)$ Shakedown load

$P_2 = P_4 = (P, 3P)$

$$(\mu^{inc})_n = 0.44271 \frac{M_{ob}}{PL}$$

$\nu = 0.3$

$$(\mu^{alt})_n = 0.72025 \frac{M_{ob}}{PL}$$

Figure 7.4: Collapse and inadaptation mechanisms for the simply supported grid.

$$F = \left(\frac{M_b}{M_{ob}}\right)^2 + \left(\frac{M_t}{M_{ot}}\right)^2 - 1 = 0$$

$P_1 = P_3 = 2P$

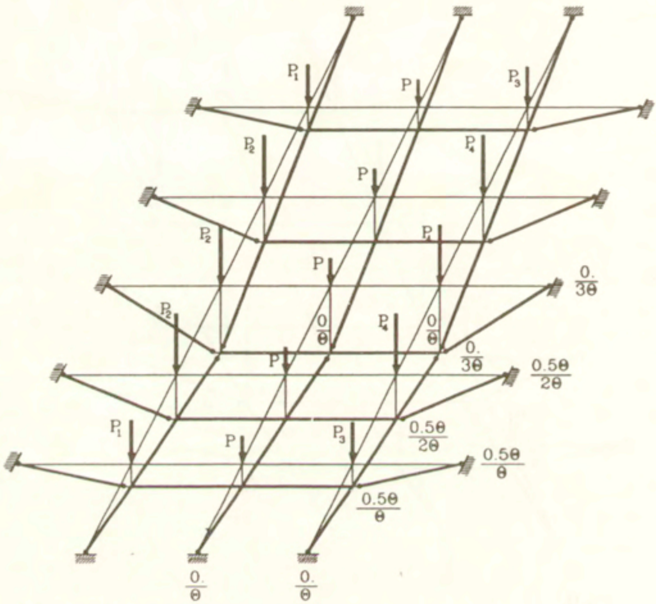
$P_2 = P_4 = 3P$

Load-carrying capacity

$(\mu^L)_t = 0.8337 \frac{M_{ob}}{PL}$

$(\mu^L)_n = 0.8329 \frac{M_{ob}}{PL}$

$\varepsilon = 0.098\%$



$P_1 = P_3 = (P, 2P)$

$P_2 = P_4 = (P, 3P)$

$\nu = 0.3$

Shakedown load

$(\mu^{inc})_n = 0.8267 \frac{M_{ob}}{PL}$

$(\mu^{alt})_n = 0.8589 \frac{M_{ob}}{PL}$

Figure 7.5: Collapse and inadaptation mechanisms for the clamped grid.

Checking the accuracy for determination of the shakedown load multiplier is much more complex, because of loads varying in prescribed limits (specified in Table 6). However, in some particular cases it is possible to do that. For example if the shakedown load multiplier differs from the collapse load multiplier then a shifting the lower load limits to the upper one should result with approaching of the shakedown load multiplier to the collapse one. Sometimes, it happens that these two multipliers are equal to each other. Such a situation occurs in the considered example.

The load multiplier μ^{inc} (5.72), which protects the structure against occurrence of the incremental collapse, is about 1.63 times smaller than the multiplier with respect to the alternating plasticity

$$(\mu^{inc})_n = (\frac{1}{\eta^{inc}})_n = 0.44271\frac{M_o b}{PL}, \qquad (\mu^{alt})_n = (\frac{1}{\eta^{alt}})_n = 0.72025\frac{M_o b}{PL}. \tag{7.36}$$

So, this multiplier is responsible for the inadaptation of the structure and it is also equal to the collapse load multiplier (7.34). From the physical point of view it means that instead of the mentioned incremental mechanism, which cause progressive accumulation of plastic strains during each load cycle, an instantaneous collapse mechanism occurs. Such a conclusion can be directly drawn from the obtained numerical solution, because the stress state at all critical points was given for the 4-th loading case.

Let us remember that the solution of the problem concerning alternating plasticity (5.73) can be considered as a purely elastic problem with symmetrical variable portion of the external loads. In this case only the elastic solution is needed to solve the problem. The shakedown load multiplier with respect to the alternating plasticity is obtained in the first iterative step of the solution procedure of the min-max problem.

Let us consider the next example of the same grid but with clamped ends (Fig. 7.5) under the loads shown in the Table 6. Similarly to the previous example, we can determine the collapse load multiplier and the corresponding kinematical plastic mechanism for the 4-th case of loading. The relative error concerning the lower and the upper bound solutions is below 0.1 percent

$$\epsilon = \frac{(\mu^L)_k - (\mu^L)_n}{(\mu^L)_k}\, 100\% = 0.098\%, \qquad \begin{array}{l} (\mu^L)_k = \frac{1}{\eta^L} = 0.8337\frac{M_{ob}}{PL}, \\ (\mu^L)_n = \frac{1}{\eta^L} = 0.8329\frac{M_{ob}}{PL}. \end{array} \tag{7.37}$$

The load multiplier, which protects the structure against incremental collapse μ^{inc} is now a slightly smaller than the collapse load multiplier. The difference between multipliers concerning the incremental collapse and the alternating plasticity is also smaller than in the case of the simply supported grid.

$$(\mu^{inc})_n = \frac{1}{\eta^{inc}} = 0.8267\frac{M_{ob}}{PL}, \qquad (\mu^{alt})_n = \frac{1}{\eta^{alt}} = 0.8589\frac{M_{ob}}{PL}. \tag{7.38}$$

Let us notice, that the collapse and the shakedown multipliers for the clamped grid are about 1.9 times greater than the corresponding multipliers for the simply supported grid. This fact is of course obvious for the the collapse load, whereas in the case of the shakedown analysis it does not have to be true, e.g., in the presence of imposed displacements.

7.6 Variable loads and imposed displacements

Let us consider the loading consisted of four independent sets of external forces and imposed displacements as shown in the Table 5. We shall investigate behaviour of the structure under imposed displacements depending upon the type of supports. Envelopes of the elastic and the shakedown domains are presented on the Fig. 7.6 for the simply supported and the clamped grids. The horizontal axis describes the ratio between the displacement u and the grid span L. The vertical axis represents nondimensional external force limit PL/M_{ob} (M_{ob}–denotes fully plastic bending moment).

The case of vanishing imposed displacements was analyzed in the previous section. It is worthy to stress that the limit elastic load for the clamped grid is about 3.6 times smaller than the collapse or the shakedown load. A big difference appears between the shakedown load multipliers for the simply supported and the clamped grids under variable external forces (without any imposed displacements). This situation changes considerably when two systems of imposed displacements are added. For the clamped grid the range for which the incremental collapse can occur is very small. A relatively large decrease in the load shakedown multiplier is observed due to the imposed displacements. Although, the differences between elastic and shakedown domains are very large, the imposed displacements considerably decrease the safety of the structure with this kind of support. The structure is simply too rigid.

A quite opposite situation appears for the simply supported grid. It is true that the shakedown load multiplier is about 1.9 times smaller than the highest multiplier for the clamped grid but the value of this multiplier maintains almost unchanged for relatively large range of the imposed displacements (more than three times greater than for the clamped grid). Also the character of the incremental collapse curve is different than the character of the limit elastic curve. For small imposed displacements, (from 0 to $24 \times 10^{-3} u/L$), the first curve decreases slightly faster than the second one, whereas in the interval $(24 - 36) \times 10^{-3} u/L$ this decrease is considerably slower.

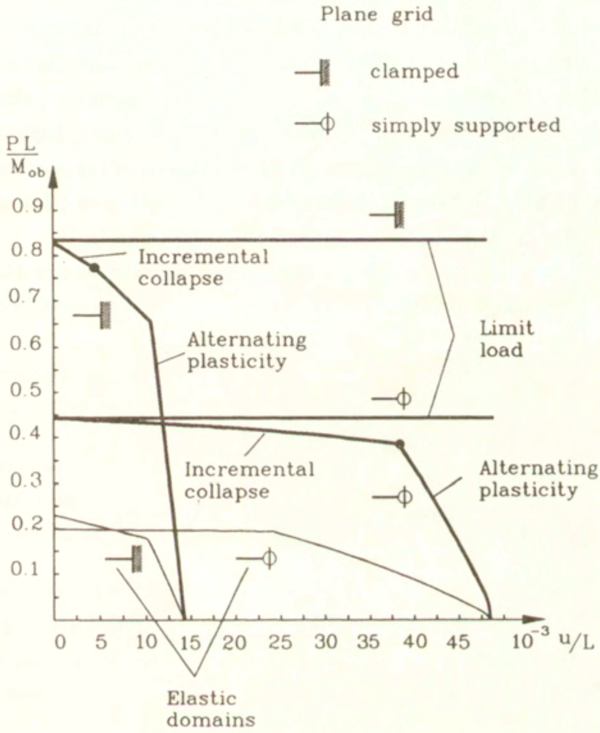


Figure 7.6: Comparison between the elastic and the shakedown envelopes for the simply supported and the clamped plane grid under imposed displacements.

Conclusions concerning the short range of the incremental collapse domain for the clamped grid and the long range for the simply supported grid could have been drawn from the analysis carried out in the previous section. Namely, in the first case of clamped grid and imposed displacements neglected, the difference between the multipliers concerning the incremental collapse and the alternating plasticity (7.38) was about 4%. In the case of simply supported grid (7.36) it attained 39%. Taking into account the fact that according to the Corollary 4.1 the action of the imposed displacements alone generate only the alternating plasticity mode of inadaptation, we arrive to the conclusion, that this type of mechanism may develop when both external forces and imposed displacements are applied. One can say that the alternating plasticity mechanism will appear the faster the smaller was the difference between the multipliers concerning incremental collapse and alternating plasticity in the case of vanishing imposed displacements.

The examples presented above show in what degree different serviceability criteria may impose contradictory requirements.

8 Conclusions and final remarks

No previous papers concerning the shakedown analysis considered in detail the subject of variable in time imposed displacements, although the problem is important from the engineering point of view. The action of this type of loading combined with variable repeated external forces appears in the analysis of floating bridges, off-shore structures and pipeline compensators.

The aim of the paper was to deliver a comprehensive analysis of the action of the variable loading consisted of external forces and imposed displacements on elastic-plastic bodies (structures). It concerns both applicability of general theorems as well as a method for solution of the problem together with a FEM formulation for bar and surface structures.

In the paper a distinction was made between the stress states produced by the external forces and by the imposed displacements (Fig. 2.3). The difference in the influence of these stresses on the behaviour of the body (structure) is pointed out. Due to the proposed stress decomposition it was possible to derive some fundamental properties for these two different types of actions. Particularly, it was shown that:

1. In the case of the imposed displacements alone (external forces are equal to zero) inadaptation of the structure appears exclusively in the form of the alternating plasticity (Corollary 4.1);

2. Addition to the action of variable imposed displacements, which produce an alternating plasticity mode of inadaptation, any time-independent system of external forces may cause occurrence of an incremental collapse mechanism even if the constant load is great deal smaller than the collapse load (Corollary 4.2, Example 4.3);

3. Adding to the body under given external loads, which produce an incremental collapse, any imposed displacements, may change the mechanism into another also an incremental one (Corollary 4.3, Example 4.1);

4. The shakedown load multiplier in the case of imposed displacements depends only on the difference between the load limits for these displacements. It means that , a rigid translation of the imposed displacements domain does not affect the value of the shakedown load multiplier (Conclusion 4.1).

The load elastically equivalent to the imposed displacements was defined. It permitted to introduce a notion of the equivalent load (consisted of external forces only) with respect to the combined load (variable external forces and imposed displacements) and to compare the safety level of the structure (body) under these two types of loading. It was pointed out, that:

1. The shakedown load multiplier obtained for the combined load which consists of the time-independent system of external forces and time-dependent imposed displacements, is at least equal or greater than the corresponding multiplier for the equivalent load (Theorem 4.2);

2. If combined load consists of variable repeated both imposed displacements and external forces, then the load shakedown multiplier for this load may sometime be smaller than the corresponding one for the equivalent load. It may happen when the structure is in danger of the alternating plasticity mode of inadaptation due to combined load (proof of Corollary 4.5).

The first of the mentioned properties may be used to bound from below the shakedown load multiplier for combined load provided that we consider the equivalent load consisted of the systems of external forces only (variable and time-independent). If imposed displacements are also time-independent this property becomes a particular case of the limit analysis theorem. It states that an introduction of some additional kinematical constraints does not decrease the carrying capacity of the body.

Next part of the dissertation is devoted to some numerical aspects. Namely, an equivalence of the min-max formulation and the classical shakedown approach for a continuum was demostrated. Moreover, this formulation was extended to the case of generalized variables (Theorem 5.4). It leads to a unified description of both inadaptation mechanisms and gives a possibility of the solution of these two problems by means of the same numerical algorithm (see Section 5.6).

Employing the classical variational approach to bar and surface structures (see Washizu [85]) a generalized description for arbitrary structures was introduced by means of the differential operators. Using a standard finite element discretization, a method for determination of generalized residual stresses, which are next used in the numerical procedure, was presented. A method of solution of the previously formulated min-max problem was also proposed. The particular attention was focused on the proper choice of a direction vector, which improves the solution on a given iterative step. The presented min-max formulation and the proposed method of the solution seem to be very promising in analysis of the problems with a complex stress state (several components of generalized stresses in the cross-section). Due to this fact, further investigations concerning increasing efficiency of the convergence are recommended. Accuracy of the presented method was examined on some grid structures subjected to four independent systems of external forces and imposed displacements.

The results obtained in the dissertation give a possibility to continue the future research in the following topics:

1. Determination of the best direction vector which improves the solution on a given iterative step;

2. Employing the min-max formulation to the analysis of a steady state of stresses for the cyclic loads exceeding the shakedown values;

3. Investigation of the possibility of extending the obtained results to more complex material models with hardening and to thermal effects.

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