THE LEAST GRADIENT PROBLEM IN THE FREE MATERIAL DESIGN

W. Górny, P. Rybka, and A. Sabra

Institute of Applied Mathematics & Mechanics, the University of Warsaw, Warsaw, Poland

e-mail: rybka@mimuw.edu.pl

Stating the least gradient problem

A version of the Free Material Design maybe stated as follows: given region $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ a load at the boundary consistent with the equilibrium, i.e. $\int_{\partial\Omega} g\, dS = 0$ find the optimal distribution p of the material. By optimality we mean that

$$(1) \qquad \int_\Omega |p| = \inf\{\int_\Omega |q| : \; q \in L^1(\Omega, \mathbb{R}^d),\; \operatorname{div} q = 0,\; q \cdot \nu|_{\partial\Omega} = g\}.$$

Here, ν is the outer normal to $\partial\Omega$. It obvious from the statement of (1) that one should expect to find a solution in the space of Radon measures, $\mathcal{M}$, on Ω.

One can look for a dimension reduction of (1), which is simple, when $d = 2$. We notice that (1) is equivalent to

$$(2) \qquad \int_\Omega |Du| = \inf\{\int_\Omega |Dv| : \; v \in BV(\Omega),\; v|_{\partial\Omega} = f\},$$

where $BV(\Omega)$ is space of functions with bounded total variation and $\frac{\partial f}{\partial \tau} = g$ and τ is a tangent vector to $\partial\Omega$. The equivalence is given by the mapping $BV(\Omega) \ni u \mapsto QDu \in \mathcal{M}$, where Q is the rotation by $\frac{\pi}{2}$, for details see [3].

Existence of solution in strictly convex domains for different boundary conditions

It is well-known fact that if $f \in C(\partial\Omega)$ and $\Omega \subset \mathbb{R}^2$ is strictly convex, then there exists a unique solution to (2), see [5]. For more general data neither existence, nor uniqueness is obvious. A part of the problem is that the problem (2) is ill-posed, because the following functional $\Phi : L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$, given by $\Phi(u) = \int_\Omega |Du|$, if and only if $u \in BV(\Omega)$ and $u|_{\partial\Omega} = f$, otherwise $\Phi(u) = +\infty$, is not lower semicontinuous. Nonetheless, we can show

Theorem 1. (see [2], [3])
If $\Omega \subset \mathbb{R}^2$ is strictly convex, $f \in BV(\partial\Omega)$, then problem (2) has at least one solution. □

Here is an **Example** of a solution, [3]. If $\partial\Omega$ is parametrized by arclength, $[0, L) \ni s \mapsto x(s) \in \partial\Omega$, then we take $f = (\alpha_1 + \alpha_2)\chi_{[s_2,s_2)} + \chi_{[s_2,L)}$, $s \in [s_2, L)$. The solution, u, takes three values, 0, α_1, $\alpha_1 + \alpha_2$ and it is depicted on Fig. 1.

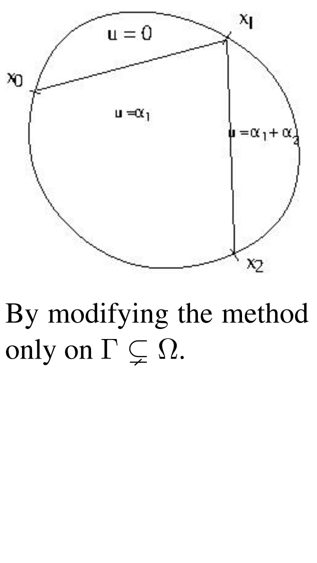

Fig. 1

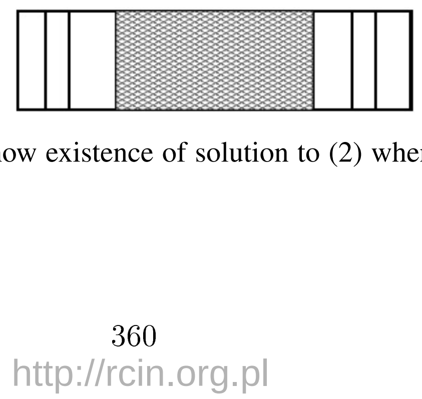

Fig. 2

By modifying the method of [5] we can show existence of solution to (2) when continuous data are specified only on $\Gamma \subsetneq \Omega$.

Theorem 2. (see [3])
If $\Omega \subset \mathbb{R}^2$ is strictly convex, $\Gamma \subsetneq \Omega$ is a smooth arc, $f \in C(\bar{\Gamma})$, then problem (2), when $u|_\Gamma = f$ is in place of $u|_{\partial\Omega} = f$, has at least one solution. □

Existence of solutions in convex but not strictly convex domains

The main problem for existence is presence of nontrivial line segments ℓ in the boundary of Ω, we call them *flat parts*. We shall say that a continuous function $f \in C(\partial\Omega)$ *satisfies the admissibility condition #1* on a flat part ℓ iff f restricted to ℓ is monotone.

We associate with f on a flat piece of the boundary, ℓ, a family of closed intervals $\{I_i\}_{i\in\mathcal{I}}$ such that $I_i = [a_i, b_i]$ is contained in the interior of ℓ relative to $\partial\Omega$. On each I_i function f attains a local maximum or minimum on each ℓ and each I_i is maximal with this property. We also set $e_i = f(I_i)$, $i \in \mathcal{I}$. For the sake of making the notation concise we will call I_i a *hump*.

After this preparation we state the admissibility condition for non-monotone functions. A continuous function f, which is not monotone on a flat part ℓ, *satisfies the admissibility condition #2* iff for each hump $I_i = [a_i, b_i] \subset \ell$ and $e_i := f([a_i, b_i])$, $i \in \mathcal{I}$ the following inequality holds,

$$\text{dist}\,(a_i, f^{-1}(e_i) \cap (\partial\Omega \setminus I_i)) + \text{dist}\,(b_i, f^{-1}(e_i) \cap (\partial\Omega \setminus I_i)) < |a_i - b_i|. \tag{3}$$

Theorem 3. (see [4])
Let us suppose that Ω is convex and $f \in C(\partial\Omega)$. In addition, $\partial\Omega$ has a finite number of flat parts $\{\ell_k\}_{k=1}^N$. If f satisfies the admissibility conditions #1 or #2 on each flat part $\{\ell_k\}_{k=1}^N$ of $\partial\Omega$, then there is a continuous solution to the least gradient problem. □

We can extend this result also to the case $f \in BV(\partial\Omega)$ or an infinite number of flat parts of $\partial\Omega$.

Example
We define $\Omega = (-L, L) \times (-1, 1)$, $L > 2$. We take, $f_i \in C(\partial\Omega)$, $i = 1, 2$ given by $f_1(x, y) = \cos(\frac{\pi}{2}y)$ and $f_2(x, y) = \cos(\frac{\pi}{2}\frac{x}{L})\chi_{|x|>L-2}(x) + \chi_{|x|\le L-2}(x)$. For f_1 problem (2) has no solution, while for f_2 there is a unique solution whose level sets are shown on Fig. 2. The shaded area is a level set of positive Lebesgue measure.

We also discuss the lack of uniqueness of solutions. We show that non-uniqueness of solutions to (2) is related to level sets of u with positive 2-d Lebesgue measure and discontinuities of f. This is done in [1].

Acknowledgments The work of AS was supported by the Research Grant 2015/19/P/ST1/02618 financed by the National Science Centre, European Union's Horizon 2020 research and innovation program under the Marie Curie grant agreement No 665778. PR was in part supported by the National Science Centre Research Grants no 2013/11/B/ST8/04436 and 2015/19/P/ST1/02618.

References

[1] W. Górny, (Non)uniqueness of minimizers in the least gradient problem, arxiv:1709.02185.
[2] W. Górny, Planar least gradient problem: existence, regularity and anisotropic case, arxiv:1608.02617.
[3] W. Górny, P.Rybka, A.Sabra, Special cases of the planar least gradient problem, *Nonlinear Analysis*, **151** (2017), 66-95.
[4] P.Rybka, A.Sabra, The planar Least Gradient problem in convex domains, arxiv:1712.07150.
[5] P.Sternberg, G.Williams, W.P.Ziemer, Existence, uniqueness, and regularity for functions of least gradient, *J. Reine Angew. Math.*, **430** (1992) 35-60.