CHAPTER XXXIX.

THEOREMS OF STOKES AND GREEN. INTRODUCTION TO HARMONIC ANALYSIS.

1780. It is proposed to give in this chapter several theorems of the Integral Calculus which are of especial service in the higher branches of Physical Analysis.

1781. STOKES' THEOREM.

Let X, Y, Z be the components referred to rectangular axes Ox, Oy, Oz of any vector quantity U. Then the line integral of this vector taken along a given path on any given surface from a fixed point A to another fixed point B is

$$I_{AB} \equiv \int \left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds} \right) ds = \int (X\,dx + Y\,dy + Z\,dz).$$

Let us deform this path into an adjacent arbitrary path from A to B on the surface.

Then $\delta X = \dfrac{\partial X}{\partial x}\,\delta x + +$, $dX = \dfrac{\partial X}{\partial x}\,dx + +$, and

$$\delta I_{AB} = \int_A^B \{(\delta X\,dx + +) + (X\,d\delta x + +\}$$

$$= \int_A^B (\delta X\,dx + +) + [X\,\delta x + +]_A^B - \int_A^B (dX\,\delta x + +)$$

$$= \int_A^B \{(\delta X\,dx - dX\,\delta x) + +\}$$

$$= \int_A^B \left\{ \left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z} \right)(\delta y\,dz - \delta z\,dy) + \left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x} \right)(\delta z\,dx - \delta x\,dz) \right.$$
$$\left. + \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right)(\delta x\,dy - \delta y\,dx) \right\}.$$

But if P, Q be adjacent points (x, y, z), $(x+dx, y+dy, z+dz)$ on the path $APQB$, and P', Q' the points to which they are deformed, having coordinates $(x+\delta x$, etc.), and to the first order $(x+dx+\delta x$, etc.), these four points are to that order the corners of a parallelogram the area of whose projection upon the plane of y-z is $\delta y\, dz - \delta z\, dy$.

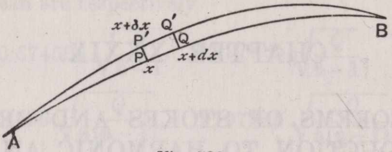

Fig. 588.

Let dS be the area of the element $PQQ'P'$; l, m, n the direction cosines of the normal to the surface at x, y, z. Then to the second order

$$\delta y\, dz - \delta z\, dy = l\, dS, \quad \delta z\, dx - \delta x\, dz = m\, dS, \quad \delta x\, dy - \delta y\, dx = n\, dS.$$

Therefore the variation in the line integral along $APQB$ by deformation into $AP'Q'B$ is

$$\delta I = \iint \left[l\left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right) + m\left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\right) + n\left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right)\right] dS,$$

the integration being for all the elements of S which lie between the two paths.

If we enlarge the strip by taking a new variation of the path $AP'Q'B$ to an adjacent path $AP''Q''B$, the extra increase is the same integral taken over the area between the second and third paths; and this process may be followed by other deformations to any extent so long as X, Y, Z and their differential coefficients remain single-valued, finite and continuous in the deformation (Fig. 589).

If then A and B be any two points upon a contour $ACBD$ drawn upon the surface within which contour X, Y, Z and their differential coefficients are at all points single-valued, finite and continuous, the difference of the line integral along ACB and that along ADB is measured by the surface integral $\iint \left[l\left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right) + + \right] dS$, taken over the whole surface bounded by the contour. Also the line integral from A to B along $ADB = -$ the line integral along BDA (Fig. 590).

Hence the line integral round the whole contour is equal to the surface integral $\int\left[l\left(\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z}\right)++\right]dS$, over the whole area bounded by the contour.

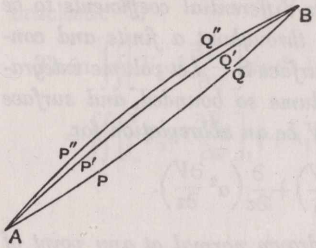

Fig. 589.

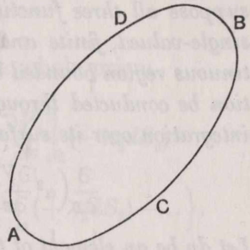

Fig. 590.

Now let R be some vector quantity whose components 2ξ, 2η, 2ζ are such that

$$2\xi=\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z},\quad 2\eta=\frac{\partial X}{\partial z}-\frac{\partial Z}{\partial x},\quad 2\zeta=\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y};$$

then we have

$$\int\left(X\frac{dx}{ds}+Y\frac{dy}{ds}+Z\frac{dz}{ds}\right)ds\left(\begin{matrix}\text{taken round}\\ \text{the contour}\end{matrix}\right)=2\iint(l\xi+m\eta+n\zeta)dS,$$

taken over the bounded surface.

But $2(l\xi+m\eta+n\zeta)$ is the component of the vector R along the normal $=R\cos\epsilon$, say, where ϵ is the angle between the normal to the surface and the direction of R; and if ϵ' be the angle between the vector U and the tangent to the contour

$$X\frac{dx}{ds}+Y\frac{dy}{ds}+Z\frac{dz}{ds}=U\cos\epsilon'.$$

Hence $\iint R\cos\epsilon\, dS=\int U\cos\epsilon'\, ds$, a result due to Stokes and of the highest importance in Higher Physics. [See Lamb, *Hydrodyn.*, Art. 33.]

It is remarkable that the surface integral is independent of the form of the surface, and depends only upon the line integral round the bounding edge, so that it is the same for all diaphragms with a given edge; provided that in the deformation from any one diaphragm to any other no point in space is passed for which X, Y, Z or any of their first order differential coefficients cease to be single-valued, finite and continuous.

1782. GREEN'S THEOREM.* LORD KELVIN'S EXTENSION.

Let V_1 and V_2 be any two functions of x, y, z, the coordinates of a point P, and a any quantity, constant for Green's Theorem, or any function of the variables for Lord Kelvin's extension, and suppose all three functions and their differential coefficients to be single-valued, finite and continuous throughout a finite and continuous region bounded by a given surface S. Let volume integration be conducted throughout the volume so bounded, and surface integration over its surface. Let $\nabla^2 V$ be an abbreviation for

$$\frac{\partial}{\partial x}\left(a^2\frac{\partial V}{\partial x}\right)+\frac{\partial}{\partial y}\left(a^2\frac{\partial V}{\partial y}\right)+\frac{\partial}{\partial z}\left(a^2\frac{\partial V}{\partial z}\right).$$

Let dn be an element of the outward drawn normal at any point of the bounding surface S. The theorem to be established is

$$\iint a^2\left(\frac{\partial V_1}{\partial x}\frac{\partial V_2}{\partial x}+\frac{\partial V_1}{\partial y}\frac{\partial V_2}{\partial y}+\frac{\partial V_1}{\partial z}\frac{\partial V_2}{\partial z}\right)dx\,dy\,dz$$

$$=\iint V_1a^2\frac{\partial V_2}{\partial n}\,dS-\iiint V_1\nabla^2V_2\,dx\,dy\,dz$$

$$=\iint V_2a^2\frac{\partial V_1}{\partial n}\,dS-\iiint V_2\nabla^2V_1\,dx\,dy\,dz.$$

Consider the term $\iiint a^2\frac{\partial V_1}{\partial x}\frac{\partial V_2}{\partial x}\,dx\,dy\,dz$. Integration by parts gives

$$\iint\left[V_2a^2\frac{\partial V_1}{\partial x}\right]dy\,dz-\iiint V_2\frac{\partial}{\partial x}\left(a^2\frac{\partial V_1}{\partial x}\right)dx\,dy\,dz.$$

Construct an elementary rectangular prism parallel to the x-axis on base $dy\,dz$ in the y-z plane, and let it intercept upon the surface S elementary areas dS_1, dS_2, dS_3, ..., at which the direction cosines of the normals are $(\lambda_1, \mu_1, \nu_1)$, $(\lambda_2, \mu_2, \nu_2)$, ..., the suffix 1 relating to the element furthest from the y-z plane and the others being in order of approach to that plane. Then

$$dy\,dz=+\lambda_1dS_1=-\lambda_2dS_2=+\lambda_3dS_3=\dots.$$

Now the limits in the first integral $\left[V_2a^2\frac{\partial V_1}{\partial x}\right]$ are those which correspond to the elements in which the elementary prism cuts the surface S, *i.e.* from the end of any intercepted

* *Math. Papers of the late George Green.* Edited by Dr. Ferrers.

portion of the prism nearest the y-z plane to the end furthest from that plane. Let the values of $V_2 a^2 \frac{\partial V_1}{\partial x}$ at the several points be denoted by the corresponding suffixes to the square brackets.

Then $\iint \left[V_2 a^2 \frac{\partial V_1}{\partial x} \right] dy\, dz$ taken for the whole prism

$$=\iint \left\{ \left[V_2 a^2 \frac{\partial V_1}{\partial x} \right]_1 (+\lambda_1 dS_1) - \left[V_2 a^2 \frac{\partial V_1}{\partial x} \right]_2 (-\lambda_2 dS_2) + \left[V_2 a^2 \frac{\partial V_1}{\partial x} \right]_3 (+\lambda_3 dS_3) - \dots \right\},$$

that is simply, when we integrate for the whole surface, summing the results for all such prisms

$$=\iint V_2 a^2 \frac{\partial V_1}{\partial x} \lambda\, dS.$$

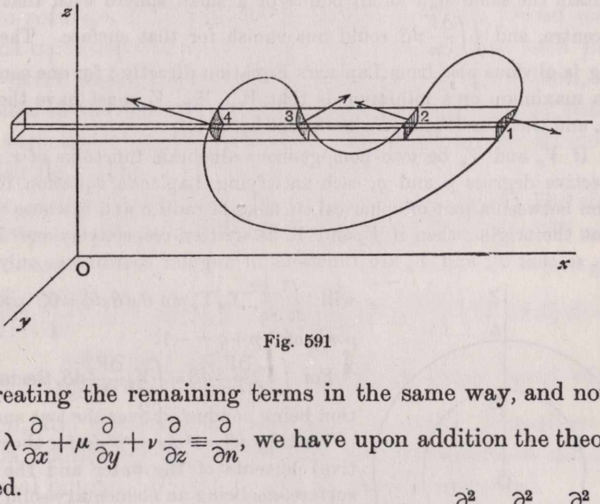

Fig. 591

Treating the remaining terms in the same way, and noting that $\lambda \frac{\partial}{\partial x} + \mu \frac{\partial}{\partial y} + \nu \frac{\partial}{\partial z} \equiv \frac{\partial}{\partial n}$, we have upon addition the theorem stated.

Green's Theorem, for which $a=1$ and $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, is

$$\iiint \left(\frac{\partial V_1}{\partial x} \frac{\partial V_2}{\partial x} + + \right) dx\, dy\, dz = \iint V_1 \frac{\partial V_2}{\partial n} dS - \iiint V_1 \nabla^2 V_2\, dx\, dy\, dz$$

$$= \iint V_2 \frac{\partial V_1}{\partial n} dS - \iiint V_2 \nabla^2 V_1\, dx\, dy\, dz.$$

1783. Various Deductions.

1. It follows that

$$\iint\left(V_1\frac{\partial V_2}{\partial n}-V_2\frac{\partial V_1}{\partial n}\right)dS=\iiint(V_1\nabla^2 V_2-V_2\nabla^2 V_1)\,dx\,dy\,dz.$$

2. If V_1 and V_2 both satisfy Laplace's Equation $\nabla^2 V=0$, we have

$$\iint V_1\frac{\partial V_2}{\partial n}dS=\iint V_2\frac{\partial V_1}{\partial n}dS.$$

3. If $V_2=$constant, $\iint\frac{\partial V_1}{\partial n}dS=\iiint\nabla^2 V_1\,dx\,dy\,dz.$ This is known as the Divergence Theorem (see Webster, *Elect. and Mag.*, p. 66).

4. If $V_2=$constant and V_1 be a function of x, y, z, viz. V, satisfying Laplace's Equation, $\iint\frac{\partial V}{\partial n}dS=0.$ It follows that V does not under such circumstances admit of a true maximum or minimum value for all directions of displacement at any point of space for which it remains finite and continuous and satisfies Laplace's Equation. For if at any point such a maximum or minimum could exist, V would be decreasing or increasing in all directions from that point, and therefore $\frac{\partial V}{\partial n}$ would maintain the same sign at all points of a small sphere with that point for centre, and $\iint\frac{\partial V}{\partial n}dS$ could not vanish for that surface. The same thing is obvious also from Laplace's Equation directly; for one condition for a maximum or a minimum is that V_{xx}, V_{yy}, V_{zz} must have the same sign, and therefore their sum could not be zero.

5. If V_p and V_q be two homogeneous algebraic functions of x, y, z of respective degrees p and q, each satisfying Laplace's equation for the region between a pair of spherical surfaces of radii a and b, whose centres are at the origin; then if V_p and V_q be written respectively as r^pY_p and r^qY_q, so that Y_p and Y_q are functions of angular coordinates only, then will $\int_0^\pi\int_0^{2\pi}Y_pY_q\sin\theta\,d\theta\,d\phi=0$, provided $p\neq q$ and $p+q\neq -1$.

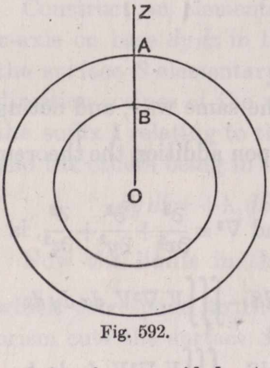

Fig. 592.

For $\int V_p\frac{\partial V_q}{\partial n}dS=\int V_q\frac{\partial V_p}{\partial n}dS$, the integration being conducted over the two surfaces. Writing $dS=a^2d\omega$ or $b^2d\omega$ for the respective elements of the outer and the inner surface, $d\omega$ being an elementary solid angle, we get

$$\int(r^pY_pqr^{q-1}Y_q-r^qY_qpr^{p-1}Y_p)dS=0,$$

and $(q-p)(a^{p+q+1}-b^{p+q+1})\int Y_pY_qd\omega=0,$

and therefore, provided $p\neq q$ and $p+q\neq -1$, $\int_0^\pi\int_0^{2\pi}Y_pY_q\sin\theta\,d\theta\,d\phi=0,$

or writing $\mu \equiv \cos\theta$, $\int_{-1}^{1}\int_{0}^{2\pi} Y_p Y_q \, d\mu \, d\phi = 0$; that is $\int V_p V_q \, dS = 0$, where the integration is taken over the surface of any sphere with centre at the origin.

The theorem is due to Laplace. The proof is Lord Kelvin's [Thomson and Tait, *Nat. Phil.* 1879, p. 180].

Note that in the proof of this general result the taking of an inner surface $r=b$ avoids the continuation of the volume integration over the immediate region of the origin at which such a solution of Laplace's Equation as $V=r^{-1}$ would become infinite, and Green's Theorem on which this result is based would be inapplicable.

6. Many other deductions will be found in works dealing with attractions, electricity and magnetism, etc.

The region bounded by the surface S is regarded as "singly connected," or capable of being made so by suitable diaphragms; so that any of the infinite number of paths from any point A to any second point B within the region are deformable into each other without crossing the boundaries of the surface.*

1784. Unique Character of Solutions of Laplace's Equation.

If a solution of Laplace's Equation has been found which is such as to assume a definite assigned value at each point of a given closed surface S bounding a given region, that solution is unique for all points within the region; and if it is such as to vanish at ∞ it is also unique for all points outside the region.

For, if two functions V_1 and V_2 could each satisfy the stated conditions at points within the surface, their difference W would vanish at all points of the surface. But Green's Theorem gives

$$\iiint\left[\left(\frac{\partial W}{\partial x}\right)^2 + + \right]dx\,dy\,dz = \iint W\frac{\partial W}{\partial n}\,dS - \iiint \nabla^2 W\,dx\,dy\,dz = 0.$$

Hence $\frac{\partial W}{\partial x}$, $\frac{\partial W}{\partial y}$, $\frac{\partial W}{\partial z}$ must vanish at every point of the region, and therefore W must be a constant throughout the region, vanishing at the surface, and therefore at all other internal points. Hence V_1 and V_2 must be identical.

Similarly for points outside the surface with the condition as to vanishing at infinity.

Hence solutions of Laplace's Equation are unique and determinate for any finite region when their values are known over its surface supposed closed.

* For the effect of Cyclosis, see Clerk Maxwell, *E. and M.*, I., page 109.

We note also that if $\dfrac{\partial V}{\partial n}$ were given at each point of the surface, we should equally have $\displaystyle\int W\frac{\partial W}{\partial n}\,dS\equiv 0$, for $\dfrac{\partial W}{\partial n}=0$.

HARMONIC ANALYSIS.

1785. **Def.** Any homogeneous function of x, y, z which satisfies the equation $\nabla^2 V=0$ is called a **Spherical Solid Harmonic.**

Denoting $x^2+y^2+z^2$ by r^2, we have $\nabla^2 r^m=m(m+1)r^{m-2}$ (*D.C.*, p. 137).

This vanishes when $m=0$, or -1, (except where $r=0$). Hence a constant is a spherical solid harmonic of degree zero, and r^{-1} is a spherical solid harmonic of degree -1.

Laplace's equation is unaffected by writing $x-x_0$, $y-y_0$, $z-z_0$ for x, y, z respectively.

Hence $\{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\}^{-\frac{1}{2}}$ is also a solution, except at (x_0, y_0, z_0), where it becomes infinite.

If V_n be any homogeneous function of degree n satisfying $\nabla^2 V=0$, then V_n/r^{2n+1} is also a solution (*D.C.*, p. 137). Its degree is $-n-1$. Therefore to any spherical solid harmonic of degree n corresponds another, viz. V_n/r^{2n+1} of degree $-n-1$.

1786. **Specimens of Spherical Solid Harmonics.**

Lord Kelvin (Thomson and Tait, *Nat. Phil.*, pp. 172-176) gives a long list of particular solutions of $\nabla^2 V=0$. We select a few typical cases, which may readily be verified.

Degree zero, $\quad \log\dfrac{r+z}{r-z},\quad \tan^{-1}\dfrac{y}{x},\quad \dfrac{rx}{x^2+y^2}.$

Degree -1, $\quad \dfrac{1}{r},\quad \dfrac{1}{r}\tan^{-1}\dfrac{y}{x},\quad \dfrac{1}{r}\log\dfrac{r+z}{r-z},\quad \dfrac{x}{x^2+y^2}.$

Degrees 1 and -2,

$$Ax+By+Cz,\quad z\tan^{-1}\frac{y}{x},\quad \frac{x}{r^3},\quad \frac{y}{r^3},\quad \frac{z}{r^3},\quad \frac{z}{r^3}\tan^{-1}\frac{y}{x},\quad \frac{x^2-y^2}{(x^2+y^2)^2}.$$

Degrees 2 and -3, $\quad 2z^2-x^2-y^2,\quad x^2-y^2,\quad Ayz+Bzx+Cxy,\quad yz/r^5.$

1787. If V_n be a spherical solid harmonic of degree n, and we write $V_n=r^nY_n$, as in Art. 1783 (5), Y_n is a function of the direction of the point x, y, z as viewed from the origin, and if we take r as a constant, Y_n is called a "**Spherical Surface Harmonic**" or a "**Laplace's Function.**"

1788. **Number of Arbitrary Constants in the General Spherical Harmonic of degree n.**

The number of coefficients in the general rational integral algebraic expression of degree n in three variables is the number of homogeneous products of degree n in x, y, z, viz.

$$\tfrac{1}{2}(n+2)(n+1).$$

When operated upon by ∇^2 we have a homogeneous function of degree $n-2$ containing $\frac{1}{2}n(n-1)$ coefficients, each of which is to vanish, which furnishes this number of relations amongst the original coefficients. Hence the number of independent arbitrary constants in V_n or Y_n is

$$\tfrac{1}{2}(n+2)(n+1)-\tfrac{1}{2}n(n-1)=2n+1.$$

Such a series as $\frac{1}{r}Y_0+\frac{a}{r^2}Y_1+\frac{a^2}{r^3}Y_2+\ldots+\frac{a^n}{r^{n+1}}Y_n$, where a is given, will therefore contain $1+3+5+\ldots+(2n+1)$, *i.e.* $(n+1)^2$, arbitrary constants, and in the case where $Y_0=0$, as for the potential of a magnetic body, the number is less than this by unity, viz. $n(n+2)$.

1789. **Construction of New Harmonics.**

Since $\nabla^2 V=0$ is a linear differential equation, when any solution V_i has been found, it is obvious that $\frac{\partial^{a+b+c}}{\partial x^a \partial y^b \partial z^c}V_i$ is another solution. So that if V_i be a spherical solid harmonic of degree i, we have another of degree $i-a-b-c$.

Moreover $\left(l\frac{\partial}{\partial x}+m\frac{\partial}{\partial y}+n\frac{\partial}{\partial z}\right)V_i$ will also be a solution; or further still, if $(l_1\ m_1,\ n_1)$, $(l_2,\ m_2,\ n_2)$, ... be any number of sets of direction cosines of arbitrary linear elements dh_1, dh_2, ... so that $\frac{\partial}{\partial h_1}\equiv l_1\frac{\partial}{\partial x}+m_1\frac{\partial}{\partial y}+n_1\frac{\partial}{\partial z}$, etc., then $\frac{\partial}{\partial h_1}\frac{\partial}{\partial h_2}\frac{\partial}{\partial h_3}\cdots\frac{\partial}{\partial h_j}V$ is also a solution of Laplace's Equation, and is a spherical solid harmonic of degree $i-j$.

1790. **Poles and Axes.** Clerk Maxwell (*E.* and *M.*, p. 162)

Consider a spherical surface of centre O and radius r, referred to three rectangular axes Ox, Oy, Oz. Let A_1, A_2, A_3, ... be fixed points on the surface, and P any other point upon the surface. Let the direction cosines of OA_1, OA_2, ... be

(l_1, m_1, n_1), $(l_2, m_2\ n_2)$, ...and x, y, z the coordinates of P. Let $\lambda_i \equiv \cos A_i\hat{O}P$, $\mu_{ij} \equiv \cos A_i\hat{O}A_j$. Let dh_1, dh_2, ... be linear elements in the directions OA_1, OA_2, ... Then the lines OA_1, OA_2, ... are called "axes"; A_1, A_2, ... are called "poles"; and the operation $\dfrac{\partial}{\partial h_i} = l_i\dfrac{\partial}{\partial x} + m_i\dfrac{\partial}{\partial y} + n_i\dfrac{\partial}{\partial z}$ is called differentiation "with regard to the axis OA_i."

Let p_i be a perpendicular from O upon a plane through P perpendicular to OA_i; then $p_i = l_ix + m_iy + n_iz = r\lambda_i$, and we have

$$\frac{\partial r}{\partial h_i} = l_i\frac{\partial r}{\partial x} + m_i\frac{\partial r}{\partial y} + n_i\frac{\partial r}{\partial z} = l_i\frac{x}{r} + m_i\frac{y}{r} + n_i\frac{z}{r} = \frac{p_i}{r} = \lambda_i,$$

$$\frac{\partial p_j}{\partial h_i} = \left(l_i\frac{\partial}{\partial x} + +\right)\left(l\,x + +\right) = l_il_j + m_im_j + n_in_j = \mu_{ij} = \mu_{ji} = \frac{\partial p_i}{\partial h_j},$$

$$\frac{\partial \lambda_j}{\partial h_i} = \frac{\partial}{\partial h_i}\left(\frac{p_j}{r}\right) = \frac{1}{r}\cdot\mu_{ij} - \frac{p_j}{r^2}\lambda_i = \frac{\mu_{ij} - \lambda_i\lambda_j}{r} = \frac{\partial \lambda_i}{\partial h_j}.$$

1791. Consider the effect of the operations

$$-\frac{\partial}{\partial h_1},\quad (-1)^2\frac{\partial}{\partial h_2}\frac{\partial}{\partial h_1},\quad (-1)^3\frac{\partial}{\partial h_3}\frac{\partial}{\partial h_2}\frac{\partial}{\partial h_1},$$

performed successively upon the function $\dfrac{1}{r}$. Let us write $\Sigma\lambda^{i-2s}\mu^s$ for the sum of all possible products consisting of $i-2s$ λ's with different suffixes and s μ's with double suffixes, each suffix 1, 2, 3, ... i occurring once and once only in each product.

Also let us write V_{-i-1} for $(-1)^i\dfrac{\partial}{\partial h_i}\dfrac{\partial}{\partial h_{i-1}}\cdots\dfrac{\partial}{\partial h_1}\cdot\dfrac{1}{r}$, and also $V_{-i-1} = \dfrac{i\,!}{r^{i+1}}Y_i = \dfrac{U_i}{r^{2i+1}}$. Then V_{-i-1}, U_i are spherical solid harmonics of respective degrees $-(i+1)$ and i. We then have

$$\frac{U_0}{r} = V_{-1} = \frac{1}{r},$$

$$\frac{U_1}{r^3} = V_{-2} = -\frac{\partial}{\partial h_1}\frac{1}{r} = \frac{1}{r^2}\frac{\partial r}{\partial h_1} = \frac{\lambda_1}{r^2},$$

$$\frac{U_2}{r^5} = V_{-3} = (-1)^2\frac{\partial}{\partial h_2}\frac{\partial}{\partial h_1}\frac{1}{r} = 2\frac{\lambda_1}{r^3}\lambda_2 - \frac{1}{r^2}\frac{\mu_{12} - \lambda_1\lambda_2}{r} = \frac{1\,.\,3}{r^3}\left(\lambda_1\lambda_2 - \tfrac{1}{3}\mu_{12}\right),$$

$$\frac{U_3}{r^7} = V_{-4} = (-1)^3\frac{\partial}{\partial h_3}\frac{\partial}{\partial h_2}\frac{\partial}{\partial h_1}\frac{1}{r} = \text{etc.} = \frac{1\,.\,3\,.\,5}{r^4}\left(\lambda_1\lambda_2\lambda_3 - \tfrac{1}{5}\Sigma\lambda^1\mu^1\right), \text{etc.}$$

1792. **The General Form is**

$$\frac{U_i}{r^{2i+1}}=V_{-i-1}=\frac{1\,.\,3\,\ldots\,(2i-1)}{r^{i+1}}\left\{\lambda_1\lambda_2\ldots\lambda_i-\frac{1}{2i-1}\Sigma\lambda^{i-2}\mu+\frac{1}{(2i-1)(2i-3)}\Sigma\lambda^{i-4}\mu^2-\ldots\right\}$$

to $\frac{i-1}{2}$ or $\frac{i}{2}$ terms, according as i is odd or even,

$$i.e.\ Y_i=\frac{1\,.\,3\,\ldots\,(2i-1)}{i!}\left\{\lambda_1\lambda_2\ldots\lambda_i-\frac{1}{2i-1}\Sigma\lambda^{i-2}\mu+\frac{1}{(2i-1)(2i-3)}\Sigma\lambda^{i-4}\mu^2-\ldots\right\}.$$

1793. This form may be established by induction (Clerk Maxwell, *E. and M.*, I., p. 161). To do so it is desirable to substitute for each λ the corresponding p/r. For differentiation of r and the p's is simpler than that of the λ's in performing the operation $-\frac{\partial}{\partial h_{i+1}}$.

1794. When all the axes coincide the λ's are all equal, and the μ's are each unity.

If we write $V_{-i-1}\equiv i!\,\frac{Y_i}{r^{i+1}}$ when the axes are different, and $i!\,\frac{Z_i}{r^{i+1}}$ when they are coincident, we have

$$Y_i=\frac{1\,.\,3\,\ldots\,(2i-1)}{i!}\left\{\lambda_1\lambda_2\ldots\lambda_i-\frac{1}{2i-1}\Sigma\lambda^{i-2}\mu+\frac{1}{(2i-1)(2i-3)}\Sigma\lambda^{i-4}\mu^2-\ldots\right\},$$

$$Z_i=\frac{1\,.\,3\,\ldots\,(2i-1)}{i!}\left\{\lambda^i-\frac{i(i-1)}{2(2i-1)}\lambda^{i-2}+\frac{i(i-1)(i-2)(i-3)}{2\,.\,4(2i-1)(2i-3)}\lambda^{i-4}-\ldots\right\}.$$

1795. In the latter case, when the i axes coincide, Z_i is a function of one variable only, viz. the angle which the vector to x, y, z makes with the fixed axis. When this angle is fixed, the value of Z_i is fixed, and the equation $Z_i=$const. gives a family of circles on the surface of the sphere, the planes of these circles being at right angles to the axis of the harmonic. The harmonic is now called a "**zonal harmonic.**"

1796. In the former case Y_i is a function of the i cosines $\lambda_1, \lambda_2, \ldots \lambda_i$ which are variables, and of the $\frac{i(i-1)}{2}$ cosines $\mu_{12}, \mu_{13}, \mu_{23}, \ldots$ which are constants. As there are in this case i arbitrary axes, and each requires three direction cosines l, m, n to fix it, between which there is an identical relation $l^2+m^2+n^2=1$, Y_i will involve $2i$ arbitrary constants. Also since the expression for Y_i may be multiplied by any arbitrary constant M, and the function $V_i \equiv i!MY_i r^i$ still satisfies Laplace's Equation, this value of V_i contains $2i+1$ arbitrary constants inclusive of M, and is the most general form of a spherical harmonic of degree i (see Art. 1788).

1797. The Zonal Surface Harmonic Z_i will contain three arbitrary constants, viz. two which fix the direction of its axis, and M. After the fixation of the axis, say to coincide with the z-axis, the only constant left is M, and if we choose $M=1$, Z_i becomes a definite numerical quantity.

If the axis OA of this zonal harmonic Z_i be in the direction (θ_0, ϕ_0), *i.e.* given by its co-latitude and azimuthal angle, and if OP be drawn in the direction (θ, ϕ), then

$$\lambda=\cos\theta\cos\theta_0+\sin\theta\sin\theta_0\cos(\phi-\phi_0).$$

If the axis be the z-axis, then $\theta_0=0$ and $\lambda=\cos\theta$.

In the former case there are two independent variables θ, ϕ, and the Zonal Spherical Surface Harmonic is known as a **Laplace's Coefficient.**

In the latter case there is but one independent variable, viz. θ, and the pole of the harmonic is the pole of the sphere which is the positive extremity of the z-axis.

1798. LEGENDRE'S COEFFICIENTS.

If we expand the function $(1-2ph+h^2)^{-\frac{1}{2}}$ in powers of h, taken as <1, as

$$(1-2ph+h^2)^{-\frac{1}{2}}=P_0+P_1h+P_2h^2+\dots+P_nh^n+\dots,$$

irrespective of what p may stand for, then P_n or $P_n(p)$ is called **Legendre's Coefficient** of order n.

If (r, θ, ϕ), (r_0, θ_0, ϕ_0) be the coordinates of points P, A and λ the cosine of the angle AOP, O being the origin, the inverse of the distance AP is $(r^2-2rr_0\lambda+r_0^2)^{-\frac{1}{2}}$, and may be written as $\frac{1}{r_0}\left(1-2\lambda\frac{r}{r_0}+\frac{r^2}{r_0^2}\right)^{-\frac{1}{2}}$ or $\frac{1}{r}\left(1-2\lambda\frac{r_0}{r}+\frac{r_0^2}{r^2}\right)^{-\frac{1}{2}}$, according as r_0 is $>$ or $<r$. Accordingly, we have

$$\frac{1}{AP}=\begin{cases}\frac{1}{r_0}\left(Q_0+Q_1\frac{r}{r_0}+Q_2\frac{r^2}{r_0^{\,2}}+\ldots+Q_n\frac{r^n}{r_0^{\,n}}+\ldots\right) \text{ for } r<r_0,\\ \frac{1}{r}\left(Q_0+Q_1\frac{r_0}{r}+Q_2\frac{r_0^{\,2}}{r^2}+\ldots+Q_n\frac{r_0^{\,n}}{r^n}+\ldots\right) \text{ for } r>r_0,\end{cases}$$

where the Q's are Legendre's Coefficients for the case when p is <1 and is a certain cosine. And for all values of r_0/r one or other of these expansions holds good.

Also $\frac{1}{AP}$ being an inverse distance is a Spherical Harmonic, and that series of the two above which is convergent is a spherical harmonic, and satisfies Laplace's Equation; and as it does so for all consistent values of r_0, each term will do so; so that one or other of the sets

$$(Q_0,\, Q_1 r,\, Q_2 r^2, \ldots),\quad \left(\frac{Q_0}{r}, \frac{Q_1}{r^2}, \frac{Q_2}{r^3}, \frac{Q_3}{r^4}, \ldots\right)$$

forms a series of spherical solid harmonics. Moreover, by Art. 1785, if one set be spherical harmonics, so also are the other set. Therefore they are all spherical harmonics; and Q_n is a spherical surface harmonic of the zonal species.

It follows therefore that a Legendre's Coefficient for which p is a cosine is a Zonal Surface Harmonic. We shall see later that it satisfies Laplace's Equation whatever p may be.

1799. The function

$$R^{-1}\equiv\{x^2+y^2+(z-c)^2\}^{-\frac{1}{2}}$$

satisfies Laplace's Equation.

Let $x^2+y^2+z^2=r^2$, and write $(x^2+y^2+z^2)^{-\frac{1}{2}}$ as $f(z)$.

Then

$$R^{-1}=f(z-c)=f(z)-c\frac{\partial f}{\partial z}+\frac{c^2}{2!}\frac{\partial^2 f}{\partial z^2}-\ldots+\frac{(-1)^n}{n!}c^n\frac{\partial^n f}{\partial z^n}+\ldots.$$

Again, writing $z=\lambda r$, $R^{-1}=(r^2-2\lambda cr+c^2)^{-\frac{1}{2}}$ and taking $r>c$,

$$R^{-1}=\frac{1}{r}\left(Q_0+Q_1\frac{c}{r}+Q_2\frac{c^2}{r^2}+\ldots+Q_n\frac{c^n}{r^n}+\ldots\right).$$

Hence $$\frac{Q_n}{r^{n+1}}=\frac{(-1)^n}{n!}\frac{\partial^n f}{\partial z^n}=\frac{(-1)^n}{n!}\frac{\partial^n}{\partial z^n}\frac{1}{r}.$$

The harmonic Q_n is therefore identified with one of those obtained in Arts. 1791 to 1794.

1800. Preliminary Remarks on Legendre's Coefficient $P_n(p)$.

The definition being

$$(1-2ph+h^2)^{-\frac{1}{2}}=P_0+P_1h+P_2h^2+\dots+P_nh^n+\dots \quad (h<1),$$

it follows that, whatever p may be,

$$P_0(p)=1,$$

$$P_n(1)=\text{coef. } h^n \text{ in } (1-h)^{-1}=1,$$

$$P_n(-1)=\text{coef. } h^n \text{ in } (1+h)^{-1}=(-1)^n,$$

$$P_n(0)=\text{coef. } h^n \text{ in } (1+h^2)^{-\frac{1}{2}}=0 \text{ or } (-1)^{\frac{n}{2}}\frac{1.3\dots(n-1)}{2.4\dots n},$$

according as n is odd or even.

If the signs of both p and h be changed, $(1-2ph+h^2)^{-\frac{1}{2}}$ is unaltered. Therefore

$$P_0(p)+P_1(p)h+\dots+P_n(p)h^n+\dots=P_0(-p)-P_1(-p)h+\dots$$
$$+(-1)^nP_n(-p)h^n+\dots.$$

Hence

$$P_0(-p)=P_0(p)\,;\ P_1(-p)=-P_1(p), \text{ etc., } P_n(-p)=(-1)^nP_n(p).$$

1801. Power Series for Legendre's Coefficient $P_n(p)$.

To obtain an expression for P_n as a power series in terms of p, we proceed directly by Expansion of $(1-2ph+h^2)^{-\frac{1}{2}}$, viz.

$$=1+\tfrac{1}{2}h(2p-h)+\dots+\frac{1.3\dots(2n-3)}{2.4\dots(2n-2)}h^{n-1}(2p-h)^{n-1}$$
$$+\frac{1.3\dots(2n-1)}{2.4\dots(2n)}h^n(2p-h)^n+\dots.$$

Picking out the coefficient of h^n, we have

$$P_n=\frac{1.3\dots(2n-1)}{n!}\left\{p^n-\frac{n(n-1)}{2(2n-1)}p^{n-2}\right.$$
$$\left.+\frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}p^{n-4}-\dots\right\}, \quad \dots\text{(A)}$$

which is in agreement with the second series of Art. 1794.

$P_n(p)$ is therefore a rational integral algebraic function of p of degree n. The highest index is n. P_n is an odd or an even function of p, according as n is odd or even; and $P_n(-p)=(-1)^nP_n(p)$, as already seen.

1802. **Rodrigues' Form.**

Applying Lagrange's Theorem [*D.C.*, p. 454],

$$(1-2ph+h^2)^{-\frac{1}{2}}=1+\frac{h}{1!}\frac{1}{2}\frac{d}{dp}(p^2-1)+\frac{h^2}{2!}\frac{1}{2^2}\frac{d^2}{dp^2}(p^2-1)^2+\ldots+\frac{h^n}{n!}\frac{1}{2^n}\frac{d^n}{dp^n}(p^2-1)^n+.$$

Hence

$$P_n(p)=\frac{1}{2^n.\,n!}\frac{d^n}{dp^n}(p^2-1)^n, \text{ a form due to Rodrigues. ....(B)}$$

1803. Rodrigues' form satisfies the differential equation

$$\frac{d}{dp}\left[(1-p^2)\frac{dP_n}{dp}\right]+n(n+1)P_n=0.$$

For writing $z=(p^2-1)^n$, and denoting by suffixes of z differentiations with regard to p, we have $z_1(p^2-1)=2npz$; and differentiating this $n+1$ times by Leibnitz' Theorem,

$$z_{n+2}(p^2-1)+2pz_{n+1}=n(n+1)z_n,$$

$$i.e.\ \frac{d}{dp}[(p^2-1)z_{n+1}]=n(n+1)z_n,$$

$$i.e.\ \frac{d}{dp}\left[(1-p^2)\frac{dP_n}{dp}\right]+n(n+1)P_n=0.$$

1804. **Expansion in Terms of Tangents of Half Angles.**

Using Rodrigues' form and putting $p+1\equiv u$, $p-1\equiv v$,

$$P_n=\frac{1}{2^n n!}\frac{d^n}{dp^n}(u^nv^n)=\frac{1}{2^n}\{u^n+{}^nC_1{}^2u^{n-1}v+{}^nC_2{}^2u^{n-2}v^2+\ldots+v^n\};\ \text{....(C)}$$

and putting $p=\cos\theta$, $u=2\cos^2\frac{\theta}{2}$, $v=-2\sin^2\frac{\theta}{2}$, we have

$$P_n=\cos^{2n}\frac{\theta}{2}\left\{1-{}^nC_1{}^2\tan^2\frac{\theta}{2}+{}^nC_2{}^2\tan^4\frac{\theta}{2}-{}^nC_3{}^2\tan^6\frac{\theta}{2}+\ldots\right\};\ \text{......(D)}$$

1805. **Expansion in a Series of Powers of tan θ.**

Regarding $(p^2-1)^n$ as a function of p^2 and applying the rule of *Diff. Calc.*, Art. 106,

$$P_n=p^n+\frac{1}{2^2}{}^nC_2{}^2C_1p^{n-2}(p^2-1)+\frac{1}{2^4}{}^nC_4{}^4C_2p^{n-4}(p^2-1)^2+\ldots;\ \text{......(E)}$$

and writing $p=\cos\theta$, we have a form homogeneous in $\cos\theta$ and $\sin\theta$,

$$P_n=\cos^n\theta-\frac{n(n-1)}{2^2}\cos^{n-2}\theta\sin^2\theta$$

$$+\frac{n(n-1)(n-2)(n-3)}{2^2.\,4^2}\cos^{n-4}\theta\sin^4\theta-\ldots,\ \text{......(F)}$$

$$i.e.\ P_n=\cos^n\theta\left[1-\frac{n(n-1)}{2^2}\tan^2\theta+\frac{n(n-1)(n-2)(n-3)}{2^2.\,4^2}\tan^4\theta-\ldots\right].\ \text{...(G)}$$

1806. These forms may also be derived by writing

$$(1-2ph+h^2)^{-\frac{1}{2}}=\{(1-ph)^2+h^2(1-p^2)\}^{-\frac{1}{2}},$$

expanding and picking out the coefficient of h^n.

[Todhunter, *F. of Laplace*, p. 12.]

1807. **Expansion in Powers of $\cos\dfrac{\theta}{2}$.**

Since
$$\begin{aligned}(p^2-1)^n&=(\overline{p+1}-2)^n(p+1)^n\\&=(-1)^n[2^n(p+1)^n-{}^nC_1 2^{n-1}(p+1)^{n+1}+{}^nC_2 2^{n-2}(p+1)^{n+2}-\dots,\end{aligned}$$
we have by Rodrigues' form, and putting $p=\cos\theta$,

$$P_n=(-1)^n\left[1-{}^{n+1}C_1\,{}^nC_1\cos^2\frac{\theta}{2}+{}^{n+2}C_2\,{}^nC_2\cos^4\frac{\theta}{2}-{}^{n+3}C_3\,{}^nC_3\cos^6\frac{\theta}{2}+\dots\right].\ \text{(H)}$$

1808. **Expansion in Terms of Cosines of Multiples of θ.**

Taking $2p=t+\dfrac{1}{t}=2\cos\theta$, we have, writing

$$(1-z)^{-\frac{1}{2}}\ \text{as}\ A_0+A_1z+A_2z^2+\dots,$$

$$\begin{aligned}V&=(1-2ph+h^2)^{-\frac{1}{2}}=(1-ht)^{-\frac{1}{2}}(1-ht^{-1})^{-\frac{1}{2}}\\&=(A_0+A_1ht+\dots+A_nh^nt^n+\dots)(A_0+A_1ht^{-1}+\dots+A_nh^nt^{-n}+\dots),\end{aligned}$$

and the coefficient of h^n is obviously

$$\begin{aligned}&A_0A_n(t^n+t^{-n})+A_1A_{n-1}(t^{n-2}+t^{-n+2})+\dots\\&=2\left[A_0A_n\cos n\theta+A_1A_{n-1}\cos(n-2)\theta+\dots+A_{\frac{n-1}{2}}\cdot A_{\frac{n+1}{2}}\cos\theta\ \text{ or }\ \tfrac{1}{2}A_{\frac{n}{2}}^2\right],\end{aligned}$$

as n is odd or even ;

$$\begin{aligned}\therefore\ P_n=2\Big\{&\frac{1.3\dots(2n-1)}{2.4\dots2n}\cos n\theta+\frac{1}{2}\cdot\frac{1.3\dots(2n-3)}{2.4\dots(2n-2)}\cos(n-2)\theta\\&+\frac{1.3}{2.4}\cdot\frac{1.3\dots(2n-5)}{2.4\dots(2n-4)}\cos(n-4)\theta+\dots\Big\}.\ \dots\dots\text{(I)}\end{aligned}$$

1809. **Limiting Values of the P's.**

The binomial coefficients in the above form of P_n are all positive, and therefore P_n cannot exceed in numerical value that for which each of the cosines is replaced by unity. And in this case the expression for $P_n=2(A_0A_n+A_1A_{n-1}+\dots)=$ coef. of ρ^n in $(1-\rho)^{-\frac{1}{2}}(1-\rho)^{-\frac{1}{2}}$, *i.e.* in $(1-\rho)^{-1}$, *i.e.* 1, *i.e.* the value of each of the P's cannot lie outside the limits $+1$ and -1.

The convergency of the series $1+P_1h+P_2h^2+\dots$ follows at once by comparison with $1+h+h^2+\dots=\dfrac{1}{1-h}$; $h<1$.

1810. **Expressions in Terms of Definite Integrals.** [Laplace, *Méc. Cél.*, XI.]

Supposing a positive and $>b$, both being real, we have

$$\int_0^\pi\frac{d\chi}{a+b\cos\chi}=\frac{\pi}{\sqrt{a^2-b^2}};$$

and writing $a=1-hp$, $b=h\sqrt{p^2-1}$, where p is positive and >1, and h negative to ensure a being positive, and both a and b real, we have

$$1-2ph+h^2=a^2-b^2=+^{\text{ve}};$$

$$\therefore \frac{\pi}{\sqrt{1-2ph+h^2}}=\int_0^\pi \frac{d\chi}{1-h(p-\sqrt{p^2-1}\cos\chi)};$$

and expanding each side in powers of h and equating coefficients, $P_n(p)=\frac{1}{\pi}\int_0^\pi (p-\sqrt{p^2-1}\cos\chi)^n d\chi$.

1811. Upon expansion of $(p-\sqrt{p^2-1}\cos\chi)^n$ and integration from 0 to π, all terms arising from odd powers of $\cos\chi$ disappear, and we are left with a rational integral algebraic function of p of degree n, which is identical with $P_n(p)$, (which is known to be a rational integral algebraic function of p of degree n), for all positive values of p greater than unity, *i.e.* for more than n values. Therefore the identity with $P_n(p)$ must hold for all values of p, though it was convenient in the last article to take p positive and >1. It will be seen that the expanded form is identical with the expansion (E) of Art. 1805.

Also, since the terms with odd powers of $\cos\chi$ contribute nothing, we have also

$$P_n(p)=\frac{1}{\pi}\int_0^\pi (p+\sqrt{p^2-1}\cos\chi)^n d\chi.$$

1812. Writing $p=\cosh\alpha$, we have

$$P_n(\cosh\alpha)=\frac{1}{\pi}\int_0^\pi (\cosh\alpha\mp\sinh\alpha\cos\chi)^n d\chi,$$

and we may transform these further by putting

$$\cos\chi=\frac{\cosh\alpha\cos u\pm\sinh\alpha}{\cosh\alpha\pm\cos u\sinh\alpha}$$

to the forms

$$P_n(\cosh\alpha)=\frac{1}{\pi}\int_0^\pi (\cosh\alpha\pm\sinh\alpha\cos u)^{-n-1}du.$$

1813. **Various Forms of Laplace's Equation.**

Before proceeding further it is convenient to collect together for reference the more useful forms which Laplace's Equation $\nabla^2 V=0$ takes when transformed to other systems of coordinates than the Cartesian, and the modifications it undergoes under various circumstances.

By direct transformation to spherical polars (r, θ, ϕ) (*D.C.*, p. 469),

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \text{ becomes}$$

$$\nabla^2 V \equiv \frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial V}{\partial \theta} + \frac{\operatorname{cosec}^2\theta}{r^2}\frac{\partial^2 V}{\partial \phi^2} = 0.$$

If $V_n = r^n Y_n$, Y_n being a function of θ and ϕ only, we have

$$\nabla^2 V_n = r^{n-2}\left[\frac{\partial^2 Y_n}{\partial \theta^2} + \cot\theta \frac{\partial Y_n}{\partial \theta} + \operatorname{cosec}^2\theta \frac{\partial^2 Y_n}{\partial \phi^2} + n(n+1) Y_n\right] = 0,$$

and any solution of this is a **Spherical Surface Harmonic** or **Laplace's Function.** See Art. 1787.

Writing μ for $\cos\theta$, this equation becomes

$$\frac{\partial}{\partial \mu}\left\{(1-\mu^2)\frac{\partial Y_n}{\partial \mu}\right\} + \frac{1}{1-\mu^2}\frac{\partial^2 Y_n}{\partial \phi^2} + n(n+1) Y_n = 0.$$

Laplace's Coefficients, which are Zonal Harmonics and are cases of Laplace's Functions, satisfy this equation. When ϕ is absent, V_n is a homogeneous function of the n^{th} degree symmetrical about the z-axis; Y_n is a function of θ alone, $= P_n$, and the equation becomes, when p is written for μ,

$$\frac{d}{dp}\left\{(1-p^2)\frac{dP_n}{dp}\right\} + n(n+1) P_n = 0.$$

Legendre's Coefficients satisfy this equation, and are the cases of Laplace's Functions for which ϕ is absent, and

$$p = \mu = \cos\theta.$$

Other forms of $\nabla^2 V = 0$ are

$$\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial V}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\sin\theta}\frac{\partial V}{\partial \phi}\right) = 0,$$

$$r\frac{\partial^2}{\partial r^2}(Vr) + \frac{\partial}{\partial \mu}\left\{(1-\mu^2)\frac{\partial V}{\partial \mu}\right\} + \frac{1}{1-\mu^2}\frac{\partial^2 V}{\partial \phi^2} = 0.$$

1814. **Method of Obtaining these Equations from Hydrodynamical Considerations.**

The readiest way to reproduce any particular form of the differential equation is not by direct transformation, but by formation of the appropriate hydrodynamic "Equation of Continuity," expressing the physical fact that in the case of any fluid motion, no creation of matter is going on in any element, any increase or decrease of mass in that element being due to what enters the element from outside or which leaves it.

For a homogeneous fluid in motion with velocity potential V, this condition may be written in the notation of Art. 789 as

$$\Sigma \frac{\partial}{\partial \rho_1}\left(\frac{h_1}{h_2 h_3}\frac{\partial V}{\partial \rho_1}\right)=0,$$

and by expressing this for Cartesians, for Cylindricals, for Spherical-polars, etc., the several forms cited are at once obtained.

1815. Reverting to the power series,

$$(1-2h\cos\gamma+h^2)^{-\frac{1}{2}}=R_0+R_1h+R_2h^2+\ldots+R_nh^n+\ldots \quad (h<1),$$

which defines a case of Legendre's Coefficients in which

$$\cos\gamma=\cos\theta\cos\theta_0+\sin\theta\sin\theta_0\cos(\phi-\phi_0) \text{ (Art. 1797)};$$

it appears that R_n being a zonal harmonic, and a function of θ and ϕ, is a solution of the equation

$$\frac{\partial^2 R_n}{\partial\theta^2}+\cot\theta\frac{\partial R_n}{\partial\theta}+\operatorname{cosec}^2\theta\frac{\partial^2 R_n}{\partial\phi^2}+n(n+1)R_n=0,$$

or, what is the same thing, if we write μ, μ_0 for $\cos\theta$ and $\cos\theta_0$, so that $\cos\gamma=\mu\mu_0+\sqrt{1-\mu^2}\sqrt{1-\mu_0^2}\cos(\phi-\phi_0)$,

$$\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial R_n}{\partial\mu}\right\}+\frac{1}{1-\mu^2}\frac{\partial^2 R_n}{\partial\phi^2}+n(n+1)R_n=0.$$

1816. **The General Solution in the Case when ϕ is absent.**

If the z-axis be taken coincident with the axis of the harmonic, $\mu_0=1$, $\cos\gamma=\mu=\cos\theta=p$, and the Laplacian equation reduces to

$$\frac{d}{dp}\left\{(1-p^2)\frac{dR_n}{dp}\right\}+n(n+1)R_n=0. \quad \ldots\ldots\ldots(1)$$

It will be noted that we usually use p instead of μ in this case.

The zonal harmonic P_n is *a* solution of this equation. To obtain the general solution put $R_n=P_nu$, and we obtain

$$u\left[(1-p^2)\frac{d^2P_n}{dp^2}-2p\frac{dP_n}{dp}+n(n+1)P_n\right]$$
$$+\left[(1-p^2)P_n\frac{d^2u}{dp^2}-2pP_n\frac{du}{dp}+2(1-p^2)\frac{dP_n}{dp}\frac{du}{dp}\right]=0,$$

in which the first bracket disappears. We therefore get

$$\frac{d^2u}{dp^2}\Big/\frac{du}{dp}=\frac{2p}{1-p^2}-\frac{2}{P_n}\frac{dP_n}{dp}, \quad \textit{i.e.} \ \frac{du}{dp}=\frac{B}{P_n^2(1-p^2)},$$

B being a constant.

The general solution of equation (1) is therefore of the form $R_n = AP_n + BQ_n$, where $Q_n = P_n \int \frac{dp}{P_n^2(1-p^2)}$, which is called a Legendre's Function "of the second kind."

If, then, we limit our solutions of equation (1) to such functions of p as give R_n a rational integral algebraic form, we take the arbitrary constant B to be zero, and therefore the most general solution of (1) of this form is $R_n = AP_n$.

1817. Since P_n is a particular form of the Spherical Surface Harmonic for which we have obtained the general result $\int_0^\pi \int_0^{2\pi} Y_m Y_n \, d\mu \, d\phi = 0$ when taken over the surface of the sphere, we have

$$\int_{-1}^1 \int_0^{2\pi} P_m P_n \, dp \, d\phi = 0, \quad \text{and} \therefore \int_{-1}^1 P_m P_n \, dp = 0, \quad (m \neq n).$$

1818. **Particular Cases of P_n expressed in Terms of p, and Positive Integral Powers of p in Terms of P's.**

The general result being

$$P_n = \frac{1 . 3 \ldots (2n-1)}{1 . 2 \ldots n} \left\{ p^n - \frac{n(n-1)}{2(2n-1)} p^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 . 4(2n-1)(2n-3)} p^{n-4} - \ldots \right\}$$

we have the particular cases

$$P_0 = 1; \quad P_1 = p; \quad P_2 = \tfrac{3}{2}p^2 - \tfrac{1}{2}; \quad P_3 = \tfrac{5}{2}p^3 - \tfrac{3}{2}p;$$

$$P_4 = \frac{5 . 7}{2 . 4} p^4 - 2\frac{3 . 5}{2 . 4} p^2 + \frac{1 . 3}{2 . 4}; \quad P_5 = \frac{7 . 9}{2 . 4} p^5 - 2\frac{5 . 7}{2 . 4} p^3 + \frac{3 . 5}{2 . 4} p; \quad \text{etc.}$$

Reversing these results, we have

$$1 = P_0; \quad p = P_1; \quad p^2 = \tfrac{2}{3}P_2 + \tfrac{1}{3}P_0; \quad p^3 = \tfrac{2}{5}P_3 + \tfrac{3}{5}P_1;$$

$$p^4 = \tfrac{8}{35}P_4 + \tfrac{4}{7}P_2 + \tfrac{1}{5}P_0; \quad \text{etc.}$$

1819. The general character of these latter results will be obvious, viz. p^n will consist of a series of Legendre's cofficients beginning with P_n, falling in order two at a time, with certain numerical coefficients; *i.e.* its form is

$$p^n = A_n P_n + A_{n-2} P_{n-2} + A_{n-4} P_{n-4} + \ldots,$$

and we shall consider in due course the law of formation of the successive A's.

We note at once that, since each of the P's becomes unity when $p=1$, we have $A_n+A_{n-2}+A_{n-4}+\ldots=1$.

Again, if $m<n$,

$$\int_{-1}^{1} p^m P_n\,dp=\int_{-1}^{1}(A_m P_m+A_{m-2}P_{m-2}+\ldots)P_n\,dp=0.$$

1820. If $f(p)$ be any rational integral algebraical function of p of lower dimensions than n, then, in the same way,

$$\int_{-1}^{1} f(p)P_n\,dp=0.$$

1821. The same result may be deduced from Rodrigues' form of P_n.

For $\int_{-1}^{1} f(p)P_n\,dp=\frac{1}{2^n n!}\int_{-1}^{1} f(p)\frac{d^n}{dp^n}(p^2-1)^n dp$

$$=\frac{1}{2^n n!}\Big[f(p)\frac{d^{n-1}}{dp^{n-1}}(p^2-1)^n-f'(p)\frac{d^{n-2}}{dp^{n-2}}(p^2-1)^n+\ldots$$
$$+(-1)^{n-1}f^{(n-1)}(p)\,.\,(p^2-1)^n\Big]_{-1}^{1}=0,$$

for after the differentiations are performed (p^2-1) is a factor of the whole.

It follows that $\int f(p)P_n\,dS=0$ when the integration is taken over the surface of the unit sphere.

1822. The theorem $\int_{-1}^{1} p^m P_n\,dp=0$, $(m<n)$, may be used to obtain the several functions P_1, P_2, P_3, ... without using the general formula.

Ex. 1. To find P_3, assume $P_3=Ap^3+Bp$. Then $A+B=1$.

Multiply by p and integrate; then $\frac{2A}{5}+\frac{2B}{3}=\int_{-1}^{1} pP_3\,dp=0$.

Hence $\quad \frac{A}{5}=\frac{B}{-3}=\frac{1}{2}$ and $P_3=\frac{5p^3-3p}{2}$.

Ex. 2. To find P_4. Assume $P_4=Ap^4+Bp^2+C$. Then $A+B+C=1$. Multiply by 1 and by p^2 and integrate.

Then $\quad \frac{A}{5}+\frac{B}{3}+\frac{C}{1}=0$ and $\frac{A}{7}+\frac{B}{5}+\frac{C}{3}=0$;

$$\therefore \frac{A}{35}=\frac{B}{-30}=\frac{C}{3}=\frac{1}{8}; \text{ and } P_4=\frac{35p^4-30p^2+3}{8}.$$

Or we might use a determinant to eliminate A, B, C.

These processes, however, speedily grow laborious by virtue of the number of equations to be solved or the order of the determinants to be evaluated. It is therefore desirable to follow another method, as we now show.

1823. Lemma.

If it be desired to solve a system of equations of form

$$\frac{x}{a+\alpha}+\frac{y}{b+\alpha}+\frac{z}{c+\alpha}+\ldots=0,\quad \frac{x}{a+\beta}+\frac{y}{b+\beta}+\frac{z}{c+\beta}+\ldots=0,$$

$$\frac{x}{a+\gamma}+\frac{y}{b+\gamma}+\frac{z}{c+\gamma}+\ldots=0\ldots,$$

one less in number than the number of unknowns, with

$$\frac{x}{a+\lambda}+\frac{y}{b+\lambda}+\frac{z}{c+\lambda}+\ldots=\frac{1}{\lambda};$$

and further to calculate such an expression as $\frac{x}{a+\theta}+\frac{y}{b+\theta}+\frac{z}{c+\theta}+\ldots$ for the values of $x, y, z, \ldots$ found from the above equations without actually calculating $x, y, z, \ldots$ themselves, we may proceed as follows. For convenience take the case of three letters x, y, z.

Then $\frac{x}{a+\theta}+\frac{y}{b+\theta}+\frac{z}{c+\theta}$ is to vanish when $\theta=\alpha$ or β and to become $\frac{1}{\lambda}$ when $\theta=\lambda$. Such requirements are obviously satisfied by

$$\frac{x}{a+\theta}+\frac{y}{b+\theta}+\frac{z}{c+\theta}=\frac{1}{\lambda}\frac{(a+\lambda)(b+\lambda)(c+\lambda)}{(a+\theta)(b+\theta)(c+\theta)}\cdot\frac{(\theta-\alpha)(\theta-\beta)}{(\lambda-\alpha)(\lambda-\beta)},$$

which is an obvious identity, for it is a *quadratic* relation in θ, and satisfied by *three* values of θ. The value of x can be found by multiplying by $a+\theta$, and putting $\theta=-a$, viz.

$$x=\frac{1}{\lambda}\frac{(a+\lambda)(b+\lambda)(c+\lambda)}{(b-a)(c-a)}\cdot\frac{(a+\alpha)(a+\beta)}{(\lambda-\alpha)(\lambda-\beta)},$$

and similarly for y and z. When λ is indefinitely large, the last of the given equations takes the form $x+y+z=1$, in which case

$$x=\frac{(a+\alpha)(a+\beta)}{(b-a)(c-a)},\quad y=\text{etc.},\quad z=\text{etc.};$$

and generally we have

$$\frac{x}{a+\theta}+\frac{y}{b+\theta}+\frac{z}{c+\theta}+\ldots=\frac{(\theta-\alpha)(\theta-\beta)(\theta-\gamma)\ldots}{(a+\theta)(b+\theta)(c+\theta)(d+\theta)\ldots},$$

there being one more factor in the denominator than in the numerator, no λ occurring.

1824. Ex. 1. Calculate P_5. Assume $P_5=Ap^5+Bp^3+Cp$.

Then $$\frac{A}{9}+\frac{B}{7}+\frac{C}{5}=0,\quad \frac{A}{7}+\frac{B}{5}+\frac{C}{3}=0,\quad A+B+C=1.$$

Take $\alpha=4$, $\beta=2$, $a=5$, $b=3$, $c=1$ in the Lemma.

Then $$A=\frac{(a+\alpha)(a+\beta)}{(b-a)(c-a)}=\frac{9\,.\,7}{2\,.\,4};\quad B=\frac{7\,.\,5}{(-2)\,.\,2};\quad C=\frac{5\,.\,3}{2\,.\,4};$$

and $$P_5=\frac{9\,.\,7}{2\,.\,4}p^5-2\,.\,\frac{7\,.\,5}{2\,.\,4}p^3+\frac{5\,.\,3}{2\,.\,4}p.$$

Ex. 2. Calculate $\int_{-1}^{1} p^7 P_5 \, dp$.

The result is clearly $\frac{2A}{13}+\frac{2B}{11}+\frac{2C}{9}$, but without calculating A, B or C, we have, putting $\theta=8$,

$$2\frac{(8-4)(8-2)}{13.11.9}=\frac{2.4.6}{9.11.13}=\frac{16}{429}.$$

1825. We have seen that $\int_{-1}^{1} p^m P_n \, dp=0$, if $m<n$. But if $m \not< n$, we can readily calculate the value as in the above example.

But first note that if m and n are one of them odd and the other even, the result is still zero. For writing

$$p^m=A_m P_m+A_{m-2}P_{m-2}+\dots,$$

$$\int_{-1}^{1} p^m P_n \, dp=\int_{-1}^{1}(A_m P_m+A_{m-2}P_{m-2}+\dots)P_n \, dp=0,$$

as no two suffixes in any of the products of the P's can be equal.

But if m and n be both even or both odd, and $m \not< n$, the result does not vanish. In this case, writing

$$P_n=Ap^n+Bp^{n-2}+Cp^{n-4}+\dots,$$

multiplying by p^k, where $k=n-2$, $n-4$, $n-6$, etc., and integrating from -1 to 1, we have a set of equations of the type $\frac{A}{k+n+1}+\frac{B}{k+n-1}+\frac{C}{k+n-3}+\dots=0$, one less in number than the coefficients to be found. Also

$$A+B+C+\dots=1,$$

and $\int_{-1}^{1} p^m P_n \, dp=\frac{2A}{m+n+1}+\frac{2B}{m+n-1}+\frac{2C}{m+n-3}+\dots$.

Hence the problem of evaluating this integral ($m \geqslant n$) is that considered above.

Here $\quad \alpha=n-1, \quad \beta=n-3, \quad \gamma=n-5\dots,$

$\quad a=n, \quad b=n-2, \quad c=n-4\dots,$

and $\quad \theta=m+1$;

and

$$\int_{-1}^{1} p^m P_n\,dp = 2\,\frac{(\overline{m+1}-\overline{n-1})(\overline{m+1}-\overline{n-3})\ldots \text{ to } \frac{n-1}{2} \text{ or } \frac{n}{2} \text{ factors}}{(m+1+n)(m+1+n-2)\ldots \text{ to } \frac{n+1}{2} \text{ or } \frac{n+2}{2} \text{ factors}}$$

$$= 2\,\frac{(m-n+2)(m-n+4)\ldots m-1 \text{ (or } m)}{(m+n+1)(m+n-1)\ldots m+2 \text{ (or } m+1)}.$$

1826. If $m=n$, we have $\int_{-1}^{1} p^m P_m\,dp = 2^{m+1}(m\,!)^2/(2m+1)\,!$.

1827. Again

$$\int_{-1}^{1}(P_0+P_1h+P_2h^2+\ldots)^2\,dp = \int_{-1}^{1}\frac{dp}{1-2ph+h^2} = \frac{1}{h}\log\frac{1+h}{1-h},$$

i.e. $$\int_{-1}^{1}(P_0{}^2+P_1{}^2h^2+P_2{}^2h^4+\ldots)\,dp = 2\left(1+\frac{h^2}{3}+\frac{h^4}{5}+\ldots\right),$$

Hence

$$\int_{-1}^{1}P_0{}^2\,dp=2\,;\quad \int_{-1}^{1}P_1{}^2\,dp=\tfrac{2}{3},\ \text{ etc.,}\ \int_{-1}^{1}P_n{}^2\,dp=\frac{2}{2n+1}.$$

Remembering that the area of an elementary belt on the unit sphere may be written as $d\sigma = 2\pi\sin\theta\,d\theta = -2\pi dp$, we have for the whole sphere

$$\int P_n{}^2\,d\sigma = \frac{4\pi}{2n+1}.$$

1828. Professor J. C. Adams has shown that *we may calculate the value of* $I_1 = \int_{-1}^{1}\frac{P_n}{R}dp$, *where* $R=\sqrt{1-2ph+h^2}$, *by means of Rodrigues' expression for* P_n, *and thence we may establish the integrals* $\int_{-1}^{1}P_mP_n\,dp = 0$ *or* $\frac{2}{2n+1}$ *according as* $m \neq n$ *or* $m=n$.

Integrating by parts, we have at once, writing X for $(p^2-1)^n$ for short,

$$2^n n!\,I_1 = \int_{-1}^{1}\frac{1}{R}\frac{d^n}{dp^n}(p^2-1)^n\,dp$$

$$=\left[\frac{1}{R}\left(\frac{d^{n-1}X}{dp^{n-1}}\right)\right]_{-1}^{1} - \left[\frac{1\,.\,h}{R^3}\left(\frac{d^{n-2}X}{dp^{n-2}}\right)\right]_{-1}^{1} + \ldots + (-1)^n 1\,.\,3\,.\,5\ldots(2n-1)h^n\int_{-1}^{1}\frac{X}{R^{2n+1}}dp$$

$$=(-1)^n 1\,.\,3\,.\,5\ldots(2n-1)h^n\int_{-1}^{1}X\frac{dp}{R^{2n+1}} = (-1)^n\,.\,1\,.\,3\,.\,5\ldots(2n+1)h^nU, \text{ say.}$$

Then $\dfrac{dU}{dh} = \displaystyle\int_{-1}^{1}(p^2-1)^n\frac{p-h}{R^{2n+3}}dp.$

Take a sphere of radius unity, OA the radius, $OH=h<1$, H lying upon OA. Draw an elementary double cone with vertex H intercepting superficial elements $d\sigma, d\sigma'$ at P and Q. Let $AHP=\psi$, $AOP=\theta$, $QOA=\theta'$, $HP=R$, $HQ=R'$. Then $d\sigma/R^2=d\sigma'/R'^2$; $p=\cos\theta=h+R\cos\psi$;

$$\sin\theta/R=\sin\psi/1,\quad dp=-\sin\theta\,d\theta,\quad d\sigma=\sin\theta\,d\theta\,d\phi,$$

ϕ being the azimuthal angle of the plane AOP;

$$\therefore\ \sin\theta\,d\theta/R^2=\sin\theta'\,d\theta'/R'^2,\quad i.e.\ dp/R^2=dp'/R'^2;$$

$$\therefore\ \frac{dU}{dh}=\int_{-1}^{1}\frac{(-\sin^2\theta)^n}{R^{2n}}\frac{R\cos\psi}{R^3}dp=(-1)^n\int_{-1}^{1}\sin^{2n}\psi\cos\psi\frac{dp}{R^2},$$

and for opposite elements at P and Q, $\sin^{2n}\psi$ and $\frac{dp}{R^2}$ have the same values, but $\cos\psi$ has an opposite sign; hence corresponding elements of the integrand cancel when the integration is effected for the whole sphere, *i.e.* $\frac{dU}{dh}=0$, and therefore U is independent of h.

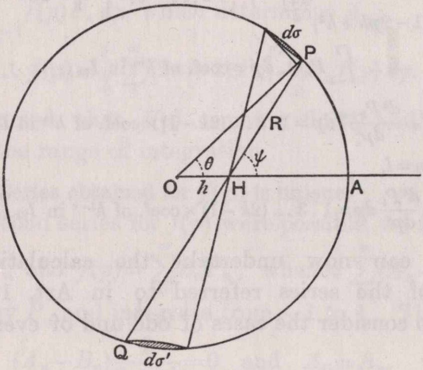

Fig 593.

Hence to evaluate U we may take $h=0$, and therefore $R=1$.

Then $(-1)^n(2n+1)U=\int_{-1}^{1}(1-p^2)^n dp=\int_{\pi}^{0}\sin^{2n}\theta(-\sin\theta\,d\theta)$

$$=2\int_0^{\frac{\pi}{2}}\sin^{2n+1}\theta\,d\theta=2^{n+1}n!/1.3.5\ldots(2n+1);$$

$$\therefore\ I_1=\frac{2}{2n+1}\cdot h^n.$$

It follows that $\int_{-1}^{1}P_n(P_0+P_1h+\ldots+P_nh^n+\ldots)dp=\frac{2h^n}{2n+1}$; whence $\int_{-1}^{1}P_mP_n\,dp=0$, $(m\neq n)$, and $\int_{-1}^{1}P_n{}^2\,dp=\frac{2}{2n+1}$, as seen before.

1829. If $I_m=\int_{-1}^{1}\frac{P_n}{R^m}dp$, where $R^2=1-2ph+h^2$, $\frac{R\,dR}{dh}=h-p$ and $2h(p-h)=1-h^2-R^2$, and we have

$$\frac{dI_m}{dh}=\int_{-1}^{1}\frac{mP_n}{R^{m+1}}\frac{p-h}{R}dp=m\int_{-1}^{1}\frac{P_n}{R^{m+2}}\frac{1-h^2-R^2}{2h}dp=m\frac{1-h^2}{2h}I_{m+2}-\frac{m}{2h}I_m.$$

Thus $I_{m+2}=\frac{2h}{1-h^2}\left(\frac{1}{2h}I_m+\frac{1}{m}\frac{dI_m}{dh}\right)$, a reduction formula for such integrals.

But $I_1=\frac{2h^n}{2n+1}$; $\therefore I_3=\frac{2h^n}{1-h^2}$; $I_5=\frac{2h^n}{3(1-h^2)^3}\{(2n+3)-(2n-1)h^2\}$; etc.

1830. Since $(1-2ph+h^2)^{-\frac{1}{2}}=P_0+P_1h+\ldots+P_kh^k+\ldots+P_{n+k}h^{n+k}+\ldots$, we have

$$1.3\ldots(2k-1)(1-2ph+h^2)^{-\frac{2k+1}{2}}=\frac{d^kP_k}{dp^k}+\frac{d^kP_{k+1}}{dp^k}h+\ldots+\frac{d^kP_{n+k}}{dp^k}h^n+\ldots;$$

and writing $(1-2ph+h^2)^{-\frac{2k+1}{2}}\equiv Q_0+Q_1h+Q_2h^2+\ldots+Q_nh^n+\ldots$, we have

$$Q_n=\frac{1}{1.3\ldots(2k-1)}\frac{d^kP_{k+n}}{dp^k}.$$

Therefore

$$I_{2k+1}=\int_{-1}^{1}\frac{P_n\,dp}{(1-2ph+h^2)^{\frac{2k+1}{2}}}=\int_{-1}^{1}P_n(Q_0+Q_1h+\ldots+Q_mh^m+\ldots)dp;$$

$$\therefore \int_{-1}^{1}P_nQ_m\,dp=\text{coef. of } h^m \text{ in } I_{2k+1},$$

i.e. $$\int_{-1}^{1}P_n.\frac{d^kP_{k+m}}{dp^k}dp=1.3\ldots(2k-1)\times\text{coef. of } h^m \text{ in } I_{2k+1};$$

or writing $k+m=l$,

$$\int_{-1}^{1}P_n\frac{d^kP_l}{dp^k}dp=1.3\ldots(2k-1)\times\text{coef. of } h^{l-k} \text{ in } I_{2k+1}.$$

1831. We can now undertake the calculation of the coefficients of the series referred to in Art. 1819. It is convenient to consider the cases of odd and of even powers of p separately.

(i) Take $p^{2m+1}=A_{2m+1}P_{2m+1}+A_{2m-1}P_{2m-1}+\ldots+A_1P_1$.

Multiply by P_{2m+1}, $P_{2m-1}\ldots$ successively, and integrate from $p=-1$ to $p=1$. We then obtain

$$\frac{2A_{2m+1}}{2(2m+1)+1}=2\frac{2.4\ldots2m}{(4m+3)(4m+1)\ldots(2m+3)};$$

$$\frac{2A_{2m-1}}{2(2m-1)+1}=2\frac{4.6\ldots2m}{(4m+1)(4m-1)\ldots(2m+3)};$$

$$\frac{2A_{2m-3}}{2(2m-3)+1}=2\frac{6.8\ldots2m}{(4m-1)(4m-3)\ldots(2m+3)}; \text{ etc.}$$

Hence writing $2m+1=n$, we have (n odd)

$$p^n=\frac{n!}{1.3\ldots(2n+1)}\Big[(2n+1)P_n+(2n-3)\frac{2n+1}{2}P_{n-2}+(2n-7)\frac{(2n+1)(2n-1)}{2.4}P_{n-4}+\ldots\Big].$$

(ii) Take $p^{2m}=A_{2m}P_{2m}+A_{2m-2}P_{2m-2}+\dots+A_0P_0$; then multiplying by P_{2m}, P_{2m-2}, etc., and proceeding as before, and writing $2m=n$, we obtain the same result.

Particular cases have already been given in Art. 1818.

It will now appear that any rational integral algebraic function of p of degree n may be expressed as a series of Legendrian coefficients, of which the order of the highest is n.

1832. **Expansion of $f(p)$ in Terms of Legendre's Coefficients.**

Supposing the expansion possible, let $f(p)=\sum_0^\infty A_nP_n$. Then multiplying by P_0, P_1, ... and integrating from -1 to 1, $\frac{2}{2n+1}A_n=\int_{-1}^1 f(p)P_n\,dp$, which determines A_n;

$$\therefore f(p)=\frac{1}{2}\sum_0^\infty(2n+1)P_n\int_{-1}^1 P_nf(p)\,dp.$$

It is assumed that $f(p)$ remains finite and continuous throughout the range of integration.

1833. **The Series obtained for $f(p)$ is unique.**

For if a second series for $f(p)$ were possible, we should have $f(p)=\sum^\infty A_nP_n$ and $f(p)=\sum_0^\infty B_nP_n$; whence $\sum_0^\infty(A_n-B_n)P_n=0$. Multiply by P_n and integrate from -1 to 1. Then

$$(A_n-B_n)\frac{2}{2n+1}=0 \quad\text{and}\quad A_n=B_n.$$

1834. **Differential Coefficients of P_n in Terms of Lower Order Legendre's Coefficients.**

P_n being a rational integral algebraic function of p of degree n, $\frac{dP_n}{dp}$ is a similar function of p of degree $n-1$, and therefore expressible in terms of P_{n-1} and lower Legendrian functions, and of form

$$\frac{dP_n}{dp}=A_{n-1}P_{n-1}+A_{n-3}P_{n-3}+A_{n-5}P_{n-5}+\dots.$$

Multiply by P_{n-1}, P_{n-3}, P_{n-5}, ... and integrate from -1 to 1.

Then, since $\int_{-1}^1 P_m\frac{dP_n}{dp}\,dp=[P_mP_n]_{-1}^1-\int_{-1}^1 P_n\frac{dP_m}{dp}\,dp$, and m having any of the values $n-1$, $n-3$, $n-5$, ..., m and n are one of them even and the other odd, we have $P_mP_n=1$ or -1

according as p is $+1$ or -1, and therefore $[P_m P_n]_{-1}^1 = 2$; and further, since $\frac{dP_m}{dp}$ cannot contain a Legendrian function of as high order as P_n the second integral vanishes. Hence in all such cases $\int_{-1}^1 P_m \frac{dP_n}{dp} dp = 2$. Hence

$$2A_{n-1}/(2n-1) = 2A_{n-3}/(2n-5) = 2A_{n-5}/(2n-9) = \ldots = 2,$$

and we have

$$\frac{dP_n}{dp} = (2n-1)P_{n-1} + (2n-5)P_{n-3} + (2n-9)P_{n-5} + \ldots + 3P_1 \text{ (or } P_0)$$

according as n is even or odd.

1835. Similarly we may write

$$\frac{d^2P_n}{dp^2} = B_{n-2}P_{n-2} + B_{n-4}P_{n-4} + B_{n-6}P_{n-6} + \ldots = \Sigma B_r P_r, \text{ say,}$$

and multiplying by P_r for $r = n-2, n-4, n-6, \ldots$, and integrating from $p = -1$ to $p = 1$ and using accents for differentiations,

$$\frac{2}{2r+1} B_r = \int_{-1}^1 P_r \frac{d^2P_n}{dp^2} dp = [P_r P_n' - P_r' P_n]_{-1}^1 + \int_{-1}^1 P_n P_r'' dp,$$

and as $r < n$ the final integral vanishes.

Also, since $(1-p^2)P_n'' - 2pP_n' + n(n+1)P_n = 0$, we have, when $p = \pm 1$,

$$P_n' = \frac{n(r+1)}{2} \frac{P_n}{p}, \text{ and therefore } [P_r P_n' - P_r' P_n]_{-1}^1 = \left\{\frac{n(n+1)}{2} - \frac{r(r+1)}{2}\right\} \left[\frac{P_n P_r}{p}\right]_{-1}^1$$

and n and r being both odd or both even, $\frac{P_n P_r}{p}$ is an odd function of p, and therefore $\left[\frac{P_n P_r}{p}\right]_{-1}^1 = 2$. Therefore $B_r = \frac{2r+1}{2}(n-r)(n+r+1)$ and

$$\frac{d^2P_n}{dp^2} = 1 \,.\, (2n-1)(2n-3)P_{n-2} + 2(2n-3)(2n-7)P_{n-4} + 3(2n-5)(2n-11)P_{n-6} + \ldots,$$

and in the same way higher order differential coefficients may be expressed.

1836. Obviously

$$\int_{-1}^1 \frac{dP_m}{dp} \cdot \frac{dP_n}{dp} dp = \int_{-1}^1 [(2m-1)P_{m-1} + \ldots][(2n-1)P_{n-1} + \ldots] dp \;;$$

and, if $m+n$ be odd, no suffixes can be the same in the two brackets, and the integral vanishes. But if $m+n$ be even, suppose $m \not> n$. Then the terms which do not vanish are

$$(2m-1)^2 \int_{-1}^1 P_{m-1}^2 dp + (2m-5)^2 \int_{-1}^1 P_{m-3}^2 dp + \ldots$$

$$= 2[(2m-1) + (2m-5) + (2m-9) + \ldots + 1 \text{ (or } 3)] \text{ as } m \text{ is odd or even;}$$

and there being $\frac{m+1}{2}$ or $\frac{m}{2}$ terms in the two cases, their sum is in either case $m(m+1)$, *i.e.* $\int_{-1}^1 \frac{dP_m}{dp} \frac{dP_n}{dp} dp = 0$ or $m(m+1)$ as $m+n$ is odd or even, m being the smaller of the two, m and n.

1837. We might also proceed directly thus ($m \leqslant n$),

$$\int_{-1}^{1} \frac{dP_m}{dp}\frac{dP_n}{dp}\,dp = [P_n P_m']_{-1}^{1} - \int_{-1}^{1} P_n P_m''\,dp\,;$$

and since n is greater than the degree of any power of p in P_m'', the terminal integral vanishes.

Again, $(1-p^2)P_m'' - 2pP_m' + m(m+1)P_m = 0$, and therefore if $p = \pm 1$ $P_m' = \dfrac{m(m+1)}{2}\dfrac{P_m}{p}$.

Now $\dfrac{P_m P_n}{p}$ is an even or an odd function of p according as $m+n$ is odd or even, and therefore $\left[\dfrac{P_m P_n}{p}\right]_{-1}^{1} = 0$ or 2 as $m+n$ is odd or even; therefore $\displaystyle\int_{-1}^{1}\frac{dP_m}{dp}\frac{dP_n}{dp}\,dp = 0$ or $m(m+1)$ according as $m+n$ is odd or even and $n \not< m$.

1838. **Differential Equation satisfied by Legendre's Functions.**

Starting again from the definition of Legendre's Coefficients, viz. $V = (1-2ph+h^2)^{-\frac{1}{2}} = \sum_0^\infty P_n h^n$, it is easy to see that they satisfy a form of Laplace's equation, without reference to the fact that when p is a cosine these coefficients are Zonal Harmonics.

For $V^2(1-2ph+h^2) = 1$ and $2\log V + \log(1-2ph+h^2) = 0$, whence

$$\frac{\partial V}{\partial p} = hV^3,\quad \frac{\partial V}{\partial h} = (p-h)V^3,\quad\text{and}\quad p\frac{\partial V}{\partial p} - h\frac{\partial V}{\partial h} = h^2V^3. \quad\ldots(1)$$

Again,

$$\left.\begin{aligned} \frac{\partial}{\partial p}\left\{(1-p^2)\frac{\partial V}{\partial p}\right\} &= -2hpV^3 + 3h^2(1-p^2)V^5,\\ \frac{\partial}{\partial h}\left(h^2\frac{\partial V}{\partial h}\right) &= (2hp-3h^2)V^3 + 3h^2(p-h)^2V^5, \end{aligned}\right\}$$

and adding, $$\frac{\partial}{\partial p}\left\{(1-p^2)\frac{\partial V}{\partial p}\right\} + \frac{\partial}{\partial h}\left(h^2\frac{\partial V}{\partial h}\right) = 0, \quad\ldots\ldots\ldots\ldots(2)$$

by virtue of $V^2(1-2ph+h^2) = 1$.

Substituting $V = \Sigma P_n h^n$, and equating to zero the coefficient of h^n,

$$\frac{d}{dp}\left\{(1-p^2)\frac{dP_n}{dp}\right\} + n(n+1)P_n = 0, \quad\ldots\ldots\ldots\ldots(3)$$

or $$(1-p^2)\frac{d^2P_n}{dp^2} - 2p\frac{dP_n}{dp} + n(n+1)P_n = 0 \ \text{(Art. 1813).} \quad\ldots(4)$$

1839. Differentiating s times, we have

$$(1-p^2)\frac{d^{s+2}P_n}{dp^{s+2}}-2(s+1)p\frac{d^{s+1}P_n}{dp^{s+1}}+\{n(n+1)-s(s+1)\}\frac{d^sP_n}{dp^s}=0, \quad \dots\dots\dots(5)$$

which is known as Ivory's Equation.

If we then take as the expansion of P_n in powers of p,

$$P_n=A_0+A_1\frac{p}{1!}+A_2\frac{p^2}{2!}+A_3\frac{p^3}{3!}+\dots,$$

it follows that

$$A_{s+2}=\{s(s+1)-n(n+1)\}A_s=-(n-s)(n+s+1)A_s, \quad s\neq n.$$

Moreover,

$$\{1-h(2p-h)\}^{-\frac{1}{2}}=\dots+\frac{1.3\dots(2n-1)}{2.4\dots 2n}h^n(2p-h)^n+\dots$$

shows that $A_n=1.3\dots(2n-1)$, also that $A_{n+1}, A_{n+2}, A_{n+3}, \dots$ are all zero, for the coefficient of h^n contains no power of p above p^n; and this coefficient containing the powers $p^n, p^{n-2}, p^{n-4}, \dots$, it is clear that $A_{n-1}, A_{n-3}, A_{n-5}, \dots$ are also all zero.

Also, as $A_s=-A_{s+2}/(n-s)(n+s+1)$, we have

$$A_n=1.3\dots(2n-1), \quad A_{n-2}=-\frac{1.3\dots(2n-1)}{2(2n-1)},$$

$$A_{n-4}=\frac{1.3\dots(2n-1)}{2.4(2n-1)(2n-3)}, \dots,$$

and we have the series of Art. 1801 (A).

1840. It appears that $\frac{d^nP_n}{dp^n}=1.3.5\dots(2n-1)$, and that all higher differential coefficients of P_n vanish.

If n be even, $=2m$, the lowest order term of P_n is an arithmetical constant, viz. what is got by putting $p=0$, *i.e.* the coefficient of h^{2m} in $(1+h^2)^{-\frac{1}{2}}$, viz. $(-1)^m\frac{1.3\dots(2m-1)}{2.4\dots 2m}$.

If n be odd, $=2m+1$, the lowest order term of P_n contains p, viz. $(-1)^m\frac{3.5\dots(2m+1)}{2.4\dots 2m}p$.

1841. Various Theorems.

Since $\dfrac{dP_{n+1}}{dp}=(2n+1)P_n+(2n-3)P_{n-2}+(2n-7)P_{n-4}+\dots,$
we have

$$\frac{dP_{n+1}}{dp}-\frac{dP_{n-1}}{dp}=(2n+1)P_n \quad\text{and}\quad P_{n+1}-P_{n-1}=(2n+1)\int_1^p P_n\,dp$$

and since $\qquad \dfrac{d}{dp}\left\{(1-p^2)\dfrac{dP_n}{dp}\right\}+n(n+1)P_n=0,$

we have $\qquad \displaystyle\int_1^p P_n\,dp=\frac{1}{n(n+1)}(p^2-1)\frac{dP_n}{dp};$

$$\therefore\ P_{n+1}-P_{n-1}=\frac{2n+1}{n(n+1)}(p^2-1)\frac{dP_n}{dp}.$$

1842. Since

$$V\equiv(1-2ph+h^2)^{-\frac{1}{2}}=\Sigma P_n h^n \quad\text{and}\quad \frac{1}{V}\frac{\partial V}{\partial h}=(p-h)V^2,$$

we have $\qquad (1-2ph+h^2)\Sigma(n+1)P_{n+1}h^n=(p-h)\Sigma P_n h^n;$
whence $\quad (n+1)P_{n+1}-2pnP_n+(n-1)P_{n-1}=pP_n-P_{n-1},$
i.e. $\qquad (n+1)P_{n+1}-(2n+1)pP_n+nP_{n-1}=0,$
which forms a difference equation connecting any three successive Legendrian Coefficients.

1843. Again

$$\frac{1}{V^2}\frac{\partial V}{\partial p}=hV,\ \textit{i.e.}\ (1-2hp+h^2)\Sigma h^{n-1}\frac{dP_n}{dp}=\Sigma h^n P_n;$$

$$\therefore\ \frac{dP_{n+1}}{dp}-2p\frac{dP_n}{dp}+\frac{dP_{n-1}}{dp}=P_n;$$

and subtracting the result $\ \dfrac{dP_{n+1}}{dp}-\dfrac{dP_{n-1}}{dp}=(2n+1)P_n,$

we have $\qquad p\dfrac{dP_n}{dp}-\dfrac{dP_{n-1}}{dp}=nP_n.$

1844. Since $\dfrac{\partial V}{\partial p}=hV^3$ and $\dfrac{\partial V}{\partial h}=(p-h)V^3$, we have

$$(p^2-1)\frac{\partial V}{\partial p}-(1-ph)\frac{\partial V}{\partial h}=-V^3p(1-2ph+h^2)=-Vp;$$

$$\therefore\ (p^2-1)\frac{\partial V}{\partial p}=\frac{\partial V}{\partial h}-p\frac{\partial}{\partial h}(Vh),$$

i.e. $\qquad (p^2-1)\Sigma h^n\dfrac{dP_n}{dp}=\Sigma nP_n h^{n-1}-p\Sigma nP_{n-1}h^{n-1}.$

$\therefore$ equating coefficients of h^{n-1}, $(p^2-1)\dfrac{dP_{n-1}}{dp}=nP_n-npP_{n-1}$,

i.e.
$$P_n-pP_{n-1}=\frac{p^2-1}{n}\frac{dP_{n-1}}{dp},$$

or
$$(p^2-1)\frac{dP_n}{dp}=(n+1)(P_{n+1}-pP_n).$$

Hence $\dfrac{p^2-1}{n+1}\dfrac{dP_n}{dp}=P_{n+1}-pP_n=\dfrac{(2n+1)pP_n-nP_{n-1}}{n+1}-pP_n$;

$$\therefore\ (p^2-1)\frac{dP_n}{dp}=n(pP_n-P_{n-1}).$$

We therefore have the two results,

$$\left.\begin{aligned} P_n-pP_{n-1}&=\frac{p^2-1}{n}P'_{n-1},\\ pP_n-P_{n-1}&=\frac{p^2-1}{n}P'_n.\end{aligned}\right\}$$

1845. We now have $P_{n+1}-pP_n=\dfrac{p^2-1}{n+1}P'_n$

$$=n\int_1^p P_n dp\ \left[\text{since } \frac{d}{dp}\left(\overline{1-p^2}\frac{dP_n}{dp}\right)+n(n+1)P_n=0\right]$$

$$=n\left(\int_0^p-\int_0^1\right)P_n dp=n\int_0^p P_n dp+C,$$

where C is a certain constant, viz. the value of P_{n+1} when $p=0$. To find C,

$$P_{n+1}=\frac{1}{2^{n+1}(n+1)!}\frac{d^{n+1}(p^2-1)^{n+1}}{dp^{n+1}}=\frac{1}{2^{n+1}(n+1)!}$$
$$\times\frac{d^{n+1}}{dp^{n+1}}[p^{2n+2}-{}^{n+1}C_1p^{2n}+{}^{n+1}C_2p^{2n-2}-\dots+(-1)^r\,{}^{n+1}C_rp^{2n-2r+2}+\dots].$$

If n be even, each term left after $(n+1)$ differentiations contains p, and therefore in this case C vanishes. If n be odd, there is a term not containing p after the differentiations, viz. when $r=\dfrac{n+1}{2}$. Hence when $p=0$, we have in this case

$$C=\frac{1}{2^{n+1}(n+1)!}(-1)^{\frac{n+1}{2}}\,{}^{n+1}C_{\frac{n+1}{2}}(n+1)!=\frac{(-1)^{\frac{n+1}{2}}}{2^{n+1}}\frac{(n+1)!}{\left(\frac{n+1}{2}!\right)^2}.$$

$\therefore\ P_{n+1}-pP_n=n\displaystyle\int_0^p P_n dp+C$, where $C=0$ or $\dfrac{(-1)^{\frac{n+1}{2}}}{2^{n+1}}\dfrac{(n+1)!}{\left(\frac{n+1}{2}!\right)^2}$ according as n is even or odd.

We also have by differentiation (and writing $n-1$ for n),

$$P'_n - pP'_{n-1} = nP_{n-1}.$$

1846. Since $(n+1)P_{n+1}-(2n+1)pP_n+nP_{n-1}=0$, we have

$$(n+1)P'_{n+1}-(2n+1)pP'_n+nP'_{n-1}=(2n+1)P_n=P'_{n+1}-P'_{n-1},$$
$$nP'_{n+1}-(2n+1)pP'_n+(n+1)P'_{n-1}=0,$$

a difference equation for the first differential coefficients of the P's.

1847. Differentiating again,

$$nP''_{n+1}-(2n+1)pP''_n+(n+1)P''_{n-1}=(2n+1)P'_n=P''_{n+1}-P''_{n-1};$$

whence $\quad (n-1)P''_{n+1}-(2n+1)pP''_n+(n+2)P''_{n-1}=0.$

Similarly $\quad (n-2)P'''_{n+1}-(2n+1)pP'''_n+(n+3)P'''_{n-1}=0,$

and so on, forming a series of difference equations for the higher differential coefficients

1848. Since $pP'_n-P'_{n-1}=nP_n$ ... (1), and $P'_n-pP'_{n-1}=nP_{n-1}$ ... (2), (Arts. 1843 and 1845), we have, by squaring and subtracting,

$$(p^2-1)(P'^2_n-P'^2_{n-1})=n^2(P^2_n-P^2_{n-1}). \quad\text{.....................(3)}$$

Writing $n^2P^2_n-(p^2-1)P'^2_n=U_n$, we have

$$U_n-U_{n-1}=\{n^2-(n-1)^2\}P^2_{n-1}=(2n-1)P^2_{n-1};$$
$$\therefore\ U_{n-1}-U_{n-2}= \qquad\qquad =(2n-3)P^2_{n-2},\ \text{etc.,}$$

and $\quad U_1=P^2_1-(p^2-1)P'^2_1 \quad =1=P^2_0.$

Hence $\quad n^2P^2_n-(p^2-1)P'^2_n=P^2_0+3P^2_1+5P^2_2+\ldots+(2n-1)P^2_{n-1}.$ ....(4)

1849. Again differentiating (1) and (2) r times, and again squaring and subtracting,

$$(p^2-1)\{(P^{(r+1)}_n)^2-(P^{(r+1)}_{n-1})^2\}=(n-r)^2(P^{(r)}_n)^2-(n+r)^2(P^{(r)}_{n-1})^2,$$

or writing $\quad V_n=(n-r)^2(P^{(r)}_n)^2-(p^2-1)(P^{(r+1)}_n)^2,$

$$V_n-V_{n-1}=\{(n+r)^2-(n-1-r)^2\}(P^{(r)}_{n-1})^2=(2n-1)(2r+1)(P^{(r)}_{n-1})^2,$$

and if $\quad n=r,\quad V_r=0;\quad$ if $n=r+1,\quad V_{r+1}=(2r+1)^2(P^{(r)}_r)^2;$

whence $\quad \dfrac{V_n}{2r+1}=(2n-1)(P^{(r)}_{n-1})^2+(2n-3)(P^{(r)}_{n-2})^2+\ldots+(2r+1)(P^{(r)}_r)^2,$

or completing the series with zero terms and reversing the order,

$$V_n/(2r+1)=(P^{(r)}_0)^2+3(P^{(r)}_1)^2+5(P^{(r)}_2)^2+\ldots+(2n-1)(P^{(r)}_{n-1})^2.$$

1850. **Illustrative Example.**

To find a series S which will assume a constant value A at all points on the surface of the unit sphere in the northern hemisphere, and a constant value B at all points of the surface in the southern hemisphere.

Suppose the series to be $S\equiv C_0+C_1P_1+C_2P_2+C_3P_3+\ldots.$

Then $S=A$ from $p=0$ to $p=1$, $S=B$ from $p=-1$ to $p=0$. Therefore multiplying by P_n,

$$\int_{-1}^{1} C_n P_n^2\,dp=\int_{-1}^{0} BP_n\,dp+\int_0^1 AP_n\,dp\,;\ \text{ and }\int_{-1}^{0} P_n\,dp=(-1)^n\int_0^1 P_n\,dp\,;$$

$$\therefore\ \frac{2}{2n+1}C_n=\{A+(-1)^nB\}\int_0^1 P_n\,dp=-\frac{A+(-1)^nB}{n(n+1)}\int_0^1\frac{d}{dp}\left\{(1-p^2)\frac{dP_n}{dp}\right\}dp$$

$$=-\frac{A+(-1)^nB}{n(n+1)}\left[(1-p^2)\frac{dP_n}{dp}\right]_0^1$$

$$=\frac{A+B}{n(n+1)}\left(\frac{dP_n}{dp}\right)_{p=0}=0,\ \text{if } n \text{ be even } (=2i) \text{ and } \neq 0,$$

or

$$=\frac{A-B}{(2i+1)(2i+2)}\cdot\frac{3\,.\,5\ldots(2i+1)}{2\,.\,4\ldots 2i}(-1)^i,\ \text{if } n \text{ be odd } (=2i+1)\,;$$

$$\therefore\ C_{2i}=0,\quad (i>0)\,;\quad C_{2i+1}=(-1)^i\frac{(4i+3)}{2}\,\frac{3\,.\,5\ldots(2i-1)}{2\,.\,4\ldots(2i+2)}\,.\,(A-B).$$

Also, if $n=0$, $C_0=\frac{1}{2}(A+B)\int_0^1 dp=\dfrac{A+B}{2}$;

if $n=1$, $C_1=\frac{3}{2}(A-B)\int_0^1 p\,dp=\frac{3}{4}(A-B)$.

Hence the series required is

$$S=\frac{A+B}{2}P_0+\frac{A-B}{2}\left\{\frac{3P_1}{1\,.\,2}-\frac{3}{2}\,.\,\frac{7P_3}{3\,.\,4}+\frac{3\,.\,5}{2\,.\,4}\,\frac{11P_5}{5\,.\,6}-\ldots\right\}.$$

1851. In case the distribution be symmetrical about some other axis than Oz, the zonal harmonics may be expressed in terms of harmonics with Oz for axis.

1852. For instance, *if we require an expression in terms of Harmonics with Oz for axis, where the value of the function is A over the whole hemisphere with OA for axis and nearer to A, and is B over the hemisphere more remote from A*, then we have just found an expression for such a function in terms of Zonal Harmonics with axis OA, viz. ΣC_nP_n. If P be any point on the spherical surface, and we put $z\hat{O}A=\alpha$, $z\hat{O}P=\theta$, $POA=\theta'$, $AzP=\phi$, we have, from the spherical triangle AzP,

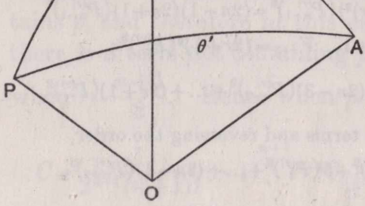

Fig. 594.

$$\cos\theta'=\cos\alpha\cos\theta+\sin\alpha\sin\theta\cos\phi,$$

and $P_n(\cos\theta')$ becomes a spherical Surface Harmonic Q_n expressed in terms of θ, ϕ, and the value of the function sought will be

$$S=\frac{A+B}{2}Q_0+\frac{A-B}{2}\left\{\frac{3Q_1}{1\,.\,2}-\frac{3}{2}\,.\,\frac{7Q_3}{3\,.\,4}+\frac{3\,.\,5}{2\,.\,4}\,\frac{11Q_5}{5\,.\,6}-\text{etc.}\right\}.$$

1853. List of Working Formulae for Legendre's Coefficients.

(Differentiations with regard to p are denoted by accents.)

1. $\dfrac{d}{dp}\{(1-p^2)P_n'\}+n(n+1)P_n=0$; $(1-p^2)P_n''-2pP_n'+n(n+1)P_n=0$;

$\dfrac{d^2P_n}{d\theta^2}+\cot\theta\,\dfrac{dP_n}{d\theta}+n(n+1)P_n=0$; $p=\mu=\cos\theta$.

2. Rodrigues' Formula; $P_n=\dfrac{1}{2^n n!}\dfrac{d^n}{dp^n}(p^2-1)^n$.

3. $$P_n=\frac{1.3.5\ldots(2n-1)}{n!}\left\{p^n-\frac{n(n-1)}{2(2n-1)}p^{n-2}\right.$$
$$\left.+\frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}p^{n-4}+\ldots\right\}.$$

4. $P_0=1,\quad P_1=p,\quad P_2=\frac{3}{2}p^2-\frac{1}{2},\quad P_3=\frac{5}{2}p^3-\frac{3}{2}p,$

$P_4=\dfrac{5.7}{2.4}p^4-2\dfrac{3.5}{2.4}p^2+\dfrac{1.3}{2.4},\quad P_5=\dfrac{7.9}{2.4}p^5-2\dfrac{5.7}{2.4}p^3+\dfrac{3.5}{2.4}p,$ etc.

5. $$p^n=\frac{n!}{1.3.5\ldots(2n+1)}\left\{(2n+1)P_n+(2n-3)\frac{2n+1}{2}P_{n-2}\right.$$
$$\left.+(2n-7)\frac{(2n+1)(2n-1)}{2.4}P_{n-4}+\ldots\right\}.$$

6. $1=P_0,\quad p=P_1,\quad p^2=\frac{1}{3}P_0+\frac{2}{3}P_2,\quad p^3=\frac{3}{5}P_1+\frac{2}{5}P_3,$

$p^4=\frac{1}{5}P_0+\frac{4}{7}P_2+\frac{8}{35}P_4,\quad p^5=\frac{3}{7}P_1+\frac{4}{9}P_3+\frac{8}{63}P_5,$ etc.

7. $P_n=\dfrac{1}{\pi}\displaystyle\int_0^\pi(p\pm\sqrt{p^2-1}\cos\chi)^n\,d\chi=\dfrac{1}{\pi}\int_0^\pi\dfrac{d\chi}{(p\mp\sqrt{p^2-1}\cos\chi)^{n+1}}$.

8. $\displaystyle\int_{-1}^1 P_mP_n\,dp=0$ if $m\neq n$, $\displaystyle\int_{-1}^1 P_n^2dp=\frac{2}{2n+1}$.

9. $P_n'=(2n-1)P_{n-1}+(2n-5)P_{n-3}+(2n-9)P_{n-5}+\ldots$ to P_0 or $3P_1$.

10. $P'_{n+1}-P'_{n-1}=(2n+1)P_n$. 11. $P_{n+1}-P_{n-1}=\dfrac{(2n+1)}{n(n+1)}(p^2-1)P_n'$.

12. $(n+1)P_{n+1}-(2n+1)pP_n+nP_{n-1}=0$.

13. $nP'_{n+1}-(2n+1)pP_n'+(n+1)P'_{n-1}=0$.

14. $pP_n'-P'_{n-1}=nP_n,\quad P_n'-pP'_{n-1}=nP_{n-1}$.

15. $P_n-pP_{n-1}=\dfrac{p^2-1}{n}P'_{n-1},\quad pP_n-P_{n-1}=\dfrac{p^2-1}{n}P_n'$.

16. $P_{n+1}-pP_n=n\displaystyle\int_0^p P_n\,dp+C$. $C=0$, if n be even, and

$$=\frac{(-1)^{\frac{n+1}{2}}}{2^{n+1}}\frac{(n+1)!}{\left\{\left(\frac{n+1}{2}\right)!\right\}^2}\text{ if } n \text{ be odd.}$$

17. $1+3P_1+5P_2+7P_3+\ldots=0$ for all values of p except $p=1$, and then is ∞. See Art. 1857.

1854. **The Roots of $P_n=0$.**

Between any two real roots of a rational algebraic equation $f(x)=0$, at least one real root of $f'(x)=0$ must lie; and if the roots of the equation $f(x)=0$ are all real, the roots of $f'(x)=0$ are all real, and separated by the roots of $f(x)=0$, and lie between the extreme roots of $f(x)=0$. The roots of $f''(x)=0$ are therefore all real and lie between the extreme roots of $f'(x)=0$, and therefore between the extreme roots of $f(x)=0$; and similarly for all the derived functions.

Hence the roots of $P_n=0$, *i.e.* of $\dfrac{d^n}{dp^n}(p^2-1)^n=0$, lie between $+1$ and -1, for the roots of $(p^2-1)^n$ are all real, and either $+1$ or -1.

Also no two roots of $P_n=0$ can be equal. For if they could, $P_n=0$ and $\dfrac{dP_n}{dp}=0$ would have a common root. But

$$(p^2-1)\frac{d^2P_n}{dp^2}+2p\frac{dP_n}{dp}=n(n+1)P_n$$

and

$$(p^2-1)\frac{d^{s+2}P_n}{dp^{s+2}}+2(s+1)p\frac{d^{s+1}P_n}{dp^{s+1}}+\{s(s+1)-n(n+1)\}\frac{d^sP_n}{dp^s}=0$$

for all positive integral values of s. So that if $P_n=0$ and $\dfrac{dP_n}{dp}=0$, we have $\dfrac{d^2P_n}{dp^2}$, $\dfrac{d^3P_n}{dp^3}$, etc., all zero. But this is contrary to the result $\dfrac{d^nP_n}{dp^n}=1.3.5\ldots(2n-1)$ (Art. 1840).

Hence the roots of $P_n=0$ are all different and lie between $+1$ and -1.

It is obvious from the forms of P_n shown in Art. 1818, that when n is odd one of the roots is zero. Also, that in any case as the powers of p are either all odd or all even, all the other roots occur in pairs, one positive and one negative, of each magnitude.

1855. **The Curves $r=aP_0$, $r=aP_1$, $r=aP_2$, etc., are readily traced.**

(1) $r=aP_0=a$ is a circle, centre at the origin and radius a (Fig. 595).

(2) $r=aP_1=a\cos\theta$ is a circle of radius $\dfrac{a}{2}$ touching the y-axis at the origin (Fig. 596).

(3) $r = aP_2 = a\dfrac{3\cos^2\theta - 1}{2}$ has max. rad. vect. $r = a$, $r = \dfrac{a}{2}$, where $\theta = 0$ or π, and $\theta = (2n+1)\dfrac{\pi}{2}$, and touches the lines $\theta = \pm\cos^{-1} 3^{-\frac{1}{2}}$ (Fig. 597).

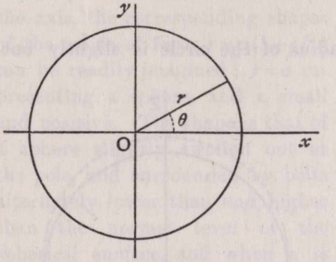

Fig. 595.

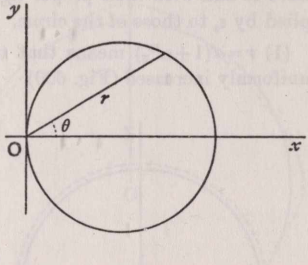

Fig. 596.

(4) $r = aP_3 = a\dfrac{5\cos^3\theta - 3\cos\theta}{2}$ has max. rad. vect. a and $a/\sqrt{5}$, where $\theta = 0$ and $\pm\cos^{-1} 5^{-\frac{1}{2}}$, and touches $\theta = \pm\cos^{-1}\sqrt{3/5}$ and $\theta = \dfrac{\pi}{2}$ (Fig. 598).

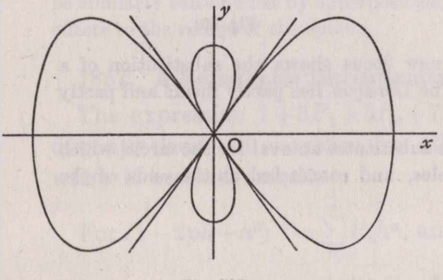

Fig. 597.

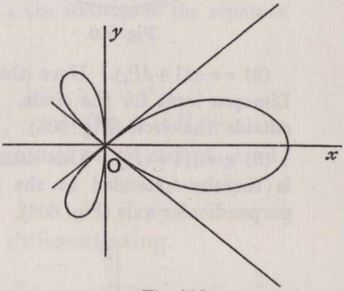

Fig. 598.

(5) $r = aP_4 = a\dfrac{35\cos^4\theta - 30\cos^2\theta + 3}{8}$ has max. rad. vect. a, where $\theta = 0$; $\dfrac{3a}{8}$, where $\theta = \dfrac{\pi}{2}$; $\dfrac{3a}{7}$ if $\theta = \cos^{-1}\sqrt{\dfrac{3}{7}}$, etc., and touches $\theta = \cos^{-1}\left\{\pm\sqrt{\dfrac{15 \pm 2\sqrt{30}}{35}}\right\}$; and so on for those of higher orders (Fig. 599).

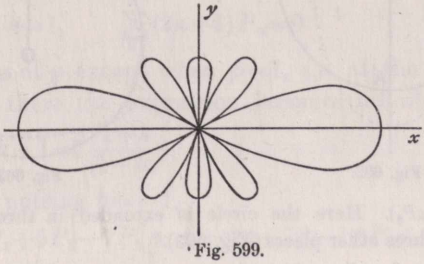

Fig. 599.

1856. We may now note the effect of a small harmonic when superposed upon the graph of a curve otherwise circular by tracing curves of the type $r=a(1+\epsilon P_n)$, where ϵ is a small positive fraction. We merely have to add with their proper signs the radii of the curves traced, multiplied by ϵ, to those of the circle.

(1) $r=a(1+\epsilon P_0)$ means that the radius of the circle is slightly but uniformly increased (Fig. 600).

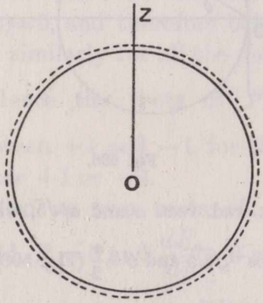

Fig. 600.

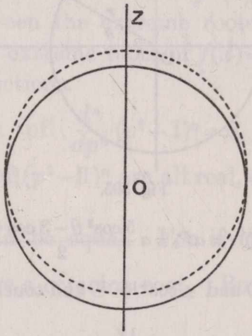

Fig. 601.

(2) $r=a(1+\epsilon P_1)$. Here the new locus shows the substitution of a Limaçon locus for the circle. The Limaçon lies partly inside and partly outside the circle (Fig. 601).

(3) $r=a(1+\epsilon P_2)$. This change substitutes an oval for the circle, which is thereby extended at the poles, and contracted at the ends of the perpendicular axis (Fig. 602).

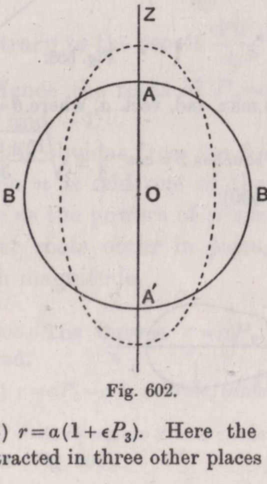

Fig. 602.

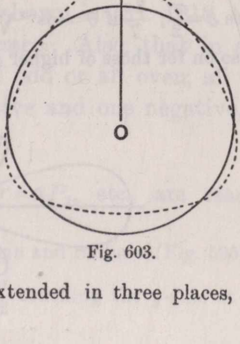

Fig. 603.

(4) $r=a(1+\epsilon P_3)$. Here the circle is extended in three places, and contracted in three other places (Fig. 603).

(5) $r=a(1+\epsilon P_4)$. Here the circle is extended in four places and contracted in four others, and so on (Fig. 604).

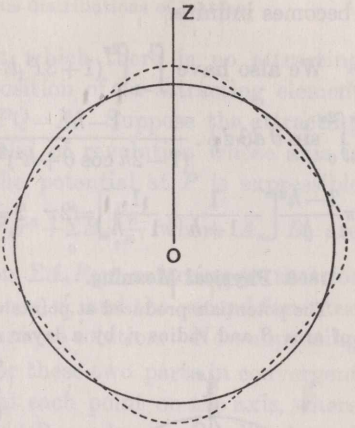

Fig. 604.

If we revolve these curves about the axis, the corresponding shapes of the solids of form $r=a(1+\epsilon P_n)$ can be readily imagined; $r=a$ representing a sphere, and ϵ small and positive. The shape is that of a sphere slightly swollen out at the pole, and surrounded by belts alternately lower than and higher than the normal level of the spherical surface, and when n is even the equatorial plane is a plane of symmetry.

If the radius of the sphere be affected by other harmonics, *e.g.* $r=a(1+\epsilon P_n+\epsilon' P_m)$, the locus can be similarly constructed by superposition, *i.e.* the addition of the separate effects to the radius of the sphere.

1857. A Remarkable Discontinuity.

The expression $1+3P_1+5P_2+7P_3+\ldots+(2n+1)P_n+\ldots$ is discontinuous. It vanishes for all values of p except $p=1$, when it becomes infinite.

For $(1-2ph+h^2)^{-\frac{1}{2}}=\sum_0^\infty P_n h^n$, and differentiating,

$$(p-h)(1-2ph+h^2)^{-\frac{3}{2}}=\sum_1^\infty nP_n h^{n-1}.$$

Multiplying the second by $2h$, and adding to the first,

$$(1-h^2)(1-2ph+h^2)^{-\frac{3}{2}}=\sum_1^\infty (2n+1)P_n h^n,$$

and putting $h=1$, $\qquad \sum_0^\infty (2n+1)P_n=0$

for all values of p except when $p=1$, *i.e.* at the pole of the sphere, and there the expression becomes infinite, being the limit when $h\to 1$ of $\dfrac{1+h}{(1-h)^2}$.

Similarly putting $h=-1$,

$$1-3P_1+5P_2-7P_3+\ldots+(2n+1)(-1)^n P_n+\ldots=0$$

except when $p=-1$, *i.e.* at the opposite pole, and there it becomes infinite.

We also have $\int_{-1}^{1}\int_{0}^{2\pi}(1+3P_1h+5P_2h^2+\dots)\,dp\,d\phi$

$$=\int_0^\pi\int_0^{2\pi}\sin\theta\,d\theta\,d\phi\,.\,\frac{1-h^2}{(1-2h\cos\theta+h^2)^{\frac{3}{2}}}=2\pi\frac{1-h^2}{h}\left[\frac{1}{(1-2h\cos\theta+h^2)^{\frac{1}{2}}}\right]_0^\pi$$

$$=2\pi\frac{1-h^2}{h}\left[-\frac{1}{1+h}+\frac{1}{1-h}\right]=2\pi\,.\,2=4\pi.$$

1858. **Physical Meaning.**

The potentials produced at points within or without a spherical surface of area S and radius r_0 by a layer of matter on the surface of surface density $(2n+1)P_n/S$ are respectively P_nr^n/r_0^{n+1} and $P_nr_0{}^n/r^{n+1}$. For both these expressions satisfy Laplace's Equation; the second vanishes at ∞ and Green's surface condition is satisfied, viz. that the difference of attractions on two points on the same normal, one just outside and one just inside, is to be $4\pi\times$surface density. And such a solution is unique.

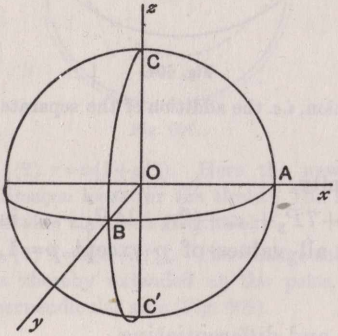

Fig. 605.

Take a particle of mass unity situated at the pole C of the sphere with centre the origin O and radius r_0. The potential produced at any point P distant r from O in colatitude $\cos^{-1}p$ is

$$(r_0{}^2-2pr_0r+r^2)^{-\frac{1}{2}}=\frac{1}{r_0}\Sigma P_n\left(\frac{r}{r_0}\right)^n \quad\text{or}\quad \frac{1}{r}\Sigma P_n\left(\frac{r_0}{r}\right)^n \text{ as } r<\text{ or }>r_0, \quad\dots\text{(I)}$$

and we have seen that an internal potential $P_n\dfrac{r^n}{r_0^{n+1}}$ and an external potential $P_n\dfrac{r_0{}^n}{r^{n+1}}$ are produced by a distribution of surface density which varies as $(2n+1)P_n$.

Hence the potentials (I) are produced by a distribution $\sum\limits_0^\infty(2n+1)P_n$.

But the distribution producing a given potential inside and outside is unique, and we have seen that a concentration into a point at the pole C does produce it. Therefore the distribution $\sum\limits_0^\infty(2n+1)P_n$ must represent a concentration of matter into a single point at the pole C, and must therefore vanish at all points of the sphere except at the pole, where it must become infinite.

This theorem is of great service in obtaining expressions for the potential in the case of discontinuous distributions of matter.

1859. Let P be a point at which there is no attracting matter, O the origin, Q the position of an attracting element of mass m; $OP=r$, $OQ=r'$, $PQ=R$. Suppose the attracting body to be a homogeneous solid of revolution whose axis is taken as the z-axis. Then the potential at P is expressible in the form $V=\Sigma\frac{m}{R}=\sum_0^\infty A_n P_n r^n+\sum_0^\infty B_n\frac{P_n}{r^n}$, where A_n, B_n are constants; the first summation $\Sigma A_n P_n r^n$ referring to that for all those particles for which $r<r'$, and the second for those for which $r>r'$, and this is a unique solution. Now supposing that the potential is known for these two parts in convergent series for each such portion at each point on the axis, where $P_n=1$, then the values of A_n and B_n are known for all values of n. Therefore, assuming that the potential at any point on the axis is expressible as $\Sigma\left(A_n r^n+\frac{B_n}{r^n}\right)$, its value at any point off the axis may be at once written as $\Sigma\left(A_n r^n+\frac{B_n}{r^n}\right)P_n$.

1860. Consider the expression

$$\sum_0^\infty (2n+1)\,P_n(\lambda)\,P_n(\mu),$$

where $P_n(\lambda)$, $P_n(\mu)$ are Zonal Harmonics and λ, μ the cosines of the colatitudes of two points.

Take the case of a circular wire of infinitesimal section. Take as origin the centre of a sphere of radius r_0 of which the wire forms a small circle, and let the z-axis be the normal to the plane of the wire. Let M be the mass of the wire considered of uniform line-density.

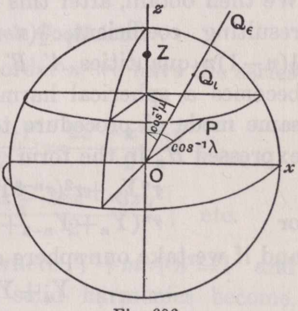

Fig. 606.

The potential of the wire at a point Z, $(0,\ 0,\ z)$ on the z-axis is $M(r_0{}^2-2\lambda r_0 z+z^2)^{-\frac{1}{2}}$, where $\cos^{-1}\lambda$ is the angular radius of the small circle, *i.e.* $\frac{M}{r_0}\sum_0^\infty P_n(\lambda)\left(\frac{z}{r_0}\right)^n$ or $\frac{M}{z}\sum_0^\infty P_n(\lambda)\left(\frac{r_0}{z}\right)^n$ as $z<$ or $>r_0$, and therefore at a point Q in colatitude $\cos^{-1}\mu$ and distant r from O, the potential is $\frac{M}{r_0}\sum_0^\infty P_n(\lambda)\,P_n(\mu)\left(\frac{r}{r_0}\right)^n$ at Q_ι, where $r<r_0$; and $\frac{M}{r}\sum_0^\infty P_n(\lambda)\,P_n(\mu)\left(\frac{r_0}{r}\right)^n$ at Q_ϵ, where $r>r_0$.

Now $(2n+1)P_n(\lambda)$ is the law of distribution of surface density giving a potential $\propto P_n r^n$ within and $\propto P_n/r^{n+1}$ without the sphere. Hence a surface density $\sum_0^\infty (2n+1)P_n(\lambda)P_n(\mu)$ will give the same potentials as it has been seen that the distribution of a uniform line density along a circular wire gives, and is unique. Therefore the expression $\sum_0^\infty (2n+1)P_n(\lambda)P_n(\mu)$ must be zero at all points of the spherical surface except for such points as lie along the small circle of angular radius $\cos^{-1}\lambda$, where the surface density is infinite but the line density finite. That is, the expression is zero except where $\lambda=\mu$, where it is infinite.

The theorem is similar to one occurring in Poisson's discussion of Fourier's Theorem, Chapter XXXV.

1861. **Practical Method of Expression of a Rational Integral Algebraic Function of x, y, z in Terms of Harmonics on Unit Sphere.**

Let $H_n \equiv Ax^n + x^{n-1}(By+Cz) + x^{n-2}(Dy^2+Eyz+Fz^2)+\dots$ be the general homogeneous expression of degree n, which contains $\frac{1}{2}(n+1)(n+2)$ coefficients. Subtract and add $(x^2+y^2+z^2)H_{n-2}$, where $H_{n-2}\equiv A'x^{n-2}+x^{n-3}(B'y+C'z)+\dots$, which contains $\frac{1}{2}(n-1)n$ coefficients $A', B', C', \dots$ to be found.

Apply the operator ∇^2 to $H_n-(x^2+y^2+z^2)H_{n-2}$, viz.

$$(A-A')x^n+\dots.$$

We then obtain, after this operation, by equating to zero each resulting coefficient, $\frac{1}{2}(n-1)n$ equations to determine the $\frac{1}{2}(n-1)n$ quantities A', B', C', etc., and $H_n-(x^2+y^2+z^2)H_{n-2}$ becomes a spherical harmonic of degree n. Next apply the same mode of procedure to H_{n-2}, and so on. We have then expressed H_n in the form

$$r^nY_n + r^2(r^{n-2}Y_{n-2}) + r^4(r^{n-4}Y_{n-4})+\dots$$

or

$$r^n(Y_n+Y_{n-2}+Y_{n-4}+\dots);$$

and if we take our sphere as $r=1$, we have

$$Y_n+Y_{n-2}+Y_{n-4}+\dots,$$

a series of surface harmonics.

If the rational integral algebraic function considered consist of groups of terms of different degrees, the same rule will apply to the terms of each group.

As a preliminary to such procedure, all terms which are obviously already solid harmonics should be laid aside, to be restored when the process is completed, amongst the other harmonics of their own degrees.

1862. Ex. Express

$$\phi = a_1x + a_2y + a_3z + b_1x^2 + b_2y^2 + b_3z^2 + b_4yz + b_5zx + b_6xy + cxyz$$

as a series in the form $r^3Y_3 + r^2Y_2 + rY_1 + Y_0$.

We only need consider the terms $b_1x^2 + b_2y^2 + b_3z^2$,

i.e. $$(b_1-\lambda)x^2 + (b_2-\lambda)y^2 + (b_3-\lambda)z^2 + \lambda(x^2+y^2+z^2),$$

and $$\nabla^2[(b_1-\lambda)x^2 + (b_2-\lambda)y^2 + (b_3-\lambda)z^2] = 2(b_1+b_2+b_3-3\lambda) = 0$$

if $\lambda = \frac{1}{3}(b_1+b_2+b_3)$;

$$\therefore\ \phi = cxyz + \left[\frac{2b_1-b_2-b_3}{3}x^2 + \frac{2b_2-b_3-b_1}{3}y^2 + \frac{2b_3-b_1-b_2}{3}z^2 + b_4yz + b_5zx + b_6xy\right] + [a_1x + a_2y + a_3z] + \frac{b_1+b_2+b_3}{3}r^2,$$

which on the surface $r=1$ is of form $Y_3 + Y_2 + Y_1 + Y_0$.

1863. If the function be not already expressed in Cartesians, it is usually best to express it so first.

Ex. Express $\sin^4\theta\sin^2 2\phi$ in terms of Surface Harmonics.

$$\sin^4\theta\sin^2 2\phi = 4(\sin\theta\cos\phi)^2(\sin\theta\sin\phi)^2 = 4x^2y^2 \quad (r=1),$$

and proceeding as before,

$$= 4\{x^2y^2 - r^2(\tfrac{4}{35}x^2 + \tfrac{4}{35}y^2 - \tfrac{1}{35}z^2)\} + \tfrac{4}{35}r^2(\tfrac{5}{3}x^2 + \tfrac{5}{3}y^2 - \tfrac{10}{3}z^2) + \tfrac{4}{35}\cdot\tfrac{7}{3}r^4;$$

and putting $x=\sin\theta\cos\phi$, $y=\sin\theta\sin\phi$, $z=\cos\theta$, and $r=1$, we have a result of the required form $Y_4 + Y_2 + Y_0$.

1864. **Change of Axis of a Legendre's Coefficient.**

If P_n be Legendre's coefficient of order n, we have the series of solid harmonics

$$P_1r = z;\quad P_2r^2 = \frac{3p^2-1}{2}r^2 = \frac{3z^2-r^2}{2} = \frac{2z^2-x^2-y^2}{2};$$

$$P_3r^3 = \frac{5p^3-3p}{2}r^3 = \frac{5z^3-3zr^2}{2} = \frac{2z^3-3zx^2-3zy^2}{2};\quad \text{etc.}$$

Writing $lX+mY+nZ$ for z, where $l^2+m^2+n^2=1$ and $x^2+y^2+z^2=X^2+Y^2+Z^2=R^2$, these solid harmonics become, when referred to new axes OX, OY, OZ, $lX+mY+nZ$;

$$\frac{3(lX+mY+nZ)^2-(X^2+Y^2+Z^2)}{2};\quad \frac{5(lX++)^3-3R^2(lX++)}{2};\ \text{etc.},$$

and the axis of this set of harmonics is $\frac{X}{l}=\frac{Y}{m}=\frac{Z}{n}$, viz. OA (Fig. 607).

If we transform to polars so that this line is given by $l=\sin\theta'\cos\phi'$, $m=\sin\theta'\sin\phi'$, $n=\cos\theta'$, and $X=R\sin\theta\cos\phi$,

$Y=R\sin\theta\sin\phi$, $Z=R\cos\theta$, the axis OA of the new set of harmonics is inclined to the new Z-axis at an angle θ' and the azimuthal angle is ϕ', and the expression

$$\frac{lX+mY+nZ}{R} \text{ is } \cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi'),$$

and is still a *cosine*, viz. the cosine of the angle between the original axis OA and the direction OP of the point X, Y, Z.

If then we take $r\equiv R=1$, and if, instead of p, we write

$$\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi'),$$

we get a more general form of Harmonic than the Legendre's Coefficients. There are now two independent variables θ and ϕ, θ' and ϕ' being regarded as known.

The Harmonics in their new form are known as **Laplace's Coefficients** and denoted by Y_1, Y_2, Y_3 .... Thus for Legendre's Coefficients the z-axis OA is taken as the axis of the system, and $AOP=\theta$. In Laplace's Coefficients the axis of the system is the line θ', ϕ', and the direction of P is θ, ϕ.

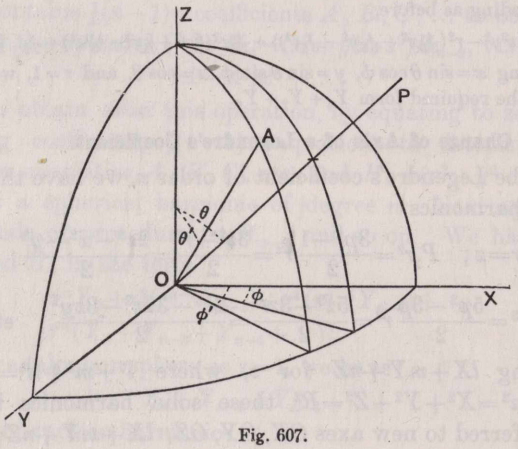

Fig. 607.

The curves for which $A\hat{O}P$ is constant are a set of parallels about the axis of the coefficient in either case, viz. $\cos\theta=$const. for a Legendre's Coefficient, and

$\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\overline{\phi-\phi'}=$const. for a Laplace's Coeff. Both sets are Zonal Surface Harmonics. When multiplied by r^n, *i.e.* OP^n, they are Zonal Solid Harmonics. If we further transform coordinates so that Z becomes the distance from any other fixed plane through O, the Solid Zonal Harmonic remains a Solid Zonal Harmonic and the Surface Zonal Harmonic remains a Surface Zonal Harmonic.

1865. **Tesseral and Sectorial Harmonics.**

Take the case of an unreal plane $Z \equiv z+\alpha(x+\iota y)$, $l=\alpha$, $m=\alpha\iota$, $n=1$, so that $l^2+m^2+n^2=1$.

Then, if $F(z)$ is a Solid Spherical Harmonic, so also is $F\{z+\alpha(x+\iota y)\}$, *i.e.*

$$F(z)+\frac{\alpha}{1!}(x+\iota y)F'(z)+\frac{\alpha^2}{2!}(x+\iota y)^2F''(z)+\ldots+\frac{\alpha^s}{s!}(x+\iota y)^sF^{(s)}(z)+\ldots$$

also satisfies Laplace's Equation $\nabla^2 V=0$ for all values of α, and the equation being linear each term of this expansion will also do so, and will itself be a Solid Spherical Harmonic; and taking either sign for ι, we have new forms of Solid Spherical Harmonics $(x\pm\iota y)^sF^{(s)}(z)$. Also their sum and difference are also Solid Spherical Harmonics. Therefore transforming to polars with $r=1$, $x=\sin\theta\cos\phi$, $y=\sin\theta\sin\phi$, $z=\cos\theta$, $\sin^s\theta\cos s\phi\, F^{(s)}(\cos\theta)$ and $\sin^s\theta\sin s\phi\, F^{(s)}(\cos\theta)$, or, what is the same thing, $(1-p^2)^{\frac{s}{2}}\cos s\phi\,\frac{d^sP_n}{d\theta^s}$ and $(1-p^2)^{\frac{s}{2}}\sin s\phi\,\frac{d^sP_n}{d\theta^s}$ are new forms of Spherical Surface Harmonic functions of θ, ϕ.

1866. These new Harmonics are called Tesseral Harmonics of degree n and order s. When $s=n$,

$$\frac{d^sP_n}{dp^s}=\frac{d^nP_n}{dp^n}=1\,.\,3\,.\,5\,\ldots\,(2n-1), \text{ a constant.}$$

Rejecting the constant, $(1-p^2)^{\frac{n}{2}}\cos n\phi$ and $(1-p^2)^{\frac{n}{2}}\sin n\phi$ are called Sectorial Harmonics of degree n.

It has been seen that in the case of a Zonal Harmonic its vanishing gives an equation of degree n in p with all its roots real, and the spherical surface is mapped out into a series of belts or zones by circular sections at right angles to the axis of the Harmonic, the angular radii of which sections are determined by the roots of this equation.

In a Sectorial Harmonic the roots $p^2=1$ give the poles in which the axis of the Harmonics cuts the sphere. But in addition we have, by the vanishing of such an Harmonic,

$\cos n\phi=0$ or $\sin n\phi=0$, as the case may be, which indicate roots $n\phi=2\lambda\pi+\frac{\pi}{2}$ or $\lambda\pi$; *i.e.* a set of great circle sections through the axis of the system of Harmonics, which therefore map out the surface of the sphere by meridians.

In the case of a Tesseral Harmonic the vanishing of $(1-p^2)^{\frac{s}{2}}\begin{matrix}\cos\\ \sin\end{matrix} s\phi\frac{d^sP_n}{dp_s}$ would give in addition to (i) the poles, (ii) the meridians (in number s), the solutions of $\frac{d^sP_n}{dp^s}=0$. This is an equation of degree $n-s$ in p determining $n-s$ small circles whose planes are at right angles to the axis of the system.

The surface is now mapped out by these meridians and small circles into a set of tile-shaped elements or tesserae. Thus to any Zonal Harmonic correspond new Harmonics, Tesseral and Sectorial, which are all species of **Laplace's Functions.**

1867. *The most general homogeneous function which is rational with respect to $x=\sin\theta\cos\phi$, $y=\sin\theta\sin\phi$, $z=\cos\theta$, and of the n^{th} degree, for which r is put $=1$, and which satisfies the equation*

$$\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial Q}{\partial\mu}\right\}+\frac{1}{1-\mu^2}\frac{\partial^2Q}{\partial\phi^2}+n(n+1)Q=0,$$

is $$Q=a_0P_n+\sum_1^n(a_k\cos k\phi+b_k\sin k\phi)\sin^k\theta\frac{\partial^kP_n}{\partial\mu^k},$$

where P_n is the Legendrian coefficient of the n^{th} order.

For considering the expression $A_k\cos k\phi+B_k\sin k\phi$, $A_k\cos k\phi$ could not be a rational integral algebraic function of $\sin\theta\sin\phi$, $\sin\theta\cos\phi$, $\cos\theta$ unless A_k itself contains a factor $\sin^k\theta$.

Put $Q\equiv\cos k\phi\sin^k\theta\,.\,v\equiv\cos k\phi\,.\,u$, say. Then the differential equation becomes $(1-\mu^2)\frac{d^2u}{d\mu^2}-2\mu\frac{du}{d\mu}+\left\{n(n+1)-\frac{k^2}{1-\mu^2}\right\}u=0$;

and writing $u=(1-\mu^2)^{\frac{k}{2}}v$, we have

$$(1-\mu^2)\frac{\partial^2v}{\partial\mu^2}-2\mu(k+1)\frac{\partial v}{\partial\mu}+\{n(n+1)-k(k+1)\}v=0,$$

which is Ivory's Equation of Art. 1839, where

$$v=\frac{\partial^k}{\partial\mu^k}\left\{AP_n+BP_n\int\frac{d\mu}{P_n^{\,2}(1-\mu^2)}\right\} \text{ (Art. 1816).}$$

But as we require the *integral* function of μ which will satisfy the general equation, we take $B=0$. Hence

$$Q=A\cos k\phi\sin^k\theta\,\frac{\partial^k P_n}{\partial\mu^n}$$

satisfies the equation. And in the same way, starting with $Q=\sin k\phi\sin^k\theta\,.\,v$, we should have arrived at a solution $Q=B\sin k\phi\sin^k\theta\,\dfrac{\partial^k P_n}{\partial\mu^n}$; and these solutions hold for all positive integral values of k. Hence the most general solution of the kind required, viz. homogeneous (with $r=1$) and a rational integral algebraic function of $\sin\theta\cos\phi$, $\sin\theta\sin\phi$, $\cos\theta$, is that stated above, viz.

$$Q=a_0P_n+\sum_1^n(a_k\cos k\phi+b_k\sin k\phi)\sin^k\theta\,\frac{\partial^k P_n}{\partial\mu^k},$$

where $\mu=\cos\theta$, and contains $2n+1$ arbitrary constants. It is clearly useless to continue the summation for values of $k>n$, for the last factor would vanish for such terms.

It thus appears directly from this form of the Laplacian Equation how the Tesseral and Sectorial Harmonics arise.

1868. **To expand any Function of μ and ϕ, say $F(\mu,\phi)$, in a Series of Laplace's Functions.**

We have seen when p is any quantity between ± 1, that with the definition $(1-2ph+h^2)^{-\frac{1}{2}}\equiv 1+P_1h+P_2h^2+\dots$, we have $1+3P_1+5P_2+\dots+(2n+1)P_n+=0$ except where $p=1$, when the sum becomes ∞. Let p stand for the cosine of the angle between the direction μ, ϕ and a fixed direction μ', ϕ', so that $p=\mu\mu'+\sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\phi-\phi')$, and consider the integral $\iint(1+3P_1+5P_2+\dots)F(\mu,\phi)\,d\mu\,d\phi$.

If we integrate over any closed region S on the sphere, which is not cut by the direction μ', ϕ', this result is evidently zero. If the integration extends over the whole surface of the sphere, the direction μ', ϕ' must be included; but no part of the integration contributes anything to the result except that included in a very small contour about the direction μ', ϕ', and in this direction $F(\mu, \phi)$ becomes $F(\mu', \phi')$. Hence the value of this double integral is $F(\mu', \phi')\iint(1+3P_1+5P_2+\ldots)\,d\mu\,d\phi$, taken over the infinitesimally small area within the small

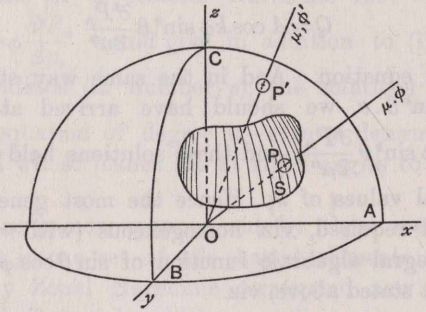

Fig. 608.

contour just enclosing μ', ϕ'. But as $1+3P_1+5P_2+\ldots$ vanishes at all other points of the sphere, this is equal to

$$F(\mu', \phi')\iint(1+3P_1+5P_2+\ldots)\,d\mu\,d\phi,$$

taken over the whole sphere, $=4\pi F(\mu', \phi')$, by Art. 1857;

$$\therefore\ F(\mu', \phi')=\frac{1}{4\pi}\sum_0^\infty(2n+1)\iint F(\mu, \phi)P_n\,d\mu\,d\phi.$$

When the integrations are effected each term is a function of μ', ϕ', which enter through the P functions alone, and each term will satisfy Laplace's Equation and be a Laplace's Function.

This proof is due to O'Brien.

When $F(\mu, \phi)$ is itself a Laplace's Function, say Y_n, we have

$$4\pi Y_n'=\sum_0^\infty(2r+1)\iint Y_nP_r\,d\mu\,d\phi,$$

where Y_n' represents the value of Y_n along the axis of the functions, *i.e.* when $\mu=\mu'$ and $\phi=\phi'$; and every term vanishes except that for which $r=n$, whence

$$\int_{-1}^1\int_0^{2\pi}Y_nP_n\,d\mu\,d\phi=\frac{4\pi Y_n'}{2n+1}.$$

1869. **The Value of the above Integral may be readily deduced by Physical Considerations.**

Take a layer of matter of surface density $\sigma = Y_n$ on the surface of the sphere (radius a). The potential at any internal point C at distance r from the centre and R from the element dS,

$$V = \int \frac{\sigma\, dS}{R} = \int \frac{\sigma\, dS}{(a^2 - 2ar\cos\theta + r^2)^{\frac{1}{2}}} = \int Y_n \frac{1}{a}\left(P_0 + P_1 \frac{r}{a} + P_2 \frac{r^2}{a^2} + \dots\right) dS,$$

i.e. $$V_i = \int Y_n P_n \frac{r^n}{a^{n+1}}\, dS.$$

Similarly, at an external point,

$$V_e = \int Y_n \frac{1}{r}\left(P_0 + P_1 \frac{a}{r} + P_2 \frac{a^2}{r^2} + \dots\right) dS,$$

i.e. $$V_e = \int Y_n P_n \frac{a^n}{r^{n+1}}\, dS.$$

But, by Green's Theorem,

$$\left(-\frac{\partial V_e}{\partial r}\right)_{r=a} - \left(-\frac{\partial V_i}{\partial r}\right)_{r=a} = 4\pi\sigma_A$$

at any point A of the surface.

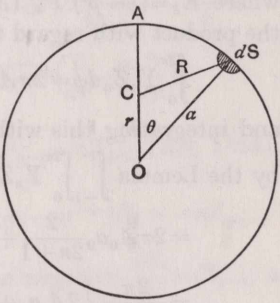

Fig. 609.

$\therefore\ \dfrac{2n+1}{a^2}\displaystyle\int Y_n P_n\, dS = 4\pi Y_n'$, and $dS = a^2 d\omega$, where $d\omega$ is the elementary solid angle subtended by dS at the centre.

Hence $$\int Y_n P_n\, d\omega = \frac{4\pi Y_n'}{2n+1}.$$

1870. **Lemma.**

If $u \equiv p+1$, $v \equiv p-1$ and $D \equiv \dfrac{d}{dp}$, we may show, by applying Leibnitz' Theorem and comparing the r^{th} non-vanishing terms on each side, that

$$u^s v^s D^{n+s} u^n v^n/(n+s)! = D^{n-s} u^n v^n/(n-s)!\,; \quad \textit{i.e.}\text{ that if } z \equiv (p^2-1),$$

$$z^{\frac{s}{2}} D^{n+s} z^n/(n+s)! = z^{-\frac{s}{2}} D^{n-s} z^n/(n-s)!\,.$$

Hence $\displaystyle\int_{-1}^{1} z^s (D^{n+s} z^n)^2\, dp$

$$= \int_{-1}^{1} z^{\frac{s}{2}} D^{n+s} z^n \,.\, z^{-\frac{s}{2}} D^{n-s} z^n\, dp \,.\, \frac{(n+s)!}{(n-s)!}$$

$= \dfrac{(n+s)!}{(n-s)!}\displaystyle\int_{-1}^{1} D^{n+s} z^n \,.\, D^{n-s} z^n\, dp$, and integrating by parts,

$$= \frac{(n+s)!}{(n-s)!}(-1)^s \int_{-1}^{1} (D^n z^n)^2\, dp$$

$$= \frac{(n+s)!}{(n-s)!}(-1)^s (2^n \,.\, n!)^2 \int_{-1}^{1} P_n^2\, dp = \frac{(n+s)!}{(n-s)!}(-1)^s (2^n \,.\, n!)^2 \,.\, \frac{2}{2n+1}.$$

1871. **Integral of Product of Two Harmonics over Unit Sphere.**

If Y_n, Z_n be two Spherical Harmonics each of degree n, viz.

$$A_0K_0+\sum_1^n(A_s\cos s\phi+B_s\sin s\phi)K_s$$

and $$a_0K_0+\sum_1^n(a_s\cos s\phi+b_s\sin s\phi)K_s,$$

where $K_s=(1-p^2)^{\frac{s}{2}}P_n^{(s)}$ (Art. 1867), we have, upon integrating the product with regard to ϕ from 0 to 2π,

$$\int_0^{2\pi}Y_nZ_n\,d\phi=2\pi A_0a_0K_0^2+\pi\sum_1^n(A_sa_s+B_sb_s)K_s^2,$$

and integrating this with regard to p from -1 to 1, we have by the Lemma $\int_{-1}^1\int_0^{2\pi}Y_nZ_n\,dp\,d\phi$

$$=2\pi A_0a_0\frac{2}{2n+1}+\sum_1^n(A_sa_s+B_sb_s)\frac{(n+s)!}{(n-s)!}\cdot\frac{2\pi}{2n+1}$$

$$=\frac{2\pi}{2n+1}\left\{2A_0a_0+\sum_1^n\frac{(n+s)!}{(n-s)!}(A_sa_s+B_sb_s)\right\}.$$

In the case when the harmonics are of different orders, viz. n and m, $\int_{-1}^1\int_0^{2\pi}Y_nZ_m\,dp\,d\phi=0$, by Art. 1783.

If the harmonics be identical, *i.e.* $Z_n\equiv Y_n$, we have

$$\int_{-1}^1\int_0^{2\pi}Y_n^2\,dp\,d\phi=\frac{2\pi}{2n+1}\left\{2A_0^2+\sum_1^n\frac{(n+s)!}{(n-s)!}(A_s^2+B_s^2)\right\}.$$

1872. If any function of μ, ϕ, say $V\equiv F(\mu,\phi)$, be expanded in a series of Laplace's Functions as $V=Y_0+Y_1+Y_2+Y_3+\dots$, which is true upon the surface of the sphere $r=a$, then at points within the sphere we shall have

$$V_i=Y_0+Y_1\frac{r}{a}+Y_2\frac{r^2}{a^2}+\dots,$$

and at points without

$$V_e=Y_0\frac{a}{r}+Y_1\frac{a^2}{r^2}+Y_2\frac{a^3}{r^3}+\dots.$$

For each term is a spherical harmonic satisfying Laplace's Equation and satisfying the conditions at the surface, and the latter vanishes at ∞; and there is but one value of V which does so.

Thus, when V is given all over the sphere, we can write down its value at any internal or any external point.

1873. Differentiation of the Zonal Harmonics

$$Z_n \equiv P_n r^n, \quad Z_{-n} \equiv \frac{P_{n-1}}{r^n}.$$

With cylindrical coordinates (ρ, ϕ, z),

$$r = \sqrt{z^2+\rho^2}, \quad \mu = \cos\theta = z/\sqrt{z^2+\rho^2},$$

$$\frac{\partial r}{\partial z} = \frac{z}{\sqrt{z^2+\rho^2}} = \mu, \quad \frac{\partial \mu}{\partial z} = \frac{1-\mu^2}{r}, \quad \frac{\partial r}{\partial \rho} = \sqrt{1-\mu^2}, \quad \frac{\partial \mu}{\partial \rho} = -\frac{\mu\sqrt{1-\mu^2}}{r}.$$

Then $\dfrac{\partial}{\partial z} \equiv \mu \dfrac{\partial}{\partial r} + \dfrac{1-\mu^2}{r}\dfrac{\partial}{\partial \mu}$; $\dfrac{\partial}{\partial \rho} = \sqrt{1-\mu^2}\left(\dfrac{\partial}{\partial r} - \dfrac{\mu}{r}\dfrac{\partial}{\partial \mu}\right)$;

$$\left.\begin{aligned} \therefore\ \frac{\partial Z_n}{\partial z} &\equiv \left\{\mu n P_n + (1-\mu^2)\frac{dP_n}{d\mu}\right\} r^{n-1} = n r^{n-1} P_{n-1} = n Z_{n-1}, \text{ (Art. 1844)}, \\ \frac{\partial Z_{-n}}{\partial z} &= \left\{-\mu n P_{n-1} + (1-\mu^2)\frac{dP_{n-1}}{d\mu}\right\} r^{-n-1} = -n r^{-n-1} P_n = -n Z_{-n-1}. \end{aligned}\right\} \text{(A)}$$

Therefore, whether i be positive or negative, $\dfrac{\partial Z_i}{\partial z} = i Z_{i-1}$, a rule analogous to the differentiation of a power. It follows that

$$\frac{\partial^2 Z_i}{\partial z^2} = i(i-1) Z_{i-2}, \ldots \frac{\partial^r Z_i}{\partial z^r} = i(i-1)\ldots(i-r+1) Z_{i-r}.$$

Again, by Arts. 1843, 1845,

$$\left.\begin{aligned} \frac{\partial Z_n}{\partial \rho} &= \sqrt{1-\mu^2}\, r^{n-1}\left(n P_n - \mu\frac{dP_n}{d\mu}\right) = -\sqrt{1-\mu^2}\, r^{n-1}\frac{dP_{n-1}}{d\mu}, \\ \frac{\partial Z_{-n}}{\partial \rho} &= -\sqrt{1-\mu^2}\, r^{-n-1}\left\{n P_{n-1} + \mu\frac{dP_{n-1}}{d\mu}\right\} = -\sqrt{1-\mu^2}\, r^{-n-1}\frac{dP_n}{d\mu}. \end{aligned}\right\} \ldots\text{(B)}$$

1874. Change of Origin of Zonal Harmonics to a New Origin O' on the same Axis Oz.

Let n be a positive integer. Taking O as the origin and Oz as the axis of the Zonal Harmonics, Z_n is a function of ρ and z alone, $=f(\rho, z)$. Then taking O' at the point $(0, 0, -a)$, the new ordinate z' of any point P, whose coordinates are x, y, z with regard to axes with origin O, is when referred to parallel

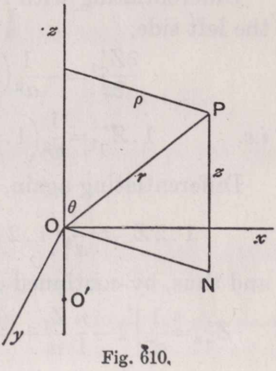

Fig. 610.

axes with origin O', $z+a$, and the corresponding Zonal Harmonic Z_n' is denoted by $f(\rho, z')$, *i.e.* $f(\rho, z+a)$; and this being of degree n in z, we have

$$Z_n'=f+a\frac{\partial f}{\partial z}+\frac{a^2}{2!}\frac{\partial^2 f}{\partial z^2}+\dots+\frac{a^n}{n!}\frac{\partial^n f}{\partial z^n},$$

the accent denoting the Zonal Harmonic of degree n with reference to the new origin. That is,

$$Z_n'=Z_n+a\frac{\partial Z_n}{\partial z}+\frac{a^2}{2!}\frac{\partial^2 Z_n}{\partial z^2}+\dots+\frac{a^n}{n!}\frac{\partial^n Z_n}{\partial z^n}$$

$$=Z_n+naZ_{n-1}+\frac{n(n-1)}{1.2}a^2Z_{n-2}+\dots+na^{n-1}Z_1+a^n.$$

Similarly, if the Zonal Harmonic be of negative order, Z_{-n} and $r>a$, we have a series in ascending powers $\frac{a}{r}$ but extending to ∞. For, as before, Z_{-n} is of form $F(\rho, z)$,

$$Z'_{-n}=F(\rho, z+a)\equiv F+a\frac{\partial F}{\partial z}+\frac{a^2}{2!}\frac{\partial^2 F}{\partial z^2}+\dots$$

$$=Z_{-n}-\frac{n}{1}aZ_{-n-1}+\frac{n(n+1)}{1.2}a^2Z_{-n-2}-\frac{n(n+1)(n+2)}{1.2.3}Z_{-n-3}+\dots.$$

But in cases where r, being measured from the first origin, is $<a$, this expansion is inadmissible. We then have

$$Z'_{-1}=\{x^2+y^2+(z+a)^2\}^{-\frac{1}{2}}=(a^2+2ar\cos\theta+r^2)^{-\frac{1}{2}}$$

$$=\frac{1}{a}\left(P_0-P_1\frac{r}{a}+P_2\frac{r^2}{a^2}-\dots\right)$$

$$=\frac{1}{a}\left(Z_0-\frac{Z_1}{a}+\frac{Z_2}{a^2}-\frac{Z_3}{a^3}+\dots\right).$$

Differentiating with regard to z, *i.e.* with regard to $z+a$ on the left side,

$$\frac{\partial Z'_{-1}}{\partial z}=-\frac{1}{a^2}\left(Z_0-\frac{2Z_1}{a}+\frac{3Z_2}{a^2}-\frac{4Z_3}{a^3}+\dots\right),$$

i.e.
$$1.Z'_{-2}=\frac{1}{a^2}\left(1.Z_0-2\frac{Z_1}{a}+3\frac{Z_2}{a^2}-4\frac{Z_3}{a^3}+\dots\right).$$

Differentiating again,

$$1.2\,Z'_{-3}=\frac{1}{a^3}\left(1.2\,Z_0-2.3\frac{Z_1}{a}+3.4\frac{Z_2}{a^2}-\dots\right),\quad \text{etc.,}$$

and thus, by continued differentiations, we arrive at

$$Z'_{-n}=\frac{1}{a^n}\left[1-\frac{n}{1}\frac{Z_1}{a}+\frac{n(n+1)}{1.2}\frac{Z_2}{a^2}-\frac{n(n+1)(n+2)}{1.2.3}\frac{Z_3}{a^3}+\dots\right].$$

PROBLEMS.

1. Show that $Ax^3+By^3+Cz^3-\frac{3}{5}(x^2+y^2+z^2)(Ax+By+Cz)$ is a spherical harmonic, and that the corresponding surface harmonic on unit sphere is

$$(A\cos^3\phi+B\sin^3\phi)\sin^3\theta+C\cos^3\theta-\tfrac{3}{5}(A\cos\phi+B\sin\phi)\sin\theta-\tfrac{3}{5}C\cos\theta.$$

2. If OA, OB, OC be three perpendicular axes cutting a unit sphere with centre O at A, B, C, and if P be any other point on the surface, show that $\cos PA\cos PB\cos PC$ is a surface harmonic.

3. ABC is a fixed quadrantal triangle on unit sphere, and a point P moves on the surface, so that

$$V\equiv a\cos^2 PA+b\cos^2 PB+c\cos^2 PC+2f\cos PB\cos PC \\ +2g\cos PC\cos PA+2h\cos PA\cos PB$$

is a surface harmonic. Show that the cone $V=0$ has three perpendicular generators.

4. If P_n be Legendre's coefficient of order n, show that

$$\int_{-1}^{1}P_1P_n(5P_2-3)\,dp=0,$$

unless $n=3$, in which case the value is 6/7.

5. Show that

$$\int_{-1}^{1}(P_0\sqrt{1}+P_1\sqrt{3}+P_2\sqrt{5}+\ldots+P_n\sqrt{2n+1})^2\,dp=2(n+1).$$

6. Show that $\int_{-1}^{1}p^4P_n\,dp=0$, except in the cases

$$\int_{-1}^{1}p^4P_0\,dp=\tfrac{2}{5},\quad \int_{-1}^{1}p^4P_2\,dp=\tfrac{8}{35},\quad \int_{-1}^{1}p^4P_4\,dp=\tfrac{16}{315}.$$

7. Show that $\int_{-1}^{1}p^5P_n\,dp=0$, except in the cases

$$\int_{-1}^{1}p^5P_1\,dp=\tfrac{1}{7},\quad \int_{-1}^{1}p^5P_3\,dp=\tfrac{8}{63},\quad \int_{-1}^{1}p^5P_5\,dp=\tfrac{16}{693}.$$

8. Show that the area of one of the larger loops of the curve $r=aP_2$ is $\dfrac{a^2}{32}\left(5\sqrt{2}+11\cos^{-1}\dfrac{1}{\sqrt{3}}\right)$.

9. Show that if ϵ be very small, the area of the nearly circular figure $r=a(1+\epsilon P_2)$ is approximately $\pi a^2(1+\frac{1}{2}\epsilon)$.

10. Show that if ϵ be very small, the volume of the nearly spherical surface $r=a(1+\epsilon P_2)$ is very approximately $\frac{4}{3}\pi a^3(1+\frac{3}{5}\epsilon^2)$

11. Show that if $R^2=1-2ax+a^2$, $R'^2=1-2\beta x+x^2$,

$$\int_{-1}^{1}\frac{dx}{RR'}=\frac{2}{\sqrt{\alpha\beta}}\tanh^{-1}\sqrt{\alpha\beta},$$

and deduce the values of

$$\int_{-1}^{1}P_m P_n\,dp,\quad m\neq n,\quad\text{and}\quad\int_{-1}^{1}P_n{}^2\,dp.$$

12. Show that

$$\frac{\sin 3\theta}{\sin\theta}=\frac{1}{3}+\frac{8}{3}P_2;\quad \frac{\sin 4\theta}{\sin\theta}=\frac{4}{5}P_1+\frac{16}{5}P_3;\quad \frac{\sin 5\theta}{\sin\theta}=\frac{1}{5}+\frac{8}{7}P_2+\frac{128}{35}P_4.$$

13. Give the rational integral function of the second degree of the three quantities, $\sin\lambda$, $\cos\lambda\sin\theta$, $\cos\lambda\cos\theta$, and put the terms of the second order under the form

$$c_1\sin^2\lambda+(c_2\sin^2\theta+c_3\sin\theta\cos\theta+c_4\cos^2\theta)\cos^2\lambda$$
$$+(c_5\cos\theta+c_6\sin\theta)\sin\lambda\cos\lambda,$$

and show that, with the addition of an arbitrary quantity c_0, it becomes a Laplace's function if $3c_0=-(c_1+c_2+c_3)$.

[SMITH'S PRIZE, 1876.]

14. For points x, y, z which lie on the sphere $x^2+y^2+z^2=1$, express Q as a series of surface harmonics, where

$$Q=x+2y+3z+4x^2+5y^2+6z^2+7yz+8zx+9xy+10x^3+11xyz.$$

15. Express $\sin^4\theta$ in a series of Legendre's coefficients as

$$\sin^4\theta=\frac{8}{15}P_0-\frac{16}{21}P_2+\frac{8}{35}P_4.$$

Why cannot $\sin^3\theta$ be expanded in a finite series of spherical harmonics?

[MATH. TRIP., 1873.]

16. If $P_n=\dfrac{1}{2^n n!}\dfrac{d^n(\mu^2-1)^n}{d\mu^n}$, prove that if $\int P_n\,d\mu$ be taken to vanish when $\mu=1$,

$$\int P_n\,d\mu=\frac{1}{n(n+1)}(\mu^2-1)\frac{dP_n}{d\mu};\quad P_{n+1}=(2n+1)\int P_n\,d\mu+P_{n-1}.$$

Show how by the help of these formulae the numerical values of P_1, P_2, P_3, ... P_n, and those of their differential coefficients, may be conveniently found for any given value of μ.

[PROF. ADAMS, S.P., 1873.]

17. Prove that

$$\log\left(1+\operatorname{cosec}\frac{\theta}{2}\right)=P_0+\tfrac{1}{2}P_1+\tfrac{1}{3}P_2+\tfrac{1}{4}P_3+\dots.$$

[COLL. EX.]

18. Obtain a solution of the differential equation

$$\frac{d}{dx}\left(\sin x \frac{d}{dx} P_n\right) + n(n+1)\sin x P_n = 0$$

in the form of a series of cosines of multiples of x.

[MATH. TRIP. II., 1888.]

19. Show that if $(1-2ax+a^2)^{-\frac{k-1}{2}} = 1 + \sum_0^\infty Q_n a^n$, then will

$$(n+2)Q_{n+2} - (2n+k+1)xQ_{n+1} + (n+k-1)Q_n = 0.$$

[E. J. ROUTH, *Proc. L.M.S.*, xxvi.

20. Prove that if

$$V \equiv (1-2ax+a^2)^{-\frac{3}{2}} = 1 + K_1 a + K_2 a^2 + \ldots + K_n a^n + \ldots,$$

(i) $x\dfrac{\partial V}{\partial x} - a\dfrac{\partial V}{da} = 3a^2 V^{\frac{5}{3}}$;

(ii) $(1-x^2)\dfrac{\partial^2 V}{\partial x^2} + a^2\dfrac{\partial^2 V}{\partial a^2} = 12a^2 V^{\frac{5}{3}}$;

(iii) $(1-x^2)K_n'' - 4xK_n' + n(n+3)K_n = 0$;

(iv) $(n+1)K_{n+1} - (2n+3)xK_n + (n+2)K_{n-1} = 0$;

(v) $K_n' = (2n+1)K_{n-1} + (2n-3)K_{n-3} + (2n-7)K_{n-5} + \ldots$

(vi) $(2n+3)\int K_n\,dx = K_{n+1} - K_{n-1} + \text{const.}$;

(vii) $K_{2n-1} = 3P_1 + 7P_3 + \ldots + (4n-1)P_{2n-1}$,

$K_{2n} = 1 + 5P_2 + 9P_4 + \ldots + (4n+1)P_{2n}$.

(viii) $\int_{-1}^1 K_m K_n\,dx = 0$ or $(n+1)(n+2)$,
according as $m+n$ is odd, or even and $m \not< n$;

21. If $V = (1-2ap+a^2)^{-\frac{2m+1}{2}} = 1 + \Sigma Q_n a^n$, show that

$$Q_n = \frac{1}{1\,.\,3\ldots(2m-1)}\left(\frac{d}{dp}\right)^m P_{m+n}.$$

22. If $V = (1-2ap+a^2)^{-\frac{2m+1}{2}} = 1 + \Sigma Q_n a^n$, prove that

(i) $\displaystyle\int_{-1}^1 Q_{2r}\,dp = 2\frac{2m(2m+1)\ldots(2m+2r-1)}{1\,.\,2\ldots(2r+1)}$; (ii) $\displaystyle\int_{-1}^1 Q_{2r+1}\,dp = 0.$

23. Show that the roots of

$$x^n - \frac{n}{1}\frac{n(n-1)}{2n(2n-1)}x^{n-2} + \frac{n(n-1)}{1\,.\,2}\frac{n(n-1)(n-2)(n-3)}{2n(2n-1)(2n-2)(2n-3)}x^{n-4} - \ldots = 0$$

are all real and unequal, and lie between 1 and -1.

24. Prove that one solution of Legendre's Equation

$$(1-x^2)y_2 - 2xy_1 + n(n+1)y = 0,$$

where n is a positive integer, is a polynomial of the n^{th} degree, and determine it.

25. Prove that a like statement is true of the equation

$$(1-x^2)y_2 + axy_1 + n(n-1-a)y = 0$$

unless $1+a-n$ be one of a series of numbers $n-2$, $n-4$, $n-6$, ... which terminate in 1 or 0, according as n is odd or even, and in that case a polynomial of degree $1+a-n$ is a solution.

[MATH. TRIP. II., 1918.]

26. $P_n(\mu)$ being the coefficient of h^n in $(1-2\mu h+h^2)^{-\frac{1}{2}}$ and m, n unequal, show that $\int_{-1}^{1} \mu^2 P_n(\mu) P_m(\mu)\,d\mu$ is zero unless m and n differ from one another by 2, and that when $m=n+2$, its value is $2(n+1)(n+2)/(2n+1)(2n+3)(2n+5)$. [MATH. TRIP. II., 1916.]

If $m=n$, show that the value is

$$2(4n^3+6n^2-1)/(2n-1)(2n+1)^2(2n+3).$$

27. Prove that

$$\text{(i)}\quad \int_{-1}^{1} (1-x^2)\,P_m'(x)\,P_n'(x)\,dx = 0 \quad (n \neq m);$$

$$\text{(ii)}\quad \int_{-1}^{1} (1-x^2)\{P_n'(x)\}^2 dx = 2n(n+1)/(2n+1).$$

[MATH. TRIP. II., 1914.]

28. Prove that $P_{n+1} - P_{n-1} = (2n+1)\int_{-1}^{p} P_n\,dp = (2n+1)\int_{1}^{p} P_n\,dp.$

29. Prove that

$$\text{(i)}\quad \int_0^{\pi} P_n(\cos\theta)\,d\theta = 0 \quad\text{or}\quad \pi\left\{\frac{1.3\ldots(n-1)}{2.4\ldots n}\right\}^2 \text{ as } n \text{ is odd or even;}$$

$$\text{(ii)}\quad \int_0^{\pi} \cos\theta P_n(\cos\theta)\,d\theta = 0 \quad\text{or}\quad \frac{n\pi}{n+1}\left\{\frac{1.3\ldots(n-2)}{2.4\ldots(n-1)}\right\}^2 \text{ as } n \text{ is even or odd.}$$

30. Show that

$$\text{(i)}\quad (1-p^2)^{-\frac{1}{2}} = \frac{\pi}{2}\left\{1 + 5\left(\frac{1}{2}\right)^2 P_2 + 9\left(\frac{1.3}{2.4}\right)^2 P_4 + 13\left(\frac{1.3.5}{2.4.6}\right)^2 P_6 + \ldots\right\};$$

$$\text{(ii)}\quad \frac{2}{\pi} = 1 - 5\left(\frac{1}{2}\right)^3 + 9\left(\frac{1.3}{2.4}\right)^3 - 13\left(\frac{1.3.5}{2.4.6}\right)^3 + \ldots;$$

$$\text{(iii)}\quad \frac{p}{\sqrt{1-p^2}} = \frac{\pi}{2}\left\{3.\frac{1}{2}P_1 + 7\left(\frac{1}{2}\right)^2.\frac{3}{4}P_3 + 11\left(\frac{1.3}{2.4}\right)^2\frac{5}{6}.P_5 + \ldots\right\}.$$

[Use formula of Art. 1813.]

[CRELLE, *Jour.* LVI.; TODHUNTER, *Functions*, p. 115.]

31. Show that $\frac{P_1^2-P_0^2}{P_1'^2-P_0'^2}, \frac{P_2^2-P_1^2}{P_2'^2-P_1'^2}, \frac{P_3^2-P_2^2}{P_3'^2-P_2'^2}, \frac{P_4^2-P_3^2}{P_4'^2-P_3'^2}$ are respectively equal to $(p^2-1)/1^2$, $(p^2-1)/2^2$, $(p^2-1)/3^2$, $(p^2-1)/4^2$, and that $P_2=P_1$, when $p=-\frac{1}{3}$ or 1; $P_3=P_2$, when $p=\frac{\pm\sqrt{6}-1}{5}$ or 1.

32. Prove that

$$P_0^2+3P_1^2+5P_2^2+\ldots+(2n+1)P_n^2=(n+1)^2P_n^2-(p^2-1)P_n'^2.$$

[MATH. TRIP., 1888.]

33. Prove that

$$P_0'^2+3P_1'^2+5P_2'^2+\ldots+(2n+1)P_n'^2=\tfrac{1}{3}\{(n+2)^2P_n'^2-(p^2-1)P_n''^2\}.$$

[MATH. TRIP., 1888.]

34. If $(1-2ax+a^2)^{-\frac{2l+1}{2}}=1+Z_1a+Z_2a^2+\ldots+Z_na^n+\ldots$, l being a positive integer, show that, accents denoting differentiations with regard to x,

(i) $\int_{-1}^{1} Z_mZ_n\,dx=0$ if $m+n$ be odd;

(ii) $(1-x^2)Z_n''-2(l+1)xZ_n'+n(n+2l+1)Z_n=0$;

(iii) $Z_n'=\{2(n+l)-1\}Z_{n-1}+\{2(n+l)-5\}Z_{n-3}+\{2(n+l)-9\}Z_{n-5}+\ldots$.

35. If $(1-2ax+a^2)^{-m}=\sum_{n=0} P_{m,n}a^n$, show that

(i) $x\frac{d}{dx}P_{m,n}-\frac{d}{dx}P_{m,n-1}=nP_{m,n}$;

(ii) $(1-x^2)\frac{d^2P_{m,n}}{dx^2}-(2m+1)x\frac{dP_{m,n}}{dx}+n(n+2m)P_{m,n}=0$;

(iii) $\int_{-1}^{1}(1-x^2)^{m-\frac{1}{2}}P_{m,n}P_{m,r}\,dx=0,\quad r\neq n$;

(iv) $\int_{-1}^{1}(1-x^2)^{m-\frac{1}{2}}P_{m,n}^2\,dx=\frac{2^{2m-1}}{m+n}\frac{\Pi(n+2m-1)}{\Pi(n)}\left\{\frac{\Pi(m-\frac{1}{2})}{\Pi(2m-1)}\right\}^2$.

36. Show that, if $k>0$ and P_λ be the Legendrian coefficient of order λ,

(i) $\int_{-1}^{1}(x^2-1)^k\frac{d^kP_m}{dx^k}\frac{d^kP_n}{dx^k}dx=0$;

(ii) $\int_0^1 x^{p+1}P_{n+1}\,dx=\frac{p+1}{p+n+3}\int_0^1 x^pP_n\,dx$;

(iii) $\int_0^1 x^pP_{n+2}\,dx=\frac{p-n}{p+n+3}\int_0^1 x^pP_n\,dx$;

m and n being different positive integers, and p any positive quantity.

[MATH. TRIP. II., 1889.]

37. Prove that $P_n(\sec\theta)=\frac{1}{\pi}\int_0^\pi \sec^n\theta(1+\sin\theta\cos\chi)^n d\chi$.

38. If $P_n(\mu)$ denote Legendre's coefficient of degree n, show that $\int_{-1}^{1}\mu(1-\mu^2)\frac{dP_n}{d\mu}\frac{dP_m}{d\mu}d\mu$ is zero unless $m \sim n$ be unity, and determine its value in these cases. [MATH. TRIP., 1896.]

39. Prove that

$$(x+\cos\phi\sqrt{x^2-1})^n = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n + \frac{1}{2^{n-1}}\sum_{m=1}^{m=n}\frac{(x^2-1)^{\frac{m}{2}}}{(n+m)!}\frac{d^{n+m}(x^2-1)^m}{dx^{n+m}}\cos m\phi,$$

and deduce the formulae

(i) $\dfrac{1}{(n-m)!}\dfrac{d^{n-m}}{dx^{n-m}}(x^2-1)^n = \dfrac{(x^2-1)^m}{(n+m)!}\cdot\dfrac{d^{n+m}}{dx^{n+m}}(x^2-1)^n$;

(ii) $P_n(x) = \dfrac{1}{\pi}\displaystyle\int_0^\pi (x+\cos\phi\sqrt{x^2-1})^n d\phi.$ [MATH. TRIP., 1887.]

40. Denoting by $P_n(\mu)$ the Legendrian coefficient of order n, prove that if $m \nless n$,

$$\int_{-1}^{1}\frac{d^2P_m}{d\mu^2}\frac{d^2P_n}{d\mu^2}d\mu = \frac{(n-1)n(n+1)(n+2)}{24}\{3m(m+1)-n(n+1)+6\},$$

if $m+n$ be even, but zero if $m+n$ be odd. [MATH. TRIP., 1897.]

41. Prove that if n be a positive integer $\left(\sinh^2 x\dfrac{d}{dx}\right)^n \operatorname{cosech}^{2n}x$ is equal to

$$(-1)^n 2^n n!\coth^n x\left\{1+\frac{n(n-1)}{2^2}\operatorname{sech}^2 x+\frac{n(n-1)(n-2)(n-3)}{2^2.4^2}\operatorname{sech}^4 x+\dots\right\},$$

and that either expression satisfies the differential equation

$$\sinh^2 x\frac{d^2y}{dx^2} = n(n+1)y.$$ [MATH. TRIP., 1897.]

42. Prove that

$$\frac{\pi}{\sqrt{2}}P_n(\cos\theta) = \int_0^\theta \frac{\cos n\phi\cos\frac{\phi}{2}}{\sqrt{\cos\phi-\cos\theta}}d\phi + \int_\theta^\pi \frac{\cos n\phi\sin\frac{\phi}{2}}{\sqrt{\cos\theta-\cos\phi}}d\phi,$$

except when $n=0$, when the right side $=\pi\sqrt{2}P_0(\cos\theta)$.

[DIRICHLET; TODHUNTER, *Functions of Laplace*, p. 35.]

43. Show that if the usual polar variables θ, ϕ be replaced by x, y defined by $\cot\dfrac{\theta}{2}.e^{\iota\phi}=x$, $\tan\dfrac{\theta}{2}.e^{\iota\phi}=-y$, the surface harmonic of order n satisfies the equation $\dfrac{\partial^2 V}{\partial x\partial y}+\dfrac{n(n+1)}{(x-y)^2}V=0.$

If V be any solution of this equation, verify that

$$\frac{\partial V}{\partial x}+\frac{\partial V}{\partial y},\quad x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y},\quad x^2\frac{\partial V}{\partial x}+y^2\frac{\partial V}{\partial y}$$

are also solutions. [MATH. TRIP. II., 1889.]

44. X_n' is the solid Zonal Harmonic of positive order n, having the axis of z for its axis and the origin of coordinates for its origin; X_m is the solid Zonal Harmonic of positive order m, having the same axis, and a point distant a from the origin for its origin; prove that

$$X_n' = X_n + naX_{n-1} + \frac{n(n-1)}{1.2}a^2X_{n-2} + \ldots + na^{n-1}X_1 + a^n.$$

The corresponding Zonal Harmonic of negative order being denoted by Y_n', prove that for points included within any sphere whose radius is less than a, and whose centre is the new origin,

$$Y_n' = \frac{1}{a^{n+1}}\left[1 - \frac{(n+1)!}{n!}\frac{X_1}{a} + \frac{(n+2)!}{2!\,n!}\frac{X_2}{a^2} - \frac{(n+3)!}{3!\,n!}\frac{X_3}{a^3} + \ldots\right].$$

Obtain the expression for Y_n' for points outside any sphere whose radius is greater than a, and whose centre is the new origin in the form

$$Y_n' = Y_n - \frac{(n+1)!}{n!}aY_{n+1} + \frac{(n+2)!}{2!\,n!}a^2Y_{n+2} - \frac{(n+3)!}{3!\,n!}a^3Y_{n+3} + \ldots.$$

[MATH. TRIP., 1885.]

45. Prove that the series

$$\tfrac{1}{2}P_1 + \sum_1^\infty (-1)^i(4i+1)\frac{1.3.5\ldots(2i-3)}{2.4.6\ldots(2i+2)}P_{2i}$$

is equal to $-\mu$ for all values of μ from -1 to 0, and to μ for all values of μ from 0 to 1. Apply this formula to calculate the potential of a hemispherical shell whose surface density varies as the density from a diametral plane at an external or internal point.

[MATH TRIP., 1878.]

46. Show that the surface

$$r = a\left[\frac{1}{2} + \frac{1}{2}\frac{5P_2}{1.4} - \frac{1.3}{2.4}\frac{9P_4}{3.6} + \frac{1.3.5}{2.4.6}\frac{13P_6}{5.8} - \ldots\right]$$

consists of two equal spheres which touch each other at the origin.

[MATH. TRIP., 1884.]

47. If $x = \operatorname{sn} x + A_3 \operatorname{sn}^3 x + A_5 \operatorname{sn}^5 x + A_7 \operatorname{sn}^7 x + \ldots$, show that

$$(2n+1)A_{2n+1} = k^n + \frac{(n+1)n}{2^2}k^{n-1}(1-k)^2$$

$$+ \frac{(n+2)(n+1)(n)(n-1)}{2^2.4^2}k^{n-2}(1-k)^4 + \text{etc.}$$

$$= \frac{2}{\pi}\int_0^{K'}\{dn(u, k')\}^{2n+1}du.$$

[MATH. TRIP. III., 1886.]

48. Prove that if $\rho^2 = x^2 + y^2$ and $r^2 = \rho^2 + z^2$, then U_i being the solid Zonal Harmonic of degree i, and P_i the corresponding Legendre's coefficient,

$$\frac{\partial^2}{\partial \rho^2} U_{i+2} = r^i [P'_{i-1} - (i^2 + i + 1) P_i],$$

and

$$\frac{\partial^2}{\partial \rho^2} \frac{U_{i-2}}{r^{2i-3}} = r^{-i-1} [P'_{i-1} - i(i-1) P_i],$$

where accents denote differentiations with regard to the cosine of the co-latitude, giving

$$r^{i+1} \frac{\partial^2 (U_{i-2}/r^{2i-3})}{\partial \rho^2} - r^{-i} \frac{\partial^2}{\partial \rho^2} U_{i+2} = (2i+1) P_i.$$

49. If $\rho = x^2 + y^2$ and V_i be the solid Zonal Harmonic of degree i, show that

$$\frac{1}{r^{2i+1}} \frac{\partial^2 V_{i+2}}{\partial \rho^2} = \frac{\partial^2}{\partial \rho^2} \frac{V_{i-2}}{r^{2i-3}},$$

where $r^2 = x^2 + y^2 + z^2$. [MATH. TRIP., 1890.]

50. Show that

$$(n - m + 1) \frac{d^m P_{n+1}}{d\mu^m} = (2n+1) \mu \frac{d^m P_n}{d\mu^m} - (n+m) \frac{d^m P_{n-1}}{d\mu^m}.$$ [S.P., 1875.]

51. Find the number of independent solutions of the equations $u_{xx} + u_{yy} + u_{zz} = 0$, $xu_x + yu_y + zu_z = nu$, and prove that if u be a solution, $u(x^2 + y^2 + z^2)^{-\frac{1}{2}(2n-1)}$ also will satisfy the first equation.

Prove that if

$\alpha + \beta\omega + \gamma\omega^2 = f(x + y\omega + z\omega^2)$ and $A + B\omega + C\omega^2 = \phi(\alpha + \beta\omega + \gamma\omega^2)$, where ω is one of the primitive cube roots of unity, then $\alpha - \beta$, $\beta - \gamma$, $\gamma - \alpha$, $A - B$, $B - C$, $C - A$ will all be spherical harmonics. [MATH. TRIP., 1876.]

52. Prove that the function which has the value $+1$ on the Northern hemisphere and -1 on the Southern is given in Zonal Harmonics by the series $\Sigma C_{2n+1} P_{2n+1}$, where

$$C_{2n+1} = (-1)^n \left\{ \frac{1.3.5 \ldots (2n-1)}{2.4.6 \ldots 2n} + \frac{1.3.5 \ldots (2n+1)}{2.4.6 \ldots (2n+2)} \right\}.$$

Hence find a function which has the values $A + B$, $A - B$ on (i) the Northern and Southern, (ii) the Eastern and Western, (iii) any two corresponding hemispheres, respectively, the axis of the Earth being permanently the axis of the harmonics. [MATH. TRIP., 1884.]

53. The polar equation of a nearly spherical surface is $r = a + bP_n$, where P_n is a zonal harmonic of the n^{th} degree, and b is a small quantity whose powers above the second may be neglected. Show

that the area of the surface exceeds the area of a sphere of radius a by $2\pi b^2(n^2+n+2)/(2n+1)$. [MATH. TRIP., 1878.]

54. In the nearly spherical surface $r=a+bP_n$, where P_n is a zonal harmonic and b is small, prove that at any point the excess of the measure of curvature above $1/a^2$ is to a first approximation

$$\frac{b}{a^3}(n^2+n-2)P_n.$$ [MATH. TRIP. III., 1886.]

55. Show that the Legendre's function Q_n of the second kind (Art. 1821) may be expressed in the form

$$Q_n=P_n\tanh^{-1}p-\left\{\frac{2n-1}{1\,.\,n}P_{n-1}+\frac{2n-5}{3(n-1)}P_{n-3}+\frac{2n-9}{5(n-2)}P_{n-5}+\dots\right\},$$

and that the general solution of John Ivory's Equation,

$$\frac{d}{dp}\left\{(1-p^2)^{s+1}\frac{d^{s+1}z}{dp^{s+1}}\right\}+\{n(n+1)-s(s+1)\}(1-p^2)^s\frac{d^sz}{dp^s}=0,$$

is given by $\frac{d^sz}{dp^s}=AP_n^{(s)}+BQ_n^{(s)}$; and further that Q_n may be expressed as $Q_n=C\left(\frac{d}{dp}\right)^{-(n+1)}(1-p^2)^{-(n+1)}$, a form corresponding to that of Rodrigues for P_n, C being a constant.

56. Find the integral of the square of a tesseral harmonic over the surface of the unit sphere.

If the general expression for a tesseral harmonic be of the form $A(1-\mu^2)^{\frac{m}{2}}\vartheta_n^{(m)}\cos m\phi$, where the coefficient of the highest power of μ in $\vartheta_n^{(m)}$ is unity, prove that

$$\vartheta_{n+1}^{(m)}=\mu\vartheta_n^{(m)}-\frac{n^2-m^2}{4n^2-1}\vartheta_{n-1}^{(m)}.$$ [MATH. TRIP.]