

CHAPTER XXXVII.

CHANCE.

1680. DEF. If an event can happen in a ways and fail in b ways, and all these ways are *equally likely* to occur, the probability of the happening is $a/(a+b)$ and of the failure to happen is $b/(a+b)$.

These measures are essentially numerical positive proper fractions. Certainty is denoted by unity. A mean value is essentially a quantity of the same kind as those of which the mean is taken. So long as a and b are finite, the theory of probability does not call for any mode of treatment other than the processes of ordinary arithmetic and algebra. If, however, a problem incurs the existence of an infinite number of ways in which an event could happen and an infinite number of ways in which it could fail to happen, all these being equally likely, the calculation of a , b and $a+b$ may call for the processes of the Integral Calculus, or at least the fundamental conceptions of the Calculus, to effect the necessary summations, though sometimes in such cases the actual labour of integration may be avoided by geometrical or other considerations.

1681. Take, for instance, the case of a material particle thrown down upon a region of area A , and which is *equally likely to fall at any point of the area*; and let us explain this phrase. Imagine the area A to be divided up into an infinite number of infinitesimally small elements of equal area, and suppose that an infinite number of trials is made. We shall also suppose that, after these trials, the particle has fallen as many times upon any one element as upon any other. Then if a be any region of finite area enclosed completely within

the contour of A , a and A contain numbers of the infinitesimal elements of area proportional to and measured by their own areas. Hence the numbers of particles which have fallen respectively upon a and upon A are measured by the respective areas of a and A , and the chance that a particle which falls upon A also falls upon a is $\frac{a}{A}$, and that it does not so fall is $1 - \frac{a}{A}$.

The chance that of two hazard throws of a particle upon A both fall upon a is $\frac{a}{A} \cdot \frac{a}{A}$. That the first does and the second does not, the chance is $\frac{a}{A} \left(1 - \frac{a}{A}\right)$. That the first does not and the second does is $\left(1 - \frac{a}{A}\right) \frac{a}{A}$; and that neither does is $\left(1 - \frac{a}{A}\right) \left(1 - \frac{a}{A}\right)$, and the sum of these is unity. And so on if there be more than two throws.

It will appear that in such cases, unless the areas be known or obtainable by some elementary means, either the Integral Calculus or some equivalent graphical method will be necessary for their evaluation. Taking any pair of rectangular axes in the plane of the region A , the chance that the throw upon A results in the particle falling upon a may be expressed as

$$\iint dx dy \text{ (taken over } a) / \iint dx dy \text{ (taken over } A).$$

1682. We note that the chance that a particle should fall upon the *perimeter* of the contour of a is infinitesimal in comparison with the chance that it should fall upon the *area* of a .

1683. We indicate by a few examples how the Integral Calculus is to be *applied* in some cases, and how the actual integration may be *evaded* in others.

1. $OA = 2a$ is the axis of a cardioid. C is the mid-point of OA . What is the chance that a random point P taken within the cardioid is further from C than C is from O ?

Drawing a circle with centre C and radius CO , P must lie without the circle but within the cardioid. The area of the cardioid

$$= 2 \cdot \frac{1}{2} \int_0^\pi a^2 (1 + \cos \theta)^2 d\theta = \frac{3}{2} \pi a^2.$$

Therefore the chance required is $(\frac{3}{2} \pi a^2 - \pi a^2) / \frac{3}{2} \pi a^2 = \frac{1}{3}$.

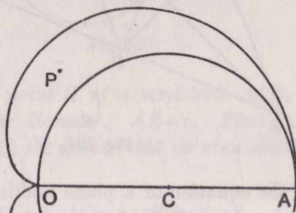


Fig. 521.

2. Given that p, q are any positive quantities of which neither is > 4 ; what is the probability that when real values are assigned to them at random, the roots of the quadratic $x^2 - px + q = 0$ shall be real?

If real, $p^2 \leq 4q$. Construct the parabola $Y^2 = 4X$. The point $(4, 4)$ lies upon it. We may then interpret the condition geometrically. A random point H is selected upon a square $ONPQ$, whose side is 4. The above parabola is drawn with axes ON, OQ . The values of p and q are denoted by the abscissa and ordinate of H . When H lies without the parabola $p^2 > 4q$. Therefore the chance that $p^2 \leq 4q$ is measured by the ratio of the area OPQ to that of the square; that is, $1/3$. (Fig. 522.)

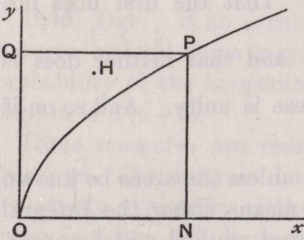


Fig. 522.

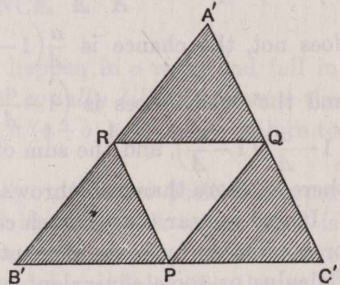


Fig. 523.

3. A rod, three feet long, is broken at random into three parts. What is the chance that we may be able to form a triangle with them?

(i) If x, y, z be the parts, $x + y + z = 1$, the unit being a yard. We are to have $y + z > x, z + x > y, x + y > z$. Interpreting x, y, z as areal coordinates, then $y + z = x$, etc.,

are the joins of the mid-points of the sides of the triangle of reference. In order that all the inequalities may be satisfied, the representative point x, y, z must lie within the triangle formed by them (unshaded, Fig. 523), which is one quarter of the whole triangle. Hence the chance is $\frac{1}{4}$.

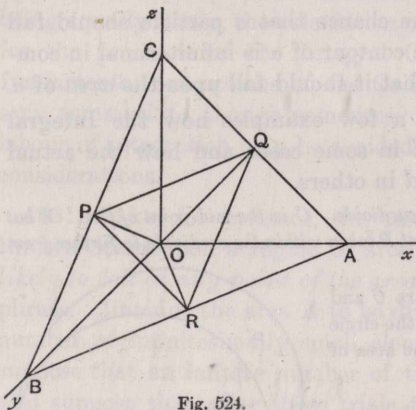


Fig. 524.

(ii) We might also regard x, y, z as the rectangular coordinates of a representative point. Taking 1 foot as unit, $x + y + z = 3$; and this

is the equation of a plane making intercepts 3, 3, 3 upon the coordinate axes. If A, B, C be the intercepted triangle; P, Q, R the mid-points of

its sides, $y+z=x$, etc., are the respective planes OQR , etc., and of all the unrestricted positions upon the triangle which the representative point x, y, z may occupy those for which $y+z > x$, etc., lie within the triangle PQR . Therefore, as before, the chance = $\frac{1}{4}$.

(iii) Again, without evasion of integration, we may proceed thus :

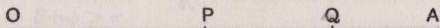


Fig. 525.

Let $OA (=a)$ be the rod, P and Q the random fractures, P being that which is nearer to O ; $OP=x, OQ=y$; $y > x$.

Then, since

$$x+(y-x) > (a-y), (y-x)+(a-y) > x, \text{ and } (a-y)+x > (y-x),$$

we have $x < \frac{a}{2}, y > \frac{a}{2}, y-x < \frac{a}{2}$. Hence the chance required is

$$\int_0^{\frac{a}{2}} \int_{\frac{a}{2}}^{\frac{a}{2}+x} dx dy / \int_0^a \int_0^y dy dx = \frac{2}{a^2} \int_0^{\frac{a}{2}} x dx = \frac{1}{4}.$$

(iv) Or still again, with the above inequalities, construct a square $OABC$ of side a, OA, OC being the x and y axes. Let P, Q, R, S (Fig. 526) be the mid-points of the sides, T that of the square. The representative point must be in some position on the triangle OBC as $y > x$, and both are positive and neither of them $> a$. The conditions $x < \frac{a}{2}, y > \frac{a}{2}, y-x < \frac{a}{2}$ restrict it further to the triangle TRS , which is obviously $\frac{1}{4}$ of OBC . Hence the chance required is $\frac{1}{4}$.

It will be noted that the integration process is merely the evaluation by that method of the areas of the triangles TRS, OBC .

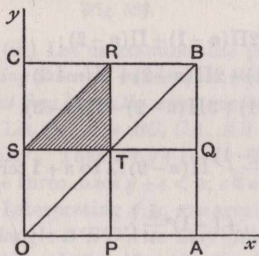


Fig. 526.

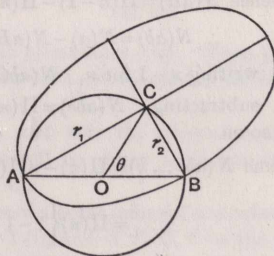


Fig. 527.

4. An ellipse has its centre at a random point C of a semicircle ACB , and two vertices at A, B the extremities of the diameter. $AB=c$. Find (i) the mean area for different positions of C ; (ii) the chance that its area shall be less than that of the circle. (Fig. 527.)

(i) Let O be the centre of the circle; $\hat{BOC} = \theta, AC=r_1, BC=r_2$.

Then Area of ellipse $= \pi r_1 r_2 = \frac{\pi c^2}{2} \sin \theta$,

and Mean area $= \frac{\pi c^2}{2} \frac{\int_0^\pi \sin \theta d\theta}{\int_0^\pi d\theta} = c^2$.

(ii) When area of ellipse = area of circle, $r_1 r_2 = \frac{1}{4} c^2$, and $\theta = 30^\circ$.

Hence, from $\theta = 30^\circ$ to $\theta = 150^\circ$, we have area of ellipse $>$ area of circle.

Therefore the chance that the area of the ellipse is less than that of the circle $= 2 \times \frac{30^\circ}{180^\circ} = \frac{1}{3}$.

5. If a quantity of homogeneous fluid contained in a vessel be thoroughly shaken up and allowed to come to rest again, prove that the chance that no particle of the fluid now occupies its original position is $1/e$.

[WHITWORTH'S PROBLEM.]

Let there be n particles a, β, γ, \dots occupying specific positions:

N the number of ways of arranging them in those positions $= \Pi(n)$,
say, $= n!$,

$N(A)$ the number of ways of arranging them with a in its original place,

$N(a)$ the number of ways of arranging them with a out of its original place,

$N(aB)$ the number of ways of arranging them with β in and a out of their original places, and so on.

Thus $N = \Pi(n)$; $N(A) = \Pi(n-1)$; $N(a) = \Pi(n) - \Pi(n-1)$.

Hence $N(aB) = \Pi(n-1) - \Pi(n-2)$;

$$\therefore N(ab) = N(a) - N(aB) = \Pi(n) - 2\Pi(n-1) + \Pi(n-2);$$

\therefore writing $n-1$ for n , $N(abC) = \Pi(n-1) - 2\Pi(n-2) + \Pi(n-3)$;

\therefore subtracting, $N(abc) = \Pi(n) - 3\Pi(n-1) + 3\Pi(n-2) - \Pi(n-3)$,
and so on.

Thus $N(abc \dots k) = \Pi(n) - n\Pi(n-1) + \frac{n(n-1)}{1 \cdot 2} \Pi(n-2) \dots$ to $n+1$ terms

$$= \Pi(n) \left\{ 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!} \right\}.$$

Hence the chance that all the particles are misplaced

$$= \lim_{n \rightarrow \infty} \frac{N(a, b, c, \dots)}{n!} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \dots = \frac{1}{e}.$$

[See the Problem of " n letters and n directed envelopes," Smith, *Algebra*, p. 293.]

In this case, although the number of cases is infinite, the problem does not call for the assistance of the Integral Calculus.

6. Find the chance that a random triangle inscribed in a circle is (i) acute angled, (ii) obtuse angled.

(i) Let ABC (Fig. 528) be the triangle; O the centre of the circle. Let the angles AOB, AOC , measured in opposite directions from OA , be called θ and ϕ .

Then $A = (2\pi - \theta - \phi)/2, B = \phi/2, C = \theta/2$, and if ABC be acute angled, $\theta < \pi, \phi < \pi, \theta + \phi > \pi$.

The chance for an acute-angled case is therefore

$$\frac{\int_0^\pi \int_{\pi-\theta}^\pi d\theta d\phi}{\int_0^{2\pi} \int_0^{2\pi-\theta} d\theta d\phi} = \frac{\int_0^\pi \theta d\theta}{\int_0^{2\pi} (2\pi - \theta) d\theta} = \frac{1}{4}.$$

(ii) The probability that A is obtuse is

$$\int_0^\pi \int_0^{\pi-\theta} d\theta d\phi / \int_0^{2\pi} \int_0^{2\pi-\theta} d\theta d\phi = \frac{1}{4}.$$

The probability that one of the three A, B or C is obtuse = $\frac{3}{4}$.

The probability that the triangle is right angled is of course infinitesimal.

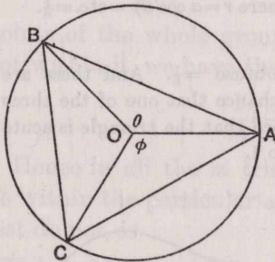


Fig. 528.

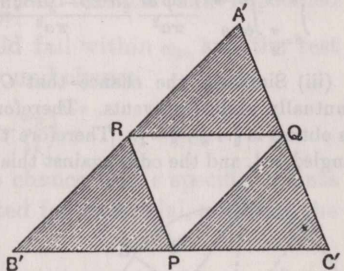


Fig. 529.

(iii) Let us examine this problem in an elementary way. Three points being taken at random on the circumference of a circle, what is the chance that they lie on the same semicircle?

Let the arcs BC, CA, AB be x, y, z ; and take the circumference as unity. Then $x + y + z = 1$. The triangle will be obtuse angled in any of the three cases $y + z < x, z + x < y, x + y < z$.

Interpreting x, y, z as areal coordinates of a point referred to a reference triangle $A'B'C'$, we may proceed as in 3 (i), and if P, Q, R be the mid-points of the sides, the chance required will be the same as the chance that an arbitrary point of the triangle $A'B'C'$ shall fall upon one of the three equal triangles $A'QR, B'RP, C'PQ$ (shaded in Fig. 529), i.e. $\frac{3}{4}$, and the chance the triangle ABC is acute angled is $\frac{1}{4}$.

(iv) A curious fallacy lies in the following argument. One pair of points, say A, B , must lie on a semicircle terminated at A . The chance that C lies on this semicircle is $\frac{1}{2}$; therefore the chance that all three lie on the same semicircle is $\frac{1}{2}$!

This is incorrect: where lies the fallacy? (Rev. T. C. Simmons, *Educ. Times*). Let the student obtain the correct result by this line of argument.

7. Two points P, Q are taken at random within a circle whose centre is C . Prove that the odds are 3 to 1 against the triangle CPQ being acute angled.

[ST. JOHN'S COLL., 1883.]

Let a be the radius; $P, (r, \phi)$, the position of one of the points.

Let a diameter ACB and a chord DPE be drawn perpendicularly to CP . Then (Fig. 530)

(i) The chance that \widehat{PCQ} is obtuse is $\frac{\text{area of a semicircle } AFB}{\text{area of circle}} = \frac{1}{2}$.

(ii) The chance that \widehat{CPQ} is obtuse is the compound chance that P should lie on the particular element $r d\phi dr$, and that if so, Q lies on the smaller segment cut off by the chord, $= \frac{r d\phi dr}{\pi a^2} \times \frac{\text{area of segment}}{\pi a^2}$. There-

fore the whole chance that wherever P lies, \widehat{CPQ} is obtuse is, with the notation indicated in the figure,

$$\int_{\theta=\frac{\pi}{2}}^{\theta=0} \int_{\phi=0}^{\phi=2\pi} \frac{r d\phi dr}{\pi a^2} \frac{\frac{1}{2}a^2 2\theta - \frac{1}{2}a^2 \sin 2\theta}{\pi a^2}, \text{ (where } r = a \cos \theta) = \text{etc.} = \frac{1}{8}.$$

(iii) Similarly the chance that \widehat{CQP} is obtuse $= \frac{1}{8}$. And these are mutually exclusive events. Therefore the chance that one of the three is obtuse is $\frac{1}{2} + \frac{1}{8} + \frac{1}{8} = \frac{3}{4}$. Therefore the chance that the triangle is acute angled is $\frac{1}{4}$, and the odds against this are 3 to 1.

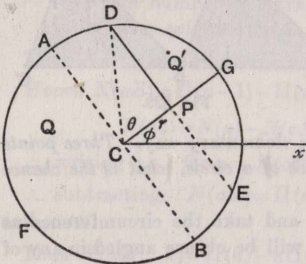


Fig. 530.

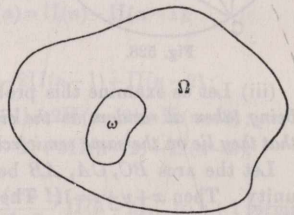


Fig. 531.

1684. We have seen that when a region Ω entirely encloses a second region ω , the chances that a random point taken within Ω should or should not lie within ω are respectively $\frac{\omega}{\Omega}$ and $1 - \frac{\omega}{\Omega}$. If n random points be taken within Ω , the chance that r specified points lie within ω , but the rest do not, is $\left(\frac{\omega}{\Omega}\right)^r \left(1 - \frac{\omega}{\Omega}\right)^{n-r}$; and if the several points be denoted

as A, B, C, \dots , the chance that some unspecified r of them lie within ω , whilst the rest do not, is nC_r times as great, that is ${}^nC_r \left(\frac{\omega}{\Omega}\right)^r \left(1 - \frac{\omega}{\Omega}\right)^{n-r}$. And the chance that *at least* r unspecified points of the whole number lie within ω is

$$\left(\frac{\omega}{\Omega}\right)^n + {}^nC_1 \left(\frac{\omega}{\Omega}\right)^{n-1} \left(1 - \frac{\omega}{\Omega}\right) + \dots + {}^nC_r \left(\frac{\omega}{\Omega}\right)^r \left(1 - \frac{\omega}{\Omega}\right)^{n-r}.$$

Now suppose that the region ω itself is variable with the different trials, and let the regions which it represents in the several trials be denoted by $\omega_1, \omega_2, \omega_3, \dots, \omega_m$, and let there be a very large number m of such trials, and that any of these ω 's may be equally likely to be selected for any particular trial of the taking of a random point P within the region Ω . The chance that at any particular trial any specified one value of ω , say ω_p , is selected is $\frac{1}{m}$, and therefore that r specified points of the whole group should fall within ω_p , and the rest not within it, we have the compound chance

$$\frac{1}{m} \left(\frac{\omega_p}{\Omega}\right)^r \left(1 - \frac{\omega_p}{\Omega}\right)^{n-r}.$$

Hence in all the m trials the chance that r specified points lie within the particular ω selected for each trial, and that the rest do not, is

$$\sum_{p=1}^{p=m} \frac{1}{m} \left(\frac{\omega_p}{\Omega}\right)^r \left(1 - \frac{\omega_p}{\Omega}\right)^{n-r} = \text{the mean value of } \left(\frac{\omega_p}{\Omega}\right)^r \left(1 - \frac{\omega_p}{\Omega}\right)^{n-r}.$$

And if the r points be not *specific* points of the group A, B, C, \dots which are to fall within the selected ω 's, the result will be the mean value of ${}^nC_r \left(\frac{\omega_p}{\Omega}\right)^r \left(1 - \frac{\omega_p}{\Omega}\right)^{n-r}$. That is, the two results are

$$M \{ \omega_p^r (\Omega - \omega_p)^{n-r} \} / \Omega^n \quad \text{or} \quad {}^nC_r M \{ \omega_p^r (\Omega - \omega_p)^{n-r} \} / \Omega^n,$$

according as the random points falling within the particular ω 's are to be specified or unspecified members of the group of random points A, B, C, \dots .

It is convenient to picture the two cases as those of n sand grains thrown at random upon the region Ω , the grains being coloured differently in the first case, uncoloured and indistinguishable in the second.

1685. Taking, for instance, the case of a rod AB of length a ; this is the region Ω . Take two points at random upon it. This marks a random region ω , viz. PQ , within Ω . Now take n other random points on AB , say differently coloured sand grains thrown at hazard upon the line. The chance that a specified group of r of these lies between P and Q , and the rest do not, $= M\{PQ^r(a-PQ)^{n-r}\}/a^n$; and if the group be unspecified, the chance will be $= {}^nC_r M\{PQ^r(a-PQ)^{n-r}\}/a^n$.

Let P be the random point which is the nearer to A ; $AP=x$, $AQ=y$.

$$\begin{aligned} \text{Then } M\{PQ^r(a-PQ)^{n-r}\} &= \int_0^a \int_0^y (y-x)^r (a-y+x)^{n-r} dy dx / \int_0^a \int_0^y dy dx \\ &= \frac{2}{a^2} \int_0^1 \int_0^{a(1-z)} a^n z^r (1-z)^{n-r} a dz d\xi \left[\text{putting } y-x=az, x=\xi, \frac{d(y, x)}{d(z, \xi)}=a \right] \\ &= 2a^n \int_0^1 z^r (1-z)^{n-r+1} dz = 2a^n \Gamma(r+1) \Gamma(n-r+2) / \Gamma(n+3) = 2a^n \frac{n-r+1}{(n+2)(n+1)} \cdot \frac{1}{nC_r} \end{aligned}$$

Therefore the chance required for r specified points, and r only, to lie between P and Q is $\frac{2(n-r+1)}{(n+2)(n+1)} \cdot \frac{1}{nC_r}$, and if the r points be unspecified $= \frac{2(n-r+1)}{(n+2)(n+1)}$.

1686. This result is obtainable directly. For the total number of points to be chosen on AB is $n+2$. The number of permutations of these is $(n+2)!$. Let us fix positions for two of these, X and Y , on the array, say the l^{th} and m^{th} . Then there are $n!$ permutations of the remaining points. Hence the chance that two particular points X and Y shall be the l^{th} and m^{th} of the array $= \frac{2 \cdot n!}{(n+2)!}$, for these two may stand in either order, either as first and $(r+2)^{\text{th}}$, second and $(r+3)^{\text{th}}$, third and $(r+4)^{\text{th}}$, ... $(n-r+1)^{\text{th}}$ and $(n+2)^{\text{th}}$, i.e. in $n-r+1$ ways, events equally likely to occur, and therefore the total chance that these two points shall find r unspecified other points between them is $\frac{2(n-r+1)}{(n+1)(n+2)}$.

1687. For instance, if there be eight indistinguishable points taken at hazard on AB after P, Q have been selected at random, the chance that three unspecified ones should lie between P and Q and five on the rest of the line AB is $\frac{2 \cdot 6}{10 \cdot 9} = \frac{2}{15}$, and the chance for three specified ones to lie between P and Q and the others on the rest of the line is

$$\frac{2}{15} \cdot \frac{1}{{}^8C_3} = \frac{2}{15} \cdot \frac{1}{56} = \frac{1}{420}.$$

1688. Random Points.

It is necessary to examine carefully what is meant when it is stated that points are taken at random within a given region.

(i) When a point P is said to be taken at random upon a line AB of length a , it is understood that AB is divided into an infinite number of equal elements, and that each element has the same chance of finding itself the recipient of the point

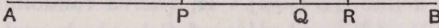


Fig. 532.

P . Thus, measuring a length x along AB from A , the chance of the random point P falling between x and $x+dx$ is dx/a .

If a random selection of several points P, Q, R be made upon the line, the chances they will severally fall between the respective distances x and $x+dx, y$ and $y+dy, z$ and $z+dz$ from A are $dx/a, dy/a$ and dz/a , and the compound chance that all three chances should concur is $\frac{dx}{a} \cdot \frac{dy}{a} \cdot \frac{dz}{a}$, dx, dy, dz denoting increments of equal length.

But if, *after* a choice of P and Q has been made at random, R is then selected at random between P and Q , the respective chances are $dx/a, dy/a, dz/(y-x)$; for now the possible region for the selection of a position for R has been restricted. The compound chance that all three things should happen is $\frac{dx}{a} \cdot \frac{dy}{a} \cdot \frac{dz}{y-x}$.

If a rod be broken simultaneously at two points at random, the chance that one fracture lies at a distance between x and $x+dx$ from A , and that the other lies between the distances y and $y+dy$ from A , is $\frac{dx}{a} \cdot \frac{dy}{a}$. But if the rod be first broken at P and then the portion AP be again broken at Q , the chance that these fractures should respectively lie at distances from A between x and $x+dx$ and between y and $y+dy$ is $\frac{dx}{a} \cdot \frac{dy}{x}$.

(ii) When a point P is said to be taken at random on a given area A or within a volume V , then, if R be the whole region in question, and if R be divided up into an infinite number of equal infinitesimally small regions $\delta R, \delta R', \delta R'', \dots$, it is understood that each element has the same chance of finding itself the recipient of the point P , and the chance

that specified points P, P', P'', \dots should occupy the respective elements $\delta R, \delta R', \delta R'', \dots$ is $\frac{\delta R}{R} \cdot \frac{\delta R'}{R} \cdot \frac{\delta R''}{R} \dots$

1689. To return to the case of a distribution of possible positions on a line $AB (=a)$. If, *after* a random selection of one point P on AB , a selection of Q be made at hazard upon

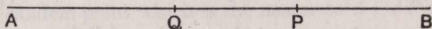


Fig. 533.

AP , it is evident that, since the number of possible positions for Q on AP is smaller than the number of possible positions for P in the whole line AB , the chance of any one element of AP distant between y and $y+dy$ from A being the recipient of Q is greater than that of the element between x and $x+dx$ being the recipient of P . The circumstance of the random choice of Q being made subsequently to the random choice of P , upon a limited range, has increased the chance of the dy element, but all equal elements between A and P have the same chance, the compound chance being, as before stated, $\frac{dx}{a} \cdot \frac{dy}{x}$.

1690. We have, then, for the total chance that AQ shall not be less than a certain length $c (<a)$,

$$\frac{\int_c^a \int_c^x \frac{dx}{a} \frac{dy}{x}}{\int_0^a \int_0^x \frac{dx}{a} \frac{dy}{x}} = \frac{\int_c^a \frac{dx}{ax} \cdot (x-c)}{\int_0^a \frac{dx}{ax} \cdot x} = \frac{a-c-c \log_e \frac{a}{c}}{a}$$

1691. Thus for a rod four feet long and AQ to exceed one foot, the chance $= (3 - \log 4)/4 = 4034\dots$

1692. It will be observed from Art. 1690 that for the compound event the chance of the element between x and $x+dx$ being the recipient of the random point P , and also being such that the subsequent random choice of Q will give a result for which $AQ < c$, is no longer $\frac{dx}{a}$ but $\frac{x-c}{x} \frac{dx}{a}$, and therefore the density of the possible positions of P on the line is not the same at various positions, but varies as $1 - \frac{c}{x}$, *i.e.* in a hyper-

bolic manner. This "density" of distribution may be represented graphically as in Fig. 534, and shows that the condensation of points P in an element dx , which can bring about a value AQ greater than c , increases from zero at $x=c$, and continues its increase as P approaches B , tending in a hyperbolic manner to an asymptote parallel to the x -axis.

Taking $\eta = k \frac{x-c}{x}$ as the equation of this graph, ηdx is a measure of the number of cases in which P lies in the element dx . That is, this number is proportional to the ordinate of the graph. And the total number of cases is measured on the same scale by the area between the x -axis, the curve and the ordinate at $x=a$. This area up to any definite ordinate is

$$\int_c^x k \frac{x-c}{x} dx = k \left\{ x - c - c \log \frac{x}{c} \right\}.$$

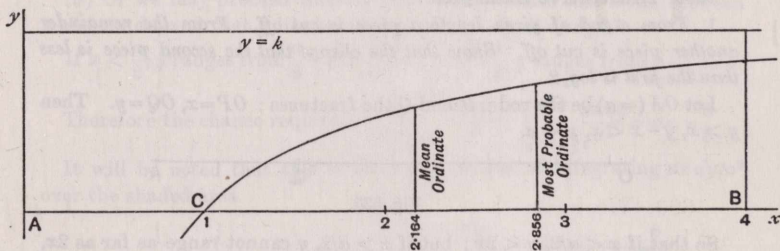


Fig. 534.

If we take an ordinate which bisects the whole area, viz. $x=x_0$, we have $k \left(x_0 - c - c \log \frac{x_0}{c} \right) = \frac{1}{2} k \left(a - c - c \log \frac{a}{c} \right)$; and this ordinate divides the whole line AB into two portions such that there are as many favourable cases for the event desired in defect of $AP (=x_0)$ as there are in excess. On these grounds the value $x=x_0$ is said to give the most probable case to secure the event.

In the case $a=4$ feet, $c=1$ foot, $x_0 - 1 - \log x_0 = \frac{1}{2}(3 - \log 4) = 0.8068$.

$\therefore x_0 - \log x_0 = 1.8068$, and by trial, or graphically, $x_0 = 2.8563$ nearly.

That is, in order that the portion AQ should exceed one-fourth of the rod, the most likely position for the first fracture to have been made is a little less than three-fourths of the length of the rod from A .

We shall call such a graph, indicating the density or condensation of points P in an element which are such that the

event may be brought to pass, the "Condensation" or "Density" graph. We shall return to it later. It is also sometimes called the "Curve of Frequency." (See Williamson, *Int. Calc.*, p. 369, ed. 8.)

In all previous cases the density or condensation has been uniform. It will now appear that many cases will arise when this is not so.

The mean value of the ordinates of the graph from $x=c$ to $x=a$ is given by

$$\int_c^a k \frac{x-c}{x} dx / \int_c^a dx = k \left(a-c - c \log \frac{a}{c} \right) / (a-c) = k - \frac{kc}{a-c} \log \frac{a}{c},$$

for which the abscissa is $\frac{a-c}{\log a - \log c}$.

In the numerical case cited, viz. $a=4, c=1, x=3/\log_e 4=2.164\dots$

1693. Illustrative Examples.

1. From a rod of given length a piece is cut off. From the remainder another piece is cut off. Show that the chance that the second piece is less than the first is $\log_e 2$.

Let $OA (=a)$ be the rod; P and Q the fractures; $OP=x, OQ=y$. Then $y > x, y-x < x, y < a$.

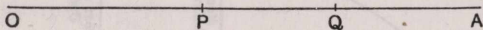


Fig. 535.

So that if $x < a/2, y < 2x$; but if $x > a/2, y$ cannot range as far as $2x$, and the inequality $y < 2x$ is necessarily satisfied and replaced by $y < a$, i.e.

when x ranges from 0 to $\frac{1}{2}a, y$ ranges from x to $2x$;

when x ranges from $\frac{1}{2}a$ to a, y ranges from x to a .

The chance of R lying between x and $x+dx$ is dx/a , and the chance of Q lying between y and $y+dy$ is $dy/(a-x)$.

Thus the chance required = $\int_0^{\frac{a}{2}} \int_x^{2x} \frac{dx}{a} \frac{dy}{a-x} + \int_{\frac{a}{2}}^a \int_x^a \frac{dx}{a} \frac{dy}{a-x} = \text{etc.} = \log_e 2$.

2. (i) Find the average distance between two points P and Q , where P is taken at random on a line AB of length a and Q is taken at random on AP .

[MATH. TRIP., 1883.]

Let $AP=x, AQ=y, x < y$.

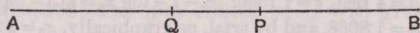


Fig. 536.

Then

$$M(QP) = \frac{\int_0^a \int_0^x (x-y) \frac{dx}{a} \frac{dy}{x}}{\int_0^a \int_0^x \frac{dx}{a} \frac{dy}{x}} = \text{etc.} = \frac{a}{4}.$$

(ii) Find the average distance between the two points P and Q when P and Q are taken at random on AB . [MATH. TRIP., 1883.]

Here Q may be on either side of P , and $x - y$ changes sign as Q passes P

$$M(\text{positive value of } QP) = \frac{\int_0^a \int_0^x (x-y) \frac{dx}{a} \frac{dy}{a} + \int_0^a \int_x^a (y-x) \frac{dx}{a} \frac{dy}{a}}{\int_0^a \int_0^a \frac{dx}{a} \frac{dy}{a}} = \text{etc.} = \frac{a}{3}.$$

3. Two lines are taken at random, each of length $< a$. Prove that the chance that, together with a line of length $\frac{1}{2}a$, they can form the three sides of a triangle is $\frac{5}{8}$. [ST. JOHN'S, 1883.]

(i) If $x, y, \frac{1}{2}a$ be the sides, we have

$$x < a, \quad y < a, \quad x + y > \frac{1}{2}a, \quad y + \frac{1}{2}a > x, \quad x + \frac{1}{2}a > y.$$

Take x, y as Cartesian coordinates of a point. Construct a square $OABC$ of side a , with OA, OC as coordinate axes. Let P, Q, R, S be the mid-points of the sides (Fig. 537). Then, of all points within the square, any point within the shaded area $PSBRQ$ will satisfy the conditions of the problem. Hence the chance required is $\frac{5}{8}$.

(ii) Or we may proceed directly thus: The chance that x lies between x and $x+dx$, and that y lies between y and $y+dy$, is $dx dy/a^2$.

If $x < \frac{a}{2}$, y ranges from $\frac{a}{2} - x$ to $\frac{a}{2} + x$; if $x > \frac{a}{2}$, y ranges from $x - \frac{a}{2}$ to a .

$$\text{Therefore the chance required} = \int_0^{\frac{a}{2}} \int_{\frac{a}{2}-x}^{\frac{a}{2}+x} \frac{dx dy}{a^2} + \int_{\frac{a}{2}}^a \int_{x-\frac{a}{2}}^a \frac{dx dy}{a^2} = \text{etc.} = \frac{5}{8}.$$

It will be noted that this is the exact process of integrating $dx dy/a^2$ over the shaded area.

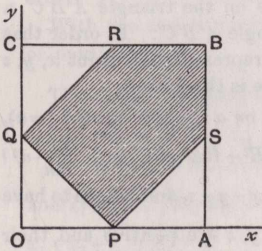


Fig. 537.

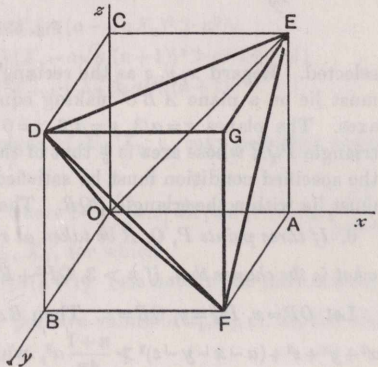


Fig. 538.

4. Three lines are chosen at random, each of length $< a$. Prove that the chance that they can form a triangle is $\frac{1}{2}$.

If x, y, z be the lengths, we must have $x < a$, etc.; $y + z > x$, etc.

Consider x, y, z the rectangular coordinates of a point. Of all points within a cube of edge a , three of whose edges coincide with the axes of

coordinates, those which give the result sought must be included between the three planes $y+z=x$, $z+x=y$, $x+y=z$, i.e. half the whole cube. Hence the chance is $\frac{1}{2}$.

5. A rod of length a is broken at random into two parts, and one of the two parts is taken at random and again broken at random. Show that for the two parts thus obtained the chance that neither is less than $\frac{1}{3}a$ is $\frac{1}{6}$.

[Ox. II. P., 1886.]

Let OQ be the part first broken off (Fig. 539), P the second fracture; $OP=x$, $PQ=y$, $QA=z$, $x+y+z=a$. Unless $x+y > 2a/3$ there is no chance that x and y shall be each $> a/3$. Therefore the larger portion must be

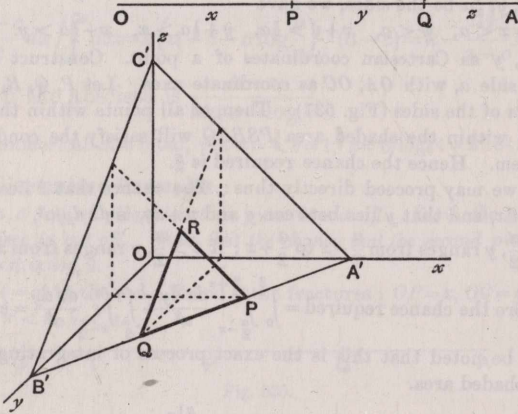


Fig. 539.

selected. Regard x, y, z as the rectangular coordinates of a point. This must lie on a plane $A'B'C'$ making equal intercepts a on the coordinate axes. The planes $x=a/3$, $y=a/3$, $z=0$ isolate on the triangle $A'B'C'$, a triangle PQR whose area is $\frac{1}{6}$ that of the triangle $A'B'C'$. In order that the specified condition must be satisfied, the representative point x, y, z must lie within the triangle PQR . The chance is therefore $\frac{1}{6}$.

6. If three points P, Q, R be taken at random on a straight line $OA (=a)$, what is the chance that, if $n > 3$, $OP^2 + PQ^2 + QR^2 + RA^2$ shall be $\geq \frac{n+1}{4n} a^2$?

Let $OP=x$, $PQ=y$, $QR=z$. Then $RA=a-x-y-z$, and we are to have $x^2 + y^2 + z^2 + (a-x-y-z)^2 \geq \frac{n+1}{4n} a^2$, whilst x, y, z are positive and their sum $< a$.

Take an orthogonal transformation in which

$$x+y+z = Z\sqrt{3} \quad \text{and} \quad x^2+y^2+z^2 = X^2+Y^2+Z^2,$$

where X, Y, Z are new variables. Then

$$X^2 + Y^2 + Z^2 + (a - Z\sqrt{3})^2 \geq \frac{n+1}{4n} a^2, \quad \text{i.e.} \quad X^2 + Y^2 + 4\left(Z - \frac{a\sqrt{3}}{4}\right)^2 \geq \frac{a^2}{4n}.$$

The whole range of the values of X, Y, Z is comprised within a spheroid of semi-axes $a/2\sqrt{n}, a/2\sqrt{n}, a/4\sqrt{n}$, which lies entirely within the tetrahedron $x=0, y=0, z=0, x+y+z=a$, provided n be large enough. The centre of the spheroid is at the point given by $x=y=z=a-x-y-z$, i.e. $(a/4, a/4, a/4)$. The minor axis lies along $x=y=z$. The perpendicular from the centre on the plane $x+y+z=a$ is $a/4\sqrt{3}$, and the minor semi-axis being $a/4\sqrt{n}$, we must have $n > 3$ in order that the spheroid shall not cut the face $x+y+z=a$. The same limitation will secure that the spheroid shall not cut any of the other faces of the tetrahedron, and must therefore be completely contained by the tetrahedron. With this limitation we therefore have

$$\text{Chance required} = \frac{\text{Vol. Spheroid}}{\text{Vol. Tetrahedron}} = \frac{\pi}{2n\sqrt{n}}.$$

7. If n random points P, Q, R be taken upon a line OA , what is the chance that the sum of the squares of the $(n+1)$ parts shall not exceed $\frac{1}{n}$ the square of the whole line?

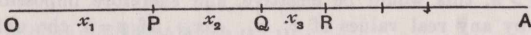


Fig. 540.

Let $x_1, x_2, x_3, \dots, x_n, a-x_1-x_2-\dots-x_n$, be the lengths of the successive parts. We are to have $x_1^2+x_2^2+\dots+(a-x_1-\dots-x_n)^2 \not\geq a^2/n$.

Take an orthogonal transformation in which $x_1+x_2+\dots+x_n=\sqrt{n}X_n$, and let X_1, X_2, \dots, X_n be the new variables. Then $\sum_1^n x_r^2 = \sum_1^n X_r^2$, and the condition becomes

$$X_1^2+X_2^2+\dots+X_n^2+(a-\sqrt{n}X_n)^2 \not\geq a^2/n.$$

i.e. $X_1^2+X_2^2+\dots+(n+1)\{X_n-a\sqrt{n}/(n+1)\}^2 \not\geq a^2/n(n+1)$

or $X_1^2+X_2^2+\dots+X_{n-1}^2+X_n^2 \not\geq a^2/n(n+1)$,

where $X_n-a\sqrt{n}/(n+1)=X_n'/\sqrt{n+1}$.

With the new variables the signs of X_1, X_2, \dots may be either positive or negative.

The chance required is N/D , where $N = \iiint \dots \int dX_1 dX_2 \dots dX_{n-1} \frac{dX_n'}{\sqrt{n+1}}$, for all values of $X_1, X_2, \dots, X_{n-1}, X_n'$, for which

$$X_1^2+X_2^2+\dots+X_{n-1}^2+X_n'^2 \not\geq a^2/n(n+1) \quad (\text{see note in the next article});$$

and $D = \iiint \dots \int dx_1 dx_2 \dots dx_n$ for positive values of x_1, x_2, \dots, x_n , for which $x_1+x_2+\dots+x_n \not\geq a$.

By Dirichlet's theorem $N = \frac{\left\{ \frac{a^2}{n(n+1)} \right\}^{\frac{n}{2}} \left\{ \Gamma\left(\frac{1}{2}\right) \right\}^n}{\Gamma\left(\frac{n}{2}+1\right)} \frac{1}{\sqrt{n+1}} 2^n$, the last

factor 2^n occurring because at each integration the result is to be doubled to take into account the negative signs of the respective variables;

$$\therefore N = \left\{ \frac{\pi a^2}{n(n+1)} \right\}^{\frac{n}{2}} \frac{1}{\sqrt{n+1} \Gamma\left(\frac{n}{2}+1\right)}, \quad \text{and} \quad D = \frac{a^n \{\Gamma(1)\}^n}{\Gamma^n \Gamma(n+1)};$$

$$\therefore \text{the chance required} = \frac{1}{\sqrt{n+1}} \left\{ \frac{\pi}{n(n+1)} \right\}^{\frac{n}{2}} \frac{\Gamma(n+1)}{\Gamma\left(\frac{n+2}{2}\right)}.$$

1694. NOTE.

Consider the equations

$$x_1^2 + x_2^2 + \dots + x_{n+1}^2 = a^2/p, \quad x_1 + x_2 + \dots + x_{n+1} = a (+ve).$$

Multiplying the second by $2a/n$ and subtracting,

$$\left(x_1 - \frac{a}{n}\right)^2 + \left(x_2 - \frac{a}{n}\right)^2 + \dots + \left(x_{n+1} - \frac{a}{n}\right)^2 = a^2 \left(\frac{1}{p} - \frac{1}{n} + \frac{1}{n^2}\right);$$

and therefore when one of the x 's is zero, say x_{n+1} ,

$$\sum_1^n \left(x_r - \frac{a}{n}\right)^2 = a^2 \left(\frac{1}{p} - \frac{1}{n}\right);$$

and if $p > n$, this would be negative, and therefore impossible to be satisfied by any real values of x_1, x_2, \dots, x_n . If $p = n$, the unique real solution would be $x_1 = x_2 = \dots = x_n = a/n$, where $x_{n+1} = 0$; and similarly if any of the other x 's were zero. We may suppose x_{n+1} as an abbreviation for $a - x_1 - x_2 - \dots - x_n$, and x_1, x_2, \dots, x_n as generalised coordinates.

(i) If $n = 2$, $x_1^2 + x_2^2 + x_3^2 = a^2/2$, where $x_3 = a - x_1 - x_2$, is a conic, and can only meet the lines $x_1 = 0, x_2 = 0, x_3 = 0$ at

$$x_1 = 0, x_2 = a/2, x_3 = a/2; \quad x_1 = a/2, x_2 = 0, x_3 = a/2; \quad x_1 = a/2, x_2 = a/2, x_3 = 0;$$

i.e. it is the ellipse which touches the lines $x_1 = 0, x_2 = 0, x_3 = 0$, at the mid-points of the sides of the triangle formed. The centre is at

$$x_1 = x_2 = x_3 = a/3,$$

and the ellipse is the maximum ellipse inscribable in the triangle. In homogeneous coordinates x_1, x_2, x_3 we may write it as

$$2(x_1^2 + \dots) = (x_1 + x_2 + x_3)^2 \quad \text{or} \quad \sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3} = 0.$$

(ii) If $n = 3$, $x_1^2 + x_2^2 + x_3^2 + x_4^2 = a^2/3$, where $x_4 \equiv a - x_1 - x_2 - x_3$, is a spheroid inscribed in the tetrahedron $x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0$, touching the faces at their several centroids.

In homogeneous coordinates x_1, x_2, x_3, x_4 ,

$$3(x_1^2 + x_2^2 + x_3^2 + x_4^2) = (x_1 + x_2 + x_3 + x_4)^2.$$

The centre is at $x_1 = x_2 = x_3 = x_4 = a/4$, and the spheroid lies entirely within the tetrahedron.

(iii) In the general case,

$$n(x_1^2 + x_2^2 + \dots + x_{n+1}^2) - (x_1 + x_2 + \dots + x_{n+1})^2 = 0$$

may be arranged as

$$(n-1)x_1^2 - 2x_1(x_2 + x_3 + \dots + x_{n+1}) + \sum_{r=3}^{r=n+1} (x_2 - x_r)^2 + \sum_4^{n+1} (x_3 - x_r)^2 + \dots + (x_n - x_{n+1})^2 = 0.$$

Hence if $n > 1$, x_1 cannot be negative unless $x_2 + x_3 + \dots + x_{n+1}$ be negative, which is impossible, since $x_1 + (x_2 + \dots + x_{n+1}) = a$, which is positive. And the same follows for each of the variables. That is, using language in analogy with the geometrical interpretations of (i) and (ii), the n -dimensional "spheroid" $x_1^2 + x_2^2 + \dots + x_{n+1}^2 = a^2/n$, in which $x_{n+1} \equiv a - x_1 - \dots - x_n$, lies entirely within the n -dimensional "region" defined by $x_1 = 0, x_2 = 0, \dots, x_{n+1} = 0$, and touches each of the "faces," viz., say, $x_1 = 0$ at $(0, \frac{a}{n}, \frac{a}{n}, \dots, \frac{a}{n})$, i.e. at its "centroid," and has its "centre" at $a/(n+1), \dots, a/(n+1)$, i.e. the "centroid" of the region, and may be written

$$\left(x_1 - \frac{a}{n+1}\right)^2 + \left(x_2 - \frac{a}{n+1}\right)^2 + \dots + \left(x_{n+1} - \frac{a}{n+1}\right)^2 = \frac{a^2}{n(n+1)}.$$

It will be seen, therefore, that in the integration of the preceding article it is proper to take the limits for X_1, X_2, \dots for all values of the variables for which $X_1^2 + \dots + X_n^2 \geq a^2/n(n+1)$; for negative values of these variables cannot imply any but positive values of the original variables x_1, x_2, \dots, x_{n+1} .

1695. GENERAL ILLUSTRATIONS.

1. If a rod be divided into p pieces at random, prove that the chance that none of the pieces shall be less than $1/m$ th of the whole, where $m > p$, is $(1 - p/m)^{p-1}$. [MATH. TRIP., 1875.]

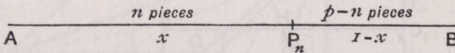


Fig. 541.

Let x be the distance of the n th point of division from one end, and let the length of the rod be taken as unity. Then, as each piece is to be $> 1/m$, we must have

$$x > n/m \text{ and } 1 - x > (p - n)/m, \text{ i.e. } 1 - (p - n)/m > x > n/m.$$

Hence each point of division, P_n , has a favourable range from $x = n/m$ to $x = 1 - p/m + n/m$, and the length of this range is $(1 - p/m)$ of the whole.

And since there are $p - 1$ points of division, the required chance is $(1 - p/m)^{p-1}$.

2. To examine the same problem by means of the Integral Calculus.

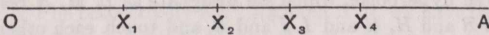


Fig. 542.

If X_1, X_2, \dots be the several points of division of the rod $OA (= a)$ at respective distances x_1, x_2 , etc., from O , we have $x_r > ra/m$ and $< x_{r+1} - a/m$ from $r = 1$ to $r = p - 1$, and $x_p = a = 1$. And the required chance is N/D , where

$$N = \int_{\frac{a}{(p-1)m}}^{a - \frac{a}{m}} \dots \int_{\frac{2a}{m}}^{x_3 - \frac{a}{m}} \int_{\frac{a}{m}}^{x_2 - \frac{a}{m}} dx_{p-1} dx_{p-2} \dots dx_1;$$

and D is the same when $m = \infty$.

Hence performing the integrations, $N/D = (1 - p/m)^{p-1}$, as before.

3. A rod $XY (= a)$ is broken at hazard into three portions. If these three parts can form the sides of a triangle, what is the chance it is acute angled?

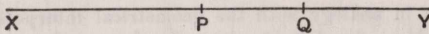


Fig. 543.

In Art. 1683, Ex. 3 (iv), it has been seen that the chance the parts form a triangle is $\frac{1}{4}$.

Let P, Q be the fractures, $XP = x$, $XQ = y$, $y > x$. As in the article cited, we must have

$$x < a/2, \quad y < x + a/2, \quad y > a/2.$$

To be acute angled, we must also have

$$(y-x)^2 + (a-y)^2 > x^2, \quad (a-y)^2 + x^2 > (y-x)^2, \quad x^2 + (y-x)^2 > (a-y)^2,$$

i.e. $y(y-x-a) + a^2/2 > 0$, $y(x-a) + a^2/2 > 0$, $(x-a)(x-y+a) + a^2/2 > 0$.

All values of x and y from $x=0$ to $x=y$, and $y=0$ to $y=a$, are equally likely. Refer to rectangular axes Ox, Oy , as before, with the same description of figure.

The region bounded by the hyperbolae $y(y-x-a) + a^2/2 = 0$, etc., includes the only positions in which the representative point (x, y) can lie to ensure that the triangle formed by the portions of the rod shall be acute

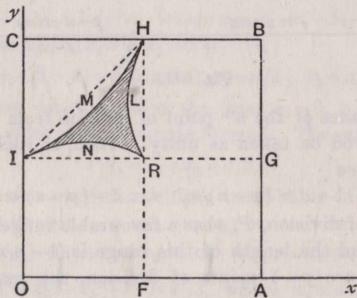


Fig. 544.

angled. These hyperbolae, which we designate as L, M, N respectively, pass through R and H , H and I , I and R , and touch each other at these points. The three segments bounded by L, M, N and their respective chords are

$$\text{for } L, \quad \int_{\frac{a}{2}}^a \left(\frac{a}{2} - x \right) dy = \int_{\frac{a}{2}}^a \left(\frac{3a}{2} - y - \frac{a^2}{2y} \right) dy = \frac{3}{8} a^2 - \frac{a^2}{2} \log 2;$$

$$\text{for } M, \quad \int_{\frac{a}{2}}^a \left\{ \left(a - \frac{a^2}{2y} \right) - \left(y - \frac{a}{2} \right) \right\} dy = \frac{3}{8} a^2 - \frac{a^2}{2} \log 2;$$

$$\text{for } N, \quad \int_0^{\frac{a}{2}} \left(y - \frac{a}{2} \right) dx = \int_0^{\frac{a}{2}} \left(x + \frac{a}{2} - \frac{a^2}{2} \frac{1}{a-x} \right) dx = \frac{3}{8} a^2 - \frac{a^2}{2} \log 2.$$

Therefore the area of the curvilinear triangle *RHI*

$$= \frac{a^2}{8} - 3\left(\frac{3}{8}a^2 - \frac{1}{2}a^2 \log 2\right) = \left(\frac{3}{2} \log 2 - 1\right) a^2.$$

Therefore the chance that the three segments of the rod form an acute-angled triangle $= \left(\frac{3}{2} \log 2 - 1\right) a^2 / \frac{1}{2} a^2 = 3 \log 2 - 2.$

The chance that any specific angle is obtuse

$$= \left(\frac{3}{8} a^2 - \frac{a^2}{2} \log 2\right) \bigg/ \frac{a^2}{2} = (3 - 4 \log 2) / 4.$$

The chance that the triangle is obtuse angled $= \frac{3}{4} (3 - 4 \log 2).$

The chance that the triangle is right angled is of course infinitesimally small.

4. *P, Q, R are random points, one on each of three equal lines X_1Y_1, X_2Y_2, X_3Y_3 ($=a$). What is the chance that the portions X_1P, X_2Q, X_3R may form an acute-angled triangle?*

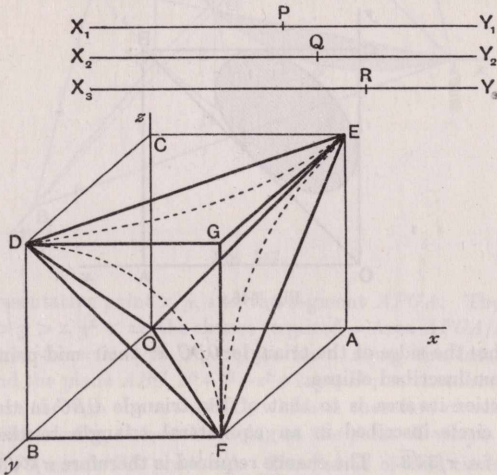


Fig. 545.

In Art. 1693, 4, the chance the parts form a triangle has been seen to be $\frac{1}{2}$. If x, y, z be respectively X_1P, X_2Q and X_3R , we have the additional conditions $y^2 + z^2 > x^2, z^2 + x^2 > y^2, x^2 + y^2 > z^2$. Referring to rectangular axes, as before, the surfaces of the right cones $y^2 + z^2 = x^2$, etc., separate the favourable positions of the representative point from the unfavourable ones. These cones touch in pairs along their common generators, which lie in the coordinate planes. The volume of the part of the cube included between them

$$= a^3 - 3 \cdot \frac{1}{3} \cdot \frac{\pi a^2}{4} \cdot a = \left(1 - \frac{\pi}{4}\right) a^3.$$

Hence the chance required $= \left(1 - \frac{\pi}{4}\right) a^3 / a^3 = 1 - \frac{\pi}{4} = .2146\dots$

5. Two points P and Q are taken at hazard upon a line $AB (= a)$, P being the nearer to A . What is the chance that the sum of the products of the segments two and two together exceeds one-fourth of the square of the line?

Let $AP = x$, $AQ = y$, $y > x$. Then x ranges from 0 to y and y from 0 to a .

The limiting case is $x(y-x) + (y-x)(a-y) + (a-y)x = \frac{a^2}{4}$.

Referring to rectangular coordinates Ox , Oy , the representative point x, y may lie anywhere within the half OBC of a square $OABC$ of side a , whose sides OA , OB are along the axes Ox , Oy ; and the favourable cases are indicated by points lying within the ellipse $x^2 - xy + y^2 - ay + \frac{a^2}{4} = 0$,

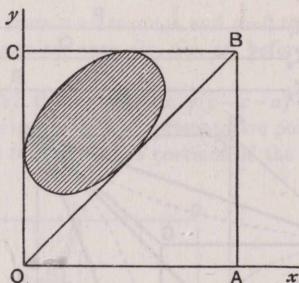
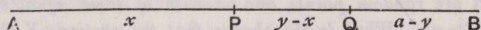


Fig. 546.

which touches the sides of the triangle OBC at their mid-points, and is the maximum inscribed ellipse.

By projection its area is to that of the triangle OBC in the ratio of that of a circle inscribed in an equilateral triangle to that of the equilateral, *i.e.* $\pi/3\sqrt{3}$. The chance required is therefore $\pi/3\sqrt{3}$.

6. A rod of length a is broken at random into three parts. What is the chance that the square on the mean segment shall be less than the rectangle contained by the other two?

Let x, y, z be the lengths of the segments. Suppose y the mean segment. Then

$$x > y > z \text{ or } x < y < z; \quad x + y + z = a; \quad y^2 < zx.$$

Refer to rectangular axes Ox, Oy, Oz . Let $OA = OB = OC = a$ (Fig. 547). Then $x + y + z = a$ is the plane ABC . Let D, E, F , be the mid-points of the sides, G the point $(a/3, a/3, a/3)$. The equations of the planes COF' and AOD are respectively $y = x$ and $y = z$.

The inequalities $y < x$ and $y < z$ for points on the plane ABC limit the region to the triangle AGF' .

The cone $y^2 = zx$ has OA and OC for generators, the coordinate planes $x=0$ and $z=0$ being tangential, and it passes through G , cutting the plane ABC in an arc $APGQC$. For points of the triangle AGB on the concave side of the arc we have $y^2 < zx$. This further limits the range

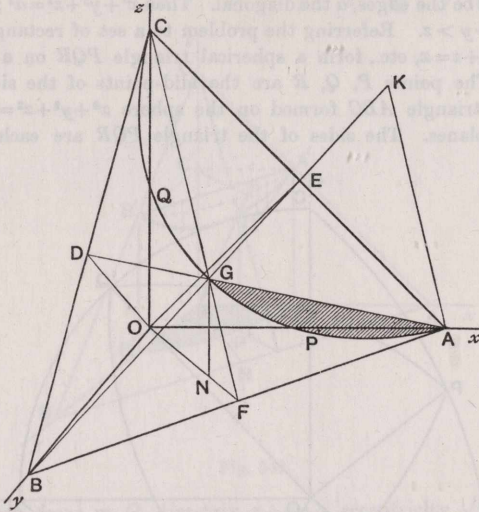


Fig. 547.

of the representative point x, y, z to the segment $APGA$. Therefore, for the case $x > y > z, y^2 < zx$, the chance required = Area $APGA$ /Area ABC .

Now, since $2xz = (a - y)^2 - x^2 - z^2$, we have along the intersection of the cone and the plane $ABC, x^2 + y^2 + z^2 + 2ay = a^2$; so that it is possible to pass a sphere through the arc $APQC$, which is therefore circular, as may be seen geometrically, the centre being at the point K where AK drawn parallel to FG meets BE produced. The radius of this circle = $a\sqrt{2/3}$;

$$\text{and Area } APGA = \frac{1}{2} \cdot \frac{2a^2}{3} \cdot \frac{\pi}{3} - \frac{1}{2} \cdot \frac{2a^2}{3} \cdot \frac{\sqrt{3}}{2} = \frac{a^2}{18} (2\pi - 3\sqrt{3}).$$

$$\text{Hence for this case the chance is } \frac{a^2}{18} (2\pi - 3\sqrt{3}) / \frac{a^2}{2} \sqrt{3} = \frac{2\pi\sqrt{3} - 9}{27}.$$

There are six such cases, viz.

$$\left. \begin{matrix} x > y > z \\ x < y < z \end{matrix} \right\} \text{ with } y^2 < zx; \quad \left. \begin{matrix} y > z > x \\ y < z < x \end{matrix} \right\} \text{ with } z^2 < xy; \quad \left. \begin{matrix} z > x > y \\ z < x < y \end{matrix} \right\} \text{ with } x^2 < yz.$$

$$\text{Therefore the total chance} = \frac{6}{27} (2\pi\sqrt{3} - 9) = \frac{2}{9} (2\pi\sqrt{3} - 9) = .418399\dots$$

If a specific segment of the line, say the middle one, is to satisfy the same conditions, we then have the two cases $x > y > z, x < y < z$, with $y^2 < zx$, and the chance is $\frac{2}{27} (2\pi\sqrt{3} - 9)$, i.e. one-third of the total chance considered above.

7. A rectangular parallelepiped is constructed with a given diagonal, and edges of any possible lengths are equally likely. What is the chance that a triangle could be constructed with its sides equal to those edges of the parallelepiped which meet in a point?

Let x, y, z be the edges, a the diagonal. Then $x^2 + y^2 + z^2 = a^2$; $y + z > x$, $z + x > y$, $x + y > z$. Referring the problem to a set of rectangular axes, the planes $y + z = x$, etc., form a spherical triangle PQR on a sphere of radius a . The points P, Q, R are the mid-points of the sides of the quadrantal triangle ABC formed on the sphere $x^2 + y^2 + z^2 = a^2$ by the coordinate planes. The sides of the triangle PQR are each $\pi/3$, and

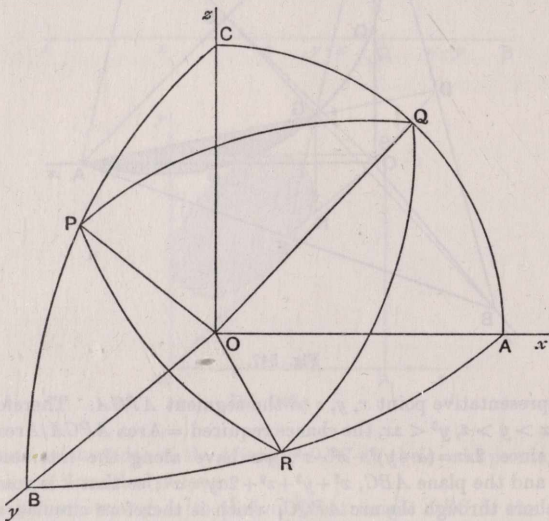


Fig. 548.

$\cos P = \cos Q = \cos R = \frac{1}{3}$. The spherical excess $= 3 \cos^{-1} \frac{1}{3} - \pi$. The area of the triangle $PQR = a^2 (3 \cos^{-1} \frac{1}{3} - \pi)$. The area of the triangle $ABC = \frac{1}{2} \pi a^2$. The "favourable" region for x, y, z consists of the three spherical triangles, AQR, BRP, CPQ , the sum of whose areas

$$= \frac{\pi a^2}{2} - a^2 \left(3 \cos^{-1} \frac{1}{3} - \pi \right) = 3a^2 \left(\frac{\pi}{2} - \cos^{-1} \frac{1}{3} \right) = 3a^2 \sin^{-1} \frac{1}{3}.$$

Hence the required chance $= \frac{6}{\pi} \sin^{-1} \frac{1}{3}$.

8. A rod AB ($=a$) is broken at hazard at two points P, Q . What is the chance that PQ shall be such that $PQ^2 \leq \frac{1}{n} (AP^2 + QB^2)$?

Let $AP = x, PQ = z, QB = y, x + y + z = a$, and we are to have $nz^2 \leq x^2 + y^2$. Refer, as before, to rectangular axes Ox, Oy, Oz . Then, of all points in

the plane $x+y+z=a$ (Fig. 549), those which lie within the right circular cone $x^2+y^2=nz^2$ are "favourable." The projection $A''B''$ of the line of intersection $A'B'$ upon the z -plane is $x^2+y^2=n(a-x-y)^2$, i.e.

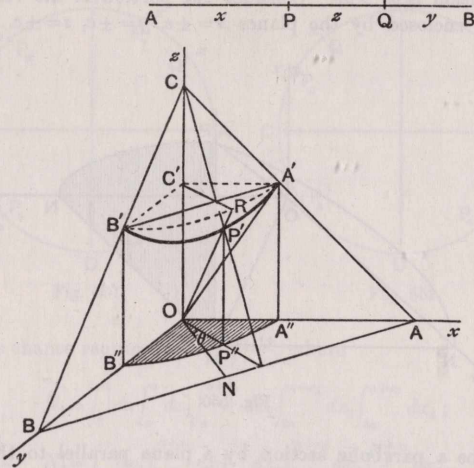


Fig. 549.

a conic with focus at O , directrix $x+y=a$, eccentricity $\sqrt{2n}$. Turning the axes round so that ON , the perpendicular upon $x+y=a$, is the new x -axis, the conic becomes $X^2+Y^2=n(a-X\sqrt{2})^2$, i.e. in polars

$$a\sqrt{n}/r = 1 + \sqrt{2n} \cos \theta.$$

The area of the portion of this conic between the radii OA'' , OB'' (Fig. 549), in the case when $n < \frac{1}{2}$, is

$$\int_{-\pi/4}^{\pi/4} r^2 d\theta = \frac{a^2 n}{2} \int_{-\pi/4}^{\pi/4} \frac{d\theta}{(1 + \sqrt{2n} \cos \theta)^2} = \text{etc.} = \frac{a^2 n}{(1 - 2n)^{3/2}} \left[\cos^{-1} \frac{1 + 2\sqrt{n}}{\sqrt{2}(1 + \sqrt{n})} - \sqrt{n} \frac{\sqrt{1 - 2n}}{1 + \sqrt{n}} \right]$$

And the chance required

$$= \text{Area } OA''B'' / \text{Area } OAB = \frac{2n}{(1 - 2n)^{3/2}} \left[\cos^{-1} \left(\frac{1 + 2\sqrt{n}}{1 + \sqrt{n}} \cdot \frac{1}{\sqrt{2}} \right) - \sqrt{n} \frac{\sqrt{1 - 2n}}{1 + \sqrt{n}} \right].$$

If $n = \frac{1}{2}$, the conic $A''B''$ is a parabola, viz. $a/r\sqrt{2} = 2 \cos^2 \frac{\theta}{2}$.

In this case, $\text{Area } OA''B'' = \frac{a^2}{8} \int_0^{\pi/4} \sec^4 \frac{\theta}{2} d\theta = \text{etc.} = \frac{a^2}{6} (4\sqrt{2} - 5)$,

and the chance required $= (4\sqrt{2} - 5)/3 = .21895 q.p.$

If $n > \frac{1}{2}$, the conic $A''B''$ is hyperbolic, and the chance required is

$$= \frac{2n}{(2n - 1)^{3/2}} \left\{ \sqrt{n} \frac{\sqrt{2n - 1}}{\sqrt{n + 1}} - \cosh^{-1} \frac{2\sqrt{n + 1}}{\sqrt{2}(\sqrt{n + 1})} \right\},$$

9. The equation $ax^2 + 2hxy + by^2 = 1$ is written down at random with real coefficients. Find the chance that it represents a hyperbola.

[Ox. II. P., 1887.]

The condition is $h^2 > ab$. Consider the portion of the volume of the cone $z^2 = xy$ enclosed by the planes $x = \pm c$, $y = \pm c$, $z = \pm c$. Let PMN

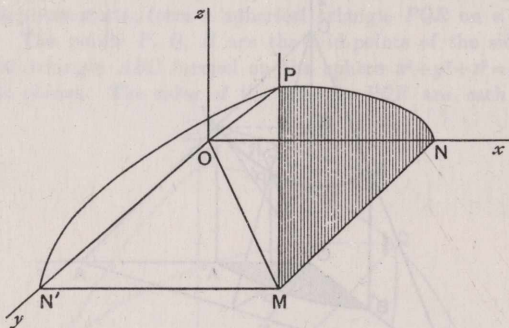


Fig. 550.

(Fig. 550) be a parabolic section by a plane parallel to the y - z plane bounded by the planes $x = y$, $z = 0$. The volume, to $x = c$,

$$= \int_0^c \frac{2}{3} MN \cdot MP dON = \frac{2}{3} c^3.$$

The volume enclosed within the cube, $x = \pm c$, $y = \pm c$, $z = \pm c$, is $8 \cdot \frac{2}{3} \cdot c^3$; and the volume of the cube $= 8c^3$.

The representative point of a , b , h , viz. x , y , z must lie outside the cone but inside the cube, however large c may be.

Hence the chance required $= 1 - \frac{2}{3} = \frac{1}{3}$.

10. Six points are taken at hazard on the circumference of a circle. What is the chance that no two consecutive selected points are separated by more than a quadrant?

It will not affect the problem if we regard one of the points, viz. A , to be at a particular point of the circle. Let AC , BD be perpendicular diameters. Let the other five selected points be P_1 , P_2 , P_3 , P_4 and Q at arcual distances x_1 , x_2 , x_3 , x_4 and x respectively from A measured counter-clockwise. One of these five must be in each quadrant, and not more than two in any one quadrant. Let P_1 , P_2 , P_3 , P_4 be the points which lie in the first, second, third and fourth quadrants, and Q the one whose quadrant is unassigned. It will be sufficient to consider the two cases, (1) when Q lies in the first quadrant, (2) when Q lies in the second quadrant, for the number of cases when two lie in the fourth or third quadrants are the same as if two lie in the first or second respectively. Also when Q lies in the first or the second quadrant, we shall suppose that point of

the two which is nearer to A to be designated as Q . Let the length of a quadrantal arc = a . Then the two cases to consider are

(1) $x < x_1 < a$ and $a < x < x_2 < 2a$ (Figs. 551, 552).

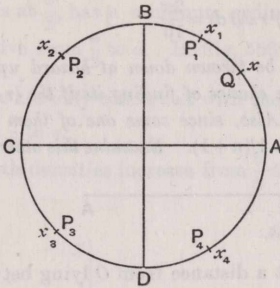


Fig. 551.

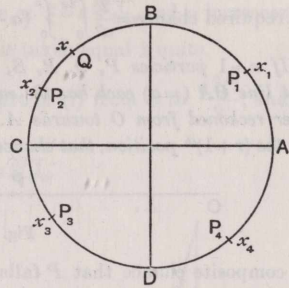


Fig. 552.

Then the chance required = $\frac{2N_1 + 2N_2}{D}$, where

$$N_1 = \int_0^a dx \int_x^a dx_1 \int_a^{a+x_1} dx_2 \int_{2a}^{a+x_2} dx_3 \int_{3a}^{a+x_3} dx_4;$$

$$N_2 = \int_0^a dx_1 \int_a^{a+x_1} dx \int_x^{2a} dx_2 \int_{2a}^{a+x_2} dx_3 \int_{3a}^{a+x_3} dx_4,$$

$$D = \int_0^{4a} dx_4 \int_0^{x_4} dx_3 \int_0^{x_3} dx_2 \int_0^{x_2} dx_1 \int_0^{x_1} dx.$$

The values of these integrals are readily shown to be $N_1 = 4a^5/5!$; $N_2 = 9a^5/5!$; $D = (4a)^5/5!$.

Hence the chance required = $\frac{26a^5/5!}{(4a)^5/5!} = \frac{13}{2^9}$.

11. Three random points L, M, N are taken within a circle of centre O and radius a . Find the chance that the circumcircle of LMN lies wholly within the original circle. [R.P.]

Let P be the centre and x the radius of the circumcircle, and $OP = r$. Take an arbitrary and indefinitely small strip of breadth k round the circumcircle. Its area = $2\pi xk$ to the first order. The chance that three random points should fall upon it = $\left(\frac{2\pi xk}{\pi a^2}\right)^3$, which we may write as $k^2 \frac{8x^3}{a^6} dx$. Integrating with regard to x from $x=0$ to $x=a-r$, which varies the size of this circle from radius zero to such a size that it will just not cut the original

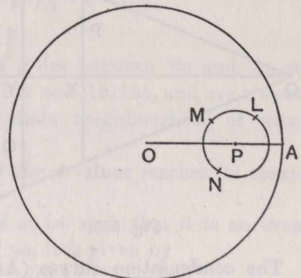


Fig. 553.

circle, we have $\frac{2k^2}{a^6} (a-r)^4$, where k^2 is an arbitrary elementary area at our

choice. We are now to sum up all such results as the above for various positions of P within the original circle. Replace k^2 by $r d\theta dr$, and integrate over the large circle.

$$\text{The required chance} = \frac{2}{a^6} \int_0^{2\pi} \int_0^a (a-r)^4 r d\theta dr = \frac{2\pi}{15}.$$

12. If $n+1$ particles P, Q, R, S, \dots be thrown down at hazard upon a straight line $OA (=a)$ each has the same chance of finding itself the $(r+1)^{\text{th}}$ in order reckoned from O towards A . Also, since some one of them must occupy the $(r+1)^{\text{th}}$ position, that chance is $1/(n+1)$. Examine this otherwise.

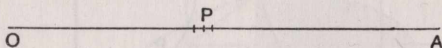


Fig. 554.

The composite chance that P falls at a distance from O lying between x and $x+dx$, and that r unspecified particles lie between O and P , and the rest between P and A , is ${}^nC_r \left(\frac{x}{a}\right)^r \left(\frac{a-x}{a}\right)^{n-r} \frac{dx}{a}$, and therefore the chance that P occupies the $(r+1)^{\text{th}}$ place irrespective of where it lies upon $OA = {}^nC_r \int_0^a x^r (a-x)^{n-r} dx / a^{n+1} = \text{etc.} = 1/(n+1)$.

13. Two points P and Q are selected at random within the volume of a right circular cone, and circular sections are drawn through them. What is the chance that the volume of the slice exceeds $1/8$ of the cone?

Take the vertex as the origin and the axis as x -axis, x and y the abscissae of the points and $y > x$. The chance that a random point has an abscissa lying between x and $x+dx$ is proportional to the volume of a slice of thickness dx , the abscissa of one of its faces being x , i.e. to $x^2 dx$. Also if a be the length of the axis, $y^3 - x^3 \leq \frac{1}{8} a^3$. The chance may then be written either as

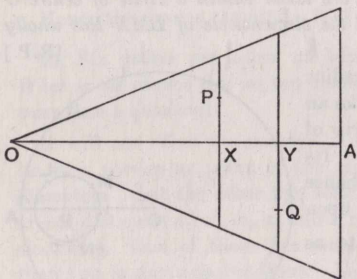


Fig. 555.

$$\frac{\int_0^{\frac{a}{2}\sqrt[3]{7}} x^2 dx \int_{\sqrt[3]{x^2 + \frac{1}{8}a^3}}^a y^2 dy}{\int_0^a x^2 dx \int_x^a y^2 dy};$$

or as

$$\frac{\int_{\frac{a}{2}}^a y^2 dy \int_0^{\sqrt[3]{y^3 - \frac{1}{8}a^3}} x^2 dx}{\int_0^a y^2 dy \int_0^y x^2 dx};$$

and each gives a result $49/64$.

The condensation curves (Art. 1692) for P -points and for Q -points, indicating the density of clustering on the x -axis of the ends of their abscissae, are

$$(i) a^4 \eta = \xi^2 \left(\frac{7}{8} a^3 - \xi^3\right) \quad \text{and} \quad (ii) a^4 \eta = \xi^2 \left(\xi^3 - \frac{a^3}{8}\right).$$

Each touches the ξ -axis at the origin ; (i) crosses the ξ -axis at $\frac{a}{2}\sqrt[3]{7}$, and has a maximum ordinate at $\xi = a\sqrt[3]{\frac{7}{20}} = a \times .70473\dots$; (ii) crosses the ξ -axis at $\frac{a}{2}$, has a minimum ordinate at $\xi = a\sqrt[3]{\frac{1}{20}}$, and η increases and is positive from $\frac{a}{2}$ to a . In Fig. 556 a is taken equal 2 units.

We are only concerned with the part of (i) from 0 to $\frac{a\sqrt[3]{7}}{2}$, and of (ii) from $\frac{a}{2}$ to a .

Both densities increase from $\frac{a}{2}$ to $a\sqrt[3]{\frac{7}{20}}$.

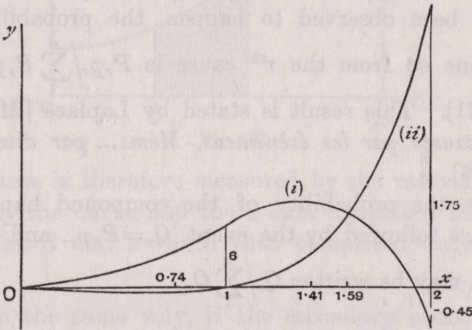


Fig. 556.

The first decreases and the second increases for the rest of the range.

If we require the chance that under the stated circumstances the point P possesses an abscissa lying between certain limits, say βa and αa , where $0 < \beta < \alpha < 1$, that chance is

$$C = \frac{\int_{\beta a}^{\alpha a} x^2 (\frac{7}{5}a^3 - x^3) dx}{\int_0^a x^2 (a^3 - x^3) dx} = (a^3 - \beta^3)(\frac{7}{4} - a^3 - \beta^3).$$

It will be found that the chances that x lies between $.6a$ and $.7a$, or between $.7a$ and $.8a$, are respectively $.151257$ and $.151255$, and are almost exactly the same. This is in the immediate neighbourhood of max. condensation.

The point at which the condensation of the x -values reaches its maximum is $a\sqrt[3]{\frac{7}{20}} = a \times .70473$.

If γa be the "most probable value" of x , i.e. such that it is an even chance whether x exceeds or falls short of γa , it is given by

$$\gamma^3(4 - \gamma^3) = \frac{1}{2} \cdot \frac{49}{84}, \quad \text{i.e. } \gamma = \frac{\sqrt[3]{7}}{2} \sqrt[3]{1 - \frac{1}{\sqrt{2}}}.$$

The ordinate at this point bisects the portion of the area in the first quadrant of the condensation curve for P -points.

1696. Inverse Probability.

Questions involving the probability of causes as deduced from observed events are called questions on "inverse" probability. Supposing $P_1, P_2, \dots P_n$, the probabilities of the existence of the several causes of an event known to have happened, and that these causes are mutually exclusive, and that these are the only causes through which the event could have happened; and further, supposing that $p_1, p_2, \dots p_n$ are the respective probabilities that when the cause exists the event will follow, then it is known that in any case when the event has been observed to happen, the probability of its having done so from the r^{th} cause is $P_r p_r / \sum_1^n P_r p_r$ (Smith, *Alg.*, p. 521). This result is stated by Laplace [*Mém. sur la prob. des causes par les évènements, Mém. ... par divers savans*, T. vi., 1774].

If Q_r be the probability of the compound happening of the r^{th} cause followed by the event, $Q_r = P_r p_r$, and the above expression may be written $Q_r / \sum_1^n Q_r$.

1697. Let the probability of the happening of a certain event A , which we may call the cause of a second event B , be x , which varies from 0 to 1. Let the happening of B depend upon the happening of A in such a manner that the compound probability of B 's happening is $\phi(x)$. It is observed that B happens. What is the chance that x lies between two assigned limits β and a ? ($0 < \beta < a < 1$.)

Let OC denote unit length on the x -axis, and let the graph of $y = \phi(x)$ be drawn (Fig. 557). The ordinates represent the probability of B happening corresponding to the abscissa which represents that of A .

Let OC be divided into n equal elements of length h , $nh = 1$. The points of division are at distances from O , $0/n$, $1/n$, $2/n$, etc., and the probability of the existence of the r^{th} cause is

$$\phi\left(\frac{r}{n}\right) / \sum_0^n \phi\left(\frac{r}{n}\right), \quad \text{i.e.} \quad \frac{1}{n} OC \phi\left(\frac{r}{n} OC\right) / \sum_{\frac{r}{n}=0}^{\frac{r}{n}=1} \frac{1}{n} OC \phi\left(\frac{r}{n} OC\right).$$

Hence the probability of the abscissa lying between x and

$x+dx$ is $\phi(x)dx / \int_0^1 \phi(x)dx$; and therefore the chance that the abscissa lies between β and a is $\int_\beta^a \phi(x)dx / \int_0^1 \phi(x)dx$.

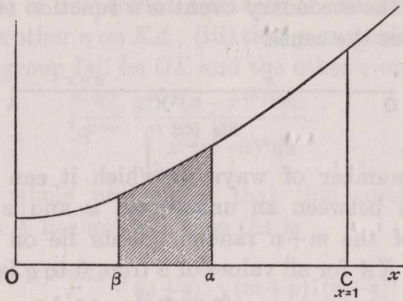


Fig. 557.

This chance is therefore measured by the ratio of the area bounded by the curve and the x -axis comprised between the ordinates $x=\beta$ and $x=a$ to that comprised between $x=0$ and $x=1$.

1698. In the same way, if the secondary event B be dependent upon two (or more) primary events A_1, A_2 , whose probabilities are represented by x_1, x_2 , whilst that of the dependent secondary event is $\phi(x_1, x_2)$, the chance that the probabilities of these primary events respectively lie between β_1 and a_1, β_2 and a_2 , where $0 < \beta_1 < a_1 < 1$ and $0 < \beta_2 < a_2 < 1$, is

$$\int_{\beta_1}^{a_1} \int_{\beta_2}^{a_2} \phi(x_1, x_2) dx_1 dx_2 / \int_0^1 \int_0^1 \phi(x_1, x_2) dx_1 dx_2,$$

with corresponding expressions if there be more than two variables.

1699. Recurring to Ex. 12, Art. 1695, we have seen that if a point X be taken at random on a line $OA=a$, and then $m+n$ other points be taken at random on the same line, the chance that m unspecified points of the group lie between O and X and the remainder between X and A is

$${}^{m+n}C_m \int_0^a \left(\frac{x}{a}\right)^m \left(\frac{a-x}{a}\right)^n \frac{dx}{a} = \frac{1}{m+n+1},$$

a fact obvious from another consideration as pointed out. We may use this problem to illustrate the result obtained in

Art. 1697. The fact that X lies at a distance x from O may be regarded as a primary event or cause from which the nature of the secondary event, viz. the particular allocation of the $m+n$ unspecified points, arises; and the chance of the happening of the secondary event is a function of the variable x which defines the cause.

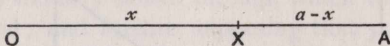


Fig. 558.

The total number of ways in which it can happen that whilst X lies between an unassigned x and $x+dx$, an unspecified m of the $m+n$ random points lie on OX and the remainder on XA for all values of x from 0 to a is measured by

$${}^{m+n}C_m a^{m+n+1} \int_0^a \left(\frac{x}{a}\right)^m \left(\frac{a-x}{a}\right)^n \frac{dx}{a};$$

and the number of ways the same thing can happen when X lies between an assigned x and $x+dx$ is measured by

$${}^{m+n}C_m a^{m+n+1} \left(\frac{x}{a}\right)^m \left(\frac{a-x}{a}\right)^n \frac{dx}{a}.$$

Therefore, when the compound event happens, the chance that x lies between x and $x+dx$ is the ratio of the second of these expressions to the first, i.e. $x^m(a-x)^n dx / \int_0^a x^m(a-x)^n dx$.

And the chance that when the compound event happens, X will lie between $x=\beta$ and $x=a$, ($0 < \beta < a < a$) is

$$\int_{\beta}^a x^m(a-x)^n dx / \int_0^a x^m(a-x)^n dx.$$

1700. Next suppose that a new group of $p+q$ random points is taken upon the line OA . What is the chance that an unspecified p of these points also lie between O and X and the remainder between X and A ?

The total number of such cases when X falls between x and $x+dx$ will be

$${}^{m+n}C_m {}^{p+q}C_p a^{m+n+p+q+1} \left(\frac{x}{a}\right)^m \left(\frac{a-x}{a}\right)^n \left(\frac{x}{a}\right)^p \left(\frac{a-x}{a}\right)^q \frac{dx}{a},$$

and the total number of cases for all positions of X , in which m unspecified points of the $m+n$ lie on OX , whilst the other

n lie on XA , whilst the $p+q$ points are distributed anywhere on the line, is ${}^{m+n}C_n a^{m+n+1} a^{p+q} \int_0^a \left(\frac{x}{a}\right)^m \left(\frac{a-x}{a}\right)^n \frac{dx}{a}$.

Therefore the compound chance that (i) X lies between x and $x+dx$; (ii) m unspecified members of the first group fall on OX and the other n on XA ; (iii) that p unspecified members of the second group fall on OX and the other q on XA , is

$$\frac{{}^{p+q}C_p x^{m+p}(a-x)^{n+q} dx}{a^{p+q} \int_0^a x^m(a-x)^n dx}$$

Hence the whole probability that this compound event happens when X lies anywhere on OA is

$$\frac{{}^{p+q}C_p \int_0^a x^{m+p}(a-x)^{n+q} dx}{a^{p+q} \int_0^a x^m(a-x)^n dx} = \frac{(p+q)!}{p!q!} \frac{(m+p)!(n+q)!}{(m+n+p+q+1)!} \frac{(m+n+1)!}{m!n!}$$

1701. The above problem forms a landmark in the History of Probability. It is associated with the names of many investigators, Bayes, Condorcet, Trembley, Laplace and others. (See Todhunter's *History*, pages 295, 383, 399, 414, 467, etc.)

It is often enunciated in a different way.

An urn is supposed to contain an infinite number of white tickets and an infinite number of black tickets, and no others, and that is all that is supposed to be known as to the tickets. These tickets correspond to possible situations of a point to the left of X or to the right of X in the foregoing problem. Then $m+n$ tickets having been drawn from the urn, m are found to be white and the remainder black. What is the probability that a further drawing of $p+q$ tickets will result in p being white and q black?

Laplace gives the required result as $\frac{\int_0^1 x^{m+p}(1-x)^{n+q} dx}{\int_0^1 x^m(1-x)^n dx}$,

which, without the factor $(p+q)!/p!q!$, supposes the tickets to have been drawn in a specific order. Todhunter quotes the following remark of Laplace: "La solution de ce problème donne une méthode directe pour déterminer la probabilité des évènements futurs d'après ceux qui sont déjà arrivés."

1702. Next suppose that on the line $OA (=a)$ several random points X_1, X_2, \dots, X_{n-1} be taken at distances x_1, x_2, \dots, x_{n-1}

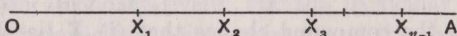


Fig. 559.

from O , in this order, and let $p_1 + p_2 + \dots + p_n$ other random points be taken upon OA . Then the compound chance that (i) X_1 lies between x_1 and $x_1 + dx_1$, X_2 between x_2 and $x_2 + dx_2$, etc.; (ii) p_1 specified points fall on OX_1 , p_2 on X_1X_2 , p_3 on X_2X_3 , etc., is

$$\left(\frac{x_1}{a}\right)^{p_1} \left(\frac{x_2 - x_1}{a}\right)^{p_2} \dots \left(\frac{a - x_{n-1}}{a}\right)^{p_n} \cdot \frac{dx_1}{a} \cdot \frac{dx_2}{a} \dots \frac{dx_{n-1}}{a}.$$

Hence, for *unspecified* groups of p_1 points between O and X_1 , p_2 between X_1 and X_2 , etc., whilst X_1, X_2, \dots, X_{n-1} lie at any points of OA , in this order, the chance is

$$\frac{(p_1 + p_2 + \dots + p_n)!}{p_1! p_2! \dots p_n!} \int_0^a \int_0^{x_{n-1}} \int_0^{x_{n-2}} \dots \int_0^{x_2} \left(\frac{x_1}{a}\right)^{p_1} \left(\frac{x_2 - x_1}{a}\right)^{p_2} \dots$$

$$\times \left(\frac{a - x_{n-1}}{a}\right)^{p_n} \frac{dx_{n-1}}{a} \frac{dx_{n-2}}{a} \dots \frac{dx_2}{a} \cdot \frac{dx_1}{a},$$

which at once reduces to $1/(\Sigma p + 1)(\Sigma p + 2) \dots (\Sigma p + n - 1)$. And this is an obvious result. For of the $p_1 + p_2 + \dots + p_n + n - 1$ points of division, the chance of the $n - 1$ points standing in the specified order in the $(p_1 + 1)^{\text{th}}$, $(p_1 + p_2 + 2)^{\text{th}}$, etc., positions is clearly

$$\frac{(p_1 + p_2 + \dots + p_n)!}{(p_1 + p_2 + \dots + n - 1)!}$$

$$= 1/(\Sigma p + 1)(\Sigma p + 2) \dots (\Sigma p + n - 1).$$

If now another group of $q_1 + q_2 + \dots + q_n$ points be chosen at random on OA , the chance that q_1 unspecified ones shall lie in the same segment as the p_1 points, q_2 in the same segment as the p_2 , and so on, will be

$$\frac{1}{a^{q_1 + \dots + q_n}} \frac{(q_1 + q_2 + \dots + q_n)!}{q_1! q_2! \dots q_n!}$$

$$\times \frac{\iint \dots \int x_1^{p_1 + q_1} (x_2 - x_1)^{p_2 + q_2} \dots (a - x_{n-1})^{p_n + q_n} dx_{n-1} dx_{n-2} \dots dx_1}{\iint \dots \int x_1^{p_1} (x_2 - x_1)^{p_2} \dots (a - x_{n-1})^{p_n} dx_{n-1} dx_{n-2} \dots dx_1},$$

the limits for x_1 being 0 to x_2 ; for x_2 , 0 to x_3 , etc.; for x_{n-1} , 0 to a , which we may evaluate as before.

1703. Ex. *From a bag containing an infinite number of tickets, each of which is known to be black or white, ten are drawn at random, and found to be four white, six black. What is the chance that a further draw of two tickets gives one white, one black?*

Here $m=4$, $n=6$, $p=1$, $q=1$, $a=1$, and the chance required

$$= {}^2C_1 \int_0^1 x^5(1-x)^7 dx / \int_0^1 x^4(1-x)^8 dx = \frac{2\Gamma(6)\Gamma(8)}{\Gamma(14)} \cdot \frac{\Gamma(12)}{\Gamma(5)\Gamma(7)} = \frac{35}{78}.$$

What would be the chance that a draw of one ticket only should yield a white one, and that a subsequent draw should yield a black one?

The chance for a white one at the next draw

$$= \int_0^1 x^5(1-x)^6 dx / \int_0^1 x^4(1-x)^6 dx = \frac{5}{12}.$$

The chance for a black to follow = $\int_0^1 x^5(1-x)^7 dx / \int_0^1 x^5(1-x)^6 dx = \frac{7}{13}.$

The chance for the two draws to result in this order = $\frac{5}{12} \cdot \frac{7}{13} = \frac{35}{156}.$

The chance that x , which represents the proportion of the number of white tickets to the whole number of tickets in the bag, should be more than $\frac{1}{2}$ of the whole is $\int_{\frac{1}{2}}^1 x^4(1-x)^6 dx / \int_0^1 x^4(1-x)^6 dx = 281/2^{10}.$

1704. **Buffon's Problem. Parallel Rulings.**

An infinite plane is ruled by an infinite system of equidistant parallel lines, whose distances apart = $2a$. A thin rod of length $2l$ ($< 2a$) is thrown at random upon the plane. What is the chance that the rod will cut one of the parallels?

Take as y -axis that one of the parallels to which the centre C of the rod falls nearest, and the x -axis perpendicular to the set. The problem is unaffected if we suppose the centre of the rod to fall upon the x -axis, for the proportion of the number of cases in which the rod cuts one of the rulings to the whole number of possible cases is not altered thereby.

Let O be the origin, $OC=x$. Let the figure represent the case in which one end of the rod lies upon the y -axis, the angle between the rod and CO being ϕ . Then $x=l \cos \phi$. Then for a given position of C , the chance of a cut

$$= 2 \cdot \frac{2\phi}{2\pi} = \frac{2}{\pi} \cos^{-1} \frac{x}{l};$$

and the chance that C lies between x and $x+dx$ on a line of

length a is dx/a , and when C falls between $x=l$ and $x=a$, there is no chance of a cut. Hence the whole chance required is

$$\frac{2}{\pi a} \int_0^l \cos^{-1} \frac{x}{l} dx = \frac{2l}{\pi a} \int_0^{\frac{\pi}{2}} \phi \sin \phi d\phi = \frac{2l}{\pi a} = \frac{\text{double the length of the rod}}{\text{circ. of a circle of radius } a}.$$

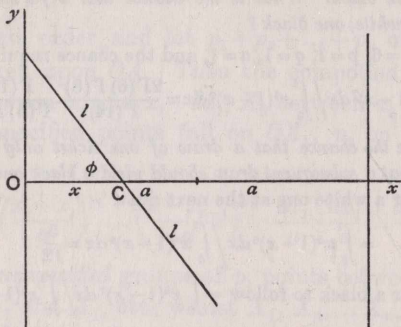


Fig. 560.

This is a particular case of a remarkable general result to be seen later. It is another landmark in the history of the subject. It was given by the naturalist Buffon in his *Essai d'Arithmétique Morale*, 1777. Also see Laplace, *Théorie de Prob.*, p. 359 (Todhunter, *History*).

1705. Rectangular Rulings.

Suppose a second system of parallel lines drawn at right angles to the former set, whose distances apart $= 2b$ ($> 2l$), thus mapping out the infinite plane into a net-work of equal parallelograms. Consider that rectangle formed by a consecutive pair of each family of rulings which finds itself the recipient of the centre of the rod. Suppose the rod to have come to rest, making an angle ϕ with the side of length $2a$. If we join the centres of the extreme positions of the rod at this inclination, an inner rectangle is formed of sides $2a - 2l \cos \phi$, $2b - 2l \sin \phi$, and no rod at this inclination, whose centre falls within this rectangle, can cut a side of the mesh, whilst those whose centres fall without it do so. Taking axes coincident with two sides of the rectangle, the angular position of the rod may range from being parallel to the x -axis to being perpendicular to it. The chance that the inclination lies between ϕ and $\phi + d\phi$ is proportional to $d\phi$, and we are

to evaluate the ratio of $\iiint \frac{dx}{a} \frac{dy}{b} d\phi$ for the favourable cases to the same integral for the whole number of cases. The integration for x and for y has been effected geometrically above.

The chance required is therefore

$$\left[\int_0^{\frac{\pi}{2}} \{2a \cdot 2b - (2a - 2l \cos \phi)(2b - 2l \sin \phi)\} d\phi \right] / \int_0^{\frac{\pi}{2}} 4ab d\phi$$

$$= \frac{2l}{\pi ab} \int_0^{\frac{\pi}{2}} (a \sin \phi + b \cos \phi - l \sin \phi \cos \phi) d\phi = \frac{l}{\pi ab} (2a + 2b - l).$$

Buffon's result $2l/\pi a$ follows at once by making b infinite.

Putting $a=b$, the result is $l(4a-l)/\pi a^2$ for square meshes.

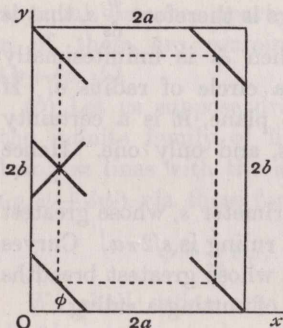


Fig. 561.

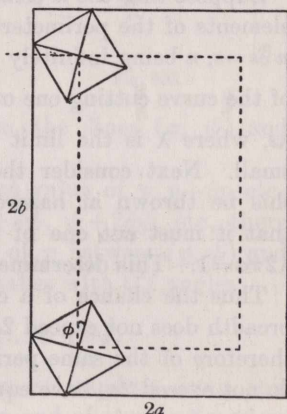


Fig. 562.

1706. Suppose a square of diagonal $2l$ to be thrown upon the above rectangular mesh-work, l being less than either a or b , and let the inclination of a diagonal to the side of length $2b$ be ϕ .

To avoid a cut, the centre of the square must lie within an inner rectangle of area $4(a-l \cos \phi)(b-l \sin \phi)$. The range for ϕ is from 0 to $\frac{\pi}{4}$, and the result $= \frac{l}{2\pi ab} \{4(a+b)\sqrt{2} - (\pi+2)l\}$.

If $b = \infty$, this becomes $\frac{\text{perimeter of square}}{\text{circumf. of circle of rad. } a}$. (See Art. 1707.)

If a circular lamina of radius r ($< a$ or b) be thrown at hazard in the same way, the chance of a cut is obviously

$$\frac{2a \cdot 2b - (2a - 2r)(2b - 2r)}{2a \cdot 2b} = \frac{r(a+b-r)}{ab}$$

And when b becomes ∞ this becomes $\frac{\text{circumf. of circle of rad. } r}{\text{circumf. of circle of rad. } a}$.

This class of problem leads us to enquire as to the chance of a hazard throw of a lamina of any shape cutting one of a system of equidistant parallels drawn upon a plane. This we proceed to consider.

1707. RANDOM LINES.

Let an infinite plane be ruled by parallel lines at distances apart $=2a$. Let n equal short lines of lengths δs , whether in rigid connection or not is immaterial, be thrown down at hazard upon the plane so ruled. Then each one has an equal chance of finding itself crossing one of the rulings. If p be that chance, the chance that some one of them crosses a ruling $=np$.

Suppose that the n elementary lines δs are the infinitesimal elements of the perimeter of some oval of perimeter s . Then $n \delta s = s$, n being infinitely great. The chance of the perimeter of the curve cutting one of the rulings is therefore $\frac{p}{\delta s} s$, that is λs , where λ is the limit of $p/\delta s$ when δs is infinitesimally small. Next consider the case of a circle of radius a . If this be thrown at hazard upon the plane, it is a certainty that it must cut one of the rulings, and only one. Hence $\lambda 2\pi a = 1$. This determines λ .

Thus the chance of a curve of perimeter s , whose greatest breadth does not exceed $2a$, cutting a ruling is $s/2\pi a$. Curves therefore of the same perimeter, and whose greatest breadths do not exceed $2a$, have equal chances of cutting a ruling.

1708. Examples.

1. If a circle of radius b ($< a$) be thrown down at hazard upon the plane, the chance of crossing a ruling $=2\pi b/2\pi a = b/a$.
2. If the contour be a square of side b ($< a\sqrt{2}$), the chance is $2b/\pi a$.
3. If the "curve" thrown down be a straight line of length $2l$ ($< 2a$), it may be considered as an ellipse of minor axis zero and perimeter $4l$, and the chance is $2l/\pi a$ (Art. 1704).
4. For a semicircle of radius b ($< a$), the chance is $(\pi + 2)b/2\pi a$.

1709. Let O be a point fixed to the contour thrown down, and OA a fixed axis on it.

Let O fall at a distance p from one of the rulings, RS , and let OA make an angle ψ with the perpendicular p . Let this contour be thrown down at random upon the ruled plane a very large number of times, and let the trace of the rulings

be marked at each throw upon the plane of the contour. Now it is immaterial whether we regard the contour as thrown down at hazard upon the ruled plane, or the ruled plane thrown at hazard upon the plane containing the contour. Take the latter case. Let a doubly infinite number of lines be drawn upon the plane of the contour according to the following plan :

(a) Let the lines be drawn parallel to a standard line

$$p = x \cos \psi + y \sin \psi,$$

which we may call the line (p, ψ) , at equal distances apart, such that

n of them are contained between the lines (p, ψ) and $(p + \delta p, \psi)$.

(b) Let us suppose drawn for *each* value of $p, p + \delta p$, etc., the infinite family of lines $\psi, \psi + \delta\psi, \psi + 2\delta\psi$, etc., there being m lines with the same value of p between (p, ψ) and $(p, \psi + \delta\psi)$, viz. those for which \hat{p} makes with OA angles

$$\psi + \frac{1}{m} \delta\psi, \quad \psi + \frac{2}{m} \delta\psi, \dots \psi + \delta\psi.$$

We shall define any line chosen at random from this double set for equal gradations of p and of ψ as a "random line."

The actual number of lines from (p, ψ) to $(p + \delta p, \psi + \delta\psi)$ is mn , and we obtain in this way the same system of lines as those obtained by the tracings of the rulings upon the plane of the contour after the contour plane is thrown down at hazard upon the ruled plane.

Taking the case of a circle of radius a and centre O , the number of such lines crossing it is

$$mn \int_0^a \int_0^{2\pi} dp d\psi = mn \cdot 2\pi a \equiv \lambda, \text{ say.}$$

Hence the number from (p, ψ) to $p + \delta p, \psi + \delta\psi$, viz. $mn \delta p \delta\psi$,

is $\frac{\lambda}{2\pi a} \delta p \delta\psi$.

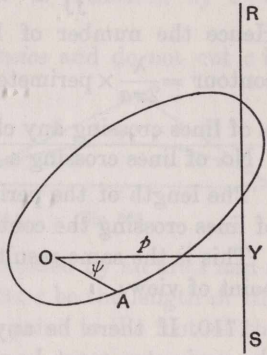


Fig. 563.

Now, if O be a point within any closed convex contour,

$$\iint dp d\psi = \int p d\psi = \text{perimeter.}$$

Hence the number of lines crossing such a closed convex contour $= \frac{\lambda}{2\pi a} \times \text{perimeter, i.e.}$

$$\frac{\text{No. of lines crossing any closed convex contour}}{\text{No. of lines crossing a circle of radius } a} = \frac{\text{perim. of curve}}{\text{perim. of circle}}$$

The length of the perimeter therefore measures the number of lines crossing the contour.

This is the same result as that of Art. 1707, from a different point of view.

1710. If there be any re-entrant portion of the contour, the perimeter must be regarded as the length of a stretched elastic band which encircles it; that is, the re-entrant portions must be excluded by double tangents. Otherwise some of the random chords will be counted more than once by the above rule.

1711. Examples.

1. If a closed convex contour of perimeter Σ completely encloses a second closed convex contour of perimeter S , the number of chords of the outer which cut the inner is $\lambda S/2\pi a$. And the total number of chords of the outer is $\lambda\Sigma/2\pi a$. Therefore the chance of a chord of the outer cutting the inner also is S/Σ .

If the outer be a circle of radius R , and the inner a square of side b , the chance is $2b/\pi R$.

2. If the inner degenerates into a straight line of length $2l$, and the outer be a circle of radius R , the chance is $4l/2\pi R = 2l/\pi R$.

3. The chance that a random chord of a circle cuts a given diameter is $2/\pi$.

1712. We may then speak of S or $\iint dp d\psi$ as "the number of lines" which cross any convex contour throughout which the integration is conducted, whenever a comparison is to be instituted between the number of lines which cut one convex contour with the number which cut another.

1713. Various Cases.

In the case of a straight line of length c , which is the limit of an ellipse of zero minor axis and perimeter $2c$, the number of random lines cutting it is then measured by $2c$.

1714. In the case of an arc of length s bounded by a chord of length c , there being no re-entrant portion, the number of random chords crossing the contour is measured by $s+c$. But the number which cross c is $2c$.

Hence the number which cross s twice and do not cut c is $s-c$.

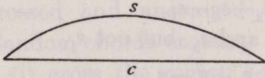


Fig. 564.

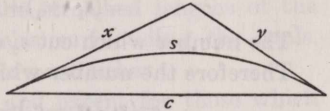


Fig. 565.

1715. In the case of the contour bounded by an arc s and a pair of tangents of lengths x and y , let c be the length of the chord; then, if s be concave at each point to the foot of the perpendicular upon the chord,

the number of random lines which cut x and y , but not c , is $x+y-c$;

the number which cut s , but not c , is $s-c$.

Therefore the number which cut x and y , but not s , is $x+y-s$.

1716. In the case of two arcs s_1, s_2 and a chord c , each arc being convex at every point to the foot of the perpendicular upon the chord, as in Fig. 566; let c_1, c_2 be the chords of the arcs s_1, s_2 respectively.

Then the number of chords cutting c_1, c_2 , but not c , $=c_1+c_2-c$. These necessarily all cut s_1 and s_2 , each once only.

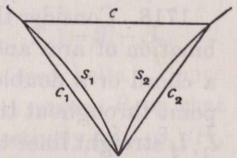


Fig. 566.

The number of those which cut s_1 twice, but not c_1 , $=s_1-c_1$.

These also cut s_2 once and c once.

The number of those which cut s_2 twice, but not c_2 , $=s_2-c_2$.

These also cut s_1 once and c once.

Hence the number which cut both s_1 and s_2

$$=(c_1+c_2-c)+(s_1-c_1)+(s_2-c_2)=s_1+s_2-c.$$

1717. In the case where the region considered is bounded by three arcs s_1, s_2, s_3 , lying within the chordal triangle c_1, c_2, c_3 , and each concave at all points to the foot of the

ordinate from the point to the chord of the arc (Fig. 567), the number of chords cutting s_1 , but not c_1 , $=s_1 - c_1$. These necessarily cut s_2 and s_3 , c_2 and c_3 .

The number of chords cutting one or other of the three arcs twice, and therefore cutting all three arcs,

$$=(s_1 - c_1) + (s_2 - c_2) + (s_3 - c_3).$$

The number which cut s_2 and $s_3 = s_2 + s_3 - c_1$.

Therefore the number which cut s_2 and s_3 , but not s_1 ,

$$=(s_2 + s_3 - c_1) - (s_1 - c_1) = s_2 + s_3 - s_1.$$

Therefore the number which cut any two of the arcs, but not the third, is

$$(s_2 + s_3 - s_1) + (s_3 + s_1 - s_2) + (s_1 + s_2 - s_3) = s_1 + s_2 + s_3.$$

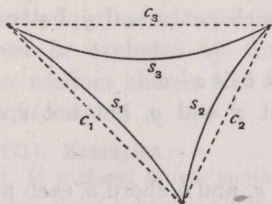


Fig. 567.

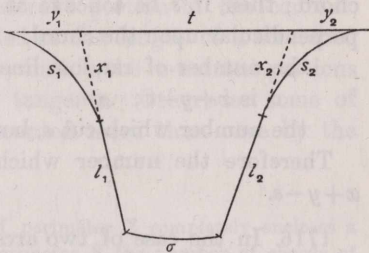


Fig. 568.

1718. Consider the case of a region bounded by such a combination of arcs and lines as exhibited in Fig. 568, where t is a chord or a double tangent; s_1, s_2 any arcs convex at each point throughout their lengths to the foot of the ordinate to t ; l_1, l_2 straight lines tangential to s_1 and s_2 , and σ an arc concave at each point to the foot of the ordinate drawn upon its own chord, which lies within the region considered, and either touching l_1 and l_2 or meeting them and lying between l_1 and l_2 produced.

The number of lines crossing this contour, but which do not cut t , with the exception of such as meet $s_1 + l_1$ or $s_2 + l_2$ twice and incidentally meet t , is

$$\{x_1 + l_1 + \sigma + l_2 + x_2 - (t - y_1 - y_2)\} - (x_1 + y_1 - s_1) - (x_2 + y_2 - s_2),$$

where the meanings of the various letters are indicated in the figure. For the first bracket includes those which cut $x_1 + l_1, y_1$, but not $s_1 + l_1$; or $x_2 + l_2, y_2$, and not $s_2 + l_2$, the number of

which cases is subtracted in the second and third brackets. The expression reduces to $s_1 + s_2 + l_1 + l_2 + \sigma - t$.

1719. In the case of two non-intersecting non-re-entrant ovals A and B , of perimeters P_A, P_B , external to each other, let the lengths of the several arcs and tangents be as indicated in Fig. 569. Let β_c and β_u be the stretched lengths of the crossed and uncrossed elastic belts surrounding the ovals. Random chords crossing both ovals must either

- (i) cross the region $s_1 x_1 x_2 \sigma_1 T_1$, and except for those which cross $s_1 + x_1$ or $\sigma_1 + x_2$ twice, not cross T_1 ; or
- (ii) cross the region $s_3 y_1 y_2 \sigma_3 T_2$, and except for those which cross $s_3 + y_1$ or $\sigma_3 + y_2$ twice, not cross T_2 .

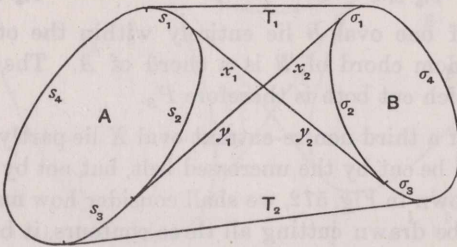


Fig. 569.

Their number is therefore

$$(s_1 + x_1 + x_2 + \sigma_1 - T_1) + (s_3 + y_1 + y_2 + \sigma_3 - T_2) = \beta_c - \beta_u,$$

i.e. the difference of the crossed and uncrossed belts. Hence the probabilities that a random chord of A crosses B , or that a random chord of B crosses A , are respectively $(\beta_c - \beta_u)/P_A$ and $(\beta_c - \beta_u)/P_B$.

1720. If the ovals touch externally $\beta_c = P_A + P_B$.

1721. If the ovals intersect, indicate the several arcs and tangents as in Fig. 570.

The chords which cut both may be classified as

- (i) those crossing s_1 and σ_1 , but which, with the exception of those cutting s_1 twice or σ_1 twice, do not cut T_1 ;
- (ii) those crossing s_2 and σ_2 , but which, with the exception of those cutting s_2 twice or σ_2 twice, do not cut T_2 ;
- (iii) those which cut the region bounded by s_3 and σ_3 .

Their number is therefore

$$(s_1 + \sigma_1 - T_1) + (s_3 + \sigma_3) + (s_2 + \sigma_2 - T_2) = P_A + P_B - \beta_u,$$

i.e. the sum of the perimeters less by the belt.

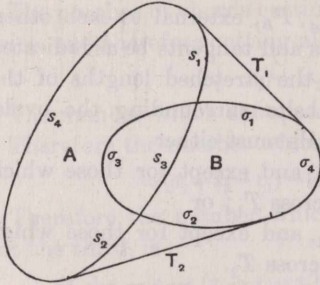


Fig. 570.

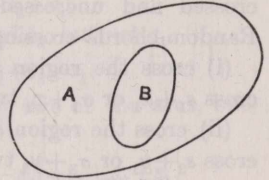


Fig. 571.

1722. If one oval B lie entirely within the other one A , every random chord of B is a chord of A . The number of chords which cut both is therefore P_B .

1723. If a third non-re-entrant oval X lie partly between A and B and be cut by the uncrossed belt, but not by the crossed belt, as shown in Fig. 572, we shall consider how many random lines can be drawn cutting all three contours, it being understood that the ovals are so situated that for all chords cutting all three the X -segment is intermediate between the other two.

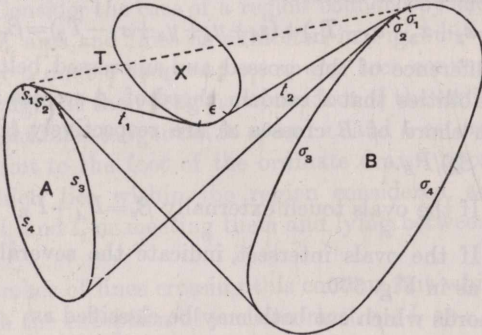


Fig. 572.

Indicating the lengths of the several arcs and tangents as in Fig. 572, all such random lines as are chords of all these regions must be chords of the region $(s_1, t_1, \epsilon, t_2, \sigma_1, T)$, but must not cross T , with the exception of those which cross $s_1 + t_1$ twice or

σ_1+t_2 twice, with an incidental crossing of T . By Art. 1718 their number is $s_1+t_1+\epsilon+t_2+\sigma_1-T$; *i.e.* the amount by which the uncrossed belt has been lengthened by X having been pushed into position from outside the belt.

1724. If in the last case the oval X has been pushed completely within the region bounded by the uncrossed belt, but still not so as to cut the crossed one, denote the various lengths of arcs and lines as in Fig. 573.

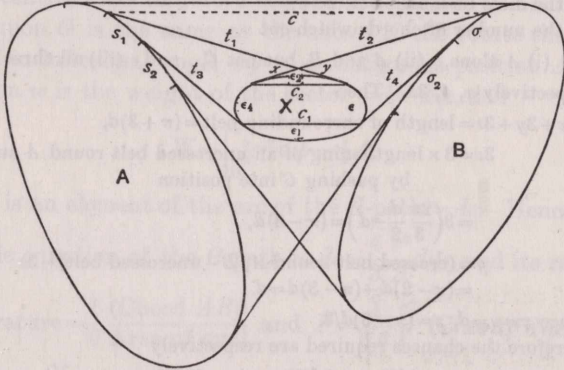


Fig. 573.

Then the number of random lines which cut all three ovals is $\alpha-\beta-\gamma+\delta$, where

(i) α is the number which cut the contour $(s_1t_3\epsilon_1t_4\sigma_1c)$, but do not cut c , with the exception of those which cut s_1+t_3 or σ_1+t_4 twice, $=s_1+t_3+\epsilon_1+t_4+\sigma_1-c$;

(ii) β is the number which cut (t_1-y, t_2-x, c) , but do not cut c , $=t_1-y+t_2-x-c$;

(iii) γ is the number which cut (x, y, c_2) , but not c_2 , $=x+y-c_2$;

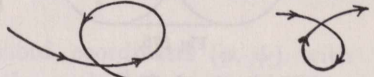


Fig. 574.

(iv) δ is the number which cut ϵ_2 twice, but not c_2 , $=\epsilon_2-c_2$.

The total, after rearranging, is

$$(s_1+t_3+\epsilon_1+\epsilon_3+\epsilon_2+\epsilon_4+\epsilon_1+t_4+\sigma_1)-(t_1+\epsilon_3+\epsilon_1+\epsilon_4+t_4),$$

which is the difference of the increases of length of the uncrossed belt caused by its being made to pass round the contour of X in opposite directions (Fig. 574).

1725. In a similar manner it is easy to examine other special cases. The last two results are due to Sylvester [*Educ. Times*], who refers for simpler cases to Czuber's *Geometrische Wahrscheinlichkeiten*.

1726. **Ex.** Three pennies of diameters d are soldered together in mutual contact at their edges.

This figure is thrown upon a table ruled with parallel lines at equal distances ($2a$) apart ($a > d$). What is the chance of 2, 4 or 6 intersections?

[BIDDLE'S PROBLEM.]

Let the discs be labelled A, B, C .

Let the number of chords which cut

(i) A alone, (ii) A and B , but not C , and (iii) all three be respectively $x, y, 3z$. Then

$$3x + 3y + 3z = \text{length of surrounding belt} = (\pi + 3)d,$$

$$3z = 3 \times \text{lengthening of an uncrossed belt round } A \text{ and } B \\ \text{by pushing } C \text{ into position}$$

$$= 3 \left(\frac{2\pi d}{3} - d \right) = (\pi - 3)d,$$

$$y = (\text{crossed belt round } A, B - \text{uncrossed belt}) - 3z \\ = (\pi - 2)d - (\pi - 3)d = d.$$

Hence $x = y = d, z = (\pi - 3)d/3$.

Therefore the chances required are respectively

$$3d/2\pi a, \quad 3d/2\pi a, \quad (\pi - 3)d/2\pi a.$$

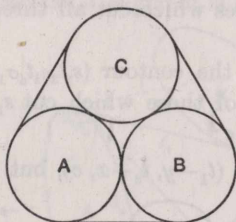


Fig. 575.

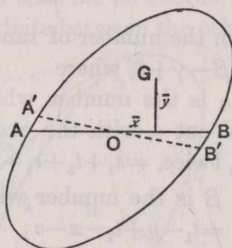


Fig. 576.

1727. Crofton's Theorem.

In any centric convex contour of area A , let AB be a diameter and G the centroid of the area of either semi-oval. Let P be the perimeter of the path of G as AB rotates; then the mean radial distance of any point within the contour from the centre O is $\frac{1}{2}P$.

If \bar{x}, \bar{y} be the coordinates of G referred to OB as x -axis, W the weight of the half oval, $AB = 2r$, and if we place two

small weights w and $-w$ at distances $\frac{2}{3}OB$ and $\frac{2}{3}OA$ from O , the new coordinates of G will be

$$\bar{x} + d\bar{x} = \left\{ W\bar{x} + w \cdot \frac{2}{3}r + (-w) \left(-\frac{2}{3}r \right) \right\} / W = \bar{x} + \frac{4}{3} \frac{w}{W} r;$$

$$\bar{y} + d\bar{y} = (W\bar{y} + 0) / W = \bar{y}.$$

Hence
$$d\bar{x} = \frac{4}{3} \frac{w}{W} r, \quad d\bar{y} = 0.$$

The centroid has therefore been moved parallel to AB . The effect upon G is the same as the above, if AB rotate through a small infinitesimal angle $d\psi$ to a contiguous position $A'OB'$, and then w is the weight of the sector $= \frac{1}{2}r^2 d\psi$, and

$$W = \int_0^\pi \frac{1}{2}r^2 d\psi = \frac{1}{2}A,$$

and $d\bar{x}$ is an element of the arc of the G -path $= ds$. Hence the intrinsic equation of the G -path is $ds = \frac{4}{3} \frac{r^3}{A} d\psi$, and its radius

of curvature $= \frac{1}{6} \frac{(\text{Chord } AB)^3}{\text{Area of oval}}$ and $P = \frac{1}{6A} \int_0^{2\pi} (\text{Chord})^3 d\psi$.

Again
$$M(r) = \frac{\iint r(r d\psi dr)}{\iint r d\psi dr} = \frac{1}{A} \iint r^2 d\psi dr = \frac{1}{24A} \int_0^{2\pi} (\text{Chord})^3 d\psi = \frac{1}{4}P.$$

Prof. Crofton's proof of this result [*Proc. Lond. Math. Soc.*, viii.] runs on different lines, but he indicates the above as a method of procedure.

1728. Useful Results for a Convex Contour of Area A and Perimeter L .

Let C be the length of a chord, coordinates (p, ψ) , with regard to an origin O within the oval, G the centroid of the oval, $OG (=c)$ the initial line from which ψ is measured, $O\xi$ a line parallel to the chord, \bar{p} the perpendicular from G upon $O\xi$; p_1 and p_2 the perpendiculars upon the tangents parallel to the chord. Then we have, taking limits from $-p_1$ to p_2 ,

$$(i) \int C dp = A; \quad (ii) \int pC dp = A\bar{p}; \quad (iii) \int p^2 C dp = A\bar{p}^2 + Ak^2,$$

where Ak^2 is the moment of inertia about a parallel through G .

Hence integrating (i) and (ii) with regard to ψ from 0 to π , which takes in all random chords,

(i) $\iint C dp d\psi = \int A d\psi = \pi A$; whence

$$M(\text{Chord}) = \frac{\iint C dp d\psi}{\iint dp d\psi} = \pi \cdot \frac{\text{Area of contour}}{\text{Perimeter}};$$

(ii) $\iint pC dp d\psi = \int A\bar{p} d\psi = Ac \int \sin \psi d\psi = 2Ac$, and in this integration it is to be noted that p changes sign as C passes through the origin.

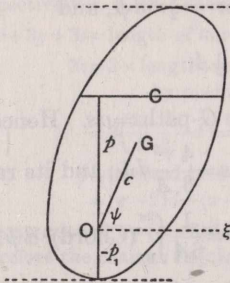


Fig. 577.

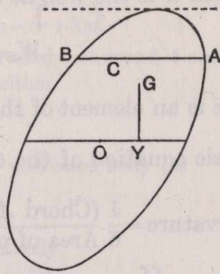


Fig. 578.

If the oval be centric and the origin be taken at the centre, we shall integrate for p from 0 to p_1 , the perpendicular upon the tangent parallel to C , and for ψ from 0 to 2π . Then

(i) $\iint C dp d\psi = \frac{1}{2}A \cdot 2\pi = A\pi$, as before;

(ii) $\iint pC dp d\psi = \frac{1}{2}A \int \bar{p} d\psi$, where \bar{p} is the perpendicular from the centroid of the half area upon a line through O parallel to the chord $(p, \psi) = \frac{1}{2}A \cdot \text{Perim. of } G\text{-path}$.

Thus $M(\triangle OAB) = \frac{\iint \frac{1}{2} pC dp d\psi}{\iint dp d\psi} = \frac{1}{4}A \cdot \frac{\text{Perim. of } G\text{-path}}{\text{Perim. of oval}}$.

1729. Mean n^{th} Power of the Distance between two Random Points within an Oval.

This mean may be expressed as an integral in terms of a chord. Let X, Y be the random points, and ψ the inclination

of XY to a given direction. Let C be the length of the chord AB through X, Y ; $ON (=p)$ the perpendicular from an origin O within the oval to AB ; $XA=r, XB=-r', XY=\rho$. Keep X fixed at first. Then the sum of all the values of ρ^n which are contained between AXB and a chord $A'XB'$, making an angle $d\psi$ with the former, each multiplied by an element of area, is

$$\int_0^r \rho^n (\rho d\psi d\rho) + \int_0^{r'} \rho^n (\rho d\psi d\rho) = \frac{r^{n+2} + r'^{n+2}}{n+2} d\psi;$$

and integrating this for all positions of X lying between the parallel chords (p, ψ) and $(p+dp, \psi)$, we have

$$\int \frac{r^{n+2} + r'^{n+2}}{n+2} d\psi dp dr,$$

$dp dr$ being the element of area in which X lies. And r varies from zero to C and $r' = C - r$. We therefore obtain

$$\frac{[r^{n+3}]_0^C - [r'^{n+3}]_C^0}{(n+2)(n+3)} d\psi dp = \frac{2C^{n+3}}{(n+2)(n+3)} d\psi dp.$$

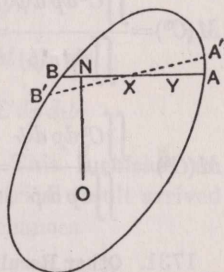


Fig. 579.

The final stage of the integration is to sum this expression for all elements $dp d\psi$ within the contour and then to divide by the number of cases, which is measured by A^2 .

$$\text{Hence } M(\rho^n) = \frac{2}{(n+2)(n+3)} \frac{1}{A^2} \iint C^{n+3} dp d\psi; \quad (n > -2).$$

1730. In the case, where $n = -1$, we have

$$M\left(\frac{1}{\rho}\right) = \frac{1}{A^2} \iint C^2 dp d\psi.$$

This may be interpreted as an expression for the mean value of the mutual potential of a pair of unit particles at random points within the contour.

$$\text{The case } n=0 \text{ gives } A^2 = \frac{1}{3} \iint C^3 dp d\psi.$$

$$\text{The case } n=1 \text{ gives } M(\rho) = \frac{1}{6A^2} \iint C^4 dp d\psi.$$

$$\text{The case } n=2 \text{ gives } M(\rho^2) = \frac{1}{10A^2} \iint C^5 dp d\psi.$$

But since $M(\rho^2) = 2k^2$, where k is the radius of gyration about the centroid,

$$A^2 k^2 = \frac{1}{20} \iint C^5 dp d\psi.$$

We obtain thus the mean values of various powers of C for cases in which the mean values of the corresponding powers of ρ have been otherwise found.

Thus, for instance,

$$M(C^3) = \frac{\iint C^3 dp d\psi}{\iint dp d\psi} = \frac{3A^2}{L} = 3 \frac{(\text{Area})^2}{\text{Perimeter}},$$

$$M(C^5) = \frac{\iint C^5 dp d\psi}{\iint dp d\psi} = \frac{20 \cdot \text{Area} \cdot (\text{Moment of In. about centroid})}{\text{Perimeter}}.$$

1731. Other Results due to Crofton.

Let ρ be the distance between any two random points X, Y within a given convex contour of area A and perimeter L . Then the probability that any random line drawn across the contour also crosses a particular position XY of the line joining the random points is $2\rho/L$.

If n be the number of cases of a random line XY , the chance that any particular one is selected is $1/n$. Therefore the chance that a particular one is selected and cut by the random chord is $2\rho/nL$; and the chance that a random chord cuts a random line XY is the sum of the values of $2\rho/nL$ for all the cases of a pair of random points (Fig. 580),

$$= \frac{2}{L} \sum \frac{\rho}{n} = \frac{2}{L} M(\rho) = \frac{1}{3A^2 L} \iint C^4 dp d\psi.$$

Again, suppose the random chord to divide A into two parts Σ and Σ' . The chance that X lies in Σ and Y in Σ' , or X in Σ' and Y in $\Sigma = 2\Sigma\Sigma'/A^2$ for any particular position of the chord. If m be the number of random chords, the chance of selection of any particular one is $1/m$, and the chance that a particular chord should be selected for which X and Y lie

on opposite sides is $\frac{1}{m} \frac{2\Sigma\Sigma'}{A^2}$; and the chance that a random chord should cut a random XY ,

$$= \frac{2}{A^2} M(\Sigma\Sigma') = \frac{2}{A^2} \frac{\iint \Sigma\Sigma' dp d\psi}{\iint dp d\psi} = \frac{2}{A^2 L} \iint \Sigma\Sigma' dp d\psi.$$

Hence, by equating the two values of the chance, we have

$$\iint C^4 dp d\psi = 6 \iint \Sigma\Sigma' dp d\psi.$$

Moreover we have two expressions for $M(\rho)$, viz.

$$\frac{1}{6A^2} \iint C^4 dp d\psi \quad \text{and} \quad \frac{1}{A^2} \iint \Sigma\Sigma' dp d\psi$$

(Crofton, *Proc. Lond. Math. Soc.*, viii.). This furnishes an interesting illustration of a difficult geometrical result arrived at by a consideration of mean values and chances.

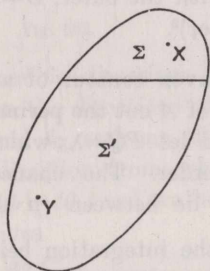


Fig. 580.

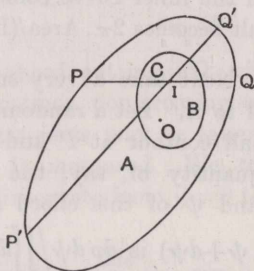


Fig. 581.

1732. A and L being respectively the area and perimeter of a given convex contour which encloses a second contour of area B , it is required to find the chance that a pair of random chords PQ , $P'Q'$ of the former should intersect within the latter. (Fig. 581.)

Take an origin O within the smaller contour, and let the random chords be denoted by the p - ψ system. Let a particular position of PQ intersect B , and suppose C the length of the chord intercepted upon it by B . The number of random lines cutting C is measured by $2C$. The number of random chords of A is measured by L . Therefore the chance that one of these cuts C is $2C/L$.

The chance that the particular chord C is one of the lines whose p and ψ lie between p and ψ , $p+dp$ and $\psi+d\psi$ is $dp d\psi / \iint dp d\psi = dp d\psi / L$, the integration being taken for the A -contour.

Therefore the chance that whilst the chord PQ lies between these limits it is met by a second random chord at a point within B is $2C dp d\psi / L^2$, and the total chance of the intersection of two random chords of A lying within B is $\frac{2}{L^2} \iint C dp d\psi$ for all values of p, ψ which can give chords intersecting B . Therefore

$$\text{the required chance} = 2\pi B / L^2 = 2\pi \cdot \text{Area of } B / (\text{Perim. of } A)^2.$$

1733. The above result is independent of the area of A or the perimeter of B , and except that it involves B and L it is independent of the shape and relative position of the ovals.

When the inner curve coincides with the outer, $B=A$, and the result becomes $2\pi \cdot \text{Area} / (\text{Perimeter})^2$.

1734. Next take a very small convex contour of area $d\sigma$ external to A . Let a random chord of A cut the perimeter of this small contour at P and Q , and let $PQ = \lambda$, which is a small quantity of, say, the first order. The chance that the p and ψ of this chord should lie between (p, ψ) and $(p+dp, \psi+d\psi)$ is $dp d\psi / \iint dp d\psi$, the integration being for the contour A , i.e. $dp d\psi / L$.

Let θ_1 and θ_2 be the angles which the tangents from P to the oval make with any specific position of PQ (Fig. 582). Then regarding the chord PQ as itself a narrow oval whose greatest breadth is an infinitesimal of the second order, the chance that a random chord of A cuts this line PQ is, by Art. 1719, (Crossed Belt—Uncrossed Belt)/ L , i.e. in the limit $(2\lambda - \lambda \cos \theta_1 - \lambda \cos \theta_2) / L$. Hence the chance that the chord of A should be selected to lie between (p, ψ) and $(p+dp, \psi+d\psi)$, and then cut by a second random chord of A within the small contour, is

$$\frac{dp d\psi}{L} \cdot \frac{\lambda}{L} (\text{vers } \theta_1 + \text{vers } \theta_2).$$

Now λ being an infinitesimal of the first order, θ_1 and θ_2 may be regarded as constant throughout $d\sigma$ for a given direction of PQ , and the integration $\int \lambda d\rho$ gives the area $d\sigma$ when taken for the small area. This integration therefore gives $d\sigma d\psi (\text{vers } \theta_1 + \text{vers } \theta_2)/L^2$. We next integrate with regard to ψ , and $\text{vers } \theta_1 + \text{vers } \theta_2 = 2 - \cos(\omega - \theta_2) - \cos \theta_2$, where ω is the angle subtended by A at the elementary area $d\sigma$.

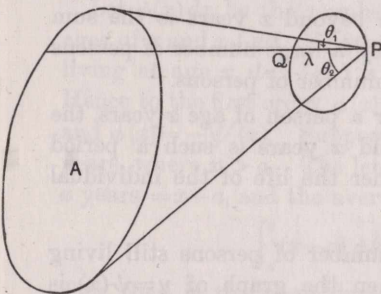


Fig. 582.

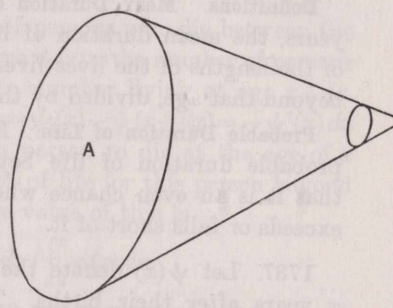


Fig. 583.

The possible directions of the chord cutting PQ will vary between the directions of the common non-crossing tangents to A and $d\sigma$, and one of these tangents may be taken as the fixed direction from which ψ is measured. We therefore have $d\psi = d\theta_2$, and we have to integrate from $\psi = 0$ to $\psi = \omega$. This gives

$$\frac{d\sigma}{L^2} \int_0^\omega [2 - \cos(\omega - \psi) - \cos \psi] d\psi = \frac{2d\sigma}{L^2} (\omega - \sin \omega).$$

We may now integrate this through any finite convex oval of area B external to A . Thus the chance that two random chords of A intersect within B is $\frac{2}{L^2} \int (\omega - \sin \omega) d\sigma$.

1735. If B be taken as the whole of space external to A , the chance of the random chords intersecting outside A must be $1 -$ the chance of intersecting within A , i.e. $1 - \frac{2\pi A}{L^2}$.

Hence we obtain the remarkable theorem that

$$2 \int (\omega - \sin \omega) d\sigma = L^2 - 2\pi A,$$

where the integration is taken over the whole plane external to A . This theorem is also due to Crofton. It is quoted by Bertrand, *Calc. Int.*, p. 491. It is another curious example (see Art. 1731) of a geometrical fact brought to light by consideration of chances.

1736. **D'Alembert's Mortality Curve.** (See Todhunter, *History*, p. 268.)

Definitions. Mean Duration of Life. For a person of age x years, the mean duration of life beyond x years is the sum of the lengths of the lives lived by a large number of persons beyond that age, divided by the number of persons.

Probable Duration of Life. For a person of age x years, the probable duration of life beyond x years is such a period that it is an even chance whether the life of the individual exceeds or falls short of it.

1737. Let $\psi(x)$ denote the number of persons still living x years after their births. Then the graph of $y = \psi(x)$ is known as the curve of mortality.

Let c years be the supreme limit of life, *i.e.* the greatest age to which any person can attain. Then $\psi(c) = 0$.

By the definition,

Mean duration for a person aged a years $= \int_a^c \psi(x) dx / \psi(a)$,

Probable duration for a person aged a years $= b$ years,

where $\psi(b) = \frac{1}{2} \psi(a)$.

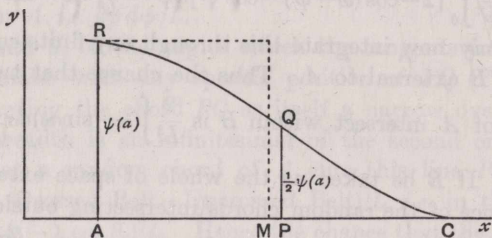


Fig. 584.

In Fig. 584, $OC = c$ is the limit of longevity, $OA = a$ years.

The ordinate AR represents the number of persons alive at age a years, AP the probable duration of life beyond the

age a for persons now of age a , the ordinate at P being half that at A . AM measures the mean duration for persons of age a years, and is such that $AR \cdot AM = \text{area } RAPCQR$.

1738. A Different View.

The usual method of estimating the mean and probable duration of life for a person aged a years is somewhat different from that explained above, but will be shown to be in agreement with it.

Let $\phi(x)dx$ be the number of persons who die between the ages of x and $x+dx$. Then, since $\psi(x) \equiv$ the number of persons living at age x , $\psi(x+dx)$ is the number living at age $x+dx$. Hence to the first order, $\phi(x)dx = \psi(x) - \psi(x+dx) = -\psi'(x)dx$ and $\phi(x) = -\psi'(x)$. Suppose a person to die at the age of x years, where $x > a$. The length of life for this person beyond a years $= x - a$, and the average value of this is

$$\int_a^c (x-a) \phi(x) dx / \int_a^c \phi(x) dx.$$

This then is the *mean* duration for persons of age a years. The *probable* duration is b years where

$$\int_a^b \phi(x) dx = \int_b^c \phi(x) dx, \quad \text{i.e.} \quad \int_a^b \phi(x) dx = \frac{1}{2} \int_a^c \phi(x) dx.$$

1739. Agreement.

The agreement of these estimates with those of D'Alembert will be clear.

$$\text{For (i) } \int_a^c \phi(x) dx = - \int_a^c \psi'(x) dx = \psi(a) - \psi(c) = \psi(a)$$

$$\begin{aligned} \text{and } \int_a^c (x-a) \phi(x) dx &= - \int_a^c (x-a) \psi'(x) dx \\ &= - \left[(x-a) \psi(x) \right]_a^c + \int_a^c \psi(x) dx = \int_a^c \psi(x) dx; \\ \therefore \int_a^c (x-a) \phi(x) dx / \int_a^c \phi(x) dx &= \int_a^c \psi(x) dx / \psi(a). \end{aligned}$$

(ii) Again, since $\int_a^b \phi(x) dx = \frac{1}{2} \int_a^c \phi(x) dx$, we have

$$\int_a^b \psi'(x) dx = \frac{1}{2} \int_a^c \psi'(x) dx;$$

$$\therefore \psi(b) - \psi(a) = \frac{1}{2} \{ \psi(c) - \psi(a) \} = -\frac{1}{2} \psi(a); \quad \psi(b) = \frac{1}{2} \psi(a).$$

1740. **Chance of Survival.**

For a person of present age a , the chance of death between the ages p and q ($p < q$) is $\frac{\psi(p) - \psi(q)}{\psi(a)}$, and $\frac{\psi(a) - \psi(c)}{\psi(a)} = 1$, and the chance of survival to at least the age of q is $\psi(q)/\psi(a)$.

The probability of death between the ages of x and $x + dx$ for a person of age a is

$$\frac{\psi(x) - \psi(x + dx)}{\psi(a)} = -\frac{\psi'(x)}{\psi(a)} dx.$$

The probability of death for a person of age x years, between the ages of x and $x + dx$, *i.e.* of almost immediate death, is $-\psi'(x) dx / \psi(x) = -d \log \psi(x)$.

1741. **Expectation of Life.**

Defining the "Expectation of Life" at a definite age of a years as the average or mean duration of life after that age, the following results were calculated by Neison (*Vital Statistics*, p. 8) from the tables of the Registrar General. (See Boole, *Finite Differences*, p. 45.)

Age	10	20	30	40	50	60	70	80	90
Expectation	47.7564	40.6910	34.0990	27.4760	20.8463	14.5854	9.2176	5.2160	2.8930

Δ (Expectation) -7.0654 -6.5920 -6.6230 -6.6297 -6.2609 -5.3678 -4.0016 -2.3230,
 Δ^2 (Expectation) .4734 -.0310 -.0067 .3688 .8931 1.3662 1.6786,
 etc.

The expectations for intervening ages may be very closely obtained by the ordinary interpolation methods, *e.g.*

$$u_{x+n} = u_x + n\Delta u_x + \frac{n(n-1)}{1.2} \Delta^2 u_x + \frac{n(n-1)(n-2)}{1.2.3} \Delta^3 u_x + \dots$$

But probably no purely algebraical law expressed as a series in powers of the age, on which supposition interpolation formulae are based, would be adequate to express the true law of expectation for all ages; particularly near the extremities of the table, for ages of very young children or for persons of very advanced years. The graph of this expectation is shown in Fig. 585.

In the decades of the first differences from 20 to 60, it will be noted that there is but small change. Hence in the graph of the expectation the fall in the value of the expectation between these ages is roughly uniform, and this portion of

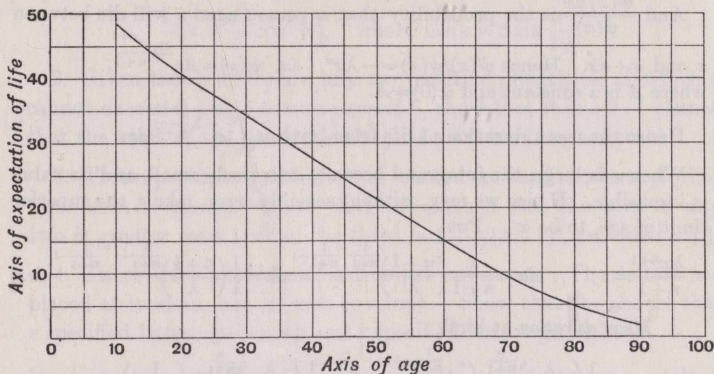


Fig. 585.

the graph is very approximately straight. From the age of 60 onwards the curvature shows a definite bending away from the axis of age, the curve becoming more definitely convex at each point to the foot of the ordinate. This is the curve

$$y = \int_x^c \psi(\xi) d\xi / \psi(x), \text{ that is } y = \int_x^c (\xi - x) \phi(\xi) d\xi / \int_x^c \phi(\xi) d\xi.$$

1742. Remarks on the Mortality Curve.

It has been remarked by Todhunter (*Hist. of Prob.*, p. 269) that the "mean duration" beyond a represents the abscissa of the "centre of gravity of a certain area," namely of that area which is bounded by the curve $y = \phi(x)$, the x -axis and its ordinate for age a , the abscissa in question being measured from $x = a$. The "probable duration" beyond a is represented by the abscissa, also measured from $x = a$, of the ordinate which bisects that area. It would appear from tables that the "mortality curve" $y = \psi(x)$ is not either always concave or always convex to the foot of the ordinate upon the x -axis, and also that the probable duration is not always greater than the mean duration. (See Todhunter's remarks on Buffon's tables and on d'Alembert's views, *History of Prob.*, p. 285.)

1743. Let us take a supposititious law that the probability of a person of present age x years dying before he is aged $x+dx$ is $\lambda x^n dx$, where λ and n are certain constants.

Let $\psi(x)$ denote the number of persons alive x years after their birth, $\phi(x) dx$ the number who die between x and $x+dx$. Then $\phi(x) = -\psi'(x)$.

And $\frac{\phi(x) dx}{\psi(x)}$ is the probability that a person aged x will die between x and $x+dx$. Hence $\psi'(x)/\psi(x) = -\lambda x^n$, i.e. $\psi(x) = A e^{-\lambda \frac{x^{n+1}}{n+1}}$, where A is a constant and $\psi(0) = A$.

Hence the mean duration of life from birth is $\int_0^c e^{-\lambda \frac{x^{n+1}}{n+1}} dx$.

When x is large, the integrand becomes extremely small, and its value is insensible. Hence we may, without sensible error, take c , the superior limit of age, to be ∞ . Put

$$\frac{\lambda x^{n+1}}{n+1} = z; \quad \therefore dx = \frac{1}{n+1} \left(\frac{n+1}{\lambda} \right)^{\frac{1}{n+1}} z^{\frac{1}{n+1}-1} dz = \frac{1}{\lambda} \left(\frac{n+1}{\lambda} \right)^{\frac{1}{n+1}-1} z^{\frac{1}{n+1}-1} dz.$$

\therefore Mean duration at birth

$$= \frac{1}{\lambda} \left(\frac{\lambda}{n+1} \right)^{\frac{n}{n+1}} \int_0^{\infty} z^{\frac{1}{n+1}-1} e^{-z} dz = \frac{1}{\lambda} \left(\frac{\lambda}{n+1} \right)^{\frac{n}{n+1}} \Gamma \left(\frac{1}{n+1} \right).$$

The Probable duration of life at birth is b years, where $e^{-\frac{\lambda b^{n+1}}{n+1}} = \frac{1}{2}$,

$$\text{i.e. } b^{n+1} = \frac{n+1}{\lambda} \log_e 2, \quad \text{i.e. } b = \left\{ \frac{n+1}{\lambda} \log_e 2 \right\}^{\frac{1}{n+1}}.$$

For a person of age a years, the probability of death within the next r years

$$= \frac{e^{-\frac{\lambda a^{n+1}}{n+1}} - e^{-\frac{\lambda(a+r)^{n+1}}{n+1}}}{e^{-\frac{\lambda a^{n+1}}{n+1}}} = 1 - e^{-\frac{\lambda a^{n+1}}{n+1} \left[\left(1 + \frac{r}{a}\right)^{n+1} - 1 \right]}.$$

If r be small in comparison with a , this becomes approximately

$$K \frac{r}{a} \left\{ 1 - \frac{K-n}{2} \frac{r}{a} \right\}, \quad \text{where } K = \lambda a^{n+1}.$$

PROBLEMS.

1. A cardioide is drawn upon a plane and a point P is taken at random within the contour; show that the chance that it is nearer to the vertex than to the cusp is

$$\frac{1}{\pi} \left(\alpha + \frac{\sqrt{5}}{3} \cos^{\frac{3}{2}} \alpha \right), \quad \text{where } \cos \alpha = 2 \sin \frac{\pi}{10}.$$

2. Given that p and q are any two positive quantities, of which q cannot exceed 9 and p cannot exceed 6, show that it is a 2 : 1 chance that the roots of the quadratic $x^2 - px + q = 0$ are imaginary.

3. Three positive quantities are chosen at random, except that their sum is known. Show that the chance that the sum of any two is greater than $1/n^{\text{th}}$ of the third is $1 - 3/(n+1)^2$, provided $n \leq 1$.

4. There are n letters and n directed envelopes. The letters are placed at random, one in each envelope. Show that the chance that r specified letters go wrong and s specified letters go right is

$$[(n-s)! - r(n-s-1)! + \frac{r(r-1)}{1 \cdot 2} (n-s-2)! - \dots + (-1)^r (n-s-r)!] / n!,$$

where $n \leq r + s$.

5. A circle of radius r lies entirely within an ellipse of semi-axes a and b ; $m+n$ random points are taken within the ellipse. What is the chance that m of them lie within the circle and the rest do not?

6. Let two points P and Q be taken at hazard in a line AB in either order, and let three other points be now taken at hazard upon the line. What is the chance that (i) all three should lie between P and Q , (ii) one should lie between P and Q and the others not so, (iii) two specified ones should fall between P and Q and the other not so?

7. A point P is chosen at random upon a line AB , and then a random point Q is taken upon AP . Show that the chance that AQ is less than $1/n^{\text{th}}$ of AB is $\log \sqrt[n]{en}$, ($n > 1$).

8. Four random points are taken upon a straight line. Show that the chance that the sum of the squares of the five parts should not exceed the square on half the line is $3\pi^2/100\sqrt{5}$.

9. A rod is divided into five pieces at random. Show that the chance that none of them is less than $1/10$ of the whole is $1/16$.

10. A rod AB is broken into three pieces AP, PQ, QB at random. Show that the chance that the sum of the squares of AP and QB shall be less than the square of $5PQ$ is $\frac{2 \cdot 5}{10 \cdot 2 \cdot 9} (35 - 6 \log 3/\sqrt{2})$.

11. A random point X is taken upon a line AB . Six other random points are then taken on AB . What is the chance that two of these will lie on AX and four on XB ?

12. From an urn containing an infinite number of balls, all of which are known to be either red or white, a group of seven is drawn out at random, and four are found to be red and three white. What is the chance that a second draw of seven shall also produce four red and three white?

13. A square ticket of side a is thrown at hazard upon a large table ruled into squares of side $2a$. Show that the chance that the ticket will cross a ruling is about 0.86.

14. A circle of radius a is thrown at hazard upon a table ruled in squares of side $3a$. Show that the chance of crossing a ruling is $5/9$.

15. A large table is ruled with parallel lines two inches apart. A one-inch equilateral triangle is thrown at hazard upon the table. Show that the chance it cuts a ruling is $3/2\pi$.

16. A letter L , with thin arms 3 inches long and at right angles to each other, is thrown at hazard upon a large table ruled with parallels 4 inches apart. Show that the chance of crossing a ruling is $3(2 + \sqrt{2})/4\pi$.

17. A cardioid of axis $2a$ inches is thrown at hazard upon a large table ruled with parallel lines at a distance $4a$ inches apart. Show that the chance it cuts a ruling is $9\sqrt{3}/8\pi$.

18. Show that the mean value of the cubes of all random chords of a circle $= \frac{3}{2} \times \text{area of circle} \times \text{radius}$.

19. Show that the mean value of the cubes of all random chords which meet an equilateral triangle of side a is $3a^3/16$.

20. Show that the mean value of the lengths of all random lines terminated by the sides of a square of side a is $\pi a/4$.

21. A circle of radius b lies entirely within a circle of radius a . Show that the chance that a pair of chords of the latter intersect within the former is $b^2/2a^2$.

22. Show that the chance that a pair of random chords of the director circle of an ellipse of semi-axes a and b should not intersect within the ellipse is $1 - ab/2(a^2 + b^2)$.

23. Evaluate the integral $\int (\omega - \sin \omega) d\sigma$ for all elements of area $d\sigma$ which lie outside a given circle of radius a , ω being the angle

between the tangents from the element $d\sigma$ to the circle. Explain the connection of this integral with the theory of chances.

24. Find the chance that if two points be taken at random within a circle of radius a the distance between them will be $< c$ where $c < 2a$.

[ST. JOHN'S, 1885.]

25. Two men, A and B , are walking at rates equally likely to be anything from 0 to a miles an hour and from 0 to b miles an hour respectively. They walk in the same direction along a straight road for a time $c/(a-b)$ hours, where c miles is the initial distance between them. What is the probability that A , who starts behind B , will overtake him?

[TRINITY, 1889.]

26. Suppose there are n sugar sticks each of length $2a$, each broken at random into two pieces. A child is promised the biggest of the $2n$ pieces. What is the value of his expectation?

[W. A. WHITWORTH, *E.T.*, 13736.]

Show that the expectation of the piece of r^{th} largest size is $\{(r+1)n+1\}/2r(n+1)$ of a whole stick.

27. If there be an infinite number of balls in an urn, each ball being known to be of one of n different colours, and if $p_1 + p_2 + \dots + p_n$ balls have been drawn and found to be p_1 of one colour, p_2 of another colour, etc., what is the chance that a further drawing of $q_1 + q_2 + q_3 + \dots + q_n$ will yield q_1 of the first colour, q_2 of the second, etc.?

[ZERR, *E.T.*, 11924.]

28. Two points are taken at random within a circle of radius r , and a chord is drawn at random. Find the chance that the chord passes between the points.

[COLLEGES β , 1888.]

29. An equilateral triangle lies entirely within a regular hexagon whose sides are equal to those of the triangle. A random chord is drawn to cut the hexagon. Show that it is an even chance that it also cuts the triangle.

30. In a circle of radius a the mean of the inverse distance between two random points within the circle is $16/3\pi a$.

[CROFTON, *London M.S. Proc.*, viii., p. 309.]

31. If the probability of a person of age x years dying before he is aged $x+dx$ be $\lambda x dx$, show that the average length of life from birth is $\sqrt{\pi/2\lambda}$. (See a problem by Stanham, *E.T.*, 13021.) Also show that the probable duration of life is $\sqrt{(2 \log 2)/\lambda}$, which is rather less than the average duration.

32. Prove that
$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\theta - \sin \theta \cos \theta) \sin \theta \cos \theta d\theta = \frac{\pi}{16} - \frac{3\sqrt{3}}{64}.$$

Two points are taken at random within a circle. Find the chance that their distance apart is less than the radius of the circle.

[Ox. I. P., 1916.]

33. Show that the mean of the cubes of all lines PQ , which are random chords drawn across the contour, are (i) for a square of side a , $3a^3/4$; (ii) for a circle of radius a , $3\pi a^3/2$; (iii) for a semicircle of radius a , $3\pi^2 a^3/4(\pi + 2)$.

34. Show that the mean of the fifth powers of all lines PQ , which are random chords drawn across the contour, are (i) for a square of side a , $5a^5/6$; (ii) for an equilateral triangle of side a and area Δ , $5a\Delta^2/9$; (iii) for a circle of radius a , $5\pi a^5$.

35. If two pennies of diameter d be soldered together by their edges so as to be in firm contact in a plane, and be thrown upon a plane ruled with equidistant parallel lines whose distance apart is a ($a > 2d$), show that the chance of both pennies being cut by a ruling is $(\pi - 2)d/\pi a$.

36. If a straight line be divided at random into four parts, prove that the chance that one of the parts shall be greater than half the line is $1/2$. Show also that the chance that three times the sum of the squares on the parts is less than the square on the whole line is $\pi\sqrt{3}/18$.

37. If a straight line be divided at random into five parts, show that the chance that four times the sum of the squares of the parts is less than the square on the whole line is $3\pi^2\sqrt{5}/500$.

[WOLSTENHOLME, *E. T.*, 2753.]

38. If random values between $\pm a^2$ be assigned to H and between $\pm(2\alpha^3 + \beta^2)$ to G in the cubic $x^3 + 3Hx + G = 0$, show that the chance of three real roots $= \frac{2}{5} \frac{a^3}{2\alpha^3 + \beta^2}$.

39. Obtain the mean value of $x^2 + y^2 + z^2$ subject to the condition $x + y + z = 0$, and that x, y, z each lie between $-c$ and $+c$.

[LAPLACE; TODHUNTER, *Hist.*, p. 411.]