CHAPTER XXXVI.

MEAN VALUES.

1640. We next exhibit the application of the principles of the Integral Calculus to the calculation of mean values. This subject and that of Chances to be considered in the following chapter are wide, and the devices and artifices numerous. The general principles and theorems are however but few, and the problems arising depend for the most part directly upon the fundamental definitions. A considerable number of illustrative examples are appended to illustrate the more important modes of procedure in the application of the Calculus, and also in the evasion of the necessity in some cases for absolute integration. Many of these are fully worked out; others are left for the reader to complete the details of the integration when it is not necessary to supply them; for it is in the formation of the proper expressions to integrate and in the assignment of the correct limits that difficulties arise rather than in the subsequent mechanical process of evaluation.

1641. Def. *The quantity* $\frac{1}{n}(a_1+a_2+\ldots+a_n)$ *is defined as the Mean Value of the n quantities* $a_1, a_2, \ldots a_n$, *supposed all of the same kind, n being a finite number.*

This is the quantity known arithmetically as the "arithmetic mean" or average value. It may be written as $\frac{1}{n}\Sigma(a)$, and denoted by $M(a)$.

1642. Combination of Means of Several Groups.

If there be several groups of quantities of the same kind, viz. $(a_1, a_2, \ldots a_p)$, $(b_1, b_2, \ldots b_q)$, $(c_1, c_2, \ldots c_r)$, ... of respective numbers p, q, r, etc., and $M(a)$, $M(b)$, $M(c)$, ... the respective means of the groups, then the mean M of the whole set is

$$M=\frac{\Sigma(a)+\Sigma(b)+\Sigma(c)+\dots}{p+q+r+\dots}=\frac{pM(a)+qM(b)+rM(c)+\dots}{p+q+r+\dots}=\frac{\Sigma pM(a)}{\Sigma p},$$

which is the same formula as that for the ordinate of the centroid of weights $p, q, r, \dots$ placed at points whose ordinates are $M(a)$, $M(b)$, $M(c)$, etc.

1643. Mean Values of Products two and two, etc.

Let there be a group of n quantities of the same kind.

Then $$\frac{(\Sigma a)^2}{n^2}=\frac{\Sigma a^2}{n^2}+\frac{2\Sigma a_r a_s}{n^2}=\frac{1}{n}\cdot\frac{\Sigma a^2}{n}+\frac{n-1}{n}\,\frac{\Sigma a_r a_s}{\frac{1}{2}n(n-1)}.$$

Hence $$\{M(a)\}^2=\frac{1}{n}M(a^2)+\frac{n-1}{n}M(a_r a_s).$$

Similarly

$$\frac{(\Sigma a)^3}{n^3}=\frac{\Sigma a^3}{n^3}+\frac{3\Sigma a_1{}^2 a_2}{n^3}+\frac{6\Sigma a_1 a_2 a_3}{n^3}=\frac{3}{n}\,\frac{\Sigma a^2}{n}\,\frac{\Sigma a}{n}-\frac{2}{n^2}\,\frac{\Sigma a^3}{n}+\frac{(n-1)(n-2)}{n^2}\,\frac{\Sigma a_1 a_2 a_3}{\frac{1}{6}n(n-1)(n-2)},$$

i.e. $$\{M(a)\}^3=\frac{3}{n}M(a^2)M(a)-\frac{2}{n^2}M(a^3)+\frac{(n-1)(n-2)}{n^2}M(a_1 a_2 a_3).$$

We may note that when n is indefinitely large, the mean of the products of pairs is the square of the mean of all quantities; and the mean of the products three at a time is the cube of the mean of them all.

These rules determine the mean values of the products, two at a time and three at a time respectively in terms of the means of the original quantities, of their squares and of their cubes.

1644. Extension of the Conception of a Mean.

If the number of the quantities a_1, a_2, etc., be very large, and their sum very large, the fraction $\frac{1}{n}\Sigma a$ tends to take the form ∞/∞. In this case suppose the several quantities a_1, a_2, etc., to be the equidistant ordinates of a continuous curve $y=\phi(x)$ corresponding to abscissae

$$x=a,\quad a+h,\ a+2h,\ \dots\ a+(n-1)h=b, \text{ say.}$$

Then the mean is

$$\frac{1}{n}\{\phi(a)+\phi(a+h)+\phi(a+2h)+\dots+\phi(a+\overline{n-1}h)\},$$

which may be written as $\sum_1^n h\phi\{a+(r-1)h\}/\Sigma h$, which when n is indefinitely increased takes the form

$$\int_a^b \phi(x)\,dx/(b-a).$$

It is assumed here that the several quantities $a_1, a_2, \dots a_n$ are such that no two consecutive ones differ by a finite difference when n is indefinitely great, but that the curve $y=\phi(x)$ is one in which there is a continuous change of the ordinates between the limits considered. Otherwise the integral expression would be meaningless.

1645. **Geometrical Meaning of the "Mean Ordinate."**

It follows that the value of the mean ordinate, taken for equidistant and indefinitely close ordinates, is represented by the area bounded by the curve, the x-axis and the terminal ordinates divided by the projection of the curve upon the x-axis.

That is the mean ordinate PN of a curve P_1Q_1, between the initial and final ordinates N_1P_1, M_1Q_1 is such that the area $P_1N_1M_1Q_1PP_1$ is equal to that of the rectangle FN_1M_1G, where FG is drawn through P parallel to the x-axis (Fig. 476). So that as much of the area of this figure lies between PG and the curve as lies between PF and the curve.

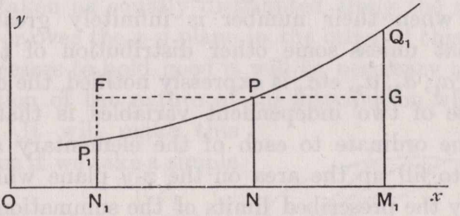


Fig. 476.

1646. **The Case when the Quantities are Functions of Several Variables. Nature of the Distribution.**

If the quantities $a_1, a_2, a_3, \dots$ be functions of several variables, first say of two, x and y, let us consider $a_1, a_2, \dots$ to be the z-ordinates of a surface $z=\phi(x, y)$. Let the plane x-y be imagined ruled by lines δx apart parallel to the y axis, and by lines δy apart parallel to the x-axis. Let one ordinate z, viz. $\phi(x, y)$, be erected at the corner x, y nearest the origin of the elementary rectangle δx, δy, and let the same be done at each of the corners nearest the origin of the remaining net-work of elementary rectangles. Then we shall understand by the "mean value" of z the limit of the fraction whose numerator is the sum of all these ordinates and whose denominator is their number, or, what is the same thing, $\iint z\,dx\,dy \Big/ \iint dx\,dy$, *i.e.* the volume bounded by the x-y plane, the surface $z=\phi(x, y)$, and cylindrical surface bounding the portion of the surface considered, whose generators are parallel to the z-axis, divided by the projection of that portion upon the x-y plane. It will be observed that the *number* of these ordinates is measured by $\iint dx\,dy$, that is the area of the projection described.

And if there be three independent variables, so that $u=\phi(x, y, z)$, we shall understand in the same way that by the "mean value" of u is meant $\iiint u\,dx\,dy\,dz \Big/ \iiint dx\,dy\,dz$, and the *number* of cases is measured by $\iiint dx\,dy\,dz$; and similarly if there be a greater number of independent variables. And as before it will be noted that it is assumed that no two contiguous quantities of the group considered differ by a finite difference when their number is infinitely great. That is to say, that unless some other distribution of the various quantities a_1, a_2, a_3, etc., is expressly notified, the distribution in the case of two independent variables is that in which there is one ordinate to each of the elementary areas $\delta x\,\delta y$, which go to fill up the area on the x-y plane which may be bounded by the prescribed limits of the summation; and that for three independent variables the region through which the summation is to be effected is divided into equal volume elements $\delta x\,\delta y\,\delta z$, and that this summation is to be taken for one value of u, viz. $\phi(x, y, z)$, for each element of volume $\delta x\,\delta y\,\delta z$.

1647. Other Systems of Variables.

Of course the elements of area and of volume expressed in the Cartesian manner as $\delta x\,\delta y$, or as $\delta x\,\delta y\,\delta z$ respectively, may be replaced at will by the corresponding expressions $r\,\delta\theta\,\delta r$ or $r^2 \sin\theta\,\delta\theta\,\delta\phi\,\delta r$, if work in polar coordinates be indicated as more convenient for the problem under consideration, or by the corresponding elements for any other system of coordinates.

And if there be more independent variables than three, so that we fail to interpret the summation by geometry of two or of three dimensions, we shall still understand the mean of the function $u \equiv \phi(x_1, x_2, x_3, \ldots x_n)$ to be

$$\iiint \ldots \int u\,dx_1\,dx_2 \ldots dx_n \Big/ \iint \ldots \int dx_1\,dx_2 \ldots dx_n,$$

and the number of cases to be measured by

$$\iint \ldots \int dx_1\,dx_2 \ldots dx_n$$

when the limits have been properly ascribed so as to effect the summations in the numerator and denominator for all values of the independent variables included in the compass of the summation to which the "mean value" refers.

1648. Nature of Various Distributions.

It will be manifest that in the case of a distribution of an infinite number of quantities such as the ordinates of a curve or of a surface, and whose mean is required, and which have so far been taken as equally distributed along the x-axis in the one case or over the x-y plane in the other, if this equable distribution ceases to hold good it will be necessary to form a clear conception of the nature of the distribution which is to be adopted. It will make this matter obvious if we take a simple example.

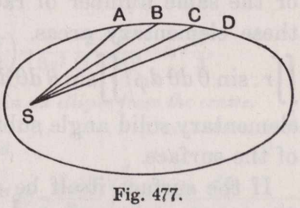


Fig. 477.

Consider the problem of finding the mean value of all focal radii vectores of an ellipse. Usually we should understand this to mean that if $A, B, C, D, \ldots$ be indefinitely close points on the circumference and S the focus from which the radii vectores are drawn, then the mean is to be taken for all the radii vectores such that the successive angles ASB, BSC, CSD, etc., are all equal infinitesimal angles $\delta\theta$. In which case, r being the radius vector for an angle θ, the mean value

$$= \int r\,d\theta \Big/ \int d\theta.$$

But it might be that the successive *arcs* AB, BC, CD, ... are to be taken as equal, or that the successive *areas* are all equal, or that the successive points $A, B, C, D, \ldots$ are defined by an equable distribution of the *feet of their ordinates* upon the x-axis, or other conceivable distributions may be adopted. The mean values in these cases are respectively

$$\int r\,ds \Big/ \int ds, \quad \int r \, . \, r^2\,d\theta \Big/ \int r^2\,d\theta, \quad \int r\,dx \Big/ \int dx,$$

and the several results are obviously not the same.

1649. **"Density" of a Distribution. General Remarks.**

It will appear therefore that in each case the nature of the distribution, or, as it may be called, the "Density," must be carefully defined. This is of primary importance.

When the distribution is one in which the angles between the successive radii vectores are equal infinitesimal angles, as in the case cited, they may be described as equally distributed about the origin from which they are drawn. This is the usual case.

In the same way, in three dimensions, when a distribution of radii vectores drawn from an origin to a surface is said to be "equable," we shall understand this to mean that a unit sphere having been drawn with centre at the origin, and its surface having been divided into equal elementary areas, one, or the same number of radii vectores, passes through each of these elementary areas. The mean value of r will then be $\iint r \, . \sin\theta\,d\theta\,d\phi \Big/ \iint \sin\theta\,d\theta\,d\phi$ or $\int r\,d\omega \Big/ \int d\omega$, where $\delta\omega$ is the elementary solid angle subtended at the origin by each element of the surface.

If the surface itself be divided into equal elementary areas δS, and the same number of radii vectores pass through each such element, the distribution may be called an "equable surface distribution," and the mean value will be $\int r\,dS \Big/ \int dS$.

If radii vectores be drawn from the origin to points within the region bounded by a given surface, it is usually understood that they are drawn to equal elements of volume. The mean is then

$$\iiint r \, . \, r^2 \sin\theta\,d\theta\,d\phi\,dr \Big/ \iiint r^2 \sin\theta\,d\theta\,d\phi\,dr.$$

1650. ILLUSTRATIVE EXAMPLES.

1. *Find the mean distance of points on the circumference of the ellipse from a focus, the density of the distribution being defined as one in which successive pairs of points subtend equal angles at the focus.*

Taking the equation as $lr^{-1}=1+e\cos\theta$, we have, b being the semi-minor axis,

$$M(r)=\frac{\int r\,d\theta}{\int d\theta}=\frac{2l\int_0^\pi (1+e\cos\theta)^{-1}d\theta}{2\pi}$$

$$=\frac{l}{\pi}\,\frac{2}{\sqrt{1-e^2}}\left[\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right)\right]_0^\pi=\frac{2b}{\pi}\cdot\frac{\pi}{2}=b.$$

2. *Find the mean inverse distance of points within an ellipse from the focus, the distribution being an equable areal one.*

Here $M\left(\dfrac{1}{r}\right)=\dfrac{\iint\frac{1}{r}\,.\,r\,d\theta\,dr}{\iint r\,d\theta\,dr}=\dfrac{\iint d\theta\,dr}{\text{Area}}=\dfrac{\int r\,d\theta}{\text{Area}}=\dfrac{2\pi b}{\pi ab}=\dfrac{2}{a}$;

a, b being the semi-axes.

3. *Find the mean distance of a point within an ellipse from a focus.* [COLLEGES α, 1886 and 1879.]

Here $M(r)=\dfrac{\iint r\,.\,r\,d\theta\,dr}{\iint r\,d\theta\,dr}=\dfrac{2}{3\pi ab}\displaystyle\int_0^\pi r^3d\theta=\dfrac{2l^3}{3\pi ab}\int_0^\pi\dfrac{d\theta}{(1+e\cos\theta)^3}$

$$=\frac{2l^3}{3\pi ab}\,\frac{1}{(1-e^2)^{\frac{5}{2}}}\int_0^\pi(1-e\cos u)^2du \quad \text{(Art. 196)}$$

$$=\frac{2l^3}{3\pi ab}\,\frac{1}{(1-e^2)^{\frac{5}{2}}}\left(\pi+2e^2\frac{1}{2}\,\frac{\pi}{2}\right)=\frac{l^3}{3a^2}\,\frac{2+e^2}{(1-e^2)^3}=a-\frac{l}{3}.$$

4. *Find the mean distance of points within an ellipse from the centre.* [COLLEGES α, 1886.]

Here, measuring θ from the minor axis,

$$\frac{1}{r^2}=\frac{\sin^2\theta}{a^2}+\frac{\cos^2\theta}{b^2}\quad\text{and}\quad M(r)=\frac{4}{3\pi ab}\int_0^{\frac{\pi}{2}}r^3d\theta=\frac{4a^2b^2}{3\pi}\int_0^{\frac{\pi}{2}}\frac{d\theta}{(a^2\cos^2\theta+b^2\sin^2\theta)^{\frac{3}{2}}}$$

$$=\frac{4b^2}{3\pi a}\int_0^{\frac{\pi}{2}}\frac{d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}=\frac{4b^2}{3\pi a}\cdot\frac{1}{1-e^2}\int_0^{\frac{\pi}{2}}(1-e^2\sin^2\theta)^{\frac{1}{2}}d\theta \quad \text{(Art. 391 (1))}$$

$$=\frac{1}{3\pi}\times(\text{Perimeter of Ellipse}) \quad \text{(Art. 567).}$$

5. *Find the mean of the distances from one of the foci of a prolate spheroid to points within the surface.* [WOLSTENHOLME, *Educ. Times.*]

Taking $lr^{-1}=1+e\cos\theta$ as the generating ellipse,

$$M(r)=\frac{\iiint r\,.\,r^2\sin\theta\,d\theta\,d\phi\,dr}{\text{Volume}}=\frac{2\pi}{\text{Vol.}}\,\frac{l^4}{4}\int_0^\pi\frac{\sin\theta}{(1+e\cos\theta)^4}d\theta=\text{etc.}=\frac{a}{4}(3+e^2).$$

6. *A particle describes an ellipse about a centre of force in the focus S. Show that its mean distance from S with regard to time is* $a\left(1+\dfrac{e^2}{2}\right)$. [R.P.]

If t be the time, then $r^2\dfrac{d\theta}{dt}=\text{const.}=h$, for equal sectorial areas are described in equal times.

$$\text{Hence}\quad M(r)=\frac{\int r\,dt}{\int dt}=\frac{\int r^3 d\theta}{\int r^2 d\theta}=\frac{\int_0^\pi r^3 d\theta}{\text{Area}}=a\left(1+\frac{e^2}{2}\right)\quad\text{(by Ex. 3).}$$

7. *Find the mean value of* r^{-2} *with regard to time under the same circumstances.*

$$M(r^{-2})=\frac{\int \frac{1}{r^2}dt}{\int dt}=\frac{\int d\theta}{\int r^2 d\theta}=\frac{2\pi}{2\,.\,\text{Area}}=\frac{1}{ab}.$$

8. *Show that the mean distance of points within a square from one of the angular points is to a side of the square in the ratio* $\{\sqrt{2}+\log(\sqrt{2}+1)\}$ *to* 3.

Take OA, OC, sides of the square $OABC$, as coordinate axes. We may confine our attention to points within the triangle OAB without altering the result. Let a be a side of the square. $OP=r$. Then (Fig. 478)

$$M(r)=\frac{\int_0^{\frac{\pi}{4}}\int_0^{a\sec\theta} r^2 d\theta\,dr}{\frac{1}{2}a^2}=\tfrac{2}{3}a\int_0^{\frac{\pi}{4}}\sec^3\theta\,d\theta=\frac{a}{3}\{\sqrt{2}+\log(\sqrt{2}+1)\}.$$

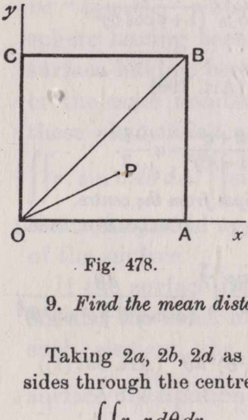


Fig. 478.

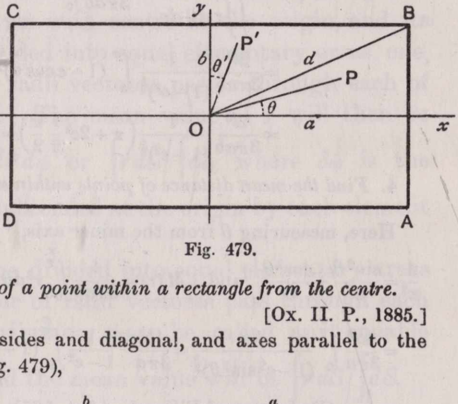


Fig. 479.

9. *Find the mean distance of a point within a rectangle from the centre.*
[Ox. II. P., 1885.]

Taking $2a$, $2b$, $2d$ as the sides and diagonal, and axes parallel to the sides through the centre (Fig. 479),

$$M(r)=\frac{\iint r\,.\,r\,d\theta\,dr}{\iint r\,d\theta\,dr}=\frac{4}{3\,.\,\text{Area}}\left\{\int_0^{\tan^{-1}\frac{b}{a}} a^3\sec^3\theta\,d\theta+\int_0^{\tan^{-1}\frac{a}{b}} b^3\sec^3\theta'\,d\theta'\right\}$$

$$=\frac{1}{6}\frac{a^2}{b}\left\{\frac{d}{a}\cdot\frac{b}{a}+\log\frac{d+b}{a}\right\}+\frac{1}{6}\frac{b^2}{a}\left\{\frac{d}{b}\cdot\frac{a}{b}+\log\frac{d+a}{b}\right\}$$

$$=\frac{d}{3}+\frac{a^2}{6b}\log\frac{d+b}{a}+\frac{b^2}{6a}\log\frac{d+a}{b}.$$

This is also obviously the result for the mean distance of a point within a rectangle of sides a, b and diagonal d from one of the angular points.

10. *Find the mean distance of points on a spherical surface from a fixed point O on the surface for an equable surface distribution of radii vectores.*

Here $M(r)=\int r\,dS\Big/\int dS$, where dS is an element of the surface, and with the notation indicated in Fig. 480,

$$M(r)=\int_0^{\frac{\pi}{2}}\int_0^{2\pi} 2a\cos\theta\,.\,2a\,d\theta\,.\,a\sin 2\theta\,d\phi/4\pi a^2=16\pi a^3/12\pi a^2=4a/3.$$

11. *Find the same mean for a distribution of radii vectores equably drawn in all directions from O.*

Here $$M(r)=\frac{\int r\,d\omega}{\int d\omega}=\frac{1}{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{2\pi} 2a\cos\theta\,.\sin\theta\,d\theta\,d\phi=a.$$

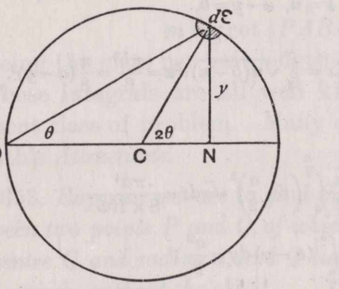


Fig. 480.

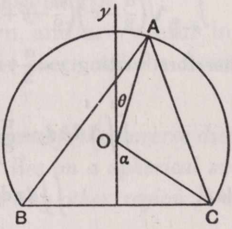


Fig. 481.

12. *Triangles are drawn on a given base a, and with a given vertical angle α. Find the average area.* [Sanjána, *Educ. Times.*]

Let A be the vertex, BC the base$=a$, O the circumcentre, $OA=R$, making an angle θ with a perpendicular to the base. Then $R=a/2\sin\alpha$.

The perpendicular from A upon $BC=R(\cos\theta+\cos\alpha)$, and if the mean be for an equable distribution of positions of OA, (Fig. 481),

$$M(\triangle ABC)=\tfrac{1}{2}aR\int_0^{\pi-\alpha}(\cos\theta+\cos\alpha)\,d\theta\Big/\int_0^{\pi-\alpha}d\theta$$

$$=\frac{1}{2}\,\frac{aR}{\pi-\alpha}\Big[\sin\theta+\theta\cos\alpha\Big]_0^{\pi-\alpha}=\frac{a^2}{4}\left(\frac{1}{\tan\alpha}+\frac{1}{\pi-\alpha}\right).$$

13. (*a*) *A person is left a triangular piece of ground whose perimeter only is known; show that he may fairly calculate that the area is to that of a circle whose radius is the known perimeter as* 1 : 105, *sides of all possible lengths being equally likely to occur.* [Math. Tripos.]

(*b*) *A straight line of length a is broken into three parts at random. If the three parts can be formed into a triangle, find its mean area.*

[ST. JOHN'S COLL., 1881.]

(*a*) and (*b*) are the same problem.

Let OA be the line, P, Q the random points of division, P being the nearer to O, $OP=x$, $OQ=y$, $OA=a$. Then

$$\Delta=\sqrt{\frac{a}{2}\left(\frac{a}{2}-x\right)\left(\frac{a}{2}-y+x\right)\left(y-\frac{a}{2}\right)},\quad \text{and}\quad M(\Delta)=\iint\Delta\,dx\,dy\Big/\iint dx\,dy.$$

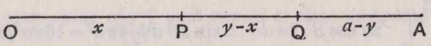


Fig. 482.

The limits of integration are to be such that

(i) $x+(y-x)\nless(a-y)$; (ii) $(y-x)+(a-y)\nless x$; (iii) $(a-y)+x\nless(y-x)$,

i.e. $y\nless\frac{a}{2}$, $x\ngtr\frac{a}{2}$, and $y\ngtr\frac{a}{2}+x$. So the limits are, for x, $y-\frac{a}{2}$ to $\frac{a}{2}$; for y, $\frac{a}{2}$ to a. Now putting $\frac{a}{2}-x=u$, $a-y=b$,

$$\int_{y-\frac{a}{2}}^{\frac{a}{2}}\sqrt{\left(\frac{a}{2}-x\right)\left(\frac{a}{2}-y+x\right)}\,dx=\int_0^b\sqrt{u(b-u)}\,du=\frac{\pi b^2}{8}=\frac{\pi}{8}(a-y)^2.$$

Therefore writing $y=\frac{a}{2}+z$,

$$\iint\Delta\,dx\,dy=\frac{\pi}{8}\sqrt{\frac{a}{2}}\int_0^{\frac{a}{2}}\left(z-\frac{a}{2}\right)^2\sqrt{z}\,dz=\frac{\pi a^4}{8\times105}.$$

Also
$$\iint dx\,dy=\int_{\frac{a}{2}}^{a}(a-y)\,dy=\frac{a^2}{8};$$

$\therefore\ M(\Delta)=\frac{\pi a^2}{105}=\frac{1}{105}$ of the area of a circle whose radius is a.

1651. The Mean Inverse Distance considered as a Potential Function.

In problems on the mean value of the inverse distance between pairs of points, much labour of integration may often be avoided if it be recognised that such problems are in fact problems on the mutual potential of two gravitating systems of material particles.

The potential at any point P of a system of gravitating particles of masses m_1, m_2, m_3, etc., at distances r_1, r_2, r_3, etc., from P is defined as $\Sigma m/r$.

The Mutual Potential of two gravitating systems of masses of two separate groups $(m_1, m_1', m_1'', \ldots)$ and $(m_2, m_2', m_2'', \ldots)$

is defined as $\Sigma m_1 m_2/r_{12}$, where r_{12} represents the distance between m_1 and m_2, etc.

But if the particles be *particles of the same group*, the mutual potential is $\frac{1}{2}\Sigma m_1 m_2/r_{12}$. [See Routh, *Attractions*, p. 29.]

1652. **Theorems in Potential required for the Problems to be considered.**

In the case of a spherical shell of mass M, the potential at an external point at a distance r from the centre is M/r. But at an internal point it is M/a, where a is the radius.

In the case of a solid sphere, the potential at an external point at a distance r from the centre is again M/r; at an internal point $\frac{2\pi\rho}{3}(3a^2-r^2)$, M being in each case the mass and ρ the uniform volume density.

The potential of a thin rod AB at any point P is

$$m \log \cot \tfrac{1}{2}P\hat{A}B \cot \tfrac{1}{2}P\hat{B}A,$$

m being the mass per unit length = mass/length.

These integrals are all well known, and are useful in the present class of problem. Many other cases will be found in Routh's *Attractions*.

1653. *Suppose we are to find the mean of the inverse distance between two points P and Q, of which P lies on a spherical surface of centre C and radius a, and Q lies in any other region R which lies entirely without the shell.*

Let dS be an element of the spherical surface, dR an element of volume of the region R.

Then

$$M\left(\frac{1}{PQ}\right)=\frac{\iint \frac{1}{PQ}\,dS\,dR}{\iint dS\,dR}.$$

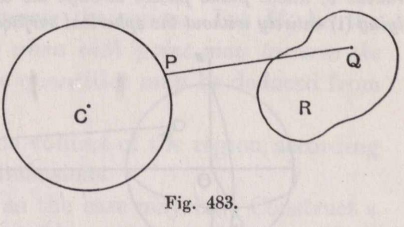


Fig. 483.

Suppose the surface and volume densities to be unity, and let $PQ=\rho$. Then

$$M\left(\frac{1}{PQ}\right)=\frac{1}{S.R}\int(\text{potential of shell at } Q)\,dR$$

$$=\frac{1}{S.R}\int S\cdot\frac{dR}{CQ}=\frac{1}{R}\cdot\text{potential of } R \text{ at } C.$$

If any portion of R lies within the shell, let R_i and R_o be the masses of the portions lying respectively within and without the shell; Q and Q' two points of the region R, the one outside, the other inside the shell. Then

$$\iint\frac{dS\,.\,dR}{PQ}=\iint\frac{dS\,.\,dR_o}{PQ}+\iint\frac{dS\,.\,dR_i}{PQ'}$$

$$=S\,.\text{ potential of }R_o\text{ at }C+S\,.\,\frac{R_i}{a}.$$

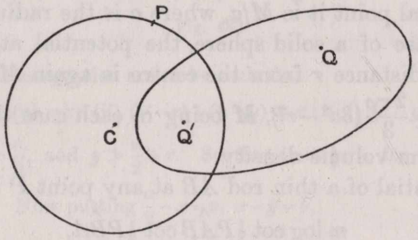


Fig. 484.

Hence $\quad M\left(\frac{1}{\rho}\right)=\frac{1}{R}\left\{\text{potential of } R_o \text{ at } C+\frac{R_i}{a}\right\}.$

(See a Theorem due to Gauss; Routh, *Attractions*, Art. 70.)

If R lies entirely inside S, $R_o=0$, $R_i=R$ and $M\left(\frac{1}{\rho}\right)=\frac{1}{a}$.

1654. Examples.

1. *Find the mean inverse distance between a point P which lies on a spherical surface of radius a, and a point Q which lies on a circular disc of radius b, whose plane passes through the centre of the sphere, and the disc lying* (i) *entirely without the spherical surface,* (ii) *entirely within.*

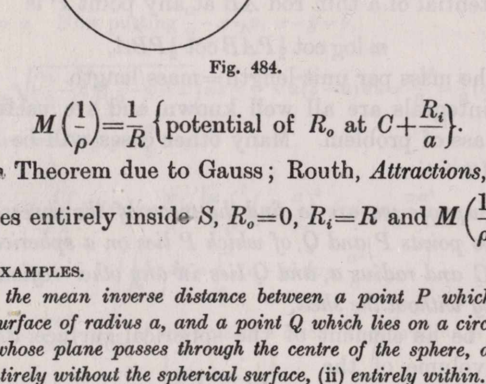


Fig. 485.

(i) Let O be the centre of the sphere, ρ the distance between a pair of the points. Then we have

$$M\left(\frac{1}{\rho}\right)=\frac{1}{\pi b^2}\cdot\text{potential of disc at } O.$$

If $c \equiv$ the distance between the centres, this may be expressed as

$$\frac{1}{\pi b^2}\int_0^{2\pi}\frac{b(b-c\cos\theta)\,d\theta}{\sqrt{b^2-2bc\cos\theta+c^2}}, \qquad \text{[MATH. TRIP., 1884.]}$$

or as
$$\frac{4c}{\pi b^2}[E_1-k'^2F_1], \quad k=\frac{b}{c}.$$

(ii) If the disc lie entirely within the spherical shell, we have at once

$$M\left(\frac{1}{\rho}\right)=\frac{1}{a}.$$

2. ***Find the mean inverse distance of two points P and Q, one within a sphere of centre A and radius a, the other within a sphere of centre B and radius b, the centres being at a distance c apart*** $(c > a+b)$.

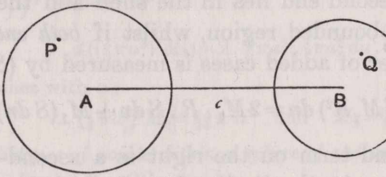


Fig. 486.

If V, V' be the respective volumes, $PQ=\rho$,

$$M\left(\frac{1}{\rho}\right)=\frac{\iint\frac{dV\,dV'}{PQ}}{VV'}=\frac{\int(\text{potential of } V \text{ at } Q)\,dV'}{VV'}=\frac{\int\frac{V}{AQ}\,dV'}{VV'}$$

$$=\frac{1}{V'}\int\frac{dV'}{AQ}=\frac{1}{V'}\cdot\text{potential of } V' \text{ at } A=\frac{1}{V'}\cdot\frac{V'}{AB}=\frac{1}{c}.$$

1655. A Useful Artifice.

Let M_1 represent the mean value of any function of the distance between two points, one *fixed on the boundary* of any region, the other *free to traverse the region.* Let M_2 be the mean of the same function *when each point may traverse the region.* Then either of these quantities may be deduced from the other.

Let A be the area, or V the volume of the region, according as it be of two or of three dimensions.

Let R stand for A or V as the case may be. Construct a parallel curve or surface by taking a length dn (a constant) upon each outward drawn normal, thus making an annulus or shell round the original region. (Fig. 487.)

By this increase of the region R, M_2 is increased by the cases in which *one or other* of the points lies in this shell, or by *both* lying in the shell.

The *number* of cases to be examined in finding M_2 is measured by R^2.

The *sum* of the cases is measured by M_2R^2.

The increase in this sum due to the increase of the normals from n to $n+dn$ is $\frac{d}{dn}(M_2R^2)\,dn$.

Again, the *number* of cases added by taking *one end* of the line on the shell and the other free to traverse the region it encloses, is measured by $R \,.\, S\,dn$, where S is the perimeter (or the surface, as the case may be) of the region. The same is true if the second end lies in the shell and the first is free to traverse the bounded region, whilst if *both ends* lie on the shell the number of added cases is measured by $(S\,dn)^2$.

Hence $$\frac{d}{dn}(M_2R^2)\,dn = 2M_1 \,.\, R \,.\, S\,dn + M_1(S\,dn)^2;$$

and as the second term on the right is a second-order infinitesimal, we have in the limit when dn is indefinitely small, $\frac{d}{dn}(M_2R^2) = 2M_1RS$, by which equation the value of either M_1 or M_2 can be deduced when the other has been found. This artifice is useful for circular areas or spherical regions, and may be used in other cases.

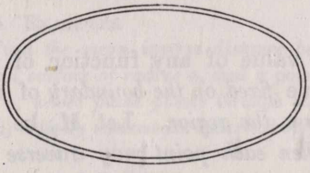


Fig. 487.

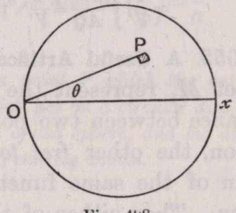


Fig. 488.

1656. Illustrative Examples.

1. (i) *Show that the mean distance of points within a circle from a fixed point in the circumference, viz.* M_1, *is* $32a/9\pi$, *a being the radius.*

(ii) *Show that the mean distance between any two points within the circle, viz.* M_2, $=128a/45\pi$. [St. John's Coll., 1885.]

Let O be the fixed point on the circumference and Ox the diameter through O. r, θ the coordinates of any point P. (Fig. 488.)

$$\text{(i)}\quad M_1 = M(OP) = \frac{\iint r^2\,d\theta\,dr}{\iint r\,d\theta\,dr} = \frac{2}{3}\,\frac{\int_0^{\frac{\pi}{2}}(2a\cos\theta)^3\,d\theta}{\int_0^{\frac{\pi}{2}}(2a\cos\theta)^2\,d\theta} = \frac{2}{3}\,.\,2a\,\frac{\frac{2}{3}}{\frac{1}{2}\,.\,\frac{\pi}{2}} = \frac{32a}{9\pi}.$$

(ii) Again $\quad d\{(\pi a^2)^2 M_2\} = 2 \,.\, \pi a^2 \,.\, 2\pi a\, da \,.\, \dfrac{32a}{9\pi} = \dfrac{128}{9}\pi a^4\, da,$

and M_2 vanishes with a.

$$\therefore\ \pi^2 a^4 M_2 = \frac{128}{45}\pi a^5 \quad\text{and}\quad M_2 = \frac{128a}{45\pi}.$$

2. (i) *Find M_1, the mean distance of a point on the surface of a sphere of radius a from internal points.*

(ii) *Find M_2, the mean distance between two points within a sphere of radius a.*

$$\text{(i)}\ M_1 = \frac{\iiint r\,.\,r^2\sin\theta\, d\theta\, d\phi\, dr}{\iiint r^2 \sin\theta\, d\theta\, d\phi\, dr} = \frac{3}{4\pi a^3}\,.\,\tfrac{1}{4}\,.\,2\pi\,.\,(2a)^4\int_0^{\frac{\pi}{2}}\cos^4\theta\sin\theta\, d\theta = \frac{6a}{5}.$$

(ii) $\quad d\{(\tfrac{4}{3}\pi a^3)^2 M_2\} = 2\,.\,\tfrac{4}{3}\pi a^3\,.\,4\pi a^2\, da\,.\,\dfrac{6a}{5},$

and M_2 vanishes with a;

$$\therefore\ (\tfrac{4}{3}\pi a^3)^2 M_2 = \tfrac{64}{35}\pi^2 a^7 \quad\text{and}\quad M_2 = \tfrac{36}{35}a.$$

3. *Mean distance of points within a sphere of radius a and centre C from a given external point O; $OC = c$.*

Let OQQ' be a chord through an internal point P, whose coordinates are r, θ with reference to O as origin, and let ϕ be the azimuthal angle of the plane OCP. Then

$$M(r) = \frac{3}{4\pi a^3}\iiint r^3\sin\theta\, d\theta\, d\phi\, dr = \frac{3}{4\pi a^3}\,.\,\frac{2\pi}{4}\int_0^{\sin^{-1}a/c}(OQ'^4 - OQ^4)\sin\theta\, d\theta.$$

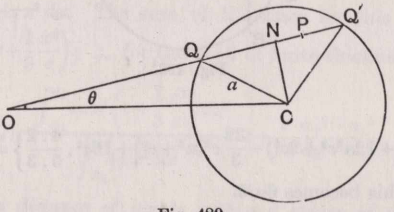


Fig. 489.

Let $QQ' = 2z$; then

$$z^2 = a^2 - c^2\sin^2\theta,\quad z\,dz = -c^2\sin\theta\cos\theta\, d\theta = -\tfrac{1}{2}(OQ + OQ')c\sin\theta\, d\theta,$$

and the limits for z are from a to 0.

$$\therefore\ M(r) = \frac{3}{8a^3}\int_0^a 2z\{4z^2 + 2(c^2 - a^2)\}\frac{2z\, dz}{c} = \frac{3}{a^3 c}\left[\frac{2}{5}z^5 + (c^2 - a^2)\frac{z^3}{3}\right]_0^a = c + \frac{1}{5}\frac{a^2}{c}.$$

4. *Mean distance of points upon the surface of the sphere from a point O without the sphere.*

The number of cases in which P can traverse the whole sphere is measured by $\tfrac{4}{3}\pi a^3$. Therefore the sum of such cases is $\tfrac{4}{3}\pi a^3\left[c + \dfrac{1}{5}\dfrac{a^2}{c}\right]$.

The change effected in this by increasing a to $a+da$ is

$$\frac{d}{da}\tfrac{4}{3}\pi a^3\left(c+\frac{1}{5}\frac{a^2}{c}\right)da=4\pi a^2\left(c+\frac{1}{3}\frac{a^2}{c}\right)da.$$

The number of these introduced cases is to the first order $4\pi a^2\,da$, the new cases being those of the points *on* the shell. Hence the mean required $=c+\frac{1}{3}\frac{a^2}{c}$.

5. *Find the mean distance of all points P within a sphere of radius a and centre C from a fixed internal point O ; OC=c.*

Here $M(OP)=\frac{1}{\text{vol.}}\iiint r^3\sin\theta\,d\theta\,d\phi\,dr=\frac{3}{4\pi a^3}\cdot\frac{2\pi}{4}\int[r^4]\sin\theta\,d\theta.$

Let QOQ' be the chord through P, AOA' a diameter and BOB' the perpendicular chord. Let $A\hat{O}Q=\theta$, $A'\hat{O}Q'=\theta'$. We may replace $[r^4]\sin\theta$ by $OQ^4\sin\theta+OQ'^4\sin\theta'$ and integrate with regard to θ $(=\theta')$ from 0 to $\frac{\pi}{2}$; for having integrated for ϕ from 0 to 2π, all elements will be thus summed. Now $OQ^2+OQ'^2=2(a^2+c^2)-4c^2\sin^2\theta$, and

$$OQ^4+OQ'^4=\{4(a^2+c^2)^2-2(a^2-c^2)^2\}-16c^2(a^2+c^2)\sin^2\theta+16c^4\sin^4\theta.$$

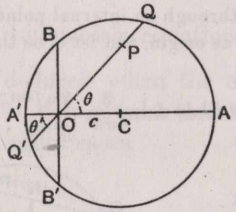


Fig. 490.

Hence

$$M(OP)=\frac{3}{8a^3}\left\{(2a^4+12a^2c^2+2c^4)-\frac{32}{3}c^2(a^2+c^2)+16c^4\cdot\frac{4\cdot 2}{5\cdot 3}\right\}=\frac{3}{4}a+\frac{1}{2}\frac{c^2}{a}-\frac{1}{20}\frac{c^4}{a^3}.$$

When $c=a$ this becomes $6a/5$.

6. *Deduce from the last result the mean distance between two random points within a sphere.*

Taking C for pole and r_1, θ_1, ϕ_1 as the coordinates of O, the sum of the cases with a given point O for an extremity is

$$\frac{4}{3}\pi a^3\left[\frac{3a}{4}+\frac{1}{2}\frac{r_1^2}{a}-\frac{1}{20}\frac{r_1^4}{a^3}\right].$$

Multiplying by $r_1^2\sin\theta_1\,d\theta_1\,d\phi_1\,dr_1$ and integrating through the sphere, we have

$$\text{Mean value required}=\frac{1}{(\frac{4}{3}\pi a^3)^2}\cdot\frac{4}{3}\pi a^3\cdot 2\pi\cdot 2\cdot\left[\frac{3a}{4}\cdot\frac{a^3}{3}+\frac{1}{2a}\cdot\frac{a^5}{5}-\frac{1}{20a^3}\cdot\frac{a^7}{7}\right]=\frac{36a}{35},$$

as otherwise in Ex. 2.

7. *Find the mean distance of a given point O within a sphere from points on the surface.*

The *sum* of the cases of distances of internal points from O being as in the last example, $\pi(a^4+\frac{2}{3}c^2a^2-\frac{1}{15}c^4)$ is increased by $\pi(4a^3+\frac{4}{3}c^2a)\,da$ by increasing the radius to $a+da$. The number of added cases is *to the first order* measured by $4\pi a^2\,da$. Therefore the mean of distances of points on the surface from the given internal point O is

$$\pi\left(4a^3+\frac{4}{3}c^2a\right)da\Big/4\pi a^2\,da=a+\frac{1}{3}\frac{c^2}{a}.$$

8. *Find the mean distance of points between the surfaces of two concentric spheres of radii a_1, a_2 from an external point P at a distance c from the centre O.*

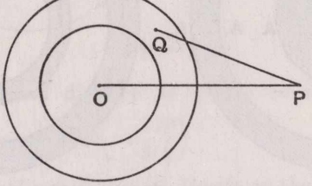


Fig. 491.

Taking Q any point of the shell distant x from the centre, the mean value of PQ is $c+\frac{1}{3}\frac{x^2}{c}$, and the number of cases between the spheres of radii x, $x+dx$ is $4\pi x^2\,dx$. The sum of the cases for this thin shell is therefore $4\pi x^2\,dx\left(c+\frac{1}{3}\frac{x^2}{c}\right)$; $\therefore$ for the shell of finite thickness,

$$M(PQ)=\frac{\int_{a_1}^{a_2}4\pi x^2\left(c+\frac{1}{3}\frac{x^2}{c}\right)dx}{\int_{a_1}^{a_2}4\pi x^2\,dx}=c+\frac{1}{5c}\frac{a_2^5-a_1^5}{a_2^3-a_1^3}.$$

9. *Find the mean distance of points within a sphere of radius a and centre O from points within an external concentric spherical shell of internal and external radii a_1 and a_2.* (Fig. 492.)

Let P and Q be two such points, Q lying within the shell, $OQ=x$. For a given position of Q, $M(PQ)=x+\frac{1}{5}\frac{a^2}{x}$. The number of cases is measured by $\frac{4}{3}\pi a^3$, and their sum by $\frac{4}{3}\pi a^3\left(x+\frac{1}{5}\frac{a^2}{x}\right)$. Now let Q traverse the shell. Let dV be an element of its volume. Then

$$M(PQ)=\frac{\int\frac{4}{3}\pi a^3\left(x+\frac{1}{5}\frac{a^2}{x}\right)dV}{\int\frac{4}{3}\pi a^3\,dV}=\frac{\int_{a_1}^{a_2}\left(x+\frac{1}{5}\frac{a^2}{x}\right)4\pi x^2\,dx}{\int_{a_1}^{a_2}4\pi x^2\,dx}=\frac{3}{4}\frac{a_2^4-a_1^4}{a_2^3-a_1^3}+\frac{3}{10}a^2\frac{a_2^2-a_1^2}{a_2^3-a_1^3}.$$

In the particular cases stated below, we have

(i) $a_1=a_2$, $M=a_1+\dfrac{1}{5}\dfrac{a^2}{a_1}$; (ii) $a_1=a_2=a$, $M=\dfrac{6a}{5}$;

(iii) $a=0$, $M=\dfrac{3}{4}\dfrac{(a_1+a_2)(a_1{}^2+a_2{}^2)}{a_1{}^2+a_1a_2+a_2{}^2}$; (iv) $a_1=a_2$ and $a=0$, $M=a_1$;

(v) $a_1=a$, $M=\dfrac{3}{20}\dfrac{(a+a_2)(7a^2+5a_2{}^2)}{a^2+aa_2+a_2{}^2}$; (vi) $a_1=a=0$, $M=\dfrac{3a_2}{4}$.

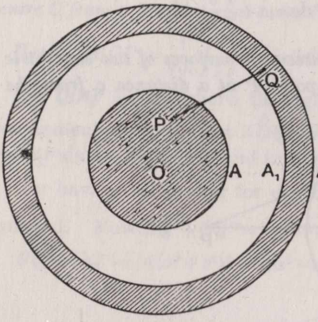


Fig. 492.

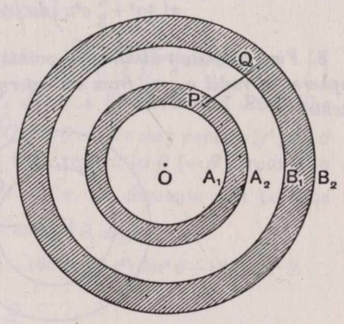


Fig. 493.

10. *Find the mean distance of a point P which lies between the surfaces of a spherical shell of inner and outer radii a_1 and a_2 from a point Q, which lies between the surfaces of a concentric spherical shell whose inner and outer radii are b_1 and b_2 ($b_2>b_1>a_2>a_1$).* (Fig. 493.)

Let O be the centre, $OQ=x$. For a fixed position of Q,

$$M(PQ)=x+\frac{1}{5x}\frac{a_2{}^5-a_1{}^5}{a_2{}^3-a_1{}^3},$$

and the number of such cases is measured by $\frac43\pi(a_2{}^3-a_1{}^3)$, and their sum by $\frac43\pi(a_2{}^3-a_1{}^3)\left[x+\frac{1}{5x}\frac{a_2{}^5-a_1{}^5}{a_2{}^3-a_1{}^3}\right]\equiv F(x)$, say. Hence when Q is free to traverse the outer shell, we have

$$M(PQ)=\frac{\int 4\pi x^2F(x)dx}{\int 4\pi x^2dx\times\frac43\pi(a_2{}^3-a_1{}^3)}=\frac{\int_{b_1}^{b_2}x^2\left(x+\frac{1}{5x}\frac{a_2{}^5-a_1{}^5}{a_2{}^3-a_1{}^3}\right)dx}{\int_{b_1}^{b_2}x^2dx}$$

$$=\frac34\frac{b_2{}^4-b_1{}^4}{b_2{}^3-b_1{}^3}+\frac{3}{10}\frac{a_2{}^5-a_1{}^5}{a_2{}^3-a_1{}^3}\cdot\frac{b_2{}^2-b_1{}^2}{b_2{}^3-b_1{}^3}.$$

11. *Mean distance of points Q within a sphere of radius a, from points P on the surface of a second of radius b external to the former.*

A and B being the respective centres and P a given point on the surface of the second sphere, the mean of distances from P of points within the first $=r+\dfrac15\dfrac{a^2}{r}$, where $AP=r$.

Hence the sum of the cases is measured by $\frac{4}{3}\pi a^3\left(r+\frac{1}{5}\frac{a^2}{r}\right)$ Hence we are to find for the second sphere $\dfrac{\int \frac{4}{3}\pi a^3\left(r+\frac{1}{5}\frac{a^2}{r}\right)dS}{\int \frac{4}{3}\pi a^3 dS}$.

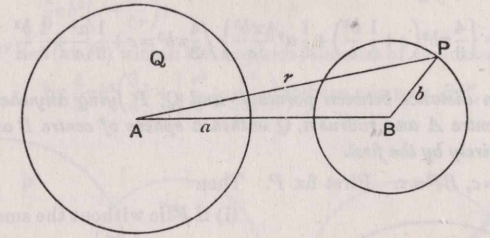


Fig. 494.

Now $\int r\,dS = 4\pi b^2 \times$ mean distance of points on the second sphere from $A = 4\pi b^2\left(c+\frac{1}{3}\frac{b^2}{c}\right)$

and $\int \frac{dS}{r}$ = potential of a shell of unit density at the point $A = \frac{4\pi b^2}{c}$;

$$\therefore \text{ mean value required} = \frac{4\pi b^2\left(c+\frac{1}{3}\frac{b^2}{c}\right)+\frac{4\pi b^2}{c}\cdot\frac{a^2}{5}}{4\pi b^2} = c+\frac{1}{3}\frac{b^2}{c}+\frac{1}{5}\frac{a^2}{c}.$$

12. *Mean distance of two points Q and P, one on each of two spherical surfaces of radii a and b, each outside the other.*

A and B being the centres, $r = AP$, the mean of the distances on the

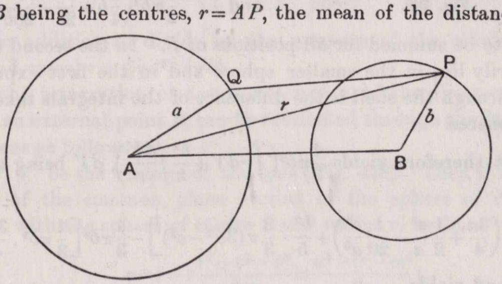


Fig. 495.

surface of the first sphere from $P = r+\frac{1}{3}\frac{a^2}{r}$, and the sum of the cases is measured by $4\pi a^2\left(r+\frac{1}{3}\frac{a^2}{r}\right)$. Hence, we have to find for the second sphere

$$\frac{\int 4\pi a^2\left(r+\frac{1}{3}\frac{a^2}{r}\right)dS}{\int 4\pi a^2 dS} = \frac{\int r\,dS}{S}+\frac{a^2}{3}\frac{\int\frac{dS}{r}}{S} = c+\frac{1}{3}\frac{b^2}{c}+\frac{1}{3}\frac{a^2}{c}.$$

13. *If each of the points in Case 12 be allowed to traverse the interior of its own sphere,*

$$M(PQ)=\frac{\int\frac{4}{3}\pi a^3\left(r+\frac{1}{5}\frac{a^2}{r}\right)dV}{\int\frac{4}{3}\pi a^3\,dV}\text{ taken through the second sphere}$$

$$=\left\{\frac{4}{3}\pi b^3\left(c+\frac{1}{5}\frac{b^2}{c}\right)+\frac{1}{5}a^2\frac{\frac{4}{3}\pi b^3}{c}\right\}\Big/\frac{4}{3}\pi b^3=c+\frac{1}{5}\frac{a^2}{c}+\frac{1}{5}\frac{b^2}{c}.$$

14. *Mean distance between points P and Q, P lying anywhere within a sphere of centre A and radius a, Q within a sphere of centre B and radius b, enclosed entirely by the first.*

Let $AB=c$, $BP=r$. First fix P. Then

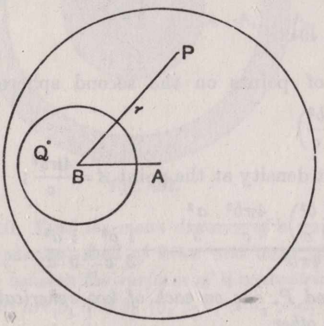


Fig. 496.

(i) if P lie without the smaller sphere $M(PQ)=r+\frac{1}{5}\frac{b^2}{r}$, and the number of such cases is measured by $\frac{4}{3}\pi b^3$;

(ii) if P lie within the smaller sphere $M(PQ)=\frac{3}{4}b+\frac{r^2}{2b}-\frac{1}{20}\frac{r^4}{b^3}$, the number of cases being, as before, measured by $\frac{4}{3}\pi b^3$.

The sums of the cases are therefore

$$\frac{4}{3}\pi b^3\left(r+\frac{1}{5}\frac{b^2}{r}\right)$$

and $$\frac{4}{3}\pi b^3\left(\frac{3}{4}b+\frac{r^2}{2b}-\frac{r^4}{20b^3}\right).$$

These are to be summed for all positions of P. In the second expression, P necessarily lies in the smaller sphere and in the first expression the integral through the shell is the difference of the integrals taken through the two spheres.

The first therefore yields $\frac{4}{3}\pi b^3\left(\int r\,dV+\frac{b^2}{5}\int\frac{dV}{r}\right)$, dV being an element of volume,

$$=\frac{4}{3}\pi b^3\left[\frac{4}{3}\pi a^3\left(\frac{3a}{4}+\frac{1}{2}\frac{c^2}{a}-\frac{1}{20}\frac{c^4}{a^3}\right)+\frac{b^2}{5}\cdot\frac{2}{3}\pi(3a^2-c^2)\right]-\frac{4}{3}\pi b^3\left[\frac{4}{3}\pi b^3\cdot\frac{3b}{4}+\frac{b^2}{5}\cdot 2\pi b^2\right].$$

The second yields

$$\int_0^\pi\int_0^{2\pi}\int_0^b\frac{4}{3}\pi b^3\left(\frac{3}{4}b+\frac{r^2}{2b}-\frac{r^4}{20b^3}\right)r^2\sin\theta\,d\theta\,d\phi\,dr=\tfrac{4}{3}\pi b^7\,.\,2\pi\,.\,2\,.\,\tfrac{12}{35}.$$

Adding and dividing by $\frac{4}{3}\pi a^3\times\frac{4}{3}\pi b^3$, the mean value required is

$$\frac{3a}{4}+\frac{c^2}{2a}-\frac{c^4}{20a^3}+\frac{3b^2}{10a}-\frac{1}{10}\frac{b^2c^2}{a^3}-\frac{3}{140}\frac{b^4}{a^3}.$$

When $c=0$ and $a=b$ this reduces to $\frac{36}{35}a$, the result Ex. 2.

15. *Mean distance PQ, where P and Q lie, one within a sphere of centre A and radius a, and the other within a sphere of centre B and radius b, the spheres intersecting, where* $AB=c\ (>a)$.

Let $BP=r$. Fix P. Then, (Fig. 497),

(i) if P lies without the b-sphere, the sum of the cases is measured by
$$\frac{4}{3}\pi b^3\left(r+\frac{b^2}{5r}\right);$$

(ii) if P lies (at P') within the b-sphere, the sum of the cases is measured by $\frac{4}{3}\pi b^3\left(\frac{3}{4}b+\frac{1}{2}\frac{r^2}{b}-\frac{r^4}{20b^3}\right)$, where r is now BP'.

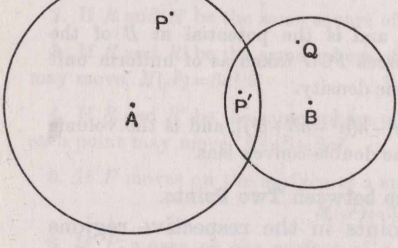


Fig. 497.

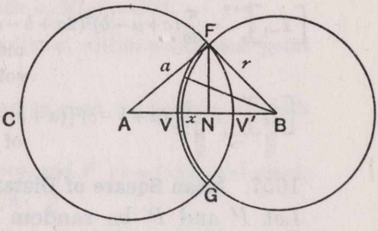


Fig. 498.

We have now to sum $\int\frac{4}{3}\pi b^3\left(r+\frac{b^2}{5r}\right)dV$ for the a-sphere, omitting the lens,

and $\int\frac{4}{3}\pi b^3\left(\frac{3}{4}b+\frac{1}{2}\frac{r^2}{b}-\frac{r^4}{20b^3}\right)dV$ for the lens,

and after addition to divide by the measure of the whole number of compound cases, viz. $\frac{4}{3}\pi a^3\,.\,\frac{4}{3}\pi b^3$.

Now the integration of any function $\phi(r)$ of the distance r of a point P' from an external point B, can be conducted through the region enclosed by the lens as follows:

Let V, V' be the vertices of the lens (Fig. 498). Then if x be distance from V of the common plane section of the sphere of radius a and centre A with the sphere of centre B and radius r, we have
$$x=r-r\frac{r^2+c^2-a^2}{2cr}=\frac{a^2-(r-c)^2}{2c};$$
and if r increases to $r+dr$, the volume of the lens increases by
$$2\pi r\frac{a^2-(r-c)^2}{2c}dr,$$
this being the volume of the added layer.

Every point of this layer is at the same distance r from B. Hence the integration of $\phi(r)$ through the lens is $\int\phi(r)\frac{\pi}{c}\{a^2-(r-c)^2\}r\,dr$ with limits $c-a$ to b; and for the rest of the a-sphere with limits from b to $c+a$. And we have

$$\int r^n dV = \frac{\pi}{c}\int r^{n+1}\{(a^2-c^2)+2cr-r^2\}\,dr$$
$$=\frac{\pi}{c}\left\{(a^2-c^2)\frac{r^{n+2}}{n+2}+2c\frac{r^{n+3}}{n+3}-\frac{r^{n+4}}{n+4}\right\}=I_n, \text{ say.}$$

Hence

$$M(PQ)=\frac{3}{4\pi a^3}\left\{\Big[I_1\Big]_b^{c+a}+\frac{b^2}{5}\Big[I_{-1}\Big]_b^{c+a}+\frac{3b}{4}\Big[I_0\Big]_{c-a}^b+\frac{1}{2b}\Big[I_2\Big]_{c-a}^b-\frac{1}{20b^3}\Big[I_4\Big]_{c-a}^b\right\}.$$

The integrals $\Big[I_{-1}\Big]_b^{c+a}$ and $\Big[I_0\Big]_{c-a}^b$ are interesting from another point of view, and reduce as follows:

$\Big[I_{-1}\Big]_b^{c+a}=\dfrac{\pi}{3c}(c+a-b)^2(2a+b-c)$, and is the potential at B of the meniscus FCG taken as of uniform unit volume density.

$\Big[I_0\Big]_{c-a}^b=\dfrac{\pi}{12c}(a+b-c)^2[(a+b+c)^2-4(a^2-ab+b^2)]$, and is the volume of the double-convex lens.

1657. **Mean Square of Distance between Two Points.**

Let P and P' be random points in the respective regions R and R', which may be one-, two- or three-dimensional. Let G, G' be the respective centroids of these regions for a uniform mass-distribution, and the line, surface or volume density, as the case may be, be taken as unity. Let H and H' be the moments of inertia with regard to the respective centroids, viz. ΣmGP^2 and $\Sigma m'GP'^2$. Then taking R, R' as the lengths, areas or volumes of the regions, as the case may be,

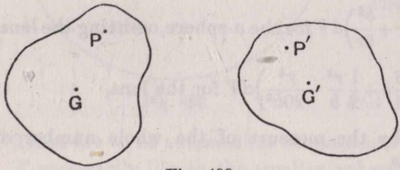


Fig. 499.

$$M(\rho^2)=GG'^2+H/R+H'/R'.$$

For $\quad M(\rho^2)=\iint PP'^2\,dR\,dR' \Big/ \iint dR\,dR'$,

and $\quad \int PP'^2 dR' = R'.PG'^2+H'$;

(Lagrange's Theorem, Routh, *A. St.*, I. 436.)

$$\iint PP'^2 dR'\,dR=\int (R'.PG'^2+H')\,dR=R'(R.GG'^2+H)+H'.R;$$

also $\quad \iint dR\,dR'=R.R; \quad \therefore\ M(\rho^2)=GG'^2+H/R+H'/R'.$

The values of H and H' are known for many elementary cases.

Cor. I. Centroids coincident, $GG'=0$, $M(\rho^2)=H/R+H'/R'$.

Cor. II. (i) Regions identical, $M(\rho^2)=2H/R$.

(ii) If the region be a plane lamina,

$$H/R=\text{sq. of radius of gyration}=k^2; \quad \therefore\ M(\rho^2)=2k^2.$$

1658. **Examples.**

1. For two ellipses, semi-axes (a, b) and (a', b'), lying in the same plane, c the distance between the centres, $M(\rho^2)=(a^2+b^2+a'^2+b'^2)/4+c^2$.

2. If R and R' be the same square of side a, $M(\rho^2)=a^2/3$.

3. If R and R' be the same sphere of radius a, within which each point may move, $M(\rho^2)=6a^2/5$.

4. If R and R' be the same sphere of radius a, on the surface of which each point may move, $M(\rho^2)=2a^2$.

5. If P moves on the surface of a sphere, and P' on a diametral plane,
$$M(\rho^2)=3a^2/2.$$

6. If P moves on the surface of a sphere, and P' on a great circle,
$$M(\rho^2)=2a^2.$$

7. If P and P' move one on each of two straight lines of lengths $2a$, $2b$, whose centres are a distance c apart, $M(\rho^2)=c^2+(a^2+b^2)/3$.

If the lines be identical, $M(\rho^2)=2a^2/3$,

with the same result if not identical, but with the same centre and of the same length.

1659. If one of the two points be fixed, say P', and P traverses a region R, then taking P' as origin O. Then

$$M(\rho^2)=\int OP^2 dR \Big/ \int dR=OG^2+H/R.$$

1660. **Examples.**

1. If O be the centre of a square of side $2a$ which P may traverse,
$$M(\rho^2)=2a^2/3.$$

2. If O be a point at distance c from the centre of a circle of radius a in any position which P may traverse, $M(\rho^2)=c^2+a^2/2$.

3. If O be the centre of an ellipsoid of semi-axes a, b, c, throughout which the free point may travel, $M(\rho^2)=(a^2+b^2+c^2)/5$.

If O be the extremity of the a-axis, $M(\rho^2)=a^2+(a^2+b^2+c^2)/5$.

4. If P lies on the circumference of a semicircle and P' on the diameter, of length $2a$,
$$M(\rho^2)=\frac{4a^2}{\pi^2}+\pi a\left(a^2-\frac{4a^2}{\pi^2}\right)\Big/\pi a+\frac{a^2}{3}=4a^2/3.$$

Otherwise :—with the notation of Fig. 500,

$$M(\rho^2)=\frac{\int_0^\pi\int_{-a}^a (a^2-2ax\cos\theta+x^2)\,d\theta\,dx}{\int_0^\pi\int_{-a}^a d\theta\,dx}=\frac{1}{2\pi a}\int_0^\pi\left(2a^3+\frac{2a^3}{3}\right)d\theta=\frac{4a^2}{3}.$$

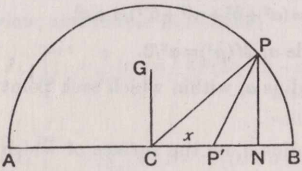


Fig. 500.

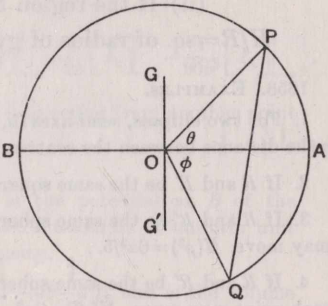


Fig. 501.

5. If P lies on the circumference of a circle, and on one side of a given diameter AB and P' on the opposite semi-circumference, $GG'=4a/\pi$;

$$\therefore\ M(\rho^2)=\frac{16a^2}{\pi^2}+2\left(a^2-\frac{4a^2}{\pi^2}\right)=\frac{2a^2}{\pi^2}(\pi^2+4).$$

Otherwise :—If O be the centre, $A\hat{O}P=\theta$, $A\hat{O}Q=\phi$, (Fig. 501),

$$M(\rho^2)=\int_0^\pi\int_0^\pi 4a^2\sin^2\frac{\theta+\phi}{2}\,d\theta\,d\phi\Big/\int_0^\pi\int_0^\pi d\theta\,d\phi=\frac{2a^2}{\pi^2}\int_0^\pi\int_0^\pi\{1-\cos(\theta+\phi)\}\,d\theta\,d\phi$$

$$=\text{etc.}=2a^2(\pi^2+4)/\pi^2.$$

1661. **Mean n^{th} Power of Distance between two points P and Q.**

EXAMPLES.

1. Let AB be a given straight line of length a ; P and Q two random points upon AB, P being the one more distant from A ; $AP=x$, $AQ=y$.

$$M(QP^n)=\int_0^a\int_0^x (x-y)^n\,dx\,dy\Big/\int_0^a\int_0^x dx\,dy=\int_0^a\left[-\frac{(x-y)^{n+1}}{n+1}\right]_{y=0}^{y=x}dx\Big/\int_0^a x\,dx$$

$$=\frac{1}{n+1}\int_0^a x^{n+1}dx\Big/\int_0^a x\,dx=2a^n/(n+1)(n+2).$$

2. If P lies on the circumference of a circle, and Q be at a fixed point O of the circumference, C the centre, (Fig. 502),

$$M(OP^n)=2\int_0^{\frac{\pi}{2}} OP^n.\,2a\,d\theta/\text{circumf.}=\frac{2}{\pi}(2a)^n\int_0^{\frac{\pi}{2}}\cos^n\theta\,d\theta=\frac{2^{n+1}a^n}{\pi}K_1;$$

where $K_1=\dfrac{(n-1)(n-3)\dots 2}{n(n-2)\dots 3}$ (n odd) or $\dfrac{(n-1)\dots 1}{n\dots 2}\cdot\dfrac{\pi}{2}$ (n even).

3. If P lie within the circle, and Q be at O, (Fig. 503),

$$M(OP^n)=2\int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta} r^n . r\,d\theta\,dr/\text{area}=\frac{2^{n+3}a^n}{(n+2)\pi}K_2,$$

where $$K_2=\int_0^{\frac{\pi}{2}}\cos^{n+2}\theta\,d\theta=\text{etc.}$$

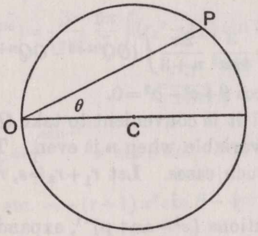


Fig. 502.

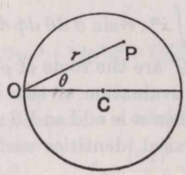


Fig. 503.

4. If P and Q both lie within a circle of radius a, $M(PQ^n)$ may be inferred from the last result. Let M be the result required. The number of cases is measured by $\pi a^2\times\pi a^2$ and their sum is measured by $M\pi^2a^4$. If the radius be increased to $a+da$, the increase in the sum $=\dfrac{d}{da}(M\pi^2a^4)\,da$. This increase is brought about by the addition of the cases in which P or Q or both lie on the annulus, and is

$$2\,.\,2\pi a\,da\,.\,\pi a^2\,\frac{2^{n+3}a^n}{(n+2)\pi}K_2+2\pi a\,da\,.\,2\pi a\,da\,.\,\frac{2^{n+1}a^n}{\pi}K_1,$$

the first factor 2 being inserted because either P or Q may lie on the annulus, and the second term arises for the case in which both lie on the annulus, but is a second-order infinitesimal.

Hence, M vanishing with a, no constant of integration is required, and

$$\frac{d}{da}(M\pi^2a^4)=\frac{2^{n+5}a^{n+3}}{n+2}\pi K_2;\quad\therefore\ M=\frac{2^{n+5}a^n}{(n+2)(n+4)}\frac{K_2}{\pi}.$$

[The result was given by the Rev. T. C. Simmons, *Educ. Times*, 7943, p. 120, vol. xliii., a different proof being adopted.]

5. If P lies on the surface of a sphere of radius a and Q is at a fixed point O of the surface, then, $(n>0)$,

$$M(OP^n)=\frac{1}{4\pi a^2}\int_0^{\frac{\pi}{2}}(2a\cos\theta)^n 2\pi(2a\sin\theta\cos\theta)\,2a\,d\theta=2(2a)^n/(n+2).$$

6. If P and Q are both free to move on the surface of the sphere and

$n>1$, $$M(PQ^n)=\iint r^n dS\,dS\Big/\iint dS\,dS=\text{etc.}=2(2a)^n/(n+2).$$

[This result might be inferred from Ex. 5.]

7. If P lies within the sphere and Q is at a fixed point O on the surface,

$$M(OP^n)=12(2a)^n/(n+3)(n+4).$$

8. If P lies within the sphere and Q be at the centre C,

$$M(CP^n)=3a^n/(n+3). \quad \text{[St. John's Coll., 1883.]}$$

9. If both P and Q lie within the sphere, proceed as in Ex. 4. Then $M(PQ^n)=2^{n+3}\,.\,3^2a^n/(n+3)(n+4)(n+6)$.

10. If one point lie within the sphere and the other lie at a fixed point O without the sphere, let OQQ' be a chord through P, C the centre, $C\hat{O}Q=\theta$, a the radius, $CO=c$, $OP=r$,

$$M(OP^n)=\iiint r^n\,.\,r^2\sin\theta\,d\theta\,d\phi\,dr/\text{vol.}=\frac{3}{4\pi a^3}\,\frac{2\pi}{n+3}\int(OQ'^{n+3}-OQ^{n+3})\sin\theta\,d\theta,$$

and OQ, OQ' are the roots of $\rho^2-2c\rho\cos\theta+c^2-a^2=0$.

For the evaluation of this integral it is convenient to take QQ' as the variable when n is odd and θ as the variable when n is even. There are two algebraical identities useful in such cases. Let $r_1+r_2=s$, $r_1-r_2=d$, $r_1r_2=p$.

Then, by putting into Partial Fractions $(x^2-sx+p)^{-1}$, expanding both sides in inverse powers of x, and equating coefficients of $1/x^{m+1}$,

$$\frac{r_1{}^m-r_2{}^m}{r_1-r_2}=s^{m-1}-(m-2)\,s^{m-3}p+\frac{(m-3)(m-4)}{1\,.\,2}\,s^{m-5}p^2-\dots.$$

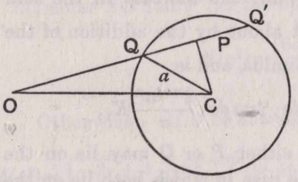


Fig. 504.

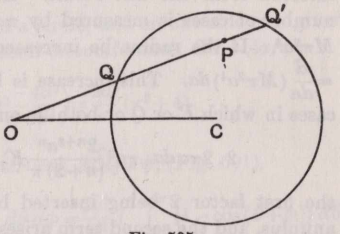


Fig. 505.

If m be odd, the indices of s are all even. Substituting for s^2 its value d^2+4p and expanding each term, the series all terminate, and we obtain

$$r_1{}^m-r_2{}^m=d^m+md^{m-2}p+\frac{m(m-3)}{1\,.\,2}\,d^{m-4}p^2+\frac{m(m-4)(m-5)}{1\,.\,2\,.\,3}\,d^{n-6}p^3+\dots. \quad \dots.(\text{A})$$

If m be even,

$$\frac{r_1{}^m-r_2{}^m}{ds}=s^{m-2}-(m-2)\,s^{m-4}p+\frac{(m-3)(m-4)}{1\,.\,2}\,s^{m-6}p^2-\dots$$

$$=(d^2+4p)^{\frac{m-2}{2}}-(m-2)\,(d^2+4p)^{\frac{m-4}{2}}\,p+\frac{(m-3)(m-4)}{1\,.\,2}\,(d^2+4p)^{\frac{m-6}{2}}\,p^2-\dots;$$

whence, expanding as before, the series all terminate and, m even,

$$r_1{}^m-r_2{}^m=sd\left\{d^{m-2}+(m-2)\,d^{m-4}p+\frac{(m-3)(m-4)}{1\,.\,2}\,d^{m-6}p^2+\dots\right\}. \quad \dots\dots(\text{B})$$

(i) Suppose, for instance, $n=3$, $m=6$. Let $QQ'=x$,

$$M(OP^3)=\frac{3}{4\pi a^3}\,.\,\frac{2\pi}{6}\int(r_1{}^6-r_2{}^6)\sin\theta\,d\theta$$

$$=\frac{1}{4a^3}\int sx\,(x^4+4px^2+3p^2)\sin\theta\,d\theta \quad \text{(from B)}.$$

Also

$$s=2c\cos\theta,\quad x^2=4(a^2-c^2\sin^2\theta),\quad p=c^2-a^2,\quad x\,dx=-4c^2\sin\theta\cos\theta\,d\theta;$$

whence $s\,.\sin\theta\,d\theta=-x\,dx/2c,$

and $$M(OP^3)=-\frac{1}{8a^3c}\int_{2a}^{0}(x^6+4px^4+3p^2x^2)\,dx=c^3+\frac{6}{5}a^2c+\frac{3}{35}\frac{a^4}{c}.$$

(ii) Suppose $n=4$, $m=7$,

$$M(OP^4)=\frac{3}{4\pi a^3}\frac{2\pi}{7}\int(r_1{}^7-r_2{}^7)\sin\theta\,d\theta$$

$$=\frac{3}{14a^3}\int_0^{\sin^{-1}\frac{a}{c}}(x^7+7px^5+14p^2x^3+7p^2x)\sin\theta\,d\theta.$$

Let $I_r=\int_0^{\sin^{-1}\frac{a}{c}}x^r\sin\theta\,d\theta.$ Put $P=x^r\cos\theta$, $x\,dx=-4c^2\sin\theta\cos\theta\,d\theta$,

$$\frac{dP}{d\theta}=\text{etc.}=-(r+1)\,x^r\sin\theta-4prx^{r-2}\sin\theta;\quad\therefore\ I_r=\frac{(2a)^r}{r+1}-\frac{4r}{r+1}pI_{r-2}.$$

Using this reduction formula, we may show that

$$I_7+7pI_5+14p^2I_3+7p^3I_1=\frac{(2a)^7}{8}+\frac{7}{2\,.\,6}(2a)^5p+\frac{7}{3\,.\,4}(2a)^3p^2,$$

and finally $M(OP^4)=c^4+2a^2c^2+\frac{3}{7}a^4.$

11. *Find the mean value of x^{2n} for all points on a spherical surface with centre at the origin and radius a, the distribution being for equal surface elements.*

$$M(x^{2n})=\frac{1}{4\pi a^2}\int_0^{\pi}(a\cos\theta)^{2n}\,.\,2\pi a\sin\theta\,.\,a\,d\theta=\frac{a^{2n}}{2n+1}.$$

$M(x^{2n+1})$ is evidently zero. For the values of x^{2n+1} for which x is negative, cancel the corresponding ones for which x is positive.

12. *Find the mean value of $(lx+my+nz)^{2p}$ taken over the same spherical surface.*

Changing the axes so that $lx+my+nz=0$ becomes the new y-z plane, $lx+\,+\,=X\sqrt{l^2+\,+\,}$, and

$$M[(lx+my+nz)^{2p}]=(l^2+m^2+n^2)^p\,a^{2p}/(2p+1).$$

13. *Find $M(x^{2p}y^{2q}z^{2r})$ over the same spherical surface.*

Let $p+q+r=k$.

Then $$\frac{(2k)!}{(2p)!\,(2q)!\,(2r)!}\int x^{2p}y^{2q}z^{2r}\,dS$$

$$=\text{coef. } l^{2p}m^{2q}n^{2r}\text{ in }(l^2+m^2+n^2)^k\,.\int X^{2k}\,dS$$

$$=\text{coef. } l^{2p}m^{2q}n^{2r}\text{ in }(l^2+m^2+n^2)^k\,.\,4\pi a^{2k+2}/(2k+1)$$

$$=\frac{k!}{p!\,q!\,r!}\frac{4\pi a^{2k+2}}{2k+1};$$

$$\therefore\ M(x^{2p}y^{2q}z^{2r})=\frac{k!}{(2k)!}\frac{(2p)!\,(2q)!\,(2r)!}{p!\,q!\,r!}\frac{a^{2(p+q+r)}}{2p+2q+2r+1}.$$

14. *Find $M(Px^{2p}y^{2q}z^{2r})$ taken over the surface of an ellipsoid of superficial area A, semi-axes a, b, c, where P is the central perpendicular on a tangent plane, the distribution being for equal elements of area.*

$M(Px^{2p}y^{2q}z^{2r})=\frac{1}{A}\int Px^{2p}y^{2q}z^{2r}\,dS$. Then writing $\frac{x}{a}=\frac{\xi}{R}$, $\frac{y}{b}=\frac{\eta}{R}$, $\frac{z}{c}=\frac{\zeta}{R}$,

$$\frac{1}{3}P\,dS=\frac{1}{3}\frac{abc}{R^3}\cdot R\,d\sigma,$$

where $d\sigma$ is the corresponding surface element on the sphere $\xi^2+\eta^2+\zeta^2=R^2$, we have as the mean value required

$$\frac{1}{A}\frac{a^{2p}b^{2q}c^{2r}}{R^{2p+2q+2r}}\cdot\frac{abc}{R^3}\cdot R\int\xi^{2p}\eta^{2q}\zeta^{2r}\,d\sigma=\frac{k!}{(2k)!}\,\frac{(2p)!\,(2q)!\,(2r)!}{p!\,q!\,r!}\,\frac{4\pi}{2k+1}\,\frac{a^{2p+1}b^{2q+1}c^{2r+1}}{A},$$

where $p+q+r=k$. (See Routh, *Rig. Dyn.*, pp. 7 and 8.)

1662. Mean Areas and Volumes.

EXAMPLES.

1. *Find the mean value of the areas of all triangles which can be found by taking at random three points on the circumference of a circle of radius R.*

Let O be the centre, ABC a specimen of the triangles; $A\hat{O}B=\theta$, $B\hat{O}C=\phi$.

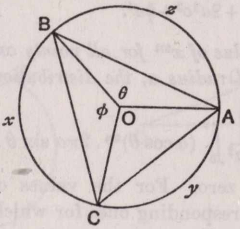


Fig. 506.

We may fix A. ϕ varies from 0 to $2\pi-\theta$, and θ from 0 to 2π. Then

$$M(\triangle ABC)=\frac{R^2}{2}\,\frac{\int_0^{2\pi}\int_0^{2\pi-\theta}\{\sin\theta+\sin\phi-\sin(\theta+\phi)\}\,d\theta\,d\phi}{\int_0^{2\pi}\int_0^{2\pi-\theta}d\theta\,d\phi}=\text{etc.}=3R^2/2\pi.$$

2. *Find the mean of the areas of all acute-angled triangles inscribable as in Ex. 1.*

Here $\theta<\pi$, $\phi<\pi$, $2\pi-\theta-\phi<\pi$. The limits are therefore $\theta=0$ to π, $\phi=\pi-\theta$ to π, and the mean $=3R^2/\pi$.

3. *Find the mean area of all right-angled triangles inscribed as before.*

Take A as the right angle. Then $\phi=\pi$ and the mean $=2R^2/\pi$, and there are the same number of cases with the same sums if B or C be the right angle. Hence the mean $=2R^2/\pi$.

4. *Find the mean area of all obtuse-angled triangles inscribed as above.*

Let A be the obtuse angle. Here $\theta<\pi$, $\phi>\pi$, $2\pi-\theta-\phi<\pi$. Then the limits for θ are 0 and π, and for ϕ, π and $2\pi-\theta$, and the mean $=R^2/\pi$.

5. *Find the mean area of all triangles formed by joining three random points on a sphere of radius a.* [MATH. TRIP., 1883.]

Let O be the centre. Consider first all the circular sections normal to a given direction OA. Let P be any point on this circle, PN a perpendicular on OA. $A\hat{O}P=\chi$. Then the mean area of all triangles inscribed in this circle $=3a^2\sin^2\chi/2\pi$, and the number of such triangles is measured by $2\pi^2$ (Ex. 1). Therefore the mean for all triangles perpendicular to the line OA for equal increments of χ is $\int_0^\pi \frac{3a^2\sin^2\chi}{2\pi}d\chi/\pi=3a^2/4\pi$, and the mean is obviously the same for all directions of OA, since the number of cases and the sum of the cases is the same for each direction of OA. (Fig. 507.)

A distribution of different nature, *e.g.* for equal increments of x, would give a different result, viz. $\frac{1}{2a}\int_{-a}^{a}\frac{3NP^2}{2\pi}dx=a^2/\pi$.

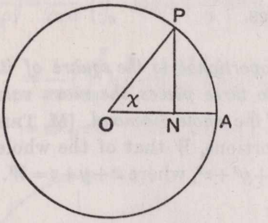


Fig. 507.

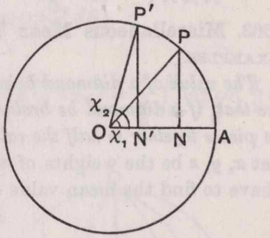


Fig. 508.

6. *Find the mean value of the volume of a tetrahedron whose angular points are four random points on a sphere of radius a.* (Fig. 508.) [MATH. TRIP., 1883.]

Without affecting the problem, we may take a set of bases fixed in direction, say normal to a given radius OA. Let one of the bases be on the circular section through the ordinate PN. Then, as the vertex of the tetrahedron travels in a circular section parallel to the base and through a second ordinate $P'N'$, the volume remains constant. Therefore the mean volume of the tetrahedron, with vertices on the plane through $P'N'$ and bases on the plane through PN

$$=\frac{1}{3}NN'\cdot\frac{3NP^2}{2\pi}.\quad \text{Let } A\hat{O}P=\chi_1,\quad A\hat{O}P'=\chi_2.$$

The measure NN' of the perpendicular height of the tetrahedron changes sign as N' passes through N. To avoid negative signs for the volumes of tetrahedra with vertices on opposite sides of their respective bases, we separate the integration into two parts. The expression for the mean volume required is then

$$\frac{\int_0^\pi\int_{\chi_1}^\pi \frac{1}{3}\cdot\frac{3NP^2}{2\pi}a(\cos\chi_1-\cos\chi_2)\,d\chi_1\,d\chi_2+\int_0^\pi\int_0^{\chi_1}\frac{1}{3}\cdot\frac{3NP^2}{2\pi}a(\cos\chi_2-\cos\chi_1)\,d\chi_1\,d\chi_2}{\int_0^\pi\int_0^\pi d\chi_1\,d\chi_2},$$

which, after integration, gives $16a^3/9\pi^3$.

The distribution here taken is for equal increments of χ_1 and χ_2.

7. *If P, Q, R be random points on the three sides BC, CA, AB of a triangle, find the mean values of the triangles AQR, BRP, CPQ, PQR.* [R. CHARTRES, *Educ. Times.*]

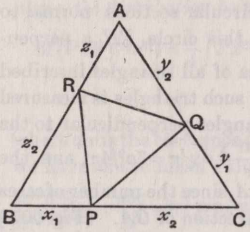


Fig. 509.

Let x_1, x_2; y_1, y_2; z_1, z_2 be the respective parts into which the sides are divided at P, Q, R; Δ the area of the triangle ABC,

$$M(AQR)=\int_0^b\int_0^c \frac{y_2z_1}{bc}\,\Delta\,dy_2\,dz_1\Big/\int_0^b\int_0^c dy_2\,dz_1=\frac{\Delta}{4}.$$

Similarly

$$M(BRP)=M(CPQ)=\frac{\Delta}{4}.$$

$$M(PQR)=\int_0^a\int_0^b\int_0^c\left(1-\frac{y_2z_1}{bc}-\frac{z_2x_1}{ca}-\frac{x_2y_1}{ab}\right)\Delta\,dx_1dy_1dz_1\Big/\int_0^a\int_0^b\int_0^c dx_1dy_1dz_1=\text{etc.}=\frac{\Delta}{4}.$$

1663. Miscellaneous Mean Values.

EXAMPLES.

1. *The value of a diamond being proportional to the square of its weight, prove that, if a diamond be broken into three pieces, the mean value of the three pieces together is half the value of the whole diamond.* [M. TRIP., 1875.]

Let x, y, z be the weights of the portions, W that of the whole. Then we have to find the mean value of $x^2+y^2+z^2$, where $x+y+z=W$. Referring to Cartesian coordinates, $x+y+z=W$ is the equation of a plane. If $d\sigma$ be an element of area of the intercepted triangle, the mean value is

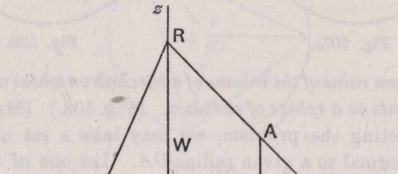


Fig. 510.

$$\int(x^2+y^2+z^2)\,d\sigma\Big/\int d\sigma=(\text{mom. of in. with respect to the origin})/\text{area}$$

$$=\tfrac{1}{2}\,(\text{the sum of the moments of in. about the axes})/\text{area}.$$

Let $3A$ be the area of the triangle. Then, concentrating A at each mid-point (Routh, *Rig. Dyn.*, Art. 35),

$$\text{Mean value}=\frac{1}{2}\cdot 3\left[A\left(\frac{W}{2}\right)^2+A\left(\frac{W}{2}\right)^2+A\left\{\left(\frac{W}{2}\right)^2+\left(\frac{W}{2}\right)^2\right\}\right]\Big/3A=\frac{1}{2}W^2.$$

2. *It is required to find the mean value of the inverse distances of points on a circle of radius a, from points on a fixed diameter AB.*

Let P be a point on the arc, Q a point on the diameter, O the centre.

$P\hat{O}B=\theta,\quad P\hat{O}A=\theta'=\pi-\theta,\quad P\hat{A}B=\phi_1,\quad P\hat{B}A=\phi_2,\quad PQ=\rho,\quad OQ=x.$

Then $\theta=2\phi_1$, $\theta'=2\phi_2$. (Fig. 511.)

$$M\left(\frac{1}{\rho}\right)=\int_0^\pi\int_{-a}^a\frac{a\,d\theta}{\rho}\,dx\bigg/\int_0^\pi\int_{-a}^a a\,d\theta\,dx.$$

Now $\int_{-a}^a\frac{dx}{\rho}$ is the potential at P of a material line AB of unit line density $=\log\cot\frac{\phi_1}{2}\cot\frac{\phi_2}{2}$ (Art. 1652).

$$\therefore M\left(\frac{1}{\rho}\right)=\frac{1}{2\pi a}\left\{\int_0^\pi\log\cot\frac{\phi_1}{2}\,d\theta+\int_0^\pi\log\cot\frac{\phi_2}{2}\,d\theta\right\}$$

$$=\frac{1}{\pi a}\left\{\int_0^{\frac{\pi}{2}}\log\cot\frac{\phi_1}{2}\,d\phi_1+\int_0^{\frac{\pi}{2}}\log\cot\frac{\phi_2}{2}\,d\phi_2\right\}=\frac{2}{\pi a}\int_0^{\frac{\pi}{2}}\log\cot\frac{\chi}{2}\,d\chi$$

$$=4s'_2/\pi a.\quad\text{(Art. 1074.)}$$

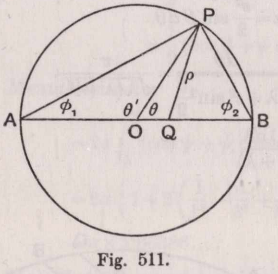


Fig. 511.

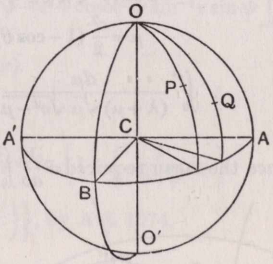


Fig. 512.

3. *O is a fixed point on the circumference of the base of a hemisphere with centre C. P and Q are random points on the surface; find the mean value of the angle between the planes OCP, OCQ.* (Fig. 512.) [Caius Coll., 1877.]

Let $AOA'O'$ be the base of the hemisphere, and B its vertex, C the centre, CA, CB, CO being taken as the rectangular coordinate axes. Let ϕ_1 and ϕ_2 be the azimuthal angles of the two planes OCP, OCQ, P being taken as the point on the plane with the greater azimuthal angle. Then if the distribution of the points P, Q be one for equal elements of area, the mean required is

$$\frac{\int_0^\pi\int_0^\pi\int_0^\pi\int_0^{\phi_1}(\phi_1-\phi_2)\sin\theta_1\sin\theta_2\,d\theta_1\,d\theta_2\,d\phi_1\,d\phi_2}{\int_0^\pi\int_0^\pi\int_0^\pi\int_0^{\phi_1}\sin\theta_1\sin\theta_2\,d\theta_1\,d\theta_2\,d\phi_1\,d\phi_2}=\text{etc.}=\pi/3.$$

4. *Prove that if 2c be the distance between the foci of an ellipse of semi-axes a and b, the mean value of* $r_1^{-2} r_2^{-2} f\{\frac{1}{4}(r_1+r_2)^2 - c^2\}$, *with respect to the area, is equal to* $\frac{1}{ab}\int_0^{b^2} \frac{f(\lambda)\,d\lambda}{\lambda(c^2+\lambda)}$; r_1, r_2 *being the focal radii of any point within the ellipse.* (Fig. 513.) [γ, 1890.]

Taking $\frac{x^2}{c^2+\lambda}+\frac{y^2}{\lambda}=1$, $\frac{x^2}{c^2-\mu}-\frac{y^2}{\mu}=1$ as confocals through the point,

$$r_1^2=(c+x)^2+y^2, \quad r_2^2=(c-x)^2+y^2, \quad r_1^2-r_2^2=4cx,$$

$$r_1+r_2=2\sqrt{c^2+\lambda}, \quad r_1-r_2=2\sqrt{c^2-\mu},$$

$$cx=\sqrt{(c^2+\lambda)(c^2-\mu)}, \quad cy=\sqrt{\lambda\mu}, \quad \tfrac{1}{4}(r_1+r_2)^2-c^2=\lambda, \quad \lambda+\mu=r_1r_2,$$

$$\frac{\partial(x,y)}{\partial(\lambda,\mu)}=\frac{1}{4}\frac{r_1r_2}{c^2xy}.$$

$$\text{Mean required}=\iint\frac{dx\,dy}{r_1^2r_2^2}f(\lambda)\Big/\iint dx\,dy=\frac{4}{\pi ab}\iint\frac{dx\,dy}{r_1^2r_2^2}f(\lambda),$$

the integration being taken through the first quadrant,

$$=\frac{4}{\pi ab}\int_0^{b^2}\int_0^{c^2}\frac{1}{4}\frac{1}{\lambda+\mu}\frac{f(\lambda)\,d\lambda\,d\mu}{\sqrt{\lambda\mu}\sqrt{(c^2+\lambda)(c^2-\mu)}}$$

$$=\frac{1}{\pi ab}\int_0^{b^2}\frac{f(\lambda)\,d\lambda}{\sqrt{\lambda}\sqrt{c^2+\lambda}}\int_0^{c^2}\frac{d\mu}{(\lambda+\mu)\sqrt{\mu}\sqrt{c^2-\mu}}.$$

Let $\mu=\frac{c^2}{2}(1-\cos\theta), \quad d\mu=\frac{c^2}{2}\sin\theta\,d\theta.$

$$\therefore \int_0^{c^2}\frac{d\mu}{(\lambda+\mu)\sqrt{\mu}\sqrt{c^2-\mu}}=\int_0^{\pi}\frac{d\theta}{\lambda+c^2\sin^2\frac{\theta}{2}}=\frac{\pi}{\sqrt{\lambda(\lambda+c^2)}}.$$

$$\text{Hence the mean required}=\frac{1}{ab}\int_0^{b^2}\frac{f(\lambda)\,d\lambda}{\lambda(c^2+\lambda)}.$$

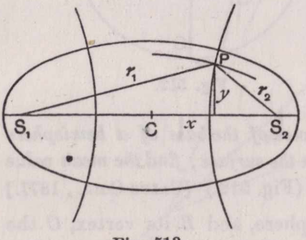


Fig. 513.

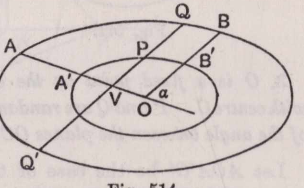


Fig. 514.

5. *Through P, any point within an ellipse, a chord QPQ′ is drawn parallel to a given semi-diameter ρ. Show that the mean value of* $\phi(QP\,.\,PQ')$ *for all points within the ellipse is*

$$2\int_0^{\frac{\pi}{2}}\phi(\rho^2\cos^2\theta)\sin\theta\cos\theta\,d\theta.$$ [δ, 1885.]

Draw a similar and similarly situated ellipse through P. (Fig. 514.)

Then $QP\,.\,PQ'$ retains the same value for all points on this ellipse, viz. $OB^2-OB'^2=\rho^2\cos^2\theta$, where $\rho=OB$ and $\sin\theta$ is the ratio $OB':OB$.

If A and A' be the areas of the larger and smaller ellipses,

$$A' = A\sin^2\theta \quad\text{and}\quad dA' = 2A\sin\theta\cos\theta\, d\theta.$$

$$\therefore\ M\{\phi(QP\,.\,PQ')\} = \frac{\int \phi(QP\,.\,PQ')\,dA'}{\int dA'} = 2\int_0^{\frac{\pi}{2}} \phi(\rho^2\cos^2\theta)\sin\theta\cos\theta\,d\theta.$$

6. *Ellipses are drawn with the same major axis $2a$ and any eccentricities; show that the mean length of their perimeters is*

$$2a\left\{1+\int_0^{\frac{\pi}{2}}\frac{\theta}{\sin\theta}d\theta\right\} = 2a\left\{1+2\left(1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\ldots\right)\right\}.$$

[St. John's, 1886.]

Taking all eccentricities as equally likely, the mean perimeter is

$$4a\int_0^1\int_0^{\frac{\pi}{2}}\sqrt{1-e^2\sin^2\psi}\,d\psi\,de\Big/\int_0^1 de. \quad \text{(Art. 567.)}$$

Now

$$\int_0^1\sqrt{1-e^2\sin^2\psi}\,de = \sin\psi\int_0^1\sqrt{\operatorname{cosec}^2\psi - e^2}\,de$$

$$= \frac{1}{2}\sin\psi\left[e\sqrt{\operatorname{cosec}^2\psi-e^2}+\operatorname{cosec}^2\psi\sin^{-1}e\sin\psi\right]_0^1$$

$$= \frac{1}{2}(\cos\psi+\psi\operatorname{cosec}\psi).$$

$\therefore$ Mean Perimeter

$$= 2a\int_0^{\frac{\pi}{2}}(\cos\psi+\psi\operatorname{cosec}\psi)\,d\psi = 2a\left\{1+\int_0^{\frac{\pi}{2}}\frac{\psi}{\sin\psi}d\psi\right\}$$

$$= 2a\left\{1+2\left(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\ldots\right)\right\}, \text{ by Art. 1074,}$$

$$= a\times 5\cdot 66386\ldots.$$

7. *Show that the average values of the lengths of the least, mean and greatest sides of all possible triangles which can be formed with lines whose lengths lie between a and $2a$ are in the ratio* $5:6:7$. [Math. Trip.]

If the sides be taken $a+x$, $a+y$, $a+z$, the ratio of their means is

$$\int_0^a dz\int_0^z dy\int_0^y dx(x+a) : \int_0^a dz\int_0^z dy\int_0^y dx(y+a) : \int_0^a dz\int_0^z dy\int_0^y dx(z+a).$$

8. *Find the mean value of xyz for points within the positive octant of the ellipsoid $a^{-2}x^2+b^{-2}y^2+c^{-2}z^2=1$.* [Ox. II., 1890.]

Use Dirichlet's integral, Art. 962. $M(xyz) = abc/8\pi$.

9. If a point be taken at random within a tetrahedron, then, of all parallelepipeds which can be described having the line joining the point to one of the angular points as diagonal and its edges parallel to the edges of the tetrahedron which meet at that point, the average volume is one twentieth that of the tetrahedron.

10. Show that for positive values of x, y, z, with condition

$$a^{-2}x^2+b^{-2}y^2+c^{-2}z^2=1, \quad \text{and } r \text{ being} > 1,$$

the mean value of $(xyz)^{r-1}$ for an equable distribution of area on the x-y plane is

$$(abc)^{r-1}\left\{\Gamma\left(\frac{r}{2}\right)\right\}^2\Gamma\left(\frac{r+1}{2}\right)\Big/\pi\Gamma\left(\frac{3r+1}{2}\right),$$

which for $r=2$ reduces to $4abc/15\pi$.

11. Find the mean value of $(xyz)^{r-1}$, $r>0$, where x, y, z are areal coordinates for points within the triangle of reference.

We require
$$\frac{\iint x^{r-1}y^{r-1}(1-x-y)^{r-1}\,dx\,dy}{\iint dx\,dy}$$
for positive values of x, y, z (see Art. 975) $=2\{\Gamma(r)\}^3/\Gamma(3r)$.

12. Show that if x, y, z, u are the tetrahedral coordinates of a point within the reference tetrahedron, $M\{(xyzu)^{r-1}\}$, $(r>0)$, $=6\{\Gamma(r)\}^4/\Gamma(4r)$.

13. Show that if $r>0$ and $x_1, x_2, \dots x_n$ be all positive and subject to the condition $x_1+x_2+\dots+x_n=1$, then

$$M\{(x_1x_2\dots x_n)^{r-1}\}=\Gamma(n)\{\Gamma(r)\}^n/\Gamma(nr).$$

14. Show that if $\iota_1, \iota_2, \dots \iota_n$ be all positive, the mean value of $x_1^{\iota_1-1}x_2^{\iota_2-1}\dots x_n^{\iota_n-1}$ for positive values of $x_1, x_2, \dots x_n$ subject to the condition $\sum_1^n x_r=1$ is $\Gamma(n)\Gamma(\iota_1)\Gamma(\iota_2)\dots\Gamma(\iota_n)/\Gamma(\sum_1^n \iota_r)$.

15. Show that the mean value of $Ayz+Bzx+Cxy$ for positive values of x, y, z subject to the condition $x+y+z=1$ is $\frac{1}{12}(A+B+C)$.

16. Show that the mean value $x^4+y^4+z^4$ for positive values of x, y, z subject to the condition $x+y+z=1$ is $\frac{1}{5}$.

17. Show that the mean value of $(A, B, C, D, E, F)(x, y, z)^2$ for positive values of x, y, z subject to the areal condition $x+y+z=1$ is

$$\tfrac{1}{6}(A+B+C+D+E+F).$$

18. *Let there be n points upon the x-axis, and let positive ordinates of increasing magnitude be erected at these points, their sum being l. Find the mean length of the r^{th} ordinate.* [LAPLACE; TODHUNTER, *Hist.*, p. 545.]

Taking as ordinates $y_1, y_1+y_2, y_1+y_2+y_3, \dots y_1+\dots+y_n$, then

$$ny_1+(n-1)y_2+(n-2)y_3+\dots+y_n=l.$$

Putting $ny_1=x_1$, $(n-1)y_2=x_2, \dots y_n=x_n$, we have $x_1+x_2+\dots+x_n=l$.

We then require
$$\frac{\iint\dots\int\left(\frac{x_1}{n}+\frac{x_2}{n-1}+\dots+\frac{x_r}{n-r+1}\right)dx_1\,dx_2\dots dx_{n-1}}{\iint\dots\int dx_1\,dx_2\dots dx_{n-1}},$$

which gives
$$\frac{l}{n}\left\{\frac{1}{n}+\frac{1}{n-1}+\frac{1}{n-2}+\dots+\frac{1}{n-r+1}\right\}.$$

19. The density at any point of a triangular lamina varies as the product of the perpendiculars on the sides. Show that the mean density is 9/20 of the density at the centre of inertia of the triangle.

1664. Certain Inequalities.

If $a, b, c, \ldots$ be any positive quantities, n in number, and $m, r, \alpha, \beta, \ldots$ positive integers and $\alpha+\beta+\ldots=m$ and $m>r$, we have

$$\text{(i)}\ \frac{\Sigma a^2}{n}>\left(\frac{\Sigma a}{n}\right)^2; \qquad \text{(ii)}\ \frac{\Sigma a^m}{n}>\frac{\Sigma a^r}{n}\cdot\frac{\Sigma a^{m-r}}{n};$$

$$\text{(iii)}\ \frac{\Sigma a^m}{n}>\frac{\Sigma a^\alpha}{n}\cdot\frac{\Sigma a^\beta}{n}\cdot\frac{\Sigma a^\gamma}{n}\cdots.$$ (Smith, *Alg.*, Art. 348.)

That is, the mean of the squares $>$ the square of the mean; the mean of the m^{th} powers $>$ the product of the means of the r^{th} and $(m-r)^{\text{th}}$ powers; and so on.

1665. If $a, b, c, \ldots$ be replaced by $\phi(a_0), \phi(a_0+h), \phi(a_0+2h), \ldots$, the values of a positive continuous single-valued function of x for equal infinitesimal increments of the variable, we have the mean value of the square of the function $>$ the square of the mean value of the function between the same limits, with other theorems of a similar nature. That is,

$$\frac{\int_p^q [\phi(x)]^2\,dx}{\int_p^q dx}>\left[\frac{\int_p^q \phi(x)\,dx}{\int_p^q dx}\right]^2;$$

$$\frac{\int_p^q [\phi(x)]^m\,dx}{\int_p^q dx}>\frac{\int_p^q [\phi(x)]^r\,dx}{\int_p^q dx}\cdot\frac{\int_p^q [\phi(x)]^{m-r}\,dx}{\int_p^q dx};\ \text{etc.}$$

1666. General Mean in Terms of Means restricted in Various Ways.

Let there be two regions Ω_1 and Ω_2 mutually exclusive. Let two random points P and Q be taken in the combined region, and let ϕ be some function of their positions, say for instance their distance apart, its square or its n^{th} power. Several cases may occur: (i) Both may lie in Ω_1; (ii) both may lie in Ω_2; (iii) and (iv) either may lie in Ω_1 and the other in Ω_2.

Let $M_{1,1}$, $M_{2,2}$, $M_{1,2}$ be the mean values of ϕ respectively in case (i), case (ii), cases (iii) and (iv), and let M be the mean value of ϕ when the positions of P and Q are unrestricted. The number of cases occurring are measured by the magnitudes of the regions, viz. Ω_1^2 if both lie in Ω_1, Ω_2^2 if both lie in Ω_2, $\Omega_1\Omega_2$ if P lies in Ω_1 and Q in Ω_2, and $\Omega_1\Omega_2$ if Q lies in Ω_1 and P in Ω_2, and $(\Omega_1+\Omega_2)^2$ if they lie in either region, unspecified.

Hence $\Omega_1^2M_{1,1}$, $\Omega_2^2M_{2,2}$, $2\Omega_1\Omega_2M_{1,2}$ and $(\Omega_1+\Omega_2)^2M$ are the sums of the several cases occurring. But the first three must make up the whole sum of the possible values of ϕ, *i.e.*

$$M=\frac{\Omega_1^2M_{1,1}+2\Omega_1\Omega_2M_{1,2}+\Omega_2^2M_{2,2}}{(\Omega_1+\Omega_2)^2}.$$

1667. Ex. If the two regions be mutually exclusive spheres of radii a and b and centres distance c apart, then for the mean distance PQ,

$$M_{1,1}=\frac{36a}{35},\quad M_{2,2}=\frac{36b}{35},\quad M_{1,2}=c+\frac{a^2+b^2}{5c}.$$

Hence the mean distance between P and Q when each may lie within either sphere or in different spheres is

$$\left[\left(\frac{4}{3}\pi a^3\right)^2\frac{36}{35}a+2\cdot\frac{4}{3}\pi a^3\cdot\frac{4}{3}\pi b^3\left(c+\frac{a^2+b^2}{5c}\right)+\left(\frac{4}{3}\pi b^3\right)^2\frac{36}{35}b\right]\Big/\left(\frac{4}{3}\pi a^3+\frac{4}{3}\pi b^3\right)^2$$

$$=\frac{36}{35}\frac{a^7+b^7}{(a^3+b^3)^2}+2\frac{a^3b^3}{(a^3+b^3)^2}c+\frac{2}{5}\frac{a^3b^3(a^2+b^2)}{(a^3+b^3)^2}\cdot\frac{1}{c}.$$

In the case where the spheres are equal and in contact, $c=2a=2b$ and $M=\frac{113}{70}a$.

1668. In the same way, if there be three or more mutually exclusive regions Ω_1, Ω_2, Ω_3, say, and ϕ be a function of the positions of three points P, Q, R which lie in one or other of these regions, then (a) all may lie in any one of the regions, (b) two may lie in one region, and one in either of the other regions, or (c) one may lie in each region.

Let $M_{3,0,0}$ be the mean value of ϕ when all lie in Ω_1, $M_{0,3,0}$ when all lie in Ω_2, $M_{2,1,0}$ when two lie in Ω_1 and one in Ω_2, and so on; and let M be the mean irrespective of where they lie. The respective *numbers* of cases are measured by Ω_1^3, Ω_2^3, $3\Omega_1^2\Omega_2$, etc., and $(\Omega_1+\Omega_2+\Omega_3)^3$, and the sums of these cases are respectively measured by

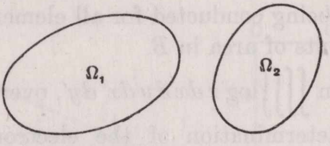
$\Omega_1^3M_{3,0,0}$, $\Omega_2^3M_{0,3,0}$, $3\Omega_1^2\Omega_2M_{2,1,0}$, etc., and $(\Omega_1+\Omega_2+\Omega_3)^3M$, and the last, being the sum of all possible values of ϕ, is equal to the sum of all the several cases previously enumerated. Hence

$$M=\frac{\Sigma\Omega_1{}^3M_{3,0,0}+3\Sigma\Omega_1{}^2\Omega_2M_{2,1,0}+6\Omega_1\Omega_2\Omega_3M_{1,1,1}}{(\Omega_1+\Omega_2+\Omega_3)^3},$$

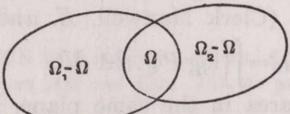
and so on if there be more than three mutually exclusive regions.

1669. Regions not mutually exclusive.

To go back to the case of two regions, suppose next that the regions Ω_1 and Ω_2 have a common region Ω. The whole region bounded is then $\Omega_1+\Omega_2-\Omega$.



(Mutually exclusive regions)



(Not mutually exclusive)

Fig. 515.

Let $M_{\Omega_1+\Omega_2-\Omega}$ be the mean value of ϕ, when the random points P, Q lie anywhere in the whole region; $M_{\Omega_1-\Omega}$ the mean when both lie in $\Omega_1-\Omega$; $M_{\Omega_2-\Omega}$ the mean when both lie in $\Omega_2-\Omega$; M the mean when one lies in Ω_1 and one in Ω_2.

The respective *numbers* of cases are $(\Omega_1+\Omega_2-\Omega)^2$, $(\Omega_1-\Omega)^2$, $(\Omega_2-\Omega)^2$ and $2\Omega_1\Omega_2-\Omega^2$; for in allowing P and Q each to range over Ω_1 and Ω_2 respectively, or Ω_2 and Ω_1 respectively, the region Ω is counted twice over.

The sum of the values of ϕ when one lies in Ω_1 and one in Ω_2 is $(2\Omega_1\Omega_2-\Omega^2)M$.

The sum when both lie in $\Omega_1-\Omega$ is $(\Omega_1-\Omega)^2M_{\Omega_1-\Omega}$.

The sum when both lie in $\Omega_2-\Omega$ is $(\Omega_2-\Omega)^2M_{\Omega_2-\Omega}$,

and the three make up the total sum $(\Omega_1+\Omega_2-\Omega)^2M_{\Omega_1+\Omega_2-\Omega}$;

$$\therefore\ M_{\Omega_1+\Omega_2-\Omega}=\frac{(\Omega_1-\Omega)^2M_{\Omega_1-\Omega}+(\Omega_2-\Omega)^2M_{\Omega_2-\Omega}+(2\Omega_1\Omega_2-\Omega^2)M}{(\Omega_1+\Omega_2-\Omega)^2}.$$

1670. Similarly more complex cases may be examined. Also the present formulae admit of considerable reduction for special cases, *e.g.* when the regions are equal or when one region is enclosed completely by the other.

1671. The Geometric Mean. Clerk Maxwell. An Integral useful in Electromagnetic Problems.

If $\log R_{AB}$ be the mean value of the logarithm of the distance between points P and Q, one in each of the areas A and B lying in the same plane, then obviously

$$\log R_{AB} = \iint \log PQ \,.\, dA\, dB \Big/ \iint dA\, dB,$$

the integrations being conducted for all elements of area in A, and for all elements of area in B.

The integration $\iiiint \log r\, dx\, dy\, dx'\, dy'$, over two such areas occurs in the determination of the electromagnetic action between two parallel straight currents flowing in conductors of given sections. (Clerk Maxwell, *E. and M.*, ii., p. 294). Clearly $A \,.\, B \,.\, \log R_{AB} = \iint \log PQ \,.\, dA\, dB.$

If C be a third area in the same plane, in which P or Q could lie, $(A+B)C \log R_{(A+B)C}$ represents on some scale the sum of the logarithms of the distances of points in C, from points in the composite area $A+B$, whilst $AC \log R_{AC}$ represents on the same scale the sum of those cases of the aforesaid group which refer to lines joining points in A with points in C; and similarly with $BC \log R_{BC}$. Hence

$$(A+B)C \log R_{(A+B)C} = AC \log R_{AC} + BC \log R_{BC}.$$

And this rule may be extended. Thus, if there be a fourth area D in the same plane,

$$\begin{aligned}(A+B+C)D \log R_{(A+B+C)D} &= (A+B)D \log R_{(A+B)D} + CD \log R_{CD}\\ &= AD \log R_{AD} + BD \log R_{BD} + CD \log R_{CD};\end{aligned}$$

and so on.

Thus, if R be found for pairs of parts of a composite figure the rule will give R for the whole figure.

Also $A, B, C, \ldots$ are not necessarily different figures.

Maxwell states the results for a number of cases. He calls the line R thus determined the Geometric mean of all the distances between such pairs of points.

1672. **Cases of Maxwell's Geometric Mean.**

I. To find R for a point C, and a finite straight line AB. (Fig. 516.)

Let CO be drawn at right angles to the direction of AB.

P a point on AB, $OA=a=x_1$, $OB=b=x_2$, $OC=p$, $OP=x$, $CP=r$, $AB=l=b-a$. $CA=r_1$, $CB=r_2$.

Then $l\log R=\int_a^b \log\sqrt{x^2+p^2}\,dx=\left[x\log\sqrt{x^2+p^2}-x+p\tan^{-1}\frac{x}{p}\right]_a^b;$

$$\therefore\ l(\log R+1)=OB\log CB-OA\log CA+OC\times \text{circ. meas. of } A\hat{C}B,$$

i.e. $$(x_2-x_1)(\log R+1)=x_2\log r_2-x_1\log r_1+p\,.\,\widehat{r_1r_2}.$$

In the case when C lies on AB produced, $p=0$, and

$$\log R+1=(x_2\log x_2-x_1\log x_1)/(x_2-x_1).$$

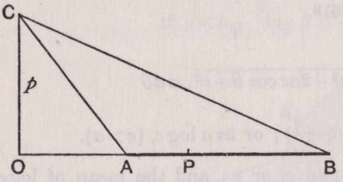


Fig. 516.

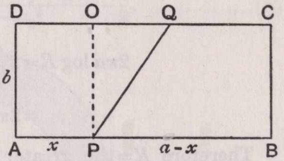


Fig. 517.

1673. II. *Let ABCD be a rectangle, $AB=a$, $AD=b$. Let P and Q be points respectively upon AB and CD. PO the perpendicular upon CD. $AP=x$.* (Fig. 517.)

For a given point P let R_1 refer to the value of R for the fixed point P,

$$a(\log R_1+1)=OD\log PD+OC\log PC+b\,C\hat{P}D$$
$$=x\log\sqrt{x^2+b^2}+(a-x)\log\sqrt{(a-x)^2+b^2}+b\left(\tan^{-1}\frac{x}{b}+\tan^{-1}\frac{a-x}{b}\right).$$

Integrating with regard to x from 0 to a,

$$a^2(\log R+1)$$
$$=\left[\frac{x^2+b^2}{2}\log\sqrt{x^2+b^2}-\frac{x^2+b^2}{4}\right]_0^a-\left[\frac{(a-x)^2+b^2}{2}\log\sqrt{(a-x)^2+b^2}-\frac{(a-x)^2+b^2}{4}\right]_0^a$$
$$+b\left[x\tan^{-1}\frac{x}{b}-b\log\sqrt{x^2+b^2}\right]_0^a-b\left[(a-x)\tan^{-1}\frac{a-x}{b}-b\log\sqrt{(a-x)^2+b^2}\right]_0^a,$$

i.e. $$a^2(\log R+\tfrac{3}{2})=(a^2-b^2)\log D+b^2\log b+2ab\tan^{-1}\frac{a}{b},$$

where D is the diagonal.

1674. III. *If P lies upon AB and Q upon AD, and R_1 as before refers to the result for a fixed point P,*

$b(\log R_1+1)=b\log\sqrt{x^2+b^2}+x\tan^{-1}\frac{b}{x}$; and integrating from 0 to a,

$$ab(\log R+1)=b\left[x\log\sqrt{x^2+b^2}-x+b\tan^{-1}\frac{x}{b}\right]_0^a+\left[\frac{x^2+b^2}{2}\tan^{-1}\frac{b}{x}+\tfrac{1}{2}bx\right]_0^a;$$

$$\therefore\ ab(\log R+\tfrac{3}{2})=ab\log D+\frac{a^2}{2}\tan^{-1}\frac{b}{a}+\frac{b^2}{2}\tan^{-1}\frac{a}{b}.$$

1675. IV. *If Q lies on the circumference of a circle of radius a, and centre O, and P be any point in its plane distant c from the centre,*

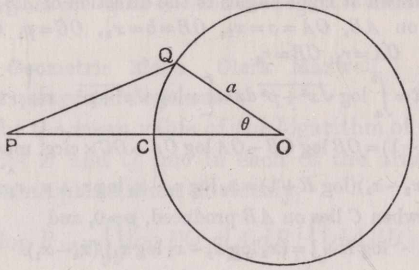


Fig. 518.

$$2\pi a \log R = 2\int_0^\pi \log \sqrt{a^2 - 2ac\cos\theta + c^2} \,.\, a\, d\theta$$

$$= 2\pi a \log a, \ (c < a)\ ; \ \text{or } 2\pi a \log c, \ (c > a).$$

Therefore R = the greater of the two a or c; and the mean of $\log r$ is accordingly

$$\log a, \ (c < a), \quad \text{or} \quad \log c, \ (c > a).$$

1676. V. *If P travels on the circumference of a second circle of radius b entirely without the former, the distance of the centres being d, and if log R stand for the mean value of log PQ,*

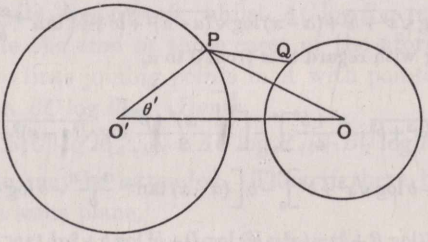


Fig. 519.

$$2\pi b \,.\, 2\pi a \log R = 2\pi a \,.\, 2\int_0^\pi \log PO \,.\, b\, d\theta'$$

$$= 2\pi a \,.\, 2\int_0^\pi \log \sqrt{b^2 - 2bd\cos\theta' + d^2}\, b\, d\theta'$$

$$= 2\pi a \,.\, 2\pi b \log d\ ; \quad \therefore \ R = d.$$

Similarly if one circle be entirely within the other.

1677. VI. *If Q lies upon a circular annulus, centre O, external and internal radii a_1 and a_2, and P be at a point distant c from O, and $\log R = M(\log PQ)$, $QO = r$, $Q\hat{O}P = \theta$,*

$$\pi(a_1^2 - a_2^2)\log R = 2\int_{a_1}^{a_2}\int_0^\pi \log\sqrt{c^2 - 2cr\cos\theta + r^2}\,.\,r\,d\theta\,dr$$

$$= 2\int_{a_2}^{a_1}\pi\log c\,.\,r\,dr,\ (c > r);\ \text{ or } = 2\int_{a_2}^{a_1}\pi\log r\,.\,r\,dr,\ (c < r),$$

$$= \pi\log c\,.\,(a_1^2 - a_2^2) \text{ if } c > a_1,$$

or $$= \pi\left[r^2\log r - \frac{r^2}{2}\right]_{a_2}^{a_1} = \pi\left(a_1^2\log a_1 - a_2^2\log a_2 - \frac{a_1^2 - a_2^2}{2}\right) \text{ if } c < a_2,$$

i.e. if $c > a_1$, $\log R = \log c$; ....(α)

if $c < a_2$, $\log R = \dfrac{a_1^2\log a_1 - a_2^2\log a_2}{a_1^2 - a_2^2} - \dfrac{1}{2}$. ....($\beta$)

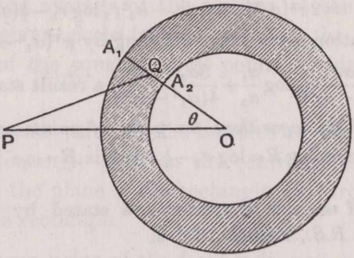


Fig. 520.

If $a_1 > c > a_2$, and P itself lies upon the annulus,

$$\pi(a_1^2 - a_2^2)\log R = \int_{a_2}^{c} 2\pi\log c\,.\,r\,dr + \int_c^{a_1} 2\pi\log r\,.\,r\,dr;$$

whence $$\log R = \frac{c^2 - a_2^2}{a_1^2 - a_2^2}\log c + \frac{a_1^2\log a_1 - c^2\log c}{a_1^2 - a_2^2} - \frac{1}{2}\,\frac{a_1^2 - c^2}{a_1^2 - a_2^2}. \quad\text{.........}(\gamma)$$

Since $R = c$ when P is without the annulus, the mean value of $\log PQ$, where P lies upon any region entirely without the annulus is the mean value of $\log PO$. And if P lies upon any region entirely within the annulus, the expression for R, in that case not containing c, is independent of the shape or position of the region.

We may deduce the result (γ) from (α) and (β) by Art. 1671. Let A and B be the regions of the annulus respectively outside and inside a concentric circle through Q. Then if C be an elementary small area in which P lies,

$$(A + B)\log R_{(A+B)C} = A\log R_{AC} + B\log R_{BC};$$

$$\therefore\ \pi(a_1^2 - a_2^2)\log R_{(A+B)C} = \pi(a_1^2 - c^2)\left\{\frac{a_1^2\log a_1 - c^2\log c}{a_1^2 - c^2} - \frac{1}{2}\right\} + \pi(c^2 - a_2^2)\log c,$$

giving the same result as before.

1678. VII. *If P be not at a fixed point within the annulus, but may travel anywhere within it,*

$$\{\pi(a_1^2-a_2^2)\}^2 \log R = \iiiint \log \sqrt{r_1^2 - 2r_1r_2\cos(\theta_1-\theta_2)+r_2^2}\,.\,r_1\,d\theta_1\,dr_1 r_2\,d\theta_2\,dr_2,$$

where r_1, θ_1 and r_2, θ_2 are the polar coordinates of P and Q.

The limits for θ_1 are θ_2 to $\theta_2+2\pi$; for θ_2, 0 to π, and double the result; for r_2 from a_2 to r_1 and r_1 to a_1; for r_1, from a_2 to a_1.

The first integration gives

$$2(\pi\log r_1)r_1r_2dr_1dr_2d\theta_2 \quad\text{or}\quad 2(\pi\log r_2)r_1r_2dr_1dr_2d\theta_2,$$

according as r_1 or r_2 is the greater.

The second merely multiplies the result by 2π.

The third gives

$$4\pi^2\int_{a_2}^{r_1} r_1r_2\log r_1\,dr_1\,dr_2 + 4\pi^2\int_{r_1}^{a_1} r_1r_2\log r_2\,dr_1\,dr_2$$

$$=2\pi^2[a_1^2\log a_1\,.\,r_1 - a_2^2r_1\log r_1 - \tfrac{1}{2}(a_1^2r_1 - r_1^3)]dr_1.$$

The final integration gives, after dividing by $\pi^2(a_1^2-a_2^2)^2$,

$$\log R = \log a_1 - \frac{a_2^4}{(a_1^2-a_2^2)^2}\log\frac{a_1}{a_2} + \frac{3a_2^2-a_1^2}{4(a_1^2-a_2^2)},$$ a result stated by Maxwell.

For the mean of the logarithms for pairs of points within any circular area, put $a_2=0$; then $\log R = \log a_1 - \frac{1}{4}$, that is $R=a_1e^{-\frac{1}{4}}$ or R is a little more than $3a/4$.

Other results of similar character are stated by Maxwell with a reference to *Trans. R.S.*, Edinb., 1871-2.

1679. Other cases of mean values will be considered in the next chapter, which are more intimately connected with the general Theory of Probability.

PROBLEMS.

1. If the sides of a rectangle may have any values between a and b, prove that the mean area $=(a+b)^2/4$. [R. P.]

2. Find the average area of a random sector whose vertex is taken at a given point on a given circle.

3. $ABCD$ is a square. Show that the average distance of A from points on BC for an equable distribution of radii vectores about A is $\dfrac{4AB}{\pi}\log\dfrac{AC+AB}{AB}$; but for an equable distribution of points on BC it is $\dfrac{AC}{2}+\dfrac{AB}{2}\log\dfrac{AC+AB}{AB}$.

4. A rod of length a is broken into two parts at random. Show that the mean value of the sum of the squares of the parts $=2a^2/3$. [Ox. II., 1886.]

5. A rod of length a is broken into two parts at random. Show that the mean value of the rectangle contained by the parts is $a^2/6$.

6. The sum of two positive numbers is given $=N$. Show that the mean value of the product of the p^{th} power of the one and the q^{th} power of the other is $p!\,q!\,N^{p+q}/(p+q+1)!$, p and q being positive integers.

7. Find the mean value of the (i) squares, (ii) cubes of all radii vectores of a cardioide for an equable angular distribution of radii vectores about the pole.

8. Given the base and the radius of the circumcircle of a triangle, determine its mean area, stating clearly what assumptions you make as to equal probability. [St. John's, 1884.]

9. Show that the average of the squares of the distances of all points within a given circle from a point on the circumference is three times that of the squares of all points within the circle from the centre. [Colleges, 1878.]

10. Find the mean value of the squares of the distances of all points within a rectangle (i) from the centre of the rectangle, (ii) from any point in the plane of the rectangle, (iii) from any point not in the plane of the rectangle.

11. Find the mean value of the focal radii vectores of a cardioide (i) for an equable angular distribution of radii, (ii) for an equable arcual distribution.

12. If a solid be formed by the revolution of a cardioide about its axis, find the mean value of the focal distances of points on the surface of the solid (i) for an equable surface distribution, (ii) for an equable solid angle distribution.

13. Find the mean value of the squares of the distances between any two points within a given (i) triangle, (ii) square, (iii) sphere, (iv) cube.

14. (i) Find the mean of the inverse distances of points within an ellipse from a focus for an equable areal distribution.

(ii) Find the mean of the inverse distances of points within a prolate spheroid from a focus for an equable volume distribution.

15. Show that the mean distance of points within a sphere of radius a from points of the surface of a shell of double the radius of the sphere is $21a/10$, and that the mean distance of points on the surface of the sphere from points on the shell is $13a/6$.

16. Show that the mean distance of all points within a sphere of radius a from a point midway between the centre and the surface is $279a/320$.

17. Show that the mean distance of a point on the external surface of a spherical shell of thickness T from points in the material of the shell is $\frac{6}{5}R+\frac{1}{5}\frac{(R-T)^3(2R-T)}{R(3R^2-3RT+T^2)}$, where R is the external radius.

18. Show that the mean distance between points P and Q, of which P lies within a sphere of radius R and Q lies between this sphere and a concentric sphere of double the radius, is $3^5R/140$.

19. There are two concentric spherical shells, the bounding surfaces of which are 1 inch, 2 inches, 3 inches, and 4 inches. Show that the average distance of points in the material of the first from points in the material of the second is $3\frac{591}{740}$ inches.

20. Two equal spherical surfaces are in contact. Show that the mean distance of points on the one surface from points on the other $=7/3$ of the radius of either.

Show further that if the points may lie anywhere within their respective spheres, their mean distance is $11/5$ of the radius of either; but that if one of the points lies within one of the spheres and the other point on the surface of the other sphere, their mean distance is $34/15$ of the radius.

21. If M_n be the mean of the n^{th} power of the distance between two points on the area bounded by a circle of diameter unity, show that
$$M_{n+2}=M_n(n+2)(n+3)/(n+4)(n+6).$$

22. If M_n be the mean of the n^{th} power of the distance between two points on the surface of a sphere of unit diameter, show that
$$M_{n+1}=M_n(n+2)/(n+3).$$

23. If M_n be the mean of the n^{th} power of the distance between two points within a sphere of diameter unity, show that
$$M_{n+1}=M_n(n+3)(n+6)/(n+5)(n+7)$$

24. A point O is taken outside a sphere with centre C and radius a. $CO=2a$. Show that the mean of the cubes of the distances of O from points within the sphere $=731a^3/70$, and that the mean of the fourth powers $=171a^4/7$.

25. Show that the mean value of $x^4y^4z^4$ over the surface of a sphere of radius a is $a^{12}/5005$.

26. Show that the mean value of $x^{p-1}y^{q-1}z^{r-1}$ for positive values of x, y, z, subject to the condition $a^{-2}x^2+b^{-2}y^2+c^{-2}z^2=1$ for an equable distribution of areas on the x-y plane, is

$$a^{p-1}b^{q-1}c^{r-1}\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)\Gamma\left(\frac{r+1}{2}\right)\Big/\pi\Gamma\left(\frac{p+q+r+1}{2}\right),$$

where p, q, r are all greater than unity.

27. On a straight line of unit length two random points are taken. Show that the mean of the square of the distance between them is 1/6 of a unit of area.

28. Circles are inscribed in the triangles formed by joining points on an ellipse of semi-axes a, b and eccentricity e to the foci. Show that the mean value of the areas of the circles for equal increments of a focal vectorial angle is

$$\pi a^2(1-e)^2(a/b-1).$$ [MATH. TRIP., 1892.]

29. Show that the mean value of the product of the three perpendiculars from any point within a triangle upon the sides is $p_1p_2p_3/60$, where p_1, p_2, p_3 are the perpendiculars from the angular points upon the opposite sides.

30. Show that the mean value of the product of the four perpendiculars from any point within a tetrahedron upon the faces is $p_1p_2p_3p_4/560$, where p_1, p_2, p_3, p_4 are the perpendiculars from the several quoins upon the opposite faces.

31. Five points, A, B, C, D, E, are taken upon a straight line AE, to which perpendiculars are drawn through these points of increasing magnitude. The sum of these five perpendiculars is 10 inches. Show that the mean length of the middle perpendicular is 47/30 of an inch.

32. Show that the mean distance of all points within a given regular polygon of side $2a$ from the centre is $\dfrac{R}{3}+\dfrac{1}{3}\dfrac{r^2}{a}\log\dfrac{R+a}{r}$, where R and r are the radii of the circumscribed and inscribed circles.

33. Show that the rectangle contained between the average value of the radius of curvature at points equally distributed along a curve and the corresponding arc is double the area contained between the curve, the evolute and the normals at the extremities of the arc.

[δ, 1883.]

34. Prove that the mean value of the radius of curvature at points equally distributed along the cardioide $r=a(1+\cos\theta)$ is $a\pi/3$, while the density distribution of the corresponding points along the pedal with respect to the pole varies at any point as the curvature at the corresponding point of the cardioide. [δ, 1883.]

35. Prove that the square of the mean value of any function of a variable between any limits of the variable is less than the mean value of the square of that function between the same limits of the variable. [ST. JOHN'S, 1883.]

36. Find the mean value of the squares of the distances from a focus of all points within an ellipse whose eccentricity is $\sqrt{3}/2$. [δ, 1881.]

37. The circumference of the auxiliary circle of an ellipse, whose axes are $ACA'=2a$, $BCB'=2b$, is divided at Q_1, Q_2, ... into a large number of equal arcs. At P_1, the point on the ellipse whose eccentric angle is ACQ_1, a circle is described so as to touch the ellipse at P_1 and to have its centre on the major axis. Show that the mean area of all such circles is $\pi b^2(a^2+b^2)/2a^2$. [α, 1881.]

38. At any point P of a catenary whose parameter is c, the ordinate PN and the normal PG are drawn to meet the directrix at N and G respectively. Prove that the mean values of the area of the triangle NPG for points proceeding by equal increments of (i) abscissa, (ii) ordinate, (iii) arc, up to a point whose coordinates are (x, y), are respectively

$$\text{(i)}\ (y^3-c^3)/6x;\quad \text{(ii)}\ c^2\left(c\sinh\frac{4x}{c}-4x\right)\Big/\,64(y-c);\quad \text{(iii)}\ (y^4-c^4)/8cs.$$

39. Find the mean of the inverse distances of two random points, one on the surface of a sphere, the other on a circular area exterior to the sphere and whose plane is at right angles to the line of centres.

40. Prove that the mean of the inverse distance between points on the surface of a sphere and points on a straight rod of length l, external to the sphere, which is bisected at right angles by a perpendicular upon it from the centre of the sphere, is $\dfrac{2}{l}\log\tan\dfrac{\pi+\alpha}{4}$, where α is the angle at the centre of the sphere subtended by the rod.

41. Prove that the mean inverse distance between points on the surface of a sphere of radius a and points on a concentric ring of radius b is b^{-1} if $b>a$ or a^{-1} if $b<a$.

42. Prove that the mean value of x for all points within the positive octant of the surface $(x/a)^{\frac{2}{3}}+(y/b)^{\frac{2}{3}}+(z/c)^{\frac{2}{3}}=1$ is $21a/128$.

43. On a given finite arc n points are drawn dividing it into equal small lengths, and n other points are taken, parallels to the normals at which divide the angle between the extreme normals into equal small angles. Prove that when n is indefinitely increased the mean of the radii of curvature at the former n points is greater than the mean of the radii of curvature at the latter n points, the curvature being supposed finite at every point of the arc. [St. John's, 1889.]

44. If $\log R$ be the mean value of the logarithm of the distance between two points P and Q which lie on a line AB of length a, show that $R=ae^{-\frac{3}{2}}$. [Clerk Maxwell, *El. and Mag.*, II., p. 296.]