CHAPTER XXXV. SECTION II.

DIRICHLET'S INVESTIGATION.

1616. **Fourier's Formulae. Dirichlet's Investigation.**

If $\phi(x)$ be a single-valued finite and continuous function of x which remains positive and either constant or continually decreasing throughout the whole range of integration from $x=0$ to $x=h$, where $0 < h \not> \pi/2$, then will

$$Lt_{\omega\to\infty}\int_0^h \frac{\sin \omega x}{\sin x}\phi(x)\,dx=\frac{\pi}{2}\phi(0).$$

This result is due to Fourier. Separating the integration range 0 to h into intervals

$$0 \text{ to } \frac{\pi}{\omega},\quad \frac{\pi}{\omega} \text{ to } \frac{2\pi}{\omega}, \ldots (n-1)\frac{\pi}{\omega} \text{ to } \frac{n\pi}{\omega},\quad \frac{n\pi}{\omega} \text{ to } h,$$

where $\frac{n\pi}{\omega}$ is the greatest multiple of $\frac{\pi}{\omega}$ contained in h, we have

$$\int_0^h \frac{\sin \omega x}{\sin x}\phi(x)\,dx=\left\{\int_0^{\frac{\pi}{\omega}}+\int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}}+\ldots+\int_{\frac{r\pi}{\omega}}^{\frac{(r+1)\pi}{\omega}}+\int_{\frac{(r+1)\pi}{\omega}}^{\frac{(r+2)\pi}{\omega}}+\ldots\right.$$
$$\left.+\int_{\frac{(n-1)\pi}{\omega}}^{\frac{n\pi}{\omega}}+\int_{\frac{n\pi}{\omega}}^{h}\right\}\frac{\sin \omega x}{\sin x}\phi(x)\,dx. \ldots(1)$$

Now as x increases from $r\pi/\omega$ to $(r+1)\pi/\omega$, ωx increases by π. Hence $\sin \omega x$ in this interval is of opposite sign to the value of $\sin \omega x$ in the next interval. But $\sin x$ and $\phi(x)$ retain the same sign. Hence the several terms in the above series are alternately positive and negative.

Again comparing corresponding elements in $\int_{\frac{r\pi}{\omega}}^{\frac{(r+1)\pi}{\omega}}(\quad)\,dx$ and $\int_{\frac{(r+1)\pi}{\omega}}^{\frac{(r+2)\pi}{\omega}}(\quad)\,dx$, write $x+\frac{\pi}{\omega}$ for x in the second, which then becomes

$$-\int_{\frac{r\pi}{\omega}}^{\frac{(r+1)\pi}{\omega}}\frac{\sin \omega x}{\sin(x+\pi/\omega)}\phi(x+\pi/\omega)\,dx.$$

And since x has increased to $x+\pi/\omega$, but is still $<\pi/2$, $\sin(x+\pi/\omega)$ is $>\sin x$, whilst $\phi(x+\pi/\omega) \not> \phi(x)$, the element in the second integral is numerically less than the corresponding element in the first.

Hence the several terms of (1) are (a) *of alternate sign*, (b) *of decreasing numerical magnitude*.

Putting $\omega x = z$,

$$Lt_{\omega\to\infty}\int_{\frac{r\pi}{\omega}}^{\frac{(r+1)\pi}{\omega}} \frac{\sin\omega x}{\sin x}\phi(x)\,dx = Lt_{\omega\to\infty}\int_{r\pi}^{(r+1)\pi}\frac{\sin z}{\omega\sin z/\omega}\phi(z/\omega)\,dz$$

$$=\phi(0)\int_{r\pi}^{(r+1)\pi}\frac{\sin z}{z}\,dz. \quad \text{(See Art. 1902.)}$$

Hence the sum of the first r terms of (1) becomes

$$\phi(0)\left[\int_0^{\pi}+\int_{\pi}^{2\pi}+\dots+\int_{(r-1)\pi}^{r\pi}\right]\frac{\sin z}{z}\,dz = \phi(0)\int_0^{r\pi}\frac{\sin z}{z}\,dz = \frac{\pi}{2}\phi(0)$$

when r is infinite.

And for the remaining terms from

$$\int_{\frac{r\pi}{\omega}}^{\frac{(r+1)\pi}{\omega}}\frac{\sin\omega x}{\sin x}\phi(x)\,dx \quad \text{to} \quad \int_{\frac{(n-1)\pi}{\omega}}^{\frac{n\pi}{\omega}}\frac{\sin\omega x}{\sin x}\phi(x)\,dx,$$

the interval of each is infinitesimally small, and the integrands are finite. Each integral is therefore infinitesimally small, they are of alternate sign and each numerically less than the preceding one. Hence their sum is less than the first of the group, which is itself infinitesimally small.

Again, as to the final integral $\int_{\frac{n\pi}{\omega}}^{h}\frac{\sin\omega x}{\sin x}\phi(x)\,dx$, it is integrated over an infinitesimal interval with a finite integrand, and therefore also vanishes.

Thus we have

$$Lt_{\omega\to\infty}\int_0^h\frac{\sin\omega x}{\sin x}\phi(x)\,dx = \frac{\pi}{2}\phi(0),$$

where $0<h \not> \frac{\pi}{2}$ under the special conditions stated as to $\phi(x)$.

The method adopted in this proof is due to Dirichlet. It is given by Bertrand, *Calc. Int.*, p. 228.

1617. If $\phi(x)$ becomes negative but not numerically greater than a definite positive constant C, remaining finite and continuous as before, then since $\phi(x)+C$ is positive and decreasing, we have

$$Lt_{\omega\to\infty}\int_0^h \frac{\sin\omega x}{\sin x}[\phi(x)+C]\,dx=[\phi(0)+C]\frac{\pi}{2}.$$

But the theorem is also true for a function which remains constant and equal to C. Hence subtracting,

$$Lt_{\omega\to\infty}\int_0^h \frac{\sin\omega x}{\sin x}\phi(x)\,dx=\frac{\pi}{2}\phi(0).$$

This has therefore been now proved whether $\phi(x)$ be positive or negative, provided it is either constant or decreasing so long as it remains finite and continuous between the limits.

1618. Further, if $\phi(x)$ be an *increasing* function, $-\phi(x)$ is a *decreasing* function to which the theorem is applicable, and therefore

$$Lt_{\omega\to\infty}\int_0^h \frac{\sin\omega x}{\sin x}\{-\phi(x)\}\,dx=\frac{\pi}{2}\{-\phi(0)\},$$

whence $$Lt_{\omega\to\infty}\int_0^h \frac{\sin\omega x}{\sin x}\phi(x)\,dx=\frac{\pi}{2}\phi(0),$$

whether $\phi(x)$ be continually either increasing or decreasing between the limits.

1619. Since the formula established is independent of h, taking p and q any two quantities between 0 and $\pi/2$, we have

$$Lt_{\omega\to\infty}\int_0^p \frac{\sin\omega x}{\sin x}\phi(x)\,dx=\frac{\pi}{2}\phi(0)=Lt_{\omega\to\infty}\int_0^q \frac{\sin\omega x}{\sin x}\phi(x)\,dx.$$

Hence if $F(x)$ be any function of x, continuous and coincident with $\phi(x)$ for the portion of $\phi(x)$ between q and p,

$$Lt_{\omega\to\infty}\int_q^p \frac{\sin\omega x}{\sin x}F(x)\,dx=0,$$

and here it is supposed that from q to p, $F(x)$ is always increasing or always decreasing, for it is coincident with $\phi(x)$ throughout that interval.

1620. **Existence of a Finite Number of Maxima and Minima.**

Suppose that there are a finite number of maxima and minima on the graph of $y=\phi(x)$ between $x=0$ and $x=h$, say at $x=x_1, x_2, x_3, \dots x_n$. Then when $\omega \to \infty$

$$Lt\int_0^h \frac{\sin \omega x}{\sin x}\phi(x)\,dx = Lt\left[\int_0^{x_1}+\int_{x_1}^{x_2}+\dots+\int_{x_n}^h\right]\frac{\sin \omega x}{\sin x}\phi(x)\,dx.$$

Now $\phi(x)$ is

continually increasing or continually decreasing from 0 to x_1,
continually decreasing or continually increasing from x_1 to x_2,
continually increasing or continually decreasing from x_2 to x_3,
etc.

The first term therefore contributes $\frac{\pi}{2}\phi(0)$. Each of the others contributes nothing by Art. 1619. So that if the number of maxima and minima be finite, the Fourier formula still holds good.

1621. **Existence of a Finite Number of Discontinuities.**

Finally, suppose a discontinuity in $\phi(x)$ occurs at a point $x=x_1$ ($<h$), where the function changes abruptly from $\phi(x_1)$ to $\psi(x_1)$, remaining finite and $\psi(x)$ retaining the property possessed by $\phi(x)$ as to continual increase or decrease throughout the remainder of the range of integration. Then

$$Lt_{\omega\to\infty}\int_0^h \frac{\sin \omega x}{\sin x}\phi(x)\,dx$$

$$=Lt_{\omega\to\infty}\int_0^{x_1}\frac{\sin \omega x}{\sin x}\phi(x)\,dx + Lt_{\omega\to\infty}\int_{x_1}^h \frac{\sin \omega x}{\sin x}\psi(x)\,dx = \frac{\pi}{2}\phi(0)+0.$$

Thus each discontinuity introduces a zero term, and provided the number of such discontinuities be finite between 0 and h, their aggregate contributes nothing to the integral.

1622. **Generalised Restatement of the Theorem.**

We may now restate the theorem thus:

Let $\phi(x)$ be any function of x with any finite number of discontinuities and any finite number of maxima and minima between $x=0$ and $x=h$, where h is positive, not infinitesimally small, and not greater than $\pi/2$; then

$$Lt_{\omega\to\infty}\int_0^h \frac{\sin \omega x}{\sin x}\phi(x)\,dx = \frac{\pi}{2}\phi(0).$$

1623. **Geometrical View of the Result.**

Drawing the graph of $y = \sin \omega x / \sin x$, the curve has a large maximum, viz. ω, at $x=0$; and crossing the x-axis at $x=\pi/\omega$, $2\pi/\omega$, $3\pi/\omega$, etc., there are successive minima and maxima, their positions being given by $\tan \omega x = \omega \tan x$.

Since $\sin \omega x$ lies between ± 1 and goes through a cycle of its numerical changes in each of the above intervals, whilst $\sin x$ is increasing throughout the whole range from $x=0$ to $x=\frac{\pi}{2}$, the excursions of the graph to one side or the other of the x-axis diminish in extent, and these subsidiary maxima and minima are relatively unimportant. The multiplication of the function by $\phi(x)$ alters the magnitude and position of the maxima and minima ordinates, but leaves the general characteristic appearance of the graph unchanged (Fig. 471).

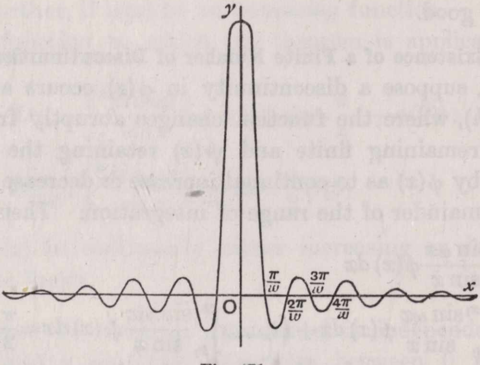

Fig. 471.

The geometrical interpretation of the formula of Art. 1622 is then as follows:

Let the graph of $y = \phi(x)\dfrac{\sin \omega x}{\sin x}$ be drawn starting from $x=0$ and extending as far as $x=h$, and also the graph of $y=\phi(x)$ extending as far as $x=\pi/2$. Let the areas enclosed by the successive portions of the former bounded by the x-axis, and, for the principal maximum, by the y-axis, and lying alternately above and below the x-axis be A_1, A_2, A_3, A_4, etc., and let B be the area of the rectangle of which two adjacent sides are the initial ordinate of the second graph, viz. $\phi(0)$ and the length $\frac{\pi}{2}$; then when ω is indefinitely increased $A_1-A_2+A_3-A_4+\dots$ tends to the limit B.

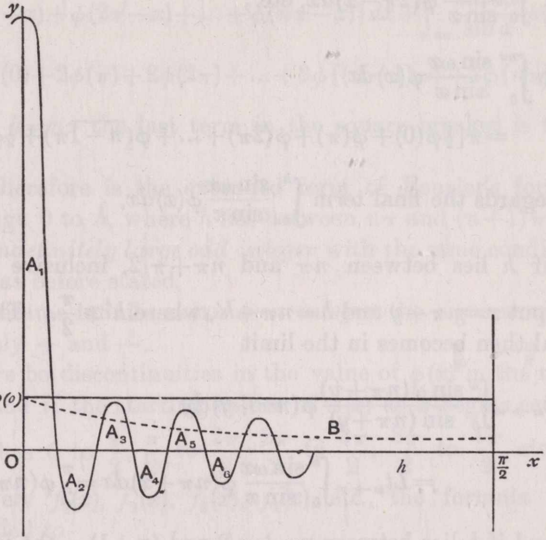

Fig. 472.

1624. Extension of Range of Integration.

If the range of integration be extended beyond $\pi/2$, and h lies between $n\pi$ and $(n+1)\pi$, we may break up the whole range into sub-ranges of extent $\pi/2$ as far as $n\pi$, and we have

$$\int_0^h \frac{\sin \omega x}{\sin x}\,\phi(x)\,dx=\left\{\int_0^{\frac{\pi}{2}}+\int_{\frac{\pi}{2}}^{\frac{2\pi}{2}}+\dots+\int_{(2n-1)\frac{\pi}{2}}^{2n\frac{\pi}{2}}+\int_{n\pi}^{h}\right\}\frac{\sin \omega x}{\sin x}\,\phi(x)\,dx.$$

In the second, third, ... $2n^{\text{th}}$ integrals replace x successively by $\pi-y,\ \pi+y,\ 2\pi-y,\ \dots\ n\pi-y$.

If we take ω to be an odd integer, these become

$$\int_{\frac{\pi}{2}}^{0}\frac{\sin \omega(\pi-y)}{\sin(\pi-y)}\,\phi(\pi-y)(-dy),\quad \int_0^{\frac{\pi}{2}}\frac{\sin \omega(\pi+y)}{\sin(\pi+y)}\,\phi(\pi+y)\,dy,$$

$$\int_{\frac{\pi}{2}}^{0}\frac{\sin \omega(2\pi-y)}{\sin(2\pi-y)}\,\phi(2\pi-y)(-dy),\ \text{etc.},$$

$$i.e.\ \int_0^{\frac{\pi}{2}} \frac{\sin \omega x}{\sin x}\phi(\pi - x)\,dx,\ \int_0^{\frac{\pi}{2}} \frac{\sin \omega x}{\sin x}\phi(\pi + x)\,dx,$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin \omega x}{\sin x}\phi(2\pi - x)\,dx, \text{ etc.};$$

whence $\displaystyle\int_0^{n\pi} \frac{\sin \omega x}{\sin x}\phi(x)\,dx$

$$= \pi\left[\tfrac{1}{2}\phi(0) + \phi(\pi) + \phi(2\pi) + \ldots + \phi(\overline{n-1}\pi) + \tfrac{1}{2}\phi(n\pi)\right].$$

As regards the final term $\displaystyle\int_{n\pi}^{h} \frac{\sin \omega x}{\sin x}\phi(x)\,dx$,

(*a*) if h lies between $n\pi$ and $n\pi + \pi/2$, inclusive of the latter, put $x = n\pi + y$ and $h = n\pi + h'$, where $h' \ngtr \frac{\pi}{2}$. The final integral then becomes in the limit

$$Lt_{\omega\to\infty} \int_0^{h'} \frac{\sin \omega(n\pi + y)}{\sin(n\pi + y)}\phi(n\pi + y)\,dy$$

$$= Lt_{\omega\to\infty} \int_0^{h'} \frac{\sin \omega x}{\sin x}\phi(n\pi + x)\,dx = \frac{\pi}{2}\phi(n\pi);$$

(*b*) and if h lies between $n\pi + \pi/2$ and $(n+1)\pi$, the integral may be written $Lt_{\omega\to\infty}\left(\displaystyle\int_{n\pi}^{n\pi+\frac{\pi}{2}} + \int_{n\pi+\frac{\pi}{2}}^{h}\right)\left\{\frac{\sin \omega x}{\sin x}\phi(x)\,dx\right\}$; and putting $x = n\pi + y$ in the first and $(n+1)\pi - y$ in the second, the first becomes $\frac{\pi}{2}\phi(n\pi)$, as has been seen, and the second becomes

$$Lt_{\omega\to\infty} \int_{\frac{\pi}{2}}^{(n+1)\pi - h} \frac{\sin \omega\{(n+1)\pi - y\}}{\sin\{(n+1)\pi - y\}}\phi\{(n+1)\pi - y\}(-dy)$$

$$= Lt_{\omega\to\infty} \int_{h'}^{\frac{\pi}{2}} \frac{\sin \omega x}{\sin x}\phi\{(n+1)\pi - x\}\,dx,$$

where $h' = (n+1)\pi - h$, which is positive and $\ngtr \frac{\pi}{2}$. Therefore this limit vanishes by Art. 1619. Hence in either case the contribution of the final integral is $\frac{\pi}{2}\phi(n\pi)$. But if $h = n\pi$ the contribution is zero.

Hence in the limit when ω is indefinitely increased,

$$Lt_{\omega\to\infty}\int_0^h \frac{\sin\omega x}{\sin x}\phi(x)\,dx = Lt_{\omega\to\infty}\int_0^{\frac{\pi}{2}}\frac{\sin\omega x}{\sin x}\{\phi(x)+\phi(\pi-x)$$

$$+\phi(\pi+x)+\phi(2\pi-x)+\ldots+\phi(n\pi-x)\}\,dx+\int_{n\pi}^h \frac{\sin\omega x}{\sin x}\phi(x)\,dx$$

$$=\frac{\pi}{2}[\phi(0)+2\phi(\pi)+2\phi(2\pi)+\ldots+2\phi\{(n-1)\pi\}+2\phi(n\pi)].$$

But if $h=n\pi$ the last term in the square bracket is to be $\phi(n\pi)$.

This therefore is the extended form of Fourier's formula for a range 0 to h, where h lies between $n\pi$ and $(n+1)\pi$, and ω is an *indefinitely large odd integer* with the same conditions for $\phi(x)$ as before stated.

If ω became infinite as *an even integer*, the signs would be alternately $+$ and $-$.

If there be discontinuities in the value of $\phi(x)$ in the range 0 to h, and if the starting values of $\phi(x)$ as x begins each of its marches 0 to $\frac{\pi}{2}$, $\frac{\pi}{2}$ to $\frac{2\pi}{2}$, $\frac{2\pi}{2}$ to $\frac{3\pi}{2}$, $\frac{3\pi}{2}$ to $\frac{4\pi}{2}$, etc., be respectively $f_1(x)$, $f_2(x)$, $f_3(x)$, $f_4(x)$, etc., the formula must be amended to

$$\frac{\pi}{2}\{f_1(0)+f_2(\pi)+f_3(\pi)+f_4(2\pi)+f_5(2\pi)+f_6(3\pi)+f_7(3\pi)$$
$$+\ldots+f_{2n}(n\pi)+f_{2n+1}(n\pi)\},$$

when ω becomes infinite as an odd integer and the number of discontinuities between 0 and h is supposed finite.

1625. If a and b be two positive quantities, $a>b$ and $m\pi<a<(m+1)\pi$, $n\pi<b<(n+1)\pi$, then

$$Lt_{\omega\to\infty}\int_0^a \frac{\sin\omega x}{\sin x}\phi(x)\,dx=\pi[\tfrac{1}{2}\phi(0)+\phi(\pi)+\phi(2\pi)+\ldots+\phi(m\pi)]$$
$$=\pi E_m,\text{ say,}$$

and

$$Lt_{\omega\to\infty}\int_0^b \frac{\sin\omega x}{\sin x}\phi(x)\,dx=\pi[\tfrac{1}{2}\phi(0)+\phi(\pi)+\phi(2\pi)+\ldots+\phi(n\pi)]$$
$$=\pi E_n,\text{ say.}$$

Then $Lt_{\omega\to\infty}\int_b^a \frac{\sin\omega x}{\sin x}\phi(x)\,dx=\pi(E_m-E_n).$

If $a-b \not> 2\pi$, so that $a \not> (n+1)\pi+2\pi$, *i.e.* $\not> (n+3)\pi$, the limit is $\pi[\phi\{(n+1)\pi\}+\phi\{(n+2)\pi\}$, $(n>0)$.

If $b<\pi$, then $a<3\pi$, and the limit is $\pi[\phi(\pi)+\phi(2\pi)]$.

Still supposing a and b both positive, and

$$a>b \quad\text{and}\quad m\pi<a<(m+1)\pi, \quad n\pi<b<(n+1)\pi,$$

consider $Lt_{\omega\to\infty}\int_0^{-b}\frac{\sin\omega x}{\sin x}\phi(x)\,dx$; write $x=-y$. Then the integral becomes

$$-Lt_{\omega\to\infty}\int_0^{b}\frac{\sin\omega y}{\sin y}\phi(-y)\,dy=-\pi[\tfrac{1}{2}\phi(0)+\phi(-\pi)+\phi(-2\pi)+\dots+\phi(-n\pi)]=-\pi E_{-n}, \text{ say.}$$

Similarly $Lt_{\omega\to\infty}\int_0^{-a}\frac{\sin\omega x}{\sin x}\phi(x)\,dx=-\pi E_{-m}$.

Thus we have

$$\left.\begin{aligned}
&Lt\int_b^a\frac{\sin\omega x}{\sin x}\phi(x)\,dx &&=\pi(E_m-E_n),\\
&Lt\int_{-b}^a\frac{\sin\omega x}{\sin x}\phi(x)\,dx=Lt\left[\int_0^a-\int_0^{-b}\right]\frac{\sin\omega x}{\sin x}\phi(x)\,dx &&=\pi(E_m+E_{-n}),\\
&Lt\int_b^{-a}\frac{\sin\omega x}{\sin x}\phi(x)\,dx=Lt\left[\int_0^{-a}-\int_0^{b}\right]\frac{\sin\omega x}{\sin x}\phi(x)\,dx &&=-\pi(E_{-m}+E_n),\\
&Lt\int_{-b}^{-a}\frac{\sin\omega x}{\sin x}\phi(x)\,dx=Lt\left[\int_0^{-a}-\int_0^{-b}\right]\frac{\sin\omega x}{\sin x}\phi(x)\,dx &&=-\pi(E_{-m}-E_{-n}),\\
& &&\qquad m>n>0.
\end{aligned}\right\}$$

In the case $0<b<a<\pi$,

$$\left.\begin{aligned}
&Lt\int_b^a\frac{\sin\omega x}{\sin x}\phi(x)\,dx=\pi[\tfrac{1}{2}\phi(0)-\tfrac{1}{2}\phi(0)] &&=0,\\
&Lt\int_{-b}^a\frac{\sin\omega x}{\sin x}\phi(x)\,dx=\pi[\tfrac{1}{2}\phi(0)+\tfrac{1}{2}\phi(0)] &&=\pi\phi(0),\\
&Lt\int_b^{-a}\frac{\sin\omega x}{\sin x}\phi(x)\,dx=\pi[-\tfrac{1}{2}\phi(0)-\tfrac{1}{2}\phi(0)] &&=-\pi\phi(0),\\
&Lt\int_{-b}^{-a}\frac{\sin\omega x}{\sin x}\phi(x)\,dx=\pi[-\tfrac{1}{2}\phi(0)+\tfrac{1}{2}\phi(0)] &&=0,
\end{aligned}\right\}$$

i.e. if the limits be of the same sign the result is zero; if the limits be of opposite signs the result is $\pi\phi(0)$ or $-\pi\phi(0)$, according as the upper limit is positive or negative.

1626. Application to the Evaluation of Fourier's Series.

Taking the identity $\dfrac{\sin(2n+1)\theta/2}{\sin\theta/2}=1+2\sum_1^n \cos p\theta$, write therein $\theta=\xi-x=2y$, $2n+1=\omega$; multiply by $f(\xi)$ and integrate with regard to ξ from β to α, where $\alpha-\beta \not> 2\pi$. We have

$$\int_\beta^\alpha f(\xi)d\xi+2\sum_1^n\int_\beta^\alpha f(\xi)\cos p(\xi-x)d\xi=2\int_{\frac{\beta-x}{2}}^{\frac{\alpha-x}{2}}\frac{\sin\omega y}{\sin y}f(x+2y)\,dy\,;$$

and increasing n without limit, $\omega\to\infty$ and

$$\frac{1}{2}\int_\beta^\alpha f(\xi)d\xi+\sum_1^\infty\int_\beta^\alpha f(\xi)\cos p(\xi-x)d\xi=Lt\int_{\frac{\beta-x}{2}}^{\frac{\alpha-x}{2}}\frac{\sin\omega y}{\sin y}f(x+2y)\,dy.$$

For the right-hand side we have the following cases:

Case.	Upper Limit.	Lower Limit.	Result.	
$\alpha>x>\beta$	$+$	$-$	$\pi f(x)$	$\alpha-\beta<2\pi.$
$\beta+2\pi>x>\alpha>\beta$	$-$	$-$	0	
$\alpha>\beta>x>\alpha-2\pi$	$+$	$+$	0	
$x=\beta$	$+$	0	$\frac{\pi}{2}f(\beta)$	
$x=\alpha$	0	$-$	$\frac{\pi}{2}f(\alpha)$	

Dividing by π, we therefore have, if $\alpha-\beta<2\pi$,

$$\left.\begin{aligned}\frac{1}{2\pi}\int_\beta^\alpha f(\xi)d\xi+\frac{1}{\pi}\sum_1^\infty\int_\beta^\alpha f(\xi)\cos p(\xi-x)d\xi&=f(x)\text{ if }\alpha>x>\beta\\ &=\tfrac{1}{2}f(\alpha)\text{ if }x=\alpha\\ &=\tfrac{1}{2}f(\beta)\text{ if }x=\beta\\ &=0\quad\text{ if }\alpha>\beta>x>\alpha-2\pi\\ &\qquad\text{or }\ 2\pi+\beta>x>\alpha>\beta.\end{aligned}\right\}$$

Again, if $\alpha-\beta=2\pi$, we have as before for the limit, $\pi f(x)$, if $\alpha>x>\beta$. But if $x=\beta$ the limit becomes

$$Lt_{\omega\to\infty}\int_0^\pi\frac{\sin\omega y}{\sin y}f(x+2y)\,dy=\frac{\pi}{2}\left[f(x+2\,.\,0)+f(x+2\pi)\right]$$

$$=\frac{\pi}{2}\left[f(\beta)+f(\alpha)\right];$$

and if $x=a$, the limit becomes

$$Lt_{\omega\to\infty}\int_{-\pi}^{0}\frac{\sin\omega y}{\sin y}f(x+2y)dy=Lt_{\omega\to\infty}\int_{0}^{\pi}\frac{\sin\omega z}{\sin z}f(x-2z)dz$$
$$=\frac{\pi}{2}[f(x-2\,.\,0)+f(x-2\pi)]=\frac{\pi}{2}[f(\alpha)+f(\beta)];$$

and dividing by π, we therefore have, if $\alpha-\beta=2\pi$,

$$\frac{1}{2\pi}\int_{\beta}^{\alpha}f(\xi)d\xi+\frac{1}{\pi}\sum_{1}^{\infty}\int_{\beta}^{\alpha}f(\xi)\cos p(\xi-x)d\xi=f(x) \text{ if } \alpha>x>\beta$$
$$=\tfrac{1}{2}[f(\alpha)+f(\beta)] \text{ if } x=\alpha \text{ or } \beta.$$

And these results are the same as those obtained otherwise in Art. 1601. It will be noted that this method of procedure is free from the objection of assuming that what is true within an immeasurably small distance of the limit is true in the limit. (See Art. 1601.)

For values of x which lie beyond $\beta+2\pi$ in the one direction or $\alpha-2\pi$ in the other, we may proceed exactly as before in Articles 1601, 1602, etc.

1627. **Cauchy's Identity.**

Taking the identity used in Art. 1626, and putting

$$\theta=2\xi \quad\text{and}\quad f(\xi)=e^{-\alpha^2\xi^2},$$

we have

$$\int_0^\infty(1+2\cos 2\xi+2\cos 4\xi+\ldots+2\cos 2n\xi)e^{-\alpha^2\xi^2}d\xi=\int_0^\infty\frac{\sin(2n+1)\xi}{\sin\xi}e^{-\alpha^2\xi^2}d\xi.$$

But $\int_0^\infty e^{-\alpha^2\xi^2}\cos 2r\xi\,d\xi=\frac{\sqrt{\pi}}{2\alpha}e^{-\frac{r^2}{\alpha^2}}$, and by Art. 1625 the limit of the right-hand side, when n is indefinitely increased, $=\frac{\pi}{2}\left(1+2\sum_1^\infty e^{-r^2\pi^2\alpha^2}\right)$.

Hence $$\frac{\sqrt{\pi}}{2\alpha}\left(1+2\sum_1^\infty e^{-\frac{r^2}{\alpha^2}}\right)=\frac{\pi}{2}\left(1+2\sum_1^\infty e^{-r^2\pi^2\alpha^2}\right);$$

and writing $\alpha=a/\pi=1/b$,

$$\sqrt{a}\left(1+2\sum_1^\infty e^{-r^2a^2}\right)=\sqrt{b}\left(1+2\sum_1^\infty e^{-r^2b^2}\right),$$

a curious and remarkable result due to Cauchy.

Series of the character here involved occur in the theory of Theta Functions, where $\Theta(u)$ may be defined by the equation

$$\Theta(u)=1-2q\cos 2x+2q^4\cos 4x-2q^9\cos 6x+\ldots,$$

where $q=e^{-\pi\frac{K'}{K}}$ and $x=\frac{\pi u}{2K}$, K and K' having their usual significations as used in Elliptic Integrals.

1628. To prove $Lt_{\omega\to\infty}\int_0^h \frac{\sin\omega x}{x}\phi(x)dx=\frac{\pi}{2}\phi(0)$.

This limiting form follows at once by writing

$$\phi(x)=\frac{x}{\sin x}\psi(x).$$

For we then have, if $0<h \ngtr \frac{\pi}{2}$,

$$Lt_{\omega\to\infty}\int_0^h \frac{\sin\omega x}{x}\phi(x)dx=Lt_{\omega\to\infty}\int_0^h \frac{\sin\omega x}{\sin x}\psi(x)dx$$

$$=\frac{\pi}{2}\psi(0)=\frac{\pi}{2}\phi(0),$$

under the same conditions as regards $\psi(x)$ as stated in Arts. 1616 to 1622.

And further, when h has a larger range, beyond $\frac{\pi}{2}$, as in Art. 1624, we have as the limit,

$$\frac{\pi}{2}\{\psi(0)+2\psi(\pi)+2\psi(2\pi)+2\psi(3\pi)+\dots\}.$$

But $\quad \psi(\pi)=\frac{\sin\pi}{\pi}\phi(\pi)=0,\quad \psi(2\pi)=\frac{\sin 2\pi}{2\pi}\phi(2\pi)=0$, etc.,

so that whatever the range of integration provided h be positive and not an infinitesimal, we have

$$Lt_{\omega\to\infty}\int_0^h \frac{\sin\omega x}{x}\phi(x)dx=\frac{\pi}{2}\phi(0).$$

In the same way the result still holds good if $\phi(x)$ presents a finite number of finite discontinuities, none of which are infinitesimally near $x=0$.

1629. Graphical Illustration.

Since $Lt_{\omega\to\infty}\int_0^x \frac{\sin\omega\xi}{\xi}\phi(\xi)d\xi=\frac{\pi}{2}\phi(0)$, putting $\xi=-\eta$,

$$Lt_{\omega\to\infty}\int_0^{-x} \frac{\sin\omega\eta}{\eta}\phi(-\eta)d\eta=-\frac{\pi}{2}\phi(0);$$

and writing $\phi(-\eta)=\psi(\eta)$,

$$Lt_{\omega\to\infty}\int_0^{-x} \frac{\sin\omega\eta}{\eta}\psi(\eta)d\eta=-\frac{\pi}{2}\psi(0);$$

and the letter denoting the function ψ being immaterial, we may replace it again by ϕ, so that

$$Lt_{\omega\to\infty}\int_0^{-x} \frac{\sin\omega\xi}{\xi}\phi(\xi)d\xi=-\frac{\pi}{2}\phi(0).$$

Also if $x=0$ the limit vanishes and there is a discontinuity. Hence the graph of

$$y=Lt_{\omega\to\infty}\int_0^x \frac{\sin\omega\xi}{\xi}\phi(\xi)\,d\xi$$

is that shown in Fig. 473 consisting of two straight lines parallel to the x-axis, with an isolated point at the origin.

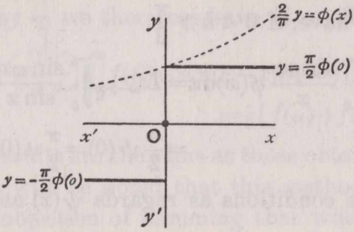

Fig. 473.

1630. Let α, β be any two positive quantities.

Then $\quad Lt_{\omega\to\infty}\int_0^\alpha \frac{\sin\omega\xi}{\xi}\phi(\xi)\,d\xi=\frac{\pi}{2}\phi(0)=Lt_{\omega\to\infty}\int_0^\beta\frac{\sin\omega\xi}{\xi}\phi(\xi)\,d\xi.$

Therefore $\quad Lt_{\omega\to\infty}\int_\beta^\alpha \frac{\sin\omega\xi}{\xi}\phi(\xi)\,d\xi=0, \quad (\alpha>\beta>0).$

Similarly $\quad Lt_{\omega\to\infty}\int_{-\beta}^{-\alpha} \frac{\sin\omega\xi}{\xi}\phi(\xi)\,d\xi=0.$

Again $Lt_{\omega\to\infty}\int_\beta^{-\alpha} \frac{\sin\omega\xi}{\xi}\phi(\xi)\,d\xi$

$$=Lt_{\omega\to\infty}\left(\int_0^{-\alpha}-\int_0^\beta\right)\frac{\sin\omega\xi}{\xi}\phi(\xi)\,d\xi=-\frac{\pi}{2}\phi(0)-\frac{\pi}{2}\phi(0)=-\pi\phi(0),$$

and $Lt_{\omega\to\infty}\int_{-\beta}^{\alpha} \frac{\sin\omega\xi}{\xi}\phi(\xi)\,d\xi$

$$=Lt_{\omega\to\infty}\left(\int_0^{\alpha}-\int_0^{-\beta}\right)\frac{\sin\omega\xi}{\xi}\phi(\xi)\,d\xi=\frac{\pi}{2}\phi(0)+\frac{\pi}{2}\phi(0)=\pi\phi(0).$$

Hence when the limits are of the same sign, the result $=0$. When of opposite sign, the result is $\pm\pi\phi(0)$, the sign being that of the upper limit. (Compare Art. 1625.)

Again $\quad \int_0^\omega \cos\xi u\,du=\left[\frac{\sin\xi u}{\xi}\right]_0^\omega=\frac{\sin\omega\xi}{\xi};$

$$\therefore\ Lt_{\omega\to\infty}\int_0^h\phi(\xi)\left\{\int_0^\omega\cos(\xi u)\,du\right\}d\xi=Lt_{\omega\to\infty}\int_0^h\frac{\sin\omega\xi}{\xi}\phi(\xi)\,d\xi,$$

i.e. $\int_0^h\int_0^\infty \phi(\xi)\cos\xi u\,d\xi\,du=\pm\frac{\pi}{2}\phi(0)$, the sign being that of h.

Further, $\int_\beta^\alpha \int_0^\infty \phi(\xi) \cos \xi u \, d\xi \, du$

$$= Lt_{\omega \to \infty} \int_\beta^\alpha \phi(\xi) \left\{ \int_0^\omega \cos(\xi u) \, du \right\} d\xi = Lt_{\omega \to \infty} \int_\beta^\alpha \frac{\sin \omega\xi}{\xi} \phi(\xi) \, d\xi$$

$= 0$, if α, β are of the same sign,

or $= \pm \pi \phi(0)$, according as α is positive or negative when β is of the opposite sign.

1631. Graphical Illustration.

Taking $\alpha > \beta > 0$ and $\xi - x = \eta$,

$$Lt_{\omega \to \infty} \int_\beta^\alpha \frac{\sin \omega(\xi - x)}{\xi - x} \phi(\xi) \, d\xi = Lt_{\omega \to \infty} \int_{\beta - x}^{\alpha - x} \frac{\sin \omega\eta}{\eta} \phi(x + \eta) \, d\eta$$

$$\left.\begin{matrix} = 0 \\ \text{if } x > \alpha > \beta \end{matrix}\right\} \text{ or } \left.\begin{matrix} = \frac{\pi}{2}\phi(\alpha) \\ \text{if } x = \alpha > \beta \end{matrix}\right\} \text{ or } \left.\begin{matrix} = \pi\phi(x) \\ \text{if } \alpha > x > \beta \end{matrix}\right\} \text{ or } \left.\begin{matrix} = \frac{\pi}{2}\phi(\beta) \\ \text{if } \alpha > x = \beta \end{matrix}\right\} \text{ or } \left.\begin{matrix} = 0 \\ \text{if } \alpha > \beta > x \end{matrix}\right\}$$

The values of this integral may be shown graphically by the heavy lines and the two isolated points in Fig. 474, in which the dotted line is the graph of $y = \pi\phi(x)$.

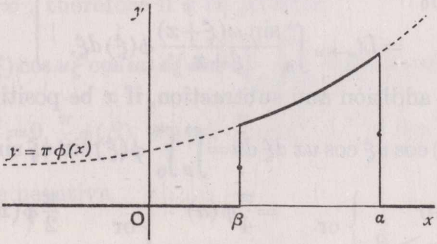

Fig. 474.

Obvious modifications will occur if α or β or both of them be negative or if $\alpha < \beta$.

1632. Still supposing that α and β are both positive and $\alpha > \beta$, and putting $\xi + x = \eta$, we have

$$Lt_{\omega \to \infty} \int_\beta^\alpha \frac{\sin \omega(\xi + x)}{\xi + x} \phi(\xi) \, d\xi = Lt_{\omega \to \infty} \int_{\beta + x}^{\alpha + x} \frac{\sin \omega\eta}{\eta} \phi(\eta - x) \, d\eta$$

$$\left.\begin{matrix} = 0 \\ \text{if } x > -\beta > -\alpha \end{matrix}\right\} \text{ or } \left.\begin{matrix} = \frac{\pi}{2}\phi(-x) = \frac{\pi}{2}\phi(\beta) \\ \text{if } x = -\beta > -\alpha \end{matrix}\right\} \text{ or } \left.\begin{matrix} = \pi\phi(-x) \\ \text{if } -\beta > x > -\alpha \end{matrix}\right\}$$

$$\text{or } \left.\begin{matrix} = \frac{\pi}{2}\phi(-x) = \frac{\pi}{2}\phi(\alpha) \\ \text{if } -\beta > x = -\alpha \end{matrix}\right\} \text{ or } \left.\begin{matrix} = 0 \\ \text{if } -\beta > -\alpha > x \end{matrix}\right\}.$$

And the graph of this integral is shown by the heavy lines and the two isolated points in Fig. 475, and is an image with regard to the y-axis of the graph of Fig. 474.

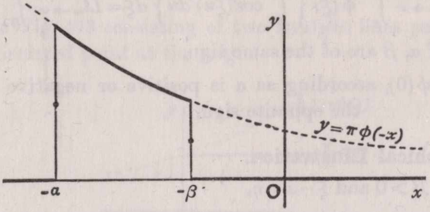

Fig. 475.

1633. Various Deductions.

Since
$$\left.\begin{aligned} &\int_\beta^\alpha \int_0^\infty \cos u(\xi-x)\,\phi(\xi)\,d\xi\,du\\ &\qquad = Lt_{\omega\to\infty}\int_\beta^\alpha \frac{\sin\omega(\xi-x)}{\xi-x}\phi(\xi)\,d\xi\\ \text{and}\quad &\int_\beta^\alpha \int_0^\infty \cos u(\xi+x)\,\phi(\xi)\,d\xi\,du\\ &\qquad = Lt_{\omega\to\infty}\int_\beta^\alpha \frac{\sin\omega(\xi+x)}{\xi+x}\phi(\xi)\,d\xi, \end{aligned}\right\}\text{whose values have been found above,}$$

we have by addition and subtraction, if x be positive,

$$\int_\beta^\alpha \int_0^\infty \phi(\xi)\cos u\xi\cos ux\,d\xi\,du = \int_\beta^\alpha \int_0^\infty \phi(\xi)\sin u\xi\sin ux\,d\xi\,du$$
$$\left.\begin{matrix} =0\\ \text{if } x>\alpha>\beta\end{matrix}\right\}\text{or}\left.\begin{matrix} =\frac{\pi}{4}\phi(\alpha)\\ \text{if } x=\alpha>\beta\end{matrix}\right\}\text{or}\left.\begin{matrix} \frac{\pi}{2}\phi(x)\\ \text{if } \alpha>x>\beta\end{matrix}\right\}$$
$$\text{or}\left.\begin{matrix} \frac{\pi}{4}\phi(\beta)\\ \text{if } \alpha>x=\beta\end{matrix}\right\}\text{or}\left.\begin{matrix} 0\\ \text{if } \alpha>\beta>x\end{matrix}\right\};$$

and if x be negative,

$$\int_\beta^\alpha \int_0^\infty \phi(\xi)\cos u\xi\cos ux\,d\xi\,du = -\int_\beta^\alpha \int_0^\infty \phi(\xi)\sin u\xi\sin ux\,d\xi\,du$$
$$\left.\begin{matrix} =0\\ \text{if } x>-\beta>-\alpha\end{matrix}\right\}\text{or}\left.\begin{matrix} =\frac{\pi}{4}\phi(\beta)\\ \text{if } x=-\beta>-\alpha\end{matrix}\right\}\text{or}\left.\begin{matrix} =\frac{\pi}{2}\phi(-x)\\ \text{if } -\beta>x>-\alpha\end{matrix}\right\}$$
$$\text{or}\left.\begin{matrix} =\frac{\pi}{4}\phi(\alpha)\\ \text{if } -\beta>x=-\alpha\end{matrix}\right\}\text{or}\left.\begin{matrix} =0\\ \text{if } -\beta>-\alpha>x\end{matrix}\right\}.$$

1634. If $\beta=0$ and $a=\infty$ and x be >0,

$$\int_0^\infty\int_0^\infty \phi(\xi)\cos u\xi\cos ux\,d\xi\,du=\int_0^\infty\int_0^\infty \phi(\xi)\sin u\xi\sin ux\,d\xi\,du$$

$$=\frac{\pi}{2}\phi(x);\quad \text{and if } x \text{ be } <0,$$

$$\int_0^\infty\int_0^\infty \phi(\xi)\cos u\xi\cos ux\,d\xi\,du=-\int_0^\infty\int_0^\infty \phi(\xi)\sin u\xi\sin ux\,d\xi\,du$$

$$=\frac{\pi}{2}\phi(-x).$$

These results are all obvious on compounding the two graphs, Figs. 474 and 475.

When $x=0$ the second integral in each case vanishes.

1635. Since the products $\cos u\xi\cos ux$ and $\sin u\xi\sin ux$ are both even functions of u, they are not affected by a change of sign of u. Hence the integration of either of them with respect to u from $-\infty$ to ∞ yields double the result of that from 0 to ∞; therefore if x be positive,

$$\int_\beta^a\int_{-\infty}^\infty \phi(\xi)\cos u\xi\cos ux\,d\xi\,du=\int_\beta^a\int_{-\infty}^\infty \phi(\xi)\sin u\xi\sin ux\,d\xi\,du$$

$$=0,\ \frac{\pi}{2}\phi(\beta),\ \pi\phi(x),\ \frac{\pi}{2}\phi(a) \text{ or } 0 \text{ in the several cases,}$$

and if x be negative,

$$\int_\beta^a\int_{-\infty}^\infty \phi(\xi)\cos u\xi\cos ux\,d\xi\,du=-\int_\beta^a\int_{-\infty}^\infty \phi(\xi)\sin u\xi\sin ux\,d\xi\,du$$

$$=0,\ \frac{\pi}{2}\phi(\beta),\ \pi\phi(-x),\ \frac{\pi}{2}\phi(a) \text{ or } 0 \text{ in the corresponding cases.}$$

1636. If $\beta=0$ and $a=\infty$, we have

$$\int_0^\infty\int_{-\infty}^\infty \phi(\xi)\cos u\xi\cos ux\,d\xi\,du=\int_0^\infty\int_{-\infty}^\infty \phi(\xi)\sin u\xi\sin ux\,d\xi\,du$$

$$=\pi\phi(x),\quad (x\ +^{\text{ve}}),\quad \ldots\ldots\ldots\ldots(1)$$

$$\int_0^\infty\int_{-\infty}^\infty \phi(\xi)\cos u\xi\cos ux\,d\xi\,du=-\int_0^\infty\int_{-\infty}^\infty \phi(\xi)\sin u\xi\sin ux\,d\xi\,du$$

$$=\pi\phi(-x),\quad (x\ -^{\text{ve}}).\quad \ldots\ldots\ldots(2)$$

1637. Fourier's Formula.

Put $\xi=-\eta$, and write ψ for ϕ. Then, as x is $+^{ve}$ or $-^{ve}$,

$$\int_{-\infty}^{0}\int_{-\infty}^{\infty}\psi(-\eta)\cos u\eta\cos ux\,d\eta\,du=\mp\int_{-\infty}^{0}\int_{-\infty}^{\infty}\psi(-\eta)\sin u\eta\sin ux\,d\eta\,du$$
$$=\pi\psi(x)\text{ or }\pi\psi(-x),\quad\text{as } x \text{ is } +^{ve}\text{ or } -^{ve}.$$

Let $\psi(-\eta)=\phi(\eta)$, and write ξ for η. Then, as x is $+^{ve}$ or $-^{ve}$,

$$\int_{-\infty}^{0}\int_{-\infty}^{\infty}\phi(\xi)\cos u\xi\cos ux\,d\xi\,du=\mp\int_{-\infty}^{0}\int_{-\infty}^{\infty}\phi(\xi)\sin u\xi\sin ux\,d\xi\,du$$
$$=\pi\phi(-x)\text{ or }\pi\phi(x),\quad\text{as } x \text{ is } +^{ve}\text{ or } -^{ve}.\quad\ldots(3)$$

Hence from equations 1, 2 and 3, whether x be $+^{ve}$ or $-^{ve}$,

$$\left.\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(\xi)\cos u\xi\cos ux\,d\xi\,du=\pi\{\phi(x)+\phi(-x)\}\\ \text{and }&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(\xi)\sin u\xi\sin ux\,d\xi\,du=\pi\{\phi(x)-\phi(-x)\}\end{aligned}\right\}$$

By addition,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(\xi)\cos u(\xi-x)\,d\xi\,du=2\pi\phi(x),$$

which is Fourier's Formula.

1638. For $+^{ve}$ values of x it follows that the graph of

$$y=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(\xi)\cos u(\xi-x)\,d\xi\,du$$

only differs from that of $y=\phi(x)$, in that all the ordinates of the latter are increased in the ratio $2\pi:1$.

Similarly for $-^{ve}$ values of x.

1639. **A Remarkable Application** (Bertrand, *Calc. Int.*, p. 238).

If in the formula $\int_0^{\infty}\int_0^{\infty}\phi(\xi)\cos u\xi\cos ux\,d\xi\,du=\frac{\pi}{2}\phi(x)$ or $\frac{\pi}{2}\phi(-x)$, as x is $+^{ve}$ or $-^{ve}$, we put $\phi(\xi)=e^{-a\xi}$, where a is $+^{ve}$; and since

$$\int_0^{\infty}e^{-a\xi}\cos(u\xi)\,d\xi=\frac{a}{a^2+u^2},$$

we have $\int_0^{\infty}\frac{\cos ux}{a^2+u^2}du=\frac{\pi}{2a}e^{-ax}$ or $\frac{\pi}{2a}e^{ax}$, according as x is $+^{ve}$ or $-^{ve}$ (Art. 1048).

PROBLEMS.

1. Find in a series a function of period $4a$ which shall be equal to $a+x$ from $x=-2a$ to $x=0$, and equal to $a-x$ from $x=0$ to $x=2a$.

[TRIN. COLL., 1881.]

2. Expand x^2 in a series of cosines of multiples of x between π and $-\pi$. What will the series so obtained represent for other values of x?

3. Find a series of sines which shall be equal to kx from $x=0$ to $x=l/2$, and equal to $k(l-x)$ from $x=l/2$ to $x=l$.

Find also a series of cosines to answer the same description.

[OX. II. P., 1900.]

4. Expand $x(\pi-x)$ in a series of sines. [OX. II. P., 1900.]

5. Find a series of sines which shall represent nkx/l from $x=0$ to $x=l/n$; k from $x=l/n$ to $x=(n-1)l/n$; and $nk(l-x)/l$ from $x=(n-1)l/n$ to $x=l$. [COLLEGES, 1878.]

6. Trace the locus of the equation

$$\frac{y}{c}=\Sigma\frac{(-1)^n}{n^2}\sin\frac{n\pi a}{2c}\sin\frac{n\pi x}{2c}.$$

[ST. JOHN'S, 1884.]

7. A function of x is equal to x^2 for values of x between $x=0$ and $x=l/2$, and vanishes when x is between $l/2$ and l; express the function by a series of sines, and also by a series of cosines of multiples of $\pi x/l$. Draw figures showing the functions represented by the two series respectively for all values of x not restricted to lie between 0 and l. What are the sums of the series for the value $x=l/2$? [γ, 1899.]

8. Show that

$$\log\operatorname{cosec} x=\log 2+\cos 2x+\tfrac{1}{2}\cos 4x+\tfrac{1}{3}\cos 6x+\dots+\frac{1}{n}\cos 2nx+\dots,$$

$$(\theta<x<\pi),$$

and deduce therefrom

$$(a)\ \int_0^{\frac{\pi}{2}}\log\sin x\,dx=\frac{\pi}{2}\log\tfrac{1}{2};\quad (b)\ \int_0^{\frac{\pi}{2}}\cos 2nx\log\sin x\,dx=-\frac{\pi}{4n}.$$

9. Prove that

$$y^2=\frac{2c^2}{3d}+\sum_1^\infty\frac{4d}{n^3\pi^3}\left\{d\sin\frac{n\pi c}{d}-n\pi c\cos\frac{n\pi c}{d}\right\}\cos\frac{n\pi x}{d}$$

represents a series of circles of radius c with their centres on the x-axis at distances $2d$ apart, and also the portions of the axis exterior to the circles, one circle having its centre at the origin. [γ, 1893.]

10. Find a series of cosines of multiples of $\pi x/l$ which shall represent a function which is equal to $x^2/4a$ for values of x between 0 and $l/2$, and is equal to $(l-x)^2/4a$ when x is between $l/2$ and l.

What does the series represent for values of x not lying between 0 and l? [COLLEGES, 1892.]

11. Find a Fourier series to be equal to x^3 between $x=\pm c$, and trace the locus
$$\frac{y}{c}=\frac{2}{\pi}\sum_1^\infty \frac{(-1)^{r-1}}{r}\left(1-\frac{6}{\pi^2 r^2}\right)\sin\frac{\pi r x}{c}.$$

12. Show by evaluation of the integral that
$$\frac{2}{\pi}\int_0^\infty \sin qx\left\{\frac{h}{q}+\tan\alpha\,\frac{\sin qb-\sin qa}{q^2}\right\}dq$$
is the ordinate of a broken line running parallel to the axis of x from $x=0$ to $x=a$ and from $x=b$ to $x=\infty$, and inclined to the axis of x at an angle α between $x=a$ and $x=b$. [MATH. TRIP., 1883.]

13. If $f(x)=\Sigma A_n \sin n\pi x/l$ and $f'(x)=B_0+\Sigma B_n \cos n\pi x/l$ for all values of x between 0 and l, prove that, provided $f(x)$ be continuous from $x=0$ to $x=l$,
$$B_n=\frac{n\pi}{l}A_n+\frac{2}{l}\{(-1)^n f(l)-f(0)\}.$$
Write down the corresponding formula if $f(x)$ be discontinuous for the value $x=a$ which lies between 0 and l. [COLLEGES, 1896.]

14. Prove that the locus represented by
$$\sum_{n=1}^{n=\infty}\frac{(-1)^{n-1}}{n^2}\sin nx \sin ny=0$$
is two systems of lines at right angles dividing the coordinate plane into squares of area π^2. [MATH. TRIP., 1895.]

15. Show that the equation
$$y=\frac{a}{2}+x-\frac{4a}{\pi^2}\left\{\cos\frac{\pi}{a}(x+y)+\frac{1}{3^2}\cos\frac{3\pi}{a}(x+y)+\frac{1}{5^2}\cos\frac{5\pi}{a}(x+y)+\text{etc.}\right\}$$
represents a staircase formed of straight lines of length a, starting from the origin and parallel, alternately, to the axes of y and x. [ST. JOHN'S COLL., 1881.]

16. If $f(\theta)$ be a finite function of θ with the period 2π, show how to find a function which, in the space between two concentric circles, is a finite and continuous solution of the equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$, with

the value $f(\theta)$ at the point of the outer circle whose polar coordinate is θ, and the value *zero* at every point of the inner circle.

[MATH. TRIP., 1896.]

[After transformation to polars,

$$u = A_0 + \sum_1^\infty (A_n r^n + B_n r^{-n}) \cos n\theta + \sum_1^\infty (C_n r^n + D_n r^{-n}) \sin n\theta$$

may be taken as the solution of this equation.]

17. If y be defined as coincident with $y = x$ from $x = 0$ to $x = \pi/2$; $y = \pi/2$ from $x = \pi/2$ to $x = 3\pi/2$; $y = 2\pi - x$ from $x = 3\pi/2$ to $x = 2\pi$, and be represented by a Fourier series of form $y = A_0 + \sum_1^\infty A_p \cos px$, show that

$$y = \frac{3\pi}{8} - \frac{2}{\pi} \sum_1^\infty \frac{\cos(2p-1)x}{(2p-1)^2} - \frac{1}{\pi} \sum_1^\infty \frac{\cos(4\pi - 2)x}{(2p-1)^2},$$

and draw a graph of this series when x is not restricted to lie between 0 and 2π.

18. Prove that the series

$$\frac{1}{l}\int_0^l \frac{f(v)+f(-v)}{2} dv + \frac{2}{l}\sum_{n=1}^{n=\infty} \cos\frac{n\pi x}{l} \int_0^l \frac{f(v)+f(-v)}{2} \cos\frac{n\pi v}{l} dv$$

$$+ \frac{2}{l}\sum_{n=1}^{n=\infty} \sin\frac{n\pi x}{l} \int_0^l \frac{f(v)-f(-v)}{2} \sin\frac{n\pi v}{l} dv$$

is equal to $f(x)$ between the limits $x = +l$ and $x = -l$; and trace the curve represented by the series for values of x outside these limits.

[MATH. TRIP., 1885.]

19. Find by Fourier's method a function of x which shall be equal to $+1$ from $x = 0$ to $x = a$, and equal to -1 from $x = a$ to $x = 2a$, and so on alternately.

20. Two uniform plates of the same substance and thickness a are in contact. The outside surface of one is impervious to heat, and that of the other is kept at zero temperature. It can be shown that if one slips over the surface of the other with constant velocity v, the friction per unit of area being F, then at any time t the temperatures of the two plates are given by

$$\theta_1 = \frac{Fv}{JC}\left\{a + \Sigma A_{2n+1} e^{-\frac{(2n+1)^2\pi^2 C^2 t}{16a^2c^2}} \cos(2n+1)\frac{\pi x}{4a}\right\},$$

$$\theta_2 = \frac{Fv}{JC}\left\{2a - x + \Sigma A_{2n+1} e^{-\frac{(2n+1)^2\pi^2 C^2 t^2}{16a^2c^2}} \cos(2n+1)\frac{\pi x}{4a}\right\},$$

respectively, at a distance x from the impervious surface, where J, C, c are certain constants. Show that, if when $t=0$, θ is zero everywhere, the coefficients A_{2n+1} are given by

$$A_{2n+1} = -\left\{\frac{4}{(2n+1)\pi}\right\}^2 a\cos(2n+1)\frac{\pi}{4}.$$

[MATH. TRIP. III., 1884.]

21. Deduce from the result $\int_0^\infty e^{-x^2}\cos 2bx\,dx = \frac{1}{2}\pi^{\frac{1}{2}}e^{-b^2}$, or otherwise obtain the result

$$e^{-x^2} + e^{-(x-a)^2} + e^{-(x+a)^2} + e^{-(x-2a)^2} + e^{-(x+2a)^2} + \text{etc.}$$

$$= \frac{\pi^{\frac{1}{2}}}{a}\left(1 + 2e^{-\frac{\pi^2}{a^2}}\cos\frac{2\pi x}{a} + 2e^{-\frac{4\pi^2}{a^2}}\cos\frac{4\pi x}{a} + 2e^{-\frac{9\pi^2}{a^2}}\cos\frac{6\pi x}{a} + \dots\right).$$

[MATH. TRIP., 1887.]

22. Prove that the equation

$$\frac{\pi^2}{24} = -\cos\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y) + \frac{1}{2^2}\cos\frac{2}{2}(x+y)\cos\frac{2}{2}(x-y)$$

$$-\frac{1}{3^2}\cos\frac{3}{2}(x+y)\cos\frac{3}{2}(x-y) + \dots$$

represents a series of circles of radius π, and trace them.

[MATH. TRIP., 1885.]

23. Show that if all effects of atmosphere be neglected, then the intensity of daylight at a given place at t o'clock true solar time at an equinox will be

$$I\left[\frac{1}{\pi} + \frac{1}{2}\cos\frac{\pi t}{12} + \frac{2}{\pi}\left\{\frac{1}{1.3}\cos\frac{\pi t}{6} - \frac{1}{3.5}\cos\frac{2\pi t}{6} + \frac{1}{5.7}\cos\frac{3\pi t}{6} - \dots\right\}\right],$$

where I is the intensity at noon. Examine the values of the above expression when (i) $t=0$, (ii) $t=6$, (iii) $t=12$. [MATH. TRIP., 1884.]

24. Prove that if

$$\sqrt{\pi}\,f(p) = \sqrt{2}\int_0^\infty \phi(x)\sin px\,dx,$$

then will $\sqrt{\pi}\,\phi(p) = \sqrt{2}\int_0^\infty f(x)\sin px\,dx.$ [MATH. TRIP., 1884.]

25. Show that, if $Ei(x) \equiv \int_{-\infty}^x \frac{e^x}{x}\,dx$, then

$$\frac{1}{q}\int_0^\infty \{e^{qx}Ei(-qx) - e^{-qx}Ei(qx)\}\sin px\,dx$$

$$= \frac{1}{p}\int_0^\infty \{e^{qx}Ei(-qx) + e^{-qx}Ei(qx)\}\cos px\,dx = -\frac{\pi}{p^2+q^2}.$$

[MATH. TRIP., 1884.]

26. Find two harmonic series, each of which shall be equal to bx/a from $x=0$ to $x=a$, one containing only harmonic functions of the form $\sin 2i\pi x/a$ and the other those of the form $\cos i\pi x/a$, where i is any integer. Trace the complete curve given by the harmonic series in each case. [MATH. TRIP., 1876.]

27. Sum the series $m\cos\theta - \frac{1}{3}m^3\cos 3\theta + \frac{1}{5}m^5\cos 5\theta - \ldots$ *ad inf.*, m being <1, and prove that it always has the same sign as $m\cos\theta$. Trace the curve

$$r = a(\cos\alpha\cos\theta - \tfrac{1}{3}\cos 3\alpha\cos 3\theta + \tfrac{1}{5}\cos 5\alpha\cos 5\theta - \ldots).$$

[MATH. TRIP., 1878.]

28. Express the doubly infinite series

$$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}(-1)^{m+n}\frac{\cos mx\cos ny}{mn(m^2+n^2)}$$

in the form of a singly infinite series of cosines of multiples of y. [S.H. PROBLEMS, 1878.]

Exhibit the result in the form

$$\sum_{n=1}^{n=\infty}\left[\left\{\phi(n)+\frac{1}{n^2}\log 2\right\}\cosh nx - \frac{1}{n^2}\log 2 + \frac{1}{n}\int_0^x \sinh n(x-u)\log\cos\frac{u}{2}\,du\right]\frac{(-1)^n\cos ny}{n}.$$

29. Deduce Fourier's formula

$$2\phi(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(\xi)\cos u(\xi - x)\,d\xi\,du$$

from the formula

$$2\phi(x) = \frac{1}{l}\int_{-l}^{l}\phi(\xi)\,d\xi + \frac{2}{l}\sum_{p=1}^{p=\infty}\int_{-l}^{l}\phi(\xi)\cos\frac{p\pi}{l}(\xi - x)\,d\xi.$$

[POISSON. See TODHUNTER, *I.C.*, Art. 332.]

30. Examine the limiting form of the curve

$$y = \frac{1}{\pi}\int_0^{\infty} e^{-kw}\,dw\left\{\int_0^1 \cos w(v-x)\,.\,v\,dv\right\}$$

when k, being positive, tends to a zero limit. [DE MORGAN, *D.C.*, p. 629.]

31. Prove the two formulae

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\cos xu\,du\int_0^{\infty} f(t)\cos ut\,dt\ ;$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\sin xu\,du\int_0^{\infty} f(t)\sin ut\,dt,$$

and point out the distinction between the two expressions for $f(x)$. [ST. JOHN'S COLL., 1881.]

32. Show that for all values of x between $-b$ and b,

$$F(x)-F(-x)=\frac{2}{\pi}\int_0^\infty \sin xu\,du\int_{-b}^{b} F(y)\sin uy\,dy.$$

[ST. JOHN'S COLL., 1881.]

33. If a uniform horizontal bar, both of whose ends are fixed, be so displaced horizontally in the direction of its length that initially one half is uniformly extended and the other uniformly compressed, and then let go, prove that the displacement y of any particle x at any time t will be

$$\frac{8nl}{\pi^2}\sum\frac{1}{(2i+1)^2}\cos(2i+1)\frac{\pi at}{2l}\cos(2i+1)\frac{\pi x}{2l},$$

$2l$ being the length of the bar, the middle point being the origin and nl the displacement of the middle point.

[The equation determining these vibrations may be assumed to be $\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}$, and a suitable form of solution of this equation is $y=\Sigma C_m\cos mx\cos mat$.

Or more generally, for an equation of type $\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}+k$, y is of the form

$$A+Bx+Ct+Dx^2+Ext+Ft^2+\Sigma L\sin\{n(at-x)+\alpha\}$$
$$+\Sigma M\sin\{n(at+x)+\beta\}$$

with certain conditions. (See Forsyth, *D. Equations.*) We are to have $y=0$ for all values of t when $x=\pm l$; and if $t=0$, $y=n(l-x)$ from $x=0$ to $x=l$, and $y=n(l+x)$ from $x=-l$ to $x=0$.]

34. A stream of uniform depth and of uniform width $2a$ flows slowly through a bridge consisting of two equal arches resting on a rectangular pier of width $2b$, the bridge being so broad that under it the water moves uniformly with velocity U. Show that after the stream has passed through the bridge the velocity potential of the motion is

$$\phi=\frac{a-b}{a}Ux+\frac{2aU}{\pi^2}\sum_1^\infty\frac{1}{i^2}\sin\frac{i\pi b}{a}\cos\frac{i\pi y}{a}e^{-\frac{i\pi x}{a}},$$

the axis of x being in the forward direction of the stream and the origin at the middle point of the pier. [MATH. TRIP., 1878.]

[The equation for ϕ is $\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$, and we are to have

$$\frac{\partial\phi}{\partial x}=\frac{a-b}{a}U \text{ when } x \text{ is infinite,}\quad \frac{\partial\phi}{\partial x}=U \text{ when } x=0,$$

except from $y=-b$ to $y=b$, where $\frac{\partial\phi}{\partial x}=0$; also $\frac{\partial\phi}{\partial y}=0$ when $y=\pm a$, and a suitable solution of the equation is

$$\phi=A_0x+\sum_1^\infty A_i\cos\frac{i\pi y}{a}e^{-\frac{i\pi x}{a}}.\Big]$$

35. Show that $\frac{\pi}{4}z=\sum_0^\infty\frac{1}{(2p+1)^2}\sin(2p+1)x\sin(2p+1)y$ represents the four sloping faces of a regular pyramid built upon a horizontal square base of side π units, two sides coinciding with the axes of coordinates, the height of the pyramid being $\pi/2$ units.

[TODHUNTER, *I.C.*, p. 304.]

36. A membrane is uniformly stretched upon a square frame to which it is attached along the edges. The centre is displaced slightly through a small distance k perpendicularly to the frame, the form being that of four planes passing through the edges of the square and a common point above the centre. The side of the square is a. The constraint is then removed. The equation to determine the subsequent vibrations is $\frac{\partial^2 w}{\partial t^2}=c^2\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)$, and a solution suitable for such a case as the above may be assumed to be

$$w=\Sigma A_{n,r}\cos\gamma t\sin\frac{n\pi(x+a)}{2a}\sin\frac{r\pi(y+a)}{2a},$$

the origin being taken at the centre of the square and the axes parallel to its sides, t being the time measured from the instant of the removal of the constraint, and n and r being integers. Also it will be noted that $x=\pm a$ and $y=\pm a$ will each give $w=0$ for all values of t.

Prove (i) $4a^2\gamma^2=c^2\pi^2(n^2+r^2)$, (ii) that n and r are odd, (iii) $A_{n,r}=0$ if $n\neq r$, (iv) $A_{n,n}=8k/n^2\pi^2$,

and

$$w=\frac{8k}{\pi^2}\sum\frac{1}{(2i+1)^2}\sin\frac{(2i+1)\pi(x+a)}{2a}\sin\frac{(2i+1)\pi(y+a)}{2a}\cos(2i+1)\frac{c\pi t}{a\sqrt{2}}.$$

37. The fixed boundary of a membrane is a square, and the centre of the membrane is displaced perpendicularly through a small space k, the membrane being made to take the form of two portions of intersecting circular cylinders. Taking the same general form of solution as before of the equation for the vibrations when the constraints are suddenly destroyed, prove that n and r are *odd* integers, and that

$$A_{n,r}=\frac{128k}{\pi^4(n^2-r^2)^2}\left(\frac{n^2+r^2}{nr}-2\sin\frac{n\pi}{2}\sin\frac{r\pi}{2}\right);$$

$$A_{n,n}=\frac{8k}{n^2\pi^2}\left(1+\frac{4}{n^2\pi^2}\right).$$

[MATH. TRIP. III., 1886.]

38. Ohm's Equation for the flux of electric current in a wire of section ω, conductivity k, and electrostatic capacity per unit length c, is $\frac{\partial V}{\partial t} = \frac{2k\omega}{c}\frac{\partial^2 V}{\partial x^2}$, giving the potential V in terms of t the time and x the distance of a point on the wire from a given origin on the wire. Assuming as a solution of this equation $V = \frac{ax}{l} + \Sigma A e^{-pt} \sin(qx + B)$, where a is the constant potential for all values of t at the battery end of the wire and x is measured from the earth end, l being the length of the wire and A, B arbitrary constants, show that

$$V = \frac{ax}{l} + \Sigma A_n e^{-\frac{2k\omega}{c}\frac{n^2\pi^2}{l^2}t} \sin\frac{n\pi x}{l};$$

and if when $t = 0$, $V = 0$ for all values of x from 0 to l, show that

$$V = \frac{ax}{l} + \frac{2a}{\pi}\sum \frac{\cos n\pi}{n} e^{-\frac{2k\omega}{c}\frac{n^2\pi^2}{l^2}t} \sin\frac{n\pi x}{l}.$$