CHAPTER XXXV. SECTION I.

FORMULAE OF LAGRANGE AND FOURIER.

1585. When a material particle is affected simultaneously by two harmonic oscillations, $a_1 \sin(n_1 t + \alpha_1)$, $a_2 \sin(n_1 t + \alpha_2)$, of the same period $2\pi/n_1$, but their amplitudes a_1 and a_2 and their phases α_1 and α_2 being different, they compound into a single simple harmonic oscillation $A \sin(n_1 t + \alpha)$ of the same period but with amplitude and phase respectively

$$\sqrt{a_1^2 + 2a_1 a_2 \cos(\alpha_1 - \alpha_2) + a_2^2} \quad \text{and} \quad \tan^{-1} \frac{a_1 \sin \alpha_1 + a_2 \sin \alpha_2}{a_1 \cos \alpha_1 + a_2 \cos \alpha_2};$$

and any number of such simple harmonic motions may be compounded in the same way, provided they all have the same periodicity.

Graphically the resultant motion may be represented by constructing the graphs of the several vibrations on the same plan and forming a new graph by the addition of their ordinates. And this always results in an ordinary "curve of sines."

1586. But if the periodicity of the two or more fundamental vibrations be different, as in

$$a_1 \sin(n_1 t + \alpha_1), \quad a_2 \sin(n_2 t + \alpha_2),$$

the above analytical process of composition breaks down but the graphical method still holds, the resulting graph, however, no longer being the simple curve of sines.

Taking for instance as a simple case the graph of

$$\frac{\pi}{4} y = \sin x - \frac{1}{3^2} \sin 3x + \frac{1}{5^2} \sin 5x - \frac{1}{7^2} \sin 7x + \dots,$$

where the periodicities of the constituent vibrations of y are respectively $2\pi/1$, $2\pi/3$, $2\pi/5$, etc., and their amplitudes $4/\pi 1^2$, $4/\pi 3^2$, $4/\pi 5^2$, etc., we have, from the first three terms only, a figure shown for the extent $x=0$ to $x=\pi/2$ in Fig. 456. And even for three terms of the series it will be

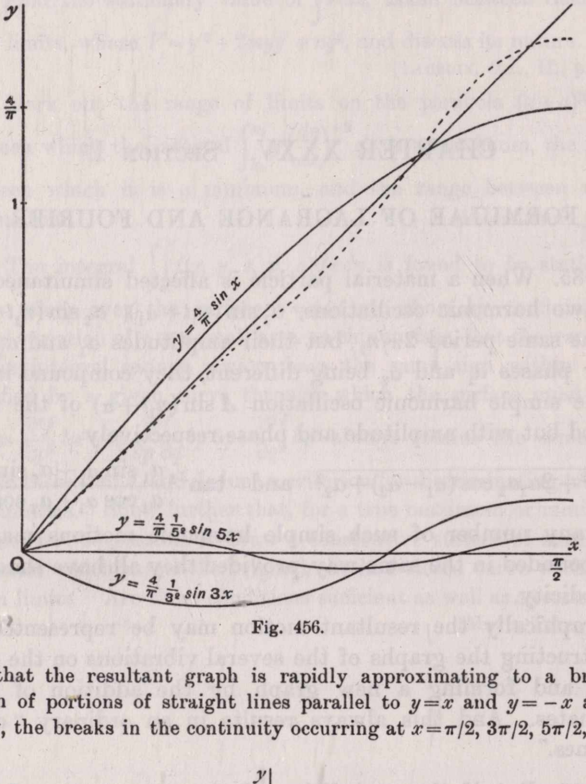


Fig. 456.

seen that the resultant graph is rapidly approximating to a broken system of portions of straight lines parallel to $y=x$ and $y=-x$ alternately, the breaks in the continuity occurring at $x=\pi/2$, $3\pi/2$, $5\pi/2$, etc.;

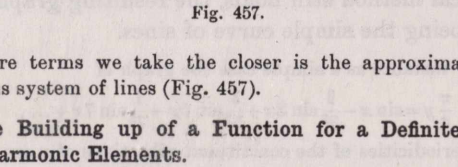


Fig. 457.

and the more terms we take the closer is the approximation to this discontinuous system of lines (Fig. 457).

1587. The Building up of a Function for a Definite Range by Means of Harmonic Elements.

Let us examine then whether it be possible to build up a function of x

viz. $f(x)$, discontinuous as regards its differential coefficients at $x=\pi/2$, $3\pi/2, 5\pi/2, \ldots$ and equal to

$$-\pi-x, \left(-\frac{3\pi}{2}<x<-\frac{\pi}{2}\right); \quad x, \left(-\frac{\pi}{2}<x<\frac{\pi}{2}\right);$$

$$\pi-x, \left(\frac{\pi}{2}<x<\frac{3\pi}{2}\right); \quad -2\pi+x, \left(\frac{3\pi}{2}<x<\frac{5\pi}{2}\right); \text{ etc.}$$

Let us assume tentatively that it is expressible as a uniformly convergent series of the form $f(x) \equiv a_0 + \sum_{p=1}^{p=\infty}(a_p \cos px + b_p \sin px)$, and let us attend to the portion $(-\pi<x<\pi)$.

Then (i) integrating from $-\pi$ to π,

$$a_0 \cdot 2\pi = \int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{-\frac{\pi}{2}}(-\pi-x)\,dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\,dx + \int_{\frac{\pi}{2}}^{\pi}(\pi-x)\,dx = 0.$$

(ii) Multiply by $\cos \pi x$, and integrate from $-\pi$ to π,

$$a_p\int_{-\pi}^{\pi}\cos^2 px\,dx = \int_{-\pi}^{-\frac{\pi}{2}}(-\pi-x)\cos px\,dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos px\,dx + \int_{\frac{\pi}{2}}^{\pi}(\pi-x)\cos px\,dx$$

$$= -\left[(\pi+x)\frac{\sin px}{p}+\frac{\cos px}{p^2}\right]_{-\pi}^{-\frac{\pi}{2}} + \left[x\frac{\sin px}{p}+\frac{\cos px}{p^2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$+\left[(\pi-x)\frac{\sin px}{p}-\frac{\cos px}{p^2}\right]_{\frac{\pi}{2}}^{\pi} = 0;$$

$$\therefore\ a_p\pi = 0 \quad\text{and}\quad a_p = 0.$$

(iii) Multiply by $\sin px$, and integrate from $-\pi$ to π,

$$b_p\int_{-\pi}^{\pi}\sin^2 px\,dx = -\left[-(\pi+x)\frac{\cos px}{p}+\frac{\sin px}{p^2}\right]_{-\pi}^{-\frac{\pi}{2}} + \left[-x\frac{\cos px}{p}+\frac{\sin px}{p^2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$+\left[-(\pi-x)\frac{\cos px}{p}-\frac{\sin px}{p^2}\right]_{\frac{\pi}{2}}^{\pi};$$

$$\therefore\ b_p\pi = \frac{4}{p^2}\sin\frac{p\pi}{2};$$

whence

$$f(x) = \sum_{p=1}^{p=\infty}\frac{4}{\pi p^2}\sin\frac{p\pi}{2}\sin px = \frac{4}{\pi}\left(\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \frac{1}{7^2}\sin 7x + \ldots\right).$$

If we write $x+2n\pi$ for x, each term of the series remains unaltered, and the result is therefore a periodic function with periodicity 2π, which is in conformity with the graph in Figs. 456 and 457.

The series is manifestly convergent for all values of x. Hence we have expressed a discontinuous function of x which takes the value $(-1)^n(x-n\pi)$ from $(2n-1)\frac{\pi}{2}$ to $(2n+1)\frac{\pi}{2}$, n being integral, as a series of sines of odd multiples of x.

1588. Functions consisting essentially of a set of simple harmonic terms are of constant occurrence in problems of Mechanical and Physical Science, *e.g.* in the vibration of a piano wire, the propagation of a signal along an electric cable, in problems on the flux of heat, or in the motion of a slide valve whose mode of travel is actuated by a system of linkages, or by a cam driven by a uniformly revolving shaft. Primarily the nature of the problem in such cases as the latter is that of the resolution of a compound motion known to be periodic, or of the function which expresses it, into its simple harmonic constituents.

A graphical method of procedure is sometimes adopted in the analysis of such a given complex periodic vibration into its simple harmonic elements useful for the practical engineer. Such methods may be found described in treatises on advanced practical mathematics. The resolution may also be performed by mechanical means.*

1589. A series of the form $a_0+\sum_{p=1}^{p=\infty}(a_p\cos px+b_p\sin px)$ may be written as $a_0+\sum_1^\infty c_p\sin(px+\alpha_p)$, where $c_p{}^2=a_p{}^2+b_p{}^2$ and $\tan\alpha_p=a_p/b_p$, in which we have half as many simple harmonics as before, but the phases are different.

That a single-valued finite and continuous function is under certain circumstances, and for a certain range of the variable, expressible by means of such a series is usually known as Fourier's Theorem.

1590. Extension of the Rules of Art. 1121.

Taking p, q and n as integers,

$$\int_\alpha^{2n\pi+\alpha}\cos px\cos qx\,dx=\frac{1}{2}\int_\alpha^{2n\pi+\alpha}\{\cos(p+q)x+\cos(p-q)x\}\,dx$$

$$=\frac{1}{2}\left[\frac{\sin(p+q)x}{p+q}+\frac{\sin(p-q)x}{p-q}\right]_\alpha^{2n\pi+\alpha}=0,\quad p\neq q,$$

$$\int_\alpha^{2n\pi+\alpha}\sin px\sin qx\,dx=\frac{1}{2}\left[-\frac{\sin(p+q)x}{p+q}+\frac{\sin(p-q)x}{p-q}\right]_\alpha^{2n\pi+\alpha}=0,\quad p\neq q,$$

$$\int_\alpha^{2n\pi+\alpha}\cos^2 px\,dx=\frac{1}{2}\int_\alpha^{2n\pi+\alpha}(1+\cos 2px)\,dx=n\pi,$$

$$\int_\alpha^{2n\pi+\alpha}\sin^2 px\,dx=\frac{1}{2}\int_\alpha^{2n\pi+\alpha}(1-\cos 2px)\,dx=n\pi,$$

$$\int_\alpha^{2n\pi+\alpha}\sin px\cos qx\,dx=\frac{1}{2}\int_\alpha^{2n\pi+\alpha}\{\sin(p+q)x+\sin(p-q)x\}\,dx$$

$$=\frac{1}{2}\left[-\frac{\cos(p+q)x}{p+q}-\frac{\cos(p-q)x}{p-q}\right]_\alpha^{2n\pi+\alpha}=0,\quad p\neq q,$$

$$\int_\alpha^{2n\pi+\alpha}\sin px\cos px\,dx=\frac{1}{2}\int_\alpha^{2n\pi+\alpha}\sin 2px\,dx=\frac{1}{4p}\Big[-\cos 2px\Big]_\alpha^{2n\pi+\alpha}=0.$$

*See Castle's *Manual* (pages 448-464); *Modern Instruments*, Messrs. Bell.

1591. We shall assume for the present that we are dealing with a function of x, $f(x)$, which is single-valued, real, finite and continuous and integrable for a range of real values of x from $x=\alpha$ to $x=\alpha+2\pi$; or that if $f(x)$ be unbounded as to the values of which it is capable in that range, that its integral for that range is absolutely convergent. Moreover, we shall assume that $f(x)$ is such that it is possible to find a series of the form $A_0+\sum_1^\infty(A_p\cos px+B_p\sin px)$ which is uniformly convergent, converging to the value $f(x)$ for each value of x within the given range, and that for such series term by term integration is a possible operation. Then the values of the several coefficients may be found as in the particular case of Art. 1587. For we have

$$\text{(i)}\quad \int_\alpha^{2\pi+\alpha} f(x)\,dx=A_0\int_\alpha^{2\pi+\alpha}dx=2\pi A_0;$$

$$\text{(ii)}\quad \int_\alpha^{2\pi+\alpha} f(x)\cos px\,dx=A_p\int_\alpha^{2\pi+\alpha}\cos^2 px\,dx=\pi A_p;$$

$$\text{(iii)}\quad \int_\alpha^{2\pi+\alpha} f(x)\sin px\,dx=B_p\int_\alpha^{2\pi+\alpha}\sin^2 px\,dx=\pi B_p.$$

Before substituting the values of the several coefficients, write ξ for x in the several integrands.

Then

$$f(x)=\frac{1}{2\pi}\int_\alpha^{2\pi+\alpha} f(\xi)\,d\xi+\frac{1}{\pi}\sum_{p=1}^{p=\infty}\left\{\cos px\int_\alpha^{2\pi+\alpha}\cos p\xi f(\xi)\,d\xi\right.$$
$$\left.+\sin px\int_\alpha^{2\pi+\alpha}\sin p\xi f(\xi)\,d\xi\right\}$$
$$=\frac{1}{2\pi}\int_\alpha^{2\pi+\alpha} f(\xi)\,d\xi+\frac{1}{\pi}\sum_{p=1}^{p=\infty}\int_\alpha^{2\pi+\alpha} f(\xi)\cos p(\xi-x)\,d\xi.$$

In the cases $\alpha=0$ and $\alpha=-\pi$, we have respectively

$$f(x)=\frac{1}{2\pi}\int_0^{2\pi} f(\xi)\,d\xi+\frac{1}{\pi}\sum_{p=1}^{p=\infty}\int_0^{2\pi} f(\xi)\cos p(\xi-x)\,d\xi,\ (2\pi>x>0),$$

and

$$f(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)\,d\xi+\frac{1}{\pi}\sum_{p=1}^{p=\infty}\int_{-\pi}^{\pi} f(\xi)\cos p(\xi-x)\,d\xi,\ (\pi>x>-\pi).$$

If we write $\xi=\frac{\pi\eta}{l}$ and $x=\frac{\pi y}{l}$, then $d\xi=\frac{\pi}{l}d\eta$. Also writing $f\left(\frac{\pi\eta}{l}\right)\equiv F(\eta)$, then $f\left(\frac{\pi y}{l}\right)=F(y)$, and we have

$$F(y)=\frac{1}{2l}\int_0^{2l}F(\eta)\,d\eta+\frac{1}{l}\sum_{p=1}^{p=\infty}\int_0^{2l}F(\eta)\cos\frac{p\pi}{l}(\eta-y)\,d\eta,\ (2l>y>0),$$

and

$$F(y)=\frac{1}{2l}\int_{-l}^{l}F(\eta)\,d\eta+\frac{1}{l}\sum_{p=1}^{p=\infty}\int_{-l}^{l}F(\eta)\cos\frac{p\pi}{l}(\eta-y)\,d\eta,\ (l>y>-l).$$

1592. This celebrated theorem was given by Fourier in 1822, in his *Théorie Analytique de la Chaleur*. A particular case had been given previously by Lagrange (*Anciens Mém. de l'Acad. de Turin*). See Thomson and Tait, *Nat. Phil.*, p. 58.

We lack space for a full discussion of the many difficulties which beset this theorem as to the propriety of integration "term by term," as to uniform convergence of the series, etc, but must refer the reader to other treatises expressly dealing with it, *e.g.* Professor Carslaw's *Introduction to the Theory of Fourier's Series and Integrals*. We only seek here to present to the student a practical working knowledge of the methods to be adopted.

1593. **The Cosine Series.**

If $f(x)$ can be expanded as *a convergent series of cosines alone*, for values of x between 0 and π, as

$$f(x)=A_0+A_1\cos x+A_2\cos 2x+A_3\cos 3x+\ldots=A_0+\sum_1^\infty A_p\cos x,$$

we have $(\pi>x>0)$,

$$\int_0^\pi f(x)\cos px\,dx=A_p\int_0^\pi\cos^2 px\,dx=\tfrac{1}{2}\pi A_p,\ \text{and}\ \int_0^\pi f(x)\,dx=\pi A_0.$$

Then

$$f(x)=\frac{1}{\pi}\int_0^\pi f(\xi)\,d\xi+\frac{2}{\pi}\sum_1^\infty\cos px\int_0^\pi f(\xi)\cos p\xi\,d\xi,\quad(\pi>x>0).$$

1594. **The Sine Series.**

Similarly, if $f(x)$ can be expanded as *a convergent series of sines alone*, for values of x between 0 and π, as

$$f(x)=B_1\sin x+B_2\sin 2x+B_3\sin 3x+\ldots=\sum_1^\infty B_p\sin px,$$

$(\pi>x>0)$,

we have $\int_0^\pi f(x)\sin px\,dx = B_p\int_0^\pi \sin^2 px\,dx = \frac{1}{2}\pi B_p.$

Thus $f(x) = \frac{2}{\pi}\sum_1^\infty \sin px\int_0^\pi f(\xi)\sin p\xi\,d\xi, \quad (\pi > x > 0),$

a theorem due to Lagrange.

1595. As before, writing $\xi = \frac{\pi}{l}\eta,\ x = \frac{\pi}{l}y,\ f\left(\frac{\pi}{l}y\right) = F(y)$, we have in the one case,

$$F(y) = \frac{1}{l}\int_0^l F(\eta)\,d\eta + \frac{2}{l}\sum_1^\infty \cos\frac{p\pi}{l}y\int_0^l F(\eta)\cos\frac{p\pi\eta}{l}\,d\eta, \quad (l > y > 0);$$

and in the other case,

$$F(y) = \frac{2}{l}\sum_1^\infty \sin\frac{p\pi}{l}y\int_0^l F(\eta)\sin\frac{p\pi\eta}{l}\,d\eta, \quad (l > y > 0).$$

1596. It will be noted that in the determination of these several Fourier coefficients as above, viz.

$$A_p = \frac{2}{\pi}\int_0^\pi f(\xi)\cos p\xi\,d\xi,\ B_p = \frac{2}{\pi}\int_0^\pi f(\xi)\sin p\xi\,d\xi,\ A_0 = \frac{1}{\pi}\int_0^\pi f(\xi)\,d\xi,$$

these coefficients are respectively the *mean* values of

$$2f(x)\cos px, \quad 2f(x)\sin px, \quad \text{and} \quad f(x)$$

taken through the period 0 to π.

1597. **A Remarkable Limiting Form.**

As a preliminary to the further consideration of the results obtained for the expansion of $f(x)$ as a Series of Simple Harmonic terms, let us examine the limit when $a \to 1$ of the integral $I \equiv \int_\beta^\alpha f(\xi)\frac{1-a^2}{1-2a\cos(\xi-x)+a^2}\,d\xi$, the range from β to α not exceeding 2π, and $f(\xi)$ being any finite function of ξ for which $f'(\xi)$ when existent is finite for all values of ξ within that range. We see at once

(i) that regarded as a function of x, I is a periodic function with periodicity 2π, for if x be increased or decreased by any multiple of 2π, I will be unchanged, and therefore will have gone through the whole cycle of values of which it is capable as x increases through 2π;

(ii) that when a approaches unity as a limit the integrand vanishes *unless the denominator vanishes at the same time,* *i.e.* unless $\xi = x,\ x \pm 2\pi,\ x \pm 4\pi,\ \dots\ x \pm 2n\pi$, where n is an integer;

(iii) that in consequence of the last fact, the only cases when the integrand can have a sensible value being in the vicinity of one of the above values of x, we may confine our integration to such limits as will just include such vicinity;

(iv) that when $\xi = x$ or $x \pm 2n\pi$, the denominator becomes $(1-a)^2$, and therefore the integrand tends to an infinite value; but its integral is not necessarily infinite;

(v) that if ξ increases through any small interval to $\xi + h$, then $f(\xi)$ becomes $f(\xi+h) = f(\xi) + hf'(\xi + \theta h)$, where θ is a positive proper fraction, provided $f'(\xi)$ be existent and remains finite throughout the interval ξ to $\xi + h$; and therefore that in that case when h is an infinitesimal, $f(\xi)$ only changes by an infinitesimal amount in the interval.

(vi) Since $\alpha - \beta \not> 2\pi$, ξ in its march from β to α *can only pass through one* of the values $x,\ x \pm 2\pi,\ x \pm 4\pi,\ \dots$, and it *may not pass through any*. But if $\alpha - \beta = 2\pi$, it *must either pass through one* of these values or *start from one and terminate at the next in order of magnitude.*

Suppose first that $\alpha - \beta < 2\pi$, and consider one cycle of the values of I, x lying intermediate between β and $\beta + 2\pi$.

First let $\alpha > x > \beta$.

Then $\int_\beta^\alpha (\quad) d\xi = \left\{ \int_\beta^{x-\epsilon_1} + \int_{x-\epsilon_1}^{x+\epsilon_2} + \int_{x+\epsilon_2}^{\alpha} \right\} (\quad) d\xi$, where ϵ_1, ϵ_2 are any two selected very small positive quantities. It has been seen that when a is ultimately $=1$, the first and third of these integrals vanish through containing the factor $(1-a)$ in the numerator. Hence

$$I = Lt_{a \to 1} \int_{x-\epsilon_1}^{x+\epsilon_2} f(\xi) \frac{1-a^2}{1 - 2a\cos(\xi - x) + a^2} d\xi,$$

and putting $\xi = x + \phi$ and remembering that $f'(\xi)$, being finite by supposition, the change in $f(\xi)$ is insensibly small between these close limits, we have

$$I = f(x)\, Lt_{a \to 1} \int_{-\epsilon_1}^{\epsilon_2} \frac{1-a^2}{1 - 2a\cos\phi + a^2} d\phi$$

$$= 2f(x)\, Lt_{a \to 1} \left[\tan^{-1} \frac{1+a}{1-a} \tan\frac{\phi}{2} \right]_{-\epsilon_1}^{\epsilon_2}$$

$$=2f(x)\,Lt_{a\to 1}\left[\tan^{-1}\frac{\phi}{1-a}\right]_{-\epsilon_1}^{\epsilon_2},\text{ since }\phi\text{ is very small,}$$

$$=2f(x)\,Lt_{a\to 1}\left\{\tan^{-1}\frac{\epsilon_2}{1-a}+\tan^{-1}\frac{\epsilon_1}{1-a}\right\}.$$

In proceeding to the limit, however small ϵ_1 and ϵ_2 may have been taken, $1-a$ becomes, in its unlimited decrease to zero, a positive infinitesimal of higher order than either ϵ_1 or ϵ_2.

Hence I converges to the limiting value

$$2f(x)\left(\frac{\pi}{2}+\frac{\pi}{2}\right),\quad\text{or}\quad 2\pi f(x).$$

Secondly, supposing x to lie beyond the limit α but $<\beta+2\pi$, *i.e.* $\beta<\alpha<x<\beta+2\pi$, then evidently $I=0$, for the denominator of the integrand never vanishes as ξ ranges from β to α.

Thirdly, supposing x to lie *at* the upper limit, *i.e.* $x=\alpha$, then $\int_\beta^\alpha(\quad)\,d\xi=\left(\int_\beta^{\alpha-\epsilon}+\int_{\alpha-\epsilon}^{\alpha}\right)(\quad)\,d\xi$, in which the first integral vanishes as before and the second becomes

$$=2f(x)\,Lt_{a\to 1}\tan^{-1}\frac{\epsilon}{1-a}=2f(x)\cdot\frac{\pi}{2}=\pi f(\alpha);\quad\therefore\ I=\pi f(\alpha).$$

In the same way if x lie *at* the lower limit, *i.e.* $x=\beta$, we have similarly $I=\pi f(\beta)$.

Fourthly, supposing $\alpha-\beta=2\pi$ and $\beta<x<\alpha$, we have, as before, $I=2\pi f(x)$. But if $x=\beta$ or $x=\alpha$, the integrand becomes infinite *at both ends* of the range, and in either case we have

$$I=2f(\alpha)\frac{\pi}{2}+2f(\beta)\frac{\pi}{2}=\pi\{f(\alpha)+f(\beta)\}.$$

Finally, supposing that at any point $x=c$ between α and β, $f(\xi)$ becomes discontinuous, suddenly changing its value from $f_1(c)$ to $f_2(c)$ as ξ passes through the value c; then

$$I=Lt_{a\to 1}\left(\int_\beta^{c-\epsilon_1}+\int_{c-\epsilon_1}^{c+\epsilon_2}+\int_{c+\epsilon_2}^{\alpha}\right)(\quad)\,d\xi$$

$$=Lt_{a\to 1}\int_{c-\epsilon_1}^{c+\epsilon_2}(\quad)\,d\xi,\text{ as in the first case,}$$

$$=L_{a\to 1}2\left\{f_2(c)\tan^{-1}\frac{\epsilon_2}{1-a}+f_1(c)\tan^{-1}\frac{\epsilon_1}{1-a}\right\}$$

$$=\qquad 2\left\{f_1(c)\cdot\frac{\pi}{2}+f_2(c)\cdot\frac{\pi}{2}\right\}=\pi\{f_1(c)+f_2(c)\}.$$

This completes the investigation of one cycle of the changes in the value of I as x increases from $x=\beta$ to $x=\beta+2\pi$.

1598. **Extension of Range of Integration.**

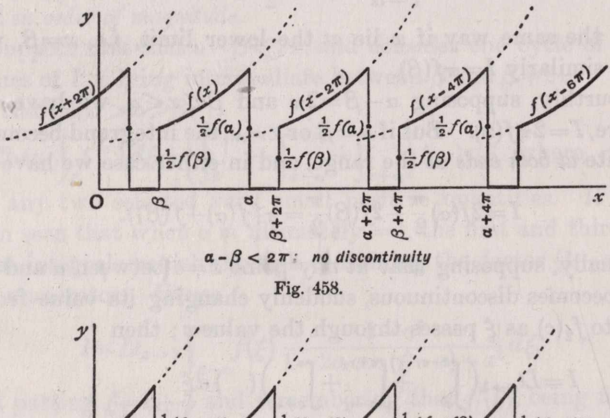
For a greater range of values of x the values found in the above cycle are merely repeated. For instance, in the next cycle, viz. $x=\beta+2\pi$ to $x=\beta+4\pi$, putting $x=2\pi+x'$, we have merely to replace $f(x)$ in the above results by $f(x')$, *i.e.* $f(x-2\pi)$, and to make no other change. If x lies between $x=\beta+2n\pi$ and $x=\beta+2(n+1)\pi$, we replace $f(x)$ by $f(x-2n\pi)$.

We exhibit in Figs. 458 to 461 graphs of

$$y=\frac{1}{2\pi}Lt_{a\to 1}\int_\beta^\alpha f(\xi)\,\frac{1-a^2}{1-2a\cos(\xi-x)+a^2}\,d\xi$$

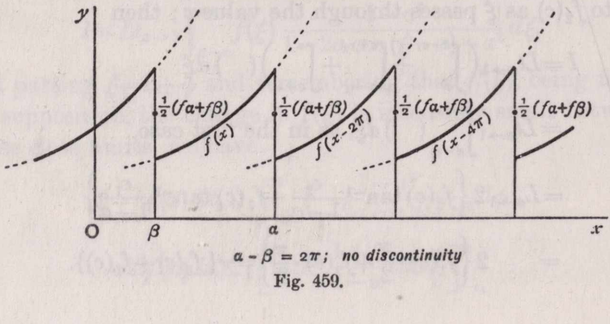
for the four cases $\alpha-\beta<2\pi$, $\alpha-\beta=2\pi$, with no discontinuity and with a discontinuity.

It will be noted that in the case of discontinuity in the ordinate of the graph of the limiting value of this integral, the value at the change is represented by half the sum of the two immediately contiguous adjacent ordinates on either side



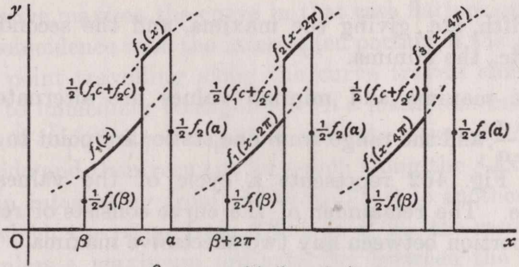
$\alpha-\beta<2\pi$; no discontinuity

Fig. 458.



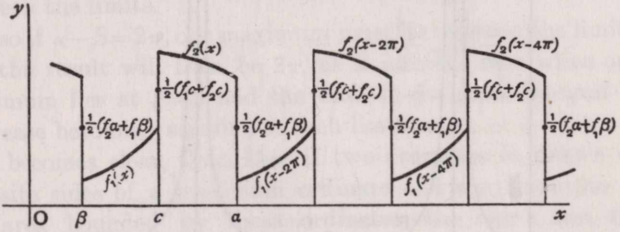
$\alpha-\beta=2\pi$; no discontinuity

Fig. 459.



$a-\beta<2\pi$; with discontinuity at $x=c$

Fig. 460.



$a-\beta=2\pi$; discontinuity at $x=c$

Fig. 461.

of the discontinuity. The graphs consist then of an infinite series of equal arcs or lines, together with an infinite series of isolated points.

1599. Geometrical Examination of the above Results.

Consider the nature of the curve $\eta=\dfrac{1-a^2}{1-2a\cos(\xi-x)+a^2}$ referred to axes $O\xi$, $O\eta$, or, what is the same thing,

$$\eta=\frac{1-a}{1+a}\,\frac{\sec^2\dfrac{\xi-x}{2}}{\left(\dfrac{1-a}{1+a}\right)^2+\tan^2\dfrac{\xi-x}{2}},$$

where x is kept constant and a positive and not greater than unity.

The curve is obviously of periodic character, for η is unaltered if we write $\xi\pm 2n\pi$ in place of ξ, n being an integer.

The maximum and minimum ordinates occur when

$$\sin(\xi-x)=0,$$

i.e. at the points $\xi=x,\ \pi+x,\ 2\pi+x,\ 3\pi+x$, etc.; the first, third, fifth, etc., giving the maxima, and the second, fourth, sixth, etc., the minima.

These maxima and minima values are alternately $\frac{1+a}{1-a}$ and $\frac{1-a}{1+a}$, and the range from one stationary point to the next is π. Fig. 462 represents a cycle of the values of the ordinate. The remainder of the curve consists of repetitions of the portion between any two successive maxima.

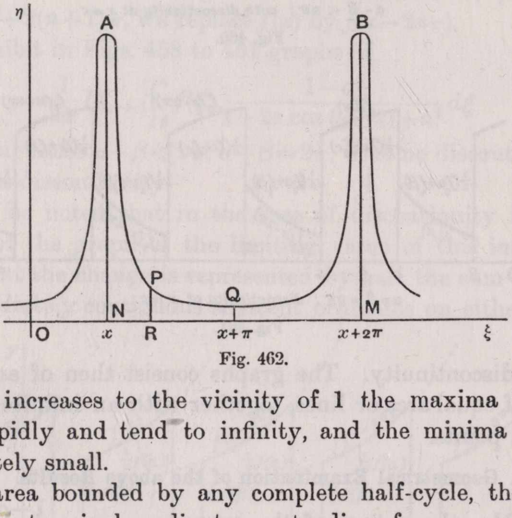


Fig. 462.

As a increases to the vicinity of 1 the maxima increase very rapidly and tend to infinity, and the minima become indefinitely small.

The area bounded by any complete half-cycle, the x-axis and the terminal ordinates, extending from a maximum ordinate to the next minimum, is

$$\int_x^{x+\pi}\eta\,d\xi=2\left[\tan^{-1}\left(\frac{1+a}{1-a}\right)\tan\frac{\xi-x}{2}\right]_x^{x+\pi}=2\tan^{-1}\left(\frac{1+a}{1-a}\tan\frac{\pi}{2}\right)=\pi$$

for any of the values of the parameter a.

Thus, in Fig. 462, the area $ANMBQA=2\pi$.

Let PR be an ordinate with abscissa $x+\epsilon$. The area of the portion $ANRP$ is $\int_x^{x+\epsilon}\eta\,d\xi=2\tan^{-1}\left(\frac{1+a}{1-a}\tan\frac{\epsilon}{2}\right)$, and evidently, however small ϵ may have been taken, when $1-a$, which is decreasing indefinitely, has become an infinitesimal of higher order than ϵ, this converges to the value π. Hence it appears that the descent of the curve on each side of a maximum ordinate is very rapid when a is nearly unity, and that between two successive maxima the curve in that case flattens out into ultimate coincidence with the intercepted portion of the ξ-axis, so that a point travelling along the curve travels along the ξ-axis up to immediate contiguity with a maximum ordinate, then travels to infinity along that ordinate, descends on the opposite side and then resumes its march along the ξ-axis.

Hence in integrating from any value $\xi=\beta$ to another limit $\xi=a$, in which the range from β to a is $< 2\pi$, the result will be zero unless a maximum ordinate lies between the limits, and the result will be 2π if a maximum ordinate does lie between the limits.

Also if $a-\beta=2\pi$, one maximum must lie between the limits, and the result will then be 2π, as is also the case when one maximum lies at $\xi=\beta$ and the next at $\xi=a$, the integral in that case becoming sensible at each limit.

It becomes clear, then, that if two ordinates be drawn on opposite sides of a maximum ordinate and contiguous to it, the area bounded by these ordinates, the curve and the intercepted portion of the x-axis tends to the limit 2π when a is made sufficiently near unity, however closely the ordinates are made to approach the maximum ordinate.

1600. Further, the presence of any *finite* factor $f(\xi)$ in the integrand for which the integral takes the form $\int \eta f(\xi)\,d\xi$ will only affect the value of the integral when the value of η is sensible, even if at any point $\xi=x$ between the limits $f(\xi)$ be discontinuous and suddenly changes its value from $f_1(x)$ to $f_2(x)$ at such point, provided that both $f_1(x)$ and $f_2(x)$ be finite. So that $\int_\beta^a \eta f(\xi)\,d\xi$ is zero when the range from β to a does not include one of the maximum η-values. In case a maximum of η *does* occur between the limits, say, between $\xi=x-\epsilon_1$ and $\xi=x+\epsilon_2$, where ϵ_1 and ϵ_2 are very small, let A and B be respectively the greatest and least of the values of $f(\xi)$ in this range. Then

$$\int_\beta^a \eta A\,d\xi > \int_\beta^a \eta f(\xi)\,d\xi > \int_\beta^a \eta B\,d\xi,$$

i.e. $\int_\beta^a \eta f(\xi)\,d\xi$ lies between $2\pi A$ and $2\pi B$.

Now, if $f(\xi)$ be single valued, finite and continuous, as ξ passes from $\xi=x-\epsilon_1$ to $\xi=x+\epsilon_2$, where ϵ_1 and ϵ_2 are made infinitesimally small, the change in $f(\xi)$ in passing from ξ to $\xi+h$ intermediate between these limits has been shown to be infinitesimal, provided $f'(\xi)$ be finite. That is, A and B are ultimately equal when ϵ_1 and ϵ_2 are taken sufficiently small.

Therefore $\int_\beta^\alpha \eta f(\xi)\,d\xi = 2\pi f(x)$.

But if whilst the range β to α includes one of the maximum η-values there be at the same point a discontinuity, $f(\xi)$ changing from $f_1(x)$ to $f_2(x)$ as ξ passes through $\xi=x$, we have

$$\int_\beta^\alpha \eta f(\xi)\,d\xi = \int_\beta^{x-\epsilon_1} \eta f(\xi)\,d\xi + \int_{x-\epsilon_1}^{x} \eta f(\xi)\,d\xi + \int_x^{x+\epsilon_2} \eta f(\xi)\,d\xi + \int_{x+\epsilon_2}^{\alpha} \eta f(\xi)\,d\xi$$
$$= 0 + \pi f_1(x) + \pi f_2(x) + 0 = \pi\{f_1(x)+f_2(x)\}.$$

[See Donkin, *Acoustics*, pages 60-66.]

1601. **Consideration of Fourier's Series from the Point of View of a Summation. Poisson's Method of Investigation, mainly of Historical Interest.**

We may now turn to the consideration of the formulae of Art. 1591, from the point of view of a summation of the series, supposed to be uniformly convergent,

$$\int_\beta^\alpha f(\xi)\,d\xi + 2\sum_{p=1}^{p=\infty} \int_\beta^\alpha f(\xi)\cos p\,(\xi - x)\,d\xi, \quad \ldots\ldots\ldots\ldots\ldots(1)$$

and endeavour to discover what such series represents in the various cases: (i) $\beta < x < \alpha$; (ii) $x=\beta$ or $x=\alpha$; (iii) x outside these limits; (iv) when $f(\xi)$ presents discontinuities.

Starting with the identity

$$1 + 2a\cos\theta + 2a^2\cos 2\theta + 2a^3\cos 3\theta + \ldots = \frac{1-a^2}{1-2a\cos\theta + a^2},$$

in which the left-hand member preserves its uniform convergency for any range of values of θ so long as $|a|<1$, put $\theta = \xi - x$, multiply by $f(\xi)$ and integrate from $\xi=\beta$ to $\xi=\alpha$, where $\alpha-\beta \not> 2\pi$.

We then get

$$\int_\beta^\alpha f(\xi)\,d\xi + 2\sum_{p=1}^{p=\infty} a^p \int_\beta^\alpha f(\xi)\cos p\,(\xi - x)\,d\xi$$
$$= \int_\beta^\alpha f(\xi)\frac{1-a^2}{1-2a\cos(\xi-x)+a^2}\,d\xi. \quad \ldots\ldots(2)$$

If we then make a approach indefinitely near to unity, the left side tends indefinitely closely to the value of the series (1).

The right-hand member of the equality (2) under the same circumstances tends to a limit which has been discussed in the previous articles.

If we *assume* the uniform convergency of series (1) and that *what is true within any infinitesimal distance of the limit, of however high an order of smallness that distance may be, is true in the limit*, we have

$$\frac{1}{2\pi}\int_\beta^\alpha f(\xi)d\xi+\frac{1}{\pi}\sum_{p=1}^{p=\infty}\int_\beta^\alpha f(\xi)\cos p(\xi-x)d\xi$$

$$\left.\begin{array}{lll} =f(x) & \text{if} & \alpha>x>\beta \\ \text{or } =\frac{1}{2}f(\alpha) & \text{if} & x=\alpha \text{ or } \frac{1}{2}f(\beta) \text{ if } x=\beta \\ \text{or } =0 & \text{if} & 2\pi+\beta>x>\alpha \end{array}\right\}\alpha-\beta<2\pi,$$

$$\left.\begin{array}{lll} \text{or } =f(x) & \text{if} & \alpha>x>\beta \\ \text{or } =\frac{1}{2}\{f(\alpha)+f(\beta)\} & \text{if} & x=\alpha \text{ or } x=\beta \end{array}\right\}\alpha-\beta=2\pi.$$

The assumption made in Poisson's investigation in the words italicised will be avoided in the method of investigation adopted by Dirichlet and discussed later.

In either case, if there be a discontinuity at $x=c$, where the value of $f(x)$ changes abruptly from $f_1(c)$ to $f_2(c)$, both being finite, the value is $\frac{1}{2}\{f_1(c)+f_2(c)\}$ for such value of x.

If x lie outside the limits β and α, say between $\beta+2n\pi$ and $\beta+2(n+1)\pi$, $f(x)$ in the above results is to be replaced by $f(x-2n\pi)$.

1602. Important Cases.

The most important cases are (i) $\beta=0$, $\alpha=2\pi$; (ii) $\beta=-\pi$, $\alpha=\pi$; (iii) $\beta=0$, $\alpha=\pi$, and in these we have respectively

$$\text{(i)}\quad \frac{1}{2\pi}\int_0^{2\pi} f(\xi)d\xi+\frac{1}{\pi}\sum_1^\infty\int_0^{2\pi} f(\xi)\cos p(\xi-x)d\xi$$

$$\begin{array}{lll} =f(x) & \text{if} & 2\pi>x>0; \\ \text{or } =\frac{1}{2}\{f(0)+f(2\pi)\} & \text{if} & x=0 \text{ or } 2\pi \text{ or } 2n\pi; \\ \text{or } =f(x-2n\pi) & \text{if} & 2(n+1)\pi>x>2n\pi. \end{array}$$

$$\text{(ii)}\quad \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)d\xi+\frac{1}{\pi}\sum_1^\infty\int_{-\pi}^{\pi} f(\xi)\cos p(\xi-x)d\xi$$

$$\begin{array}{lll} =f(x) & \text{if} & \pi>x>-\pi; \\ \text{or } =\frac{1}{2}\{f(-\pi)+f(\pi)\} & \text{if} & x=-\pi \text{ or } \pi \text{ or } (2n+1)\pi; \\ \text{or } =f(x-2n\pi) & \text{if} & (2n+1)\pi>x>(2n-1)\pi. \end{array}$$

(iii) $\frac{1}{2\pi}\int_0^\pi f(\xi)\,d\xi+\frac{1}{\pi}\sum_1^\infty\int_0^\pi f(\xi)\cos p\,(\xi-x)\,d\xi$

$$\begin{aligned}
&=f(x) &&\text{if } \pi>x>0;\\
\text{or } &=0 &&\text{if } 2\pi>x>\pi;\\
\text{or } &=\tfrac{1}{2}f(0) &&\text{if } x=0 \text{ or } 2n\pi;\\
\text{or } &=\tfrac{1}{2}f(\pi) &&\text{if } x=\pi \text{ or } (2n+1)\pi;\\
\text{or } &=f(x-2n\pi) &&\text{if } (2n+1)\pi>x>2n\pi;\\
\text{or } &=0 &&\text{if } 2n\pi>x>(2n-1)\pi.
\end{aligned}$$

1603. The same results may be exhibited in another form with limits in terms of l instead of π by changing the variables so that $\xi=\frac{\pi}{l}\eta,\ x=\frac{\pi}{l}y$. Then

$$d\xi=\frac{\pi}{l}d\eta \quad\text{and}\quad f(\xi)=f\left(\frac{\pi}{l}\eta\right)=F(\eta),\text{ say.}$$

Then the result

$$\frac{1}{2\pi}\int_\beta^\alpha f(\xi)\,d\xi+\frac{1}{\pi}\sum_1^\infty\int_\beta^\alpha f(\xi)\cos p\,(\xi-x)\,d\xi=f(x)$$

becomes $\frac{1}{2l}\int_{\frac{\beta l}{\pi}}^{\frac{\alpha l}{\pi}}F(\eta)+\frac{1}{l}\sum_1^\infty\int_{\frac{\beta l}{\pi}}^{\frac{\alpha l}{\pi}}F(\eta)\cos\frac{p\pi}{l}(\eta-y)\,d\eta=F(y).$

And the particular results (i), (ii), (iii) become, if we finally replace η by ξ, y by x and F by f to preserve conformity in the notation,

(i) $\frac{1}{2l}\int_0^{2l}f(\xi)\,d\xi+\frac{1}{l}\sum_1^\infty\int_0^{2l}f(\xi)\cos\frac{p\pi}{l}(\xi-x)\,d\xi$

$$\begin{aligned}
&=f(x) &&\text{if } 2l>x>0;\\
\text{or } &=\tfrac{1}{2}\{f(0)+f(2l)\} &&\text{if } x=0,\ 2l \text{ or } 2nl;\\
\text{or } &=f(x-2nl) &&\text{if } 2(n+1)l>x>2nl.
\end{aligned}$$

(ii) $\frac{1}{2l}\int_{-l}^{l}f(\xi)\,d\xi+\frac{1}{l}\sum_1^\infty\int_{-l}^{l}f(\xi)\cos\frac{p\pi}{l}(\xi-x)\,d\xi$

$$\begin{aligned}
&=f(x) &&\text{if } l>x>-l;\\
\text{or } &=\tfrac{1}{2}\{f(-l)+f(l)\} &&\text{if } x=-l \text{ or } l \text{ or } (2n+1)l;\\
\text{or. } &=f(x-2nl) &&\text{if } (2n+1)l>x>(2n-1)l.
\end{aligned}$$

(iii) $\frac{1}{2l}\int_0^{l}f(\xi)\,d\xi+\frac{1}{l}\sum_1^\infty\int_0^{l}f(\xi)\cos\frac{p\pi}{l}(\xi-x)\,d\xi$

$$\begin{aligned}
&=f(x) &&\text{if } l>x>0;
\end{aligned}$$

$$\begin{array}{lll} \text{or} & =0 & \text{if}\quad 2l>x>l\,; \\ \text{or} & =\tfrac{1}{2}f(0) & \text{if}\quad x=0 \text{ or } 2nl\,; \\ \text{or} & =\tfrac{1}{2}f(l) & \text{if}\quad x=l \text{ or } (2n+1)l\,; \\ \text{or} & =f(x-2nl) & \text{if}\quad (2n+1)l>x>2nl\,; \\ \text{or} & =0 & \text{if}\quad 2nl>x>(2n-1)l. \end{array}$$

If, in Art. 1601, we had written $\xi+x$ for θ instead of $\xi-x$, equation (iii) above would have been replaced by

$$\frac{1}{2l}\int_0^l f(\xi)\,d\xi+\frac{1}{l}\sum_1^\infty\int_0^l f(\xi)\cos\frac{p\pi}{l}(\xi+x)\,d\xi$$

$$\begin{array}{lll} & =0 & \text{if}\quad l>x>0\,; \\ \text{or} & =\tfrac{1}{2}f(0) & \text{if}\quad x=0\,; \\ \text{or} & =\tfrac{1}{2}f(l) & \text{if}\quad x=l. \end{array}$$

Hence adding,

$$\frac{1}{2l}\int_0^l f(\xi)\,d\xi+\frac{1}{l}\sum_1^\infty\int_0^l f(\xi)\cos\frac{p\pi\xi}{l}\cos\frac{p\pi x}{l}\,d\xi$$

$$\begin{array}{lll} & =\tfrac{1}{2}f(x) & \text{if}\quad l>x>0\,; \\ \text{or} & =\tfrac{1}{2}f(0) & \text{if}\quad x=0\,; \\ \text{or} & =\tfrac{1}{2}f(l) & \text{if}\quad x=l\,; \end{array}$$

i.e. the formula holds *inclusive of the values at the limits*, viz.

$$\frac{1}{l}\int_0^l f(\xi)\,d\xi+\frac{2}{l}\sum_1^\infty\cos\frac{p\pi x}{l}\int_0^l f(\xi)\cos\frac{p\pi\xi}{l}\,d\xi=f(x)$$

from $x=0$ to $x=l$ *inclusive.*

If we change the sign of x the left side is unaltered. The right side must then be written $f(-x)$. From $x=l$ to $x=2l$, putting $x=2l-x'$, $\cos\dfrac{p\pi x}{l}=\cos\dfrac{p\pi}{l}(2l-x')=\cos\dfrac{p\pi x'}{l}$, and the result is $f(x')$ or $f(2l-x)$; and so on. So that the results are

$$\left.\begin{array}{c}-l \text{ to } 0\\ f(-x)\end{array}\right\},\ \left.\begin{array}{c}0 \text{ to } l\\ f(x)\end{array}\right\},\ \left.\begin{array}{c}l \text{ to } 2l\\ f(2l-x)\end{array}\right\},\ \left.\begin{array}{c}2l \text{ to } 3l\\ f(x-2l)\end{array}\right\},\ \left.\begin{array}{c}3l \text{ to } 4l\\ f(4l-x)\end{array}\right\},\ \left.\begin{array}{c}4l \text{ to } 5l\\ f(x-4l)\end{array}\right\},$$

and so on, as illustrated in Fig. 463.

1604. If we subtract the same integrals, we get

$$\frac{2}{l}\sum_1^\infty\sin\frac{p\pi x}{l}\int_0^l f(\xi)\sin\frac{p\pi\xi}{l}\,d\xi=f(x) \text{ if } l>x>0\,;$$
$$\text{or}\quad =0 \quad \text{if } x=0 \text{ or } l.$$

Hence in this case the values for $x=0$ and $x=l$ are *excluded.*

Moreover, a change in the sign of x changes the sign of the left side. Hence if x lie between $-l$ and 0, we have

$$\frac{2}{l}\sum_1^\infty \sin\frac{p\pi x}{l}\int_0^l f(\xi)\sin\frac{p\pi\xi}{l}\,d\xi=-f(-x).$$

The graph of the several changes is exhibited in Fig. 464.

1605. **Graphical Representation of the Previous Results.**

Let $S\equiv\frac{1}{l}\int_0^l f(\xi)\,d\xi+\frac{2}{l}\sum_1^\infty \cos\frac{p\pi x}{l}\int_0^l f(\xi)\cos\frac{p\pi\xi}{l}\,d\xi$ for *any value of* x.

Then if $l \geqslant x \geqslant 0$, $S=f(x)$.

(α) Consider $2l \geqslant x \geqslant l$.

Put $x=2l-x'$; then $l\geqslant x'\geqslant 0$, $\cos\frac{p\pi x}{l}=\cos\frac{p\pi x'}{l}$.

Then $S=f(x')=f(2l-x)$.

(β) Consider $3l\geqslant x\geqslant 2l$.

Put $x=2l+x''$; then $l\geqslant x''\geqslant 0$, $\cos\frac{p\pi x}{l}=\cos\frac{p\pi x''}{l}$.

Then $S=f(x'')=f(x-2l)$.

(γ) Consider $4l\geqslant x\geqslant 3l$.

Put $x=4l-x'''$; then $l\geqslant x'''\geqslant 0$, $\cos\frac{p\pi x}{l}=\cos\frac{p\pi x'''}{l}$.

Then $S=f(x''')=f(4l-x)$. And so on.

Also since a change of sign in x does not affect the value of S, the y-axis is an axis of symmetry of its graph.

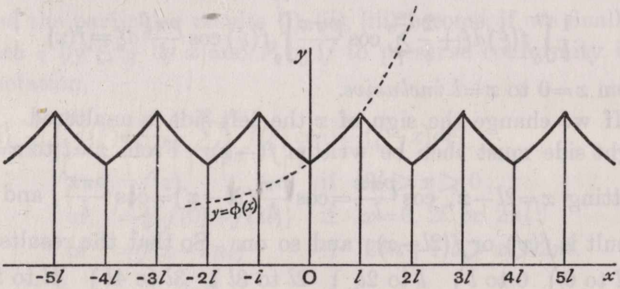


Fig. 463.

The graph of $y=S$ therefore consists of a succession of repetitions of the alternate arcs of $y=f(-x)$ from $-l$ to 0, and of $y=f(x)$ from 0 to l, coinciding with the graph of $y=f(x)$ only from 0 to l and with its image with respect to the y-axis from $-l$ to 0.

1606. Let $S'\equiv\frac{2}{l}\sum_1^\infty \sin\frac{p\pi x}{l}\int_0^l \sin\frac{p\pi\xi}{l}\,d\xi$ for *all values of* x.

Then if $x=0$, $S'=0$; if $l>x>0$, $S'=f(x)$; if $x=l$, $S'=0$.

(α) Consider $2l>x>l$.

Put $x=2l-x'$; then $l>x'>0$, $\sin\frac{p\pi x}{l}=-\sin\frac{p\pi x'}{l}$.

Then $S'=-f(x')=-f(2l-x)$; and if $x=2l$ or l, $S'=0$.

(β) Consider $3l>x>2l$.

Put $x=2l+x''$; then $l>x''>0$, $\sin\frac{p\pi x}{l}=\sin\frac{p\pi x''}{l}$.

Then $S'=f(x'')=f(x-2l)$; and if $x=3l$ or $2l$, $S'=0$.

(γ) Consider $4l>x>3l$.

Put $x=4l-x'''$; then $l>x'''>0$, $\sin\frac{p\pi x}{l}=-\sin\frac{p\pi x'''}{l}$.

Then $S'=-f(x''')=-f(4l-x)$; and if $x=4l$ or $3l$, $S'=0$. And so on

Also S' changes sign with x. Therefore the y-axis is no longer an axis of symmetry, but the origin is a centre of symmetry for the graph of S'.

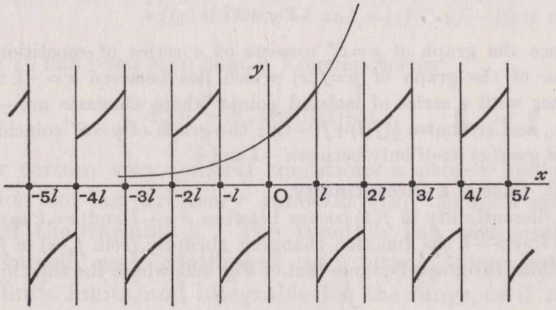


Fig. 464.

The graph of $y=S'$ therefore consists of a succession of repetitions of the alternate arcs of $y=-f(-x)$ from $-l$ to 0 and of $y=f(x)$ from 0 to l, coinciding with the graph of $y=f(x)$ only from 0 to l, together with a series of isolated points on the x-axis equably distributed at distances $=l$, starting with the origin.

The effect of a *discontinuity* in $f(x)$ existing between 0 and l would be similar to that shown in Fig. 461 at C in the segment from β to α, with a corresponding change in each of the other segments in Fig. 464.

1607. Let $S''\equiv\frac{1}{2l}\int_{-l}^{l}f(\xi)\,d\xi+\frac{1}{l}\int_{-l}^{l}f(\xi)\cos\frac{p\pi}{l}(\xi-x)\,d\xi$ for *all values of* x.

Then if $x=-l$, $S''=\frac{1}{2}\{f(l)+f(-l)\}$; if $-l<x<l$, $S''=f(x)$; if $x=l$, $S''=\frac{1}{2}\{f(l)+f(-l)\}$.

(α) Consider $3l>x>l$.

Put $x=2l+x'$; then $-l<x'<l$, $\cos\frac{p\pi}{l}(\xi-x)=\cos\frac{p\pi}{l}(\xi-x')$.

Then $S''=f(x')=f(x-2l)$; and if $x=l$ or $3l$, $S''=\frac{1}{2}\{f(l)+f(-l)\}$.

(β) Consider $5l > x > 3l$.

Put $x = 4l + x''$; then $-l < x'' < l$, $\cos\frac{p\pi}{l}(\xi - x) = \cos\frac{p\pi}{l}(\xi - x'')$.

Then $S'' = f(x'') = f(x - 4l)$; and if $x = 3l$ or $5l$, $S'' = \frac{1}{2}\{f(l) + f(-l)\}$. And so on.

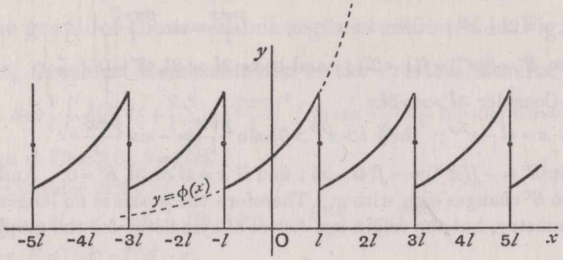


Fig. 465.

Hence the graph of $y = S''$ consists of a series of repetitions of the portion of the graph of $y = f(x)$ which lies between $x = -l$ and $x = l$, together with a series of isolated points whose abscissae are $-3l$, $-l$, l, $3l$, etc., and ordinates $\frac{1}{2}\{f(l) + f(-l)\}$; the graph of $y = S''$ coinciding with that of $y = f(x)$ itself only between $-l$ and l.

1608. Case of a Discontinuity.

If a discontinuity in $f(x)$ occurs between $x = -l$ and $x = l$, say at $x = c$, where $l > c > -l$, the function changing abruptly from $f_1(x)$ to $f_2(x)$, say, both finite, the graph becomes that of Fig. 466, where the thick line shows

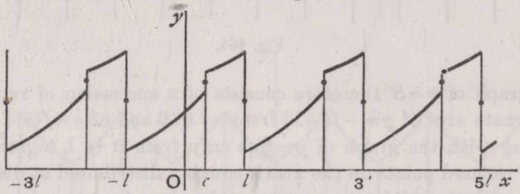


Fig. 466.

the variation of the expression S'' for different values of x and the dots, the values at $-l$, c, l, $2l + c$, $3l$, etc. The graph of $y = S''$ only coincides with that of $y = f_1(x)$ from $-l$ to c, and with that of $y = f_2(x)$ from c to l.

1609. Another Form of the Result.

Writing $-\xi$ for ξ in the formula

$$\frac{1}{2l}\int_{-l}^{l} f(\xi)\,d\xi + \frac{1}{l}\sum_1^\infty \int_{-l}^{l} f(\xi)\cos\frac{p\pi}{l}(\xi - x)\,d\xi = f(x) \text{ between } -l \text{ and } l,$$
$$\text{or } = \tfrac{1}{2}\{f(l) + f(-l)\} \text{ at } x = \pm l,$$

we have

$$\frac{1}{2l}\int_{-l}^{l} f(-\xi)\,d\xi + \frac{1}{l}\sum_1^\infty \int_{-l}^{l} f(-\xi)\cos\frac{p\pi}{l}(\xi + x)\,d\xi = f(x) \text{ between } -l \text{ and } l,$$
$$\text{or } = \tfrac{1}{2}\{f(l) + f(-l)\} \text{ at } x = \pm l.$$

Hence

$$\frac{1}{2l}\int_{-l}^{l}\frac{f(\xi)+f(-\xi)}{2}\,d\xi+\frac{1}{l}\sum_1^\infty\cos\frac{p\pi x}{l}\int_{-l}^{l}\frac{f(\xi)+f(-\xi)}{2}\cos\frac{p\pi\xi}{l}\,d\xi$$

$$+\frac{1}{l}\sum_1^\infty\sin\frac{p\pi x}{l}\int_{-l}^{l}\frac{f(\xi)-f(-\xi)}{2}\sin\frac{p\pi\xi}{l}\,d\xi$$

$$=f(x)\text{ if } l>x>-l \text{ and } =\tfrac{1}{2}\{f(l)+f(-l)\}\text{ if } x=\pm l$$

And the three integrals occurring between limits $-l$ and l are each double of the integrals from 0 to l.

$$\therefore\ \frac{1}{l}\int_0^l\frac{f(\xi)+f(-\xi)}{2}\,d\xi+\frac{2}{l}\sum_1^\infty\cos\frac{p\pi x}{l}\int_0^l\frac{f(\xi)+f(-\xi)}{2}\cos\frac{p\pi\xi}{l}\,d\xi$$

$$+\frac{2}{l}\sum_1^\infty\sin\frac{p\pi x}{l}\int_0^l\frac{f(\xi)-f(-\xi)}{2}\sin\frac{p\pi\xi}{l}\,d\xi$$

$$=f(x)\text{ if } l>x>-l \text{ and } =\tfrac{1}{2}\{f(l)+f(-l)\}\text{ if } x=\pm l.$$

1610. It has been seen that a Fourier-Series

$$A_0+\sum_1^\infty A_p\sin(px+\alpha_p)$$

is under certain very general conditions a proper analytical expression for an arbitrary function $f(x)$ between specific values of the variable x. The function has been assumed single valued, real, continuous and either lying between certain finite limits, and integrable for the range, or if not so bounded its integral for that range is assumed absolutely convergent. The possibility of expansion has been assumed in the method of undetermined coefficients, and the possibility of integration of the series term by term when multiplied by $f(x)$ throughout has also been assumed. With these assumptions it appears that when such a solution can be found and the convergence of the resulting series is uniform, the solution is unique.

1611. **Applications.**

(1) *Apply Art.* 1595 *to expand x in a series of sines of multiples of x* ($\pi>x>0$).

The formula is $f(x)=\frac{2}{l}\sum_1^\infty\sin\frac{p\pi x}{l}\int_0^l f(\xi)\sin\frac{p\pi\xi}{l}d\xi$ ($l>x>0$). But if $x=0$ or l, $f(x)$ on the left side must be replaced by 0.

Take $l=\pi$. Then

$$\int_0^\pi\xi\sin p\xi\,d\xi=\left[\xi\left(-\frac{\cos p\xi}{p}\right)-\left(-\frac{\sin p\xi}{p^2}\right)\right]_0^\pi=\frac{\pi}{p}(-1)^{p+1}.$$

Then

$$\frac{x}{2}=\sum_1^\infty(-1)^{p+1}\frac{\sin px}{p}=\frac{1}{1}\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\frac{1}{4}\sin 4x+\ldots \quad \ldots(\text{A})$$

for values of x *between* 0 and π. And the left side must be replaced by 0 if $x=0$ or π. The expansion holds therefore from $x=0$ (inclusive) to $x=\pi$ (exclusive).

A change in sign of x affects both sides. Hence if the theorem holds for any particular positive value of x, it holds also for the corresponding negative value of x. It therefore holds for all values of x from $-\pi$ to $+\pi$ both exclusive.

If $\pi<x<2\pi$, let $x=2\pi-x'$, *i.e.* $\pi>x'>0$.

Then the series $=-\left(\frac{1}{1}\sin x'-\frac{1}{2}\sin 2x'+\frac{1}{3}\sin 3x'-\ldots\right)=-\frac{x'}{2}=\frac{x-2\pi}{2}$.

If $2\pi<x<3\pi$, let $x=2\pi+x''$, *i.e.* $\pi>x''>0$.

Then the series $=\frac{1}{1}\sin x''-\frac{1}{2}\sin 2x''+\frac{1}{3}\sin 3x''-\ldots=\frac{x''}{2}=\frac{x-2\pi}{2}$.

If $3\pi<x<4\pi$, let $x=4\pi-x'''$. Then the series $=-\frac{1}{2}x'''=\frac{x-4\pi}{2}$, and so on, and the graph of $y=\frac{1}{1}\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\ldots$ will consist of lines through 0, 2π, 4π, etc., parallel to $2y=x$, with points on the x-axis at π, 3π, 5π, etc.

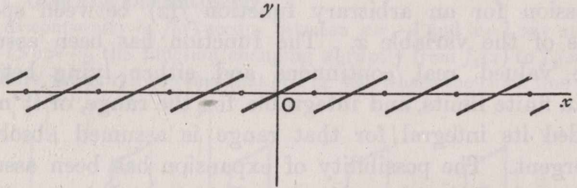


Fig. 467.

1612. (2) *Expand e^{ax} in a series of sines of multiples of x, $0<x<\pi$, and examine the series obtained.*

Taking $e^{ax}=\sum_1^\infty B_p\sin px$, we have $\int_0^\pi e^{ax}\sin px\,dx=B_p\cdot\frac{\pi}{2}$;

$$\therefore\ B_p=\frac{2}{\pi}\left[e^{ax}\frac{a\sin px-p\cos px}{a^2+p^2}\right]_0^\pi=\frac{2}{\pi}\frac{p}{a^2+p^2}\{1-(-1)^pe^{a\pi}\};$$

$$\therefore\ e^{ax}=\frac{2}{\pi}\left\{\frac{1+e^{a\pi}}{a^2+1^2}\sin x+2\frac{1-e^{a\pi}}{a^2+2^2}\sin 2x+3\frac{1+e^{a\pi}}{a^2+3^2}\sin 3x+\ldots\right\}\ (\pi>x>0).$$

But the series $=0$ at $x=0$ or $x=\pi$.

If $2\pi>x>\pi$, let $x=2\pi-x'$, *i.e.* $\pi>x'>0$. Then the series becomes

$$-\frac{2}{\pi}\left\{\frac{1+e^{a\pi}}{a^2+1^2}\sin x'+2\frac{1-e^{a\pi}}{a^2+2^2}\sin 2x'+\ldots\right\}=-e^{ax'}=-e^{a(2\pi-x)}.$$

If $3\pi>x>2\pi$, let $x=2\pi+x''$, *i.e.* $\pi>x''>0$. Then the series becomes $e^{ax''}=e^{a(x-2\pi)}$, and so on. Also at $x=0$, π, 2π, etc., the series is zero.

Hence we have for the graph of

$$y=\frac{2}{\pi}\left\{\frac{1+e^{a\pi}}{a^2+1^2}\sin x+2\,\frac{1-e^{a\pi}}{a^2+2^2}\sin 2x+3\,\frac{1+e^{a\pi}}{a^2+3^2}+\text{etc.}\right\}$$

a figure consisting of a series of arcs equal to that of the curve $y=e^{ax}$, between 0 and π, alternately above and below the x-axis, the origin being a centre of symmetry, together with the points $x=0,\ \pm\pi,\ \pm 2\pi$, etc., on the x-axis, any of which is a centre of symmetry for the whole graph (Fig. 468).

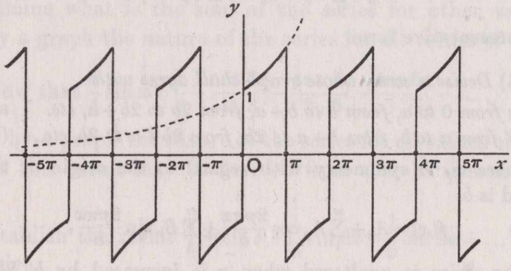


Fig. 468.

1613. (3) *To find a function of x, viz. f(x), which shall be periodic with period 2l, and shall be*

$$=\frac{l}{4}\ \textit{from}\ -l\ \textit{to}\ -\frac{l}{2};\quad =\frac{x^2}{l}\ \textit{from}\ -\frac{l}{2}\ \textit{to}\ +\frac{l}{2};\quad =\frac{l}{4}\ \textit{from}\ \frac{l}{2}\ \textit{to}\ l.$$

Let $f(x)=A_0+\sum\limits_1^\infty A_p\cos\frac{p\pi x}{l}$, the cosine series being selected because negative values and positive values of x are to give the same result.

Then $2lA_0=\int_{-l}^{-\frac{l}{2}}\frac{l}{4}\,dx+\int_{-\frac{l}{2}}^{\frac{l}{2}}\frac{x^2}{l}\,dx+\int_{\frac{l}{2}}^{l}\frac{l}{4}\,dx=\frac{l^2}{3}$; $\therefore\ A_0=\frac{l}{6}$; and

$$\int_{-l}^{l}A_p\cos^2\frac{p\pi x}{l}\,dx=\int_{-l}^{-\frac{l}{2}}\frac{l}{4}\cos\frac{p\pi x}{l}\,dx+\int_{-\frac{l}{2}}^{\frac{l}{2}}\frac{x^2}{l}\cos\frac{p\pi x}{l}\,dx+\int_{\frac{l}{2}}^{l}\frac{l}{4}\cos\frac{p\pi x}{l}\,dx;$$

whence $A_p=\frac{2l}{p^2\pi^2}\left(\cos\frac{p\pi}{2}-\frac{2}{p\pi}\sin\frac{p\pi}{2}\right)$, giving

$$f(x)=\frac{l}{6}+\frac{2l}{\pi^2}\sum_1^\infty\left(\frac{1}{p^2}\cos\frac{p\pi}{2}-\frac{2}{\pi p^3}\sin\frac{p\pi}{2}\right)\cos\frac{p\pi x}{l},$$

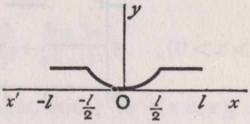


Fig. 469.

and the graph is composed of equal arcs of a parabola and straight lines of length $\frac{l}{2}$ which form prolongations of their latera recta, one cycle being exhibited in Fig. 469.

1614. **Further Remarks.**

Any series containing only cosines of multiples of x, as $A_0+\sum\limits_1^\infty A_p\cos px$, being unaffected by a change of sign of x, must have a graph for which the y-axis is an axis of symmetry. Any series containing only sines of multiples of x, as $\sum_1^\infty B_p \sin px$, changes sign with x, and the origin is therefore a centre of symmetry of the graph. Therefore if it be required to construct a series which shall represent a discontinuous system of lines or arcs of curves for which neither kind of symmetry exists, it will be necessary to assume the most general form of Fourier Series, viz.

$$A_0+\sum_1^\infty A_p \cos px+\sum_1^\infty B_p \sin px$$

as the representative form.

1615. (4) *Devise a series whose graph shall agree with*

$y=c$ *from* 0 *to* a, *from* b *to* $b+a$, *from* $2b$ *to* $2b+a$, *etc.*
and $y=c'$ *from* a *to* b, *from* $b+a$ *to* $2b$, *from* $2b+a$ *to* $3b$, *etc.*, } and so on, $(a<b)$.

Here there is no symmetry with regard to the origin or the y-axis. The period is b.

Assume $$f(x)=A_0+\sum_1^\infty A_p \cos\frac{2p\pi x}{b}+\sum_1^\infty B_p \sin\frac{2p\pi x}{b},$$

so that the series is unaltered when x is increased by b, $2b$, $3b$, etc. We have

$$A_0 b=\int_0^a c\,dx+\int_a^b c'\,dx=ca+c'(b-a)\,; \qquad \therefore\ A_0=(c-c')\frac{a}{b}+c',$$

$$A_p\frac{b}{2}=\int_0^a c\cos\frac{2p\pi x}{b}\,dx+\int_a^b c'\cos\frac{2p\pi x}{b}\,dx\,; \quad \therefore\ A_p=\frac{c-c'}{\pi p}\sin\frac{2p\pi a}{b},$$

$$B_p\frac{b}{2}=\int_0^a c\sin\frac{2p\pi x}{b}\,dx+\int_a^b c'\sin\frac{2p\pi x}{b}\,dx\,; \quad \therefore\ B_p=\frac{c-c'}{\pi p}\operatorname{vers}\frac{2p\pi a}{b}\,;$$

$$\therefore\ y\equiv f(x)=(c-c')\frac{a}{b}+c'+\frac{c-c'}{\pi}\sum_1^\infty\frac{1}{p}\sin\frac{2p\pi a}{b}\cos\frac{2p\pi x}{b}$$
$$+\frac{c-c'}{\pi}\sum_1^\infty\frac{1}{p}\operatorname{vers}\frac{2p\pi a}{b}\sin\frac{2p\pi x}{b}.$$

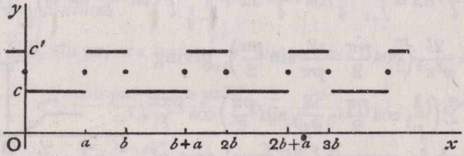


Fig. 470.

It will be seen that at the values $x=a$ or $x=b$ the series becomes $\frac{c+c'}{2}$ by virtue of the result

$$\frac{1}{1}\sin\theta+\frac{1}{2}\sin 2\theta+\frac{1}{3}\sin 3\theta+\ldots=\frac{\pi-\theta}{2} \quad (0<\theta<2\pi).$$

The graph is represented in Fig. 470.

PROBLEMS.

1. Show that from $x=0$ to $x=\pi$ exclusive

$$\frac{\pi}{4}\cos x=\frac{2}{1.3}\sin 2x+\frac{4}{3.5}\sin 4x+\frac{6}{5.7}\sin 6x+\dots$$
$$+\frac{2n}{(2n-1)(2n+1)}\sin 2nx+\dots,$$

and examine what is the sum of the series for other values of x. Show by a graph the nature of the series for all values of x.

2. Show that $\dfrac{\pi}{4}\sin x=\dfrac{1}{2}-\dfrac{\cos 2x}{1.3}-\dfrac{\cos 4x}{3.5}-\dfrac{\cos 6x}{5.7}-\dots,\quad 0<x<\pi.$

Show by a graph the nature of the series for all values of x. Show also that this result may be derived from that of question 1 or *vice versa*.

3. Establish the result $\dfrac{\pi}{4}=\sin x+\dfrac{1}{3}\sin 3x+\dfrac{1}{5}\sin 5x+\dots$ from 0 to π exclusive.

Draw a complete graph of $y=\sum\limits_0^\infty\dfrac{\sin(2p+1)x}{2p+1}$.

4. Prove that $(0<x<\pi)$

(i) $\dfrac{\pi}{2}e^{ax}=-\dfrac{1}{2a}-\dfrac{a\cos x}{a^2+1^2}-\dfrac{a\cos 2x}{a^2+2^2}-\dots+e^{a\pi}\left(\dfrac{1}{2a}-\dfrac{a\cos x}{a^2+1^2}+\dfrac{a\cos 2x}{a^2+2^2}-\dots\right).$

(ii) $\dfrac{\pi}{2}e^{ax}=\dfrac{\sin x}{a^2+1^2}+\dfrac{2\sin 2x}{a^2+2^2}+\dfrac{3\sin 3x}{a^2+3^2}+\dots+e^{a\pi}\left(\dfrac{\sin x}{a^2+1^2}-\dfrac{2\sin 2x}{a^2+2^2}+\dfrac{3\sin 3x}{a^2+3^2}-\dots\right)$

5. Prove that $(-\pi<x<\pi)$

(i) $\dfrac{\pi}{2}\cdot\dfrac{\sinh ax}{\sinh a\pi}=\dfrac{\sin x}{a^2+1^2}-\dfrac{2\sin 2x}{a^2+2^2}+\dfrac{3\sin 3x}{a^2+3^2}-\dots,$

(ii) $\dfrac{\pi}{2}\cdot\dfrac{\cosh ax}{\sinh a\pi}=\dfrac{1}{2a}-\dfrac{a\cos x}{a^2+1^2}+\dfrac{a\cos 2x}{a^2+2^2}-\dfrac{a\cos 3x}{a^2+3^2}+\dots,$

and (iii) $\dfrac{\pi}{2a}\cdot\dfrac{\cosh a(\pi-x)}{\sinh a\pi}=\dfrac{1}{2a^2}+\dfrac{\cos x}{a^2+1^2}+\dfrac{\cos 2x}{a^2+2^2}+\dfrac{\cos 3x}{a^2+3^2}+\dots,(0\leqq x<2\pi).$

6. Prove that, provided a be not an integer, and $(-\pi<x<\pi)$,

$$\frac{\pi}{2}\cdot\frac{\sin ax}{\sin a\pi}=\frac{\sin x}{1^2-a^2}-\frac{2\sin 2x}{2^2-a^2}+\frac{3\sin 3x}{3^2-a^2}-\frac{4\sin 4x}{4^2-a^2}+\dots.$$

7. Draw a graph of $y=\dfrac{1}{2a^2}+\sum\limits_1^\infty\dfrac{\cos px}{p^2+a^2}$.

8. Exhibit graphically the nature of the curve $y=\sum\limits_1^\infty\dfrac{\cos 2px}{4p^2-1}$ for all values of x.

9. Deduce other series from Examples 1, 2, 3, 4, 5 by differentiation and by integration.

10. Find a function of x in a series of sines of multiples of x which shall be equal to c_1 from 0 to a_1, c_2 from a_1 to a_2, c_3 from a_2 to a_3, and trace the graph for all values of x.

11. Find a function of x which shall be equal to c_1 from 0 to a_1, c_2 from a_1 to a_2, c_3 from a_2 to a_3, c_1 from a_3 to a_3+a_1, c_2 from a_3+a_1 to a_3+a_2, c_3 from a_3+a_2 to $2a_3$, and so on. Trace the graph completely.

12. Trace the complete graph of $\frac{y}{c}=\sum_1^\infty \frac{1}{n}\operatorname{vers}\frac{n\pi a}{l}\sin\frac{n\pi x}{l}$ for all real values of x.

13. Show that if $f(x)=x$, a, $\pi-x$ in the respective intervals 0 to a, a to $\pi-a$ and $\pi-a$ to π, then

$$\frac{\pi}{4}f(x)=\sum_0^\infty \frac{\sin(2p+1)a\sin(2p+1)x}{(2p+1)^2},$$

and give a geometrical interpretation.

14. Prove that

$$\frac{x(\pi^2-x^2)}{12}=\frac{\sin x}{1^3}-\frac{\sin 2x}{2^3}+\frac{\sin 3x}{3^3}-\dots,\quad (-\pi<x<\pi),$$

and examine the graph of $y=\sum_1^\infty(-1)^{p-1}\frac{\sin px}{p^3}$ for all values of x.

15. Show that

$$f(x)=\frac{1}{4l}\int_0^l f(\xi)d\xi+\frac{1}{2l}\sum_{p=1}^{p=\infty}f(\xi)\cos\frac{p\pi}{2l}(\xi-x)\,d\xi,\quad (0<x<l);$$

but that if $x=0$, this expression $=\frac{1}{2}f(0)$, and if $x=l$, $\frac{1}{2}f(l)$.

16. Show that

$$(a)\quad \frac{1}{l}\sum_{p=1}^{p=\infty}\int_0^l f(\xi)\cos\frac{(2p-1)\pi}{2l}(\xi-x)\,d\xi=f(x),\quad (0<x<l);$$

$$\text{or}\quad =\tfrac{1}{2}f(0),\ (x=0);\quad \text{or}\quad =\tfrac{1}{2}f(l),\ (x=l).$$

$$(b)\quad \frac{1}{l}\sum_{p=1}^{p=\infty}\int_0^l f(\xi)\cos\frac{(2p-1)\pi}{2l}(\xi+x)\,d\xi=0,\quad (0<x<l);$$

$$\text{or}\quad =\tfrac{1}{2}f(0),\ (x=0);\quad \text{or}\quad =-\tfrac{1}{2}f(l),\ (x=l).$$

[Todhunter, *I.C.*, p. 306.]

17. Assuming that $f(x)$ can be expanded in a Fourier's series of sines and cosines of multiples of x in the interval $\pi>x>-\pi$, obtain a series *of sines only* which shall represent the function in the interval $\pi>x>0$.

If $f(x)=0,\ 1,\ 0$ in the respective intervals $(l/2-b>x>0)$, $(l/2+b>x>l/2-b)$ and $(l>x>l/2+b)$, prove that throughout the interval $(l>x>0)$

$$f(x)=\frac{4}{\pi}\sum_{n=0}^{n=\infty}\frac{(-1)^n}{2n+1}\sin\frac{(2n+1)\pi b}{l}\sin\frac{(2n+1)\pi x}{l}.$$

What are the values of the series when x has the values $l/2-b$ and $l/2+b$?

18. Show that

$$\frac{2}{l}\sum_{p=1}^{p=\infty}\cos\frac{(2p-1)\pi}{2l}x\int_0^l f(\xi)\cos\frac{(2p-1)\pi}{2l}\xi\,d\xi=f(x),\quad (0\leqslant x<l);$$
$$\text{or}\quad =0,\quad (x=l).$$

$$\frac{2}{l}\sum_{p=1}^{p=\infty}\sin\frac{(2p-1)\pi}{2l}x\int_0^l f(\xi)\sin\frac{(2p-1)\pi}{2l}\xi\,d\xi=f(x),\quad (0<x\leqslant l);$$
$$\text{or}\quad =0,\quad (x=0).$$

Apply these theorems in the case $f(x)=x$.

[Todhunter, *I.C.*, p. 307.]

Exhibit by means of graphs the values of the above series for values of x beyond the limits 0 and l.

Also examine in each case the effect of a discontinuity at a point c between 0 and l in the value of the function $f(\xi)$.

19. Show that a function defined as equal to l when $-2l<x<-l$; $=-x$ when $-l<x<0$; $=x$ when $0<x<l$; $=l$ when $l<x<2l$; can be represented by

$$\frac{3l}{4}-\frac{4l}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}\cos(2m+1)\frac{\pi x}{2l}-\frac{2l}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}\cos(2m+1)\frac{\pi x}{l}.$$

[I.C.S., 1899.]

20. Prove that the graph of the function $f(x)=\dfrac{2}{\pi}\displaystyle\int_0^\infty\frac{\sin^3 t\cos xt}{t}\,dt$ consists of parts of the lines $4y=-1$, $y=0$, $2y=1$, together with four isolated points.

[Math. Trip. II., 1916.]

21. If the function defined by $y=x^2$ from 0 to $\frac{1}{2}\pi$ and by $y=0$ from $\frac{1}{2}\pi$ to π be represented by a series of sines of multiples of x, show that the coefficient of $\sin nx$ is

$$\left(\frac{4}{\pi n^3}-\frac{\pi}{2n}\right)\cos\frac{1}{2}n\pi+\frac{2}{n^2}\sin\frac{1}{2}n\pi-\frac{4}{\pi n^3}.$$

To what value does the series converge at the point $x=\frac{1}{2}\pi$? Sketch the graph of the function represented by the series for values of x not restricted to lie between 0 and π; and also indicate the graph of the cosine series which represents the same function in the interval 0 to π.

[Math. Trip. II., 1916.]