CHAPTER XXXIV. (*Continued*). SECTION II.

DOUBLE INTEGRALS, ETC. CULVERWELL'S METHOD OF DISCRIMINATION.

1547. Double Integrals. The Case of two Independent Variables.

Suppose there are two independent variables and a dependent one z which is a function of x and y, but of unspecified form. Let (p, q), (r, s, t), (u, v, w, m), etc., be the partial differential coefficients of z with regard to x and y, of the first, second, third, etc., orders. That is,

$$p\equiv\frac{\partial z}{\partial x},\quad q\equiv\frac{\partial z}{\partial y};\quad r\equiv\frac{\partial^2 z}{\partial x^2},\quad s\equiv\frac{\partial^2 z}{\partial x\,\partial y},\quad t=\frac{\partial^2 z}{\partial y^2};\quad u=\frac{\partial^3 z}{\partial x^3},\ \text{etc.}$$

We shall also use capital letters with the following signification, viz.:

$$X\equiv\frac{\partial V}{\partial x},\quad Y\equiv\frac{\partial V}{\partial y},\quad Z\equiv\frac{\partial V}{\partial z},\quad P\equiv\frac{\partial V}{\partial p},\quad Q\equiv\frac{\partial V}{\partial q},\quad R\equiv\frac{\partial V}{\partial r},\ \text{etc.,}$$

and the notation

$$P_x\equiv\frac{\partial\,.\,P}{\partial x},\quad Q_y\equiv\frac{\partial\,.\,Q}{\partial y},\quad R_{xx}\equiv\frac{\partial^2.\,R}{\partial x^2},\ \text{etc.,}$$

the dots being intended as a reminder to the reader that the letters x and y not only occur explicitly in the several subjects of partial differentiation, but also implicitly through the presence of z and its partial differential coefficients.

1548. We propose to discuss the variation of $\iint V\,dx\,dy$, where V is a function of $x, y, z;\ p, q;\ r, s, t;\ u, v, w, m;$ etc., and the integration ranges over the region bounded by a given contour in the x-y plane. Moreover, we shall assume that V and the several differential coefficients occurring remain finite, continuous, and single valued at all points of the region bounded, and at all points lying upon its contour.

For each point x, y we shall suppose an infinitesimally small variation of position arbitrary from point to point and denoted as before by δx, δy.

Now x and y being independent, δx ought not to vary *in consequence of changes in* y, nor should δy vary *in consequence of changes in* x. We should therefore have $\frac{\partial}{\partial y}\delta x=0$, $\frac{\partial}{\partial x}\delta y=0$.*

For convenience in the analysis, then, we shall suppose the variation δx in x to be the same for all points which lie on the same ordinate in the x-y plane, and similarly the variation δy in y to be the same for points which lie on the same line parallel to the x-axis. The variations being quite at our choice from point to point, we are entitled to do this. In other words, we shall assume δx and δy to be respectively independent of y and x. And this supposition in no way limits the results arrived at. The supposition that δx and δy might be functions of both x and y is discussed by Poisson (*Mém. de l'Institut*, T. xii.), and the investigation there given leads to precisely the same result as that obtained by the supposition here made. [See De Morgan, *D. and I.C.*, p. 454.]

1549. **Preliminary Considerations.**

If any function $\chi(x, y)$ be varied by changing x to $x+\delta x$, we have, as in Art. 1492,

$$\delta\chi_x\equiv\delta\frac{\partial\chi}{\partial x}=\frac{\partial}{\partial x}\delta\chi-\frac{\partial\chi}{\partial x}\frac{d\,\delta x}{dx}=\frac{\partial}{\partial x}(\delta\chi-\chi_x\,\delta x-\chi_y\,\delta y)+\chi_{xx}\,\delta x+\chi_{xy}\,\delta y,$$

$$\text{i.e.}\qquad \delta\chi_x-\chi_{xx}\,\delta x-\chi_{xy}\,\delta y=\frac{\partial}{\partial x}(\delta\chi-\chi_x\,\delta x-\chi_y\,\delta y).$$

Thus, if we write ω for $\delta z-p\,\delta x-q\,\delta y$, we have

$$\delta p-r\,\delta x-s\,\delta y=\omega_x,\quad \delta q-s\,\delta x-t\,\delta y=\omega_y;\quad \delta r-u\,\delta x-v\,\delta y=\omega_{xx},$$
$$\delta s-v\,\delta x-w\,\delta y=\omega_{xy},\quad \delta t-w\,\delta x-m\,\delta y=\omega_{yy};\ \text{etc.}$$

equations similar to those of Art. 1492 for one independent variable.

* Lacroix, *C.D. et I.*, T. ii., p. 679.

Again, to the first order,

$$\delta V \;=X\,\delta x+Y\,\delta y+Z\,\delta z+P\,\delta p+Q\,\delta q+R\,\delta r+S\,\delta s+T\,\delta t+\dots,$$

whilst
$$\frac{\partial . V}{\partial x}=X \qquad +Zp\;+Pr\;+Qs\;+Ru\;+Sv\;+Tw+\dots,$$

$$\frac{\partial . V}{\partial y}= \qquad Y\;+Zq\;+Ps\;+Qt\;+Rv\;+Sw\;+Tm+\dots;$$

$$\therefore\ \delta V-\frac{\partial . V}{\partial x}\delta x-\frac{\partial . V}{\partial y}\delta y=Z\omega+P\omega_x+Q\omega_y+R\omega_{xx}+S\omega_{xy}+T\omega_{yy}+\dots,$$

to the first order.

1550. Variation of $\iint V\,dx\,dy$.

Let the region of integration be bounded by any specific closed contour, consisting either of one closed curve or of a system of arcs of different curves in the x-y plane, each of

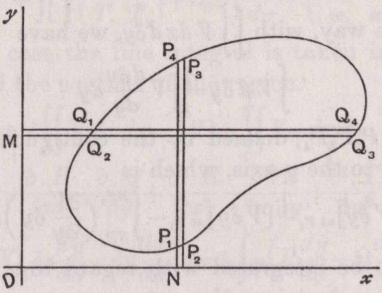

Fig. 449.

such arcs being itself subject to variation. Let the region in question be such as shown in Fig. 449. We have

$$\delta\iint V\,dx\,dy=\iint\delta(V\,dx\,dy)=\iint\delta V\,dx\,dy+\iint Vd\,\delta x\,dy+\iint V\,dx\,d\,\delta y.$$

Now
$$\iint Vd\,\delta x\,dy=\int\left[\int V\frac{d\,\delta x}{dx}dx\right]dy.$$

Integrating $\int V\frac{d\,\delta x}{dx}dx$ for a strip $Q_2Q_3Q_4Q_1$ defined by contiguous lines MQ_2Q_3, Q_1Q_4 parallel to the x-axis, we have

$$[V\,\delta x]_{\text{at }Q_3}-[V\,\delta x]_{\text{at }Q_2}-\int_{MQ_2}^{MQ_3}\left(\frac{\partial V}{\partial x}\,\delta x\right)dx,$$

and this is to be integrated with regard to y to add up the results for all such strips. Let $d\sigma$ be an element of the arc of the contour; then

$$\int\{[V\,\delta x]_{\text{at }Q_3}-[V\,\delta x]_{\text{at }Q_2}\}\,dy=\int\left\{\left[V\,\delta x\frac{dy}{d\sigma}\right]_{\text{at }Q_3}+\left[V\,\delta x\frac{dy}{d\sigma}\right]_{\text{at }Q_2}\right\}d\sigma,$$

for, if we integrate with regard to σ travelling in the positive or counter-clockwise direction, the value of dy in passing from Q_1 to Q_2 is of opposite sign to that of dy in passing from Q_3 to Q_4. Thus, this integration yields $\int\left(V\,\delta x\frac{dy}{d\sigma}\right)d\sigma$ taken round the perimeter. Hence, double integration referring to integration for the whole area bounded by the contour, and single integration to that taken in a positive direction round the perimeter,

$$\iint V\,d\,\delta x\,dy=\int\left(V\,\delta x\frac{dy}{d\sigma}\right)d\sigma-\iint\left(\frac{\partial V}{\partial x}\,\delta x\right)dx\,dy.$$

In the same way, with $\iint V\,dx\,d\,\delta y$, we have

$$\int V\,d\,\delta y=\int V\frac{d\,\delta y}{dy}dy$$

for a strip $P_1P_2P_3P_4$, defined by the contiguous lines NP_1P_4, P_2P_3, parallel to the y-axis, which is

$$[V\,\delta y]_{\text{at }P_4}-[V\,\delta y]_{\text{at }P_1}-\int_{NP_1}^{NP_4}\left(\frac{\partial V}{\partial y}\,\delta y\right)dy,$$

and this is to be integrated with regard to x to add up the results for all such strips; then

$$\int\{[V\,\delta y]_{\text{at }P_4}-[V\,\delta y]_{\text{at }P_1}\}\,dx=-\int\left\{\left[V\,\delta y\frac{dx}{d\sigma}\right]_{\text{at }P_4}+\left[V\,\delta y\frac{dx}{d\sigma}\right]_{\text{at }P_1}\right\}d\sigma$$

$$=-\int\left(V\,\delta y\frac{dx}{d\sigma}\right)d\sigma \text{ round the perimeter.}$$

Hence $$\iint V\,dx\,d\,\delta y=-\int\left(V\,\delta y\frac{dx}{d\sigma}\right)d\sigma-\iint\left(\frac{\partial V}{\partial y}\,\delta y\right)dx\,dy.$$

Therefore the total result of the variation is to the first order

$$\delta\iint V\,dx\,dy=\int V\left(\delta x\frac{dy}{d\sigma}-\delta y\frac{dx}{d\sigma}\right)d\sigma+\iint\left(\delta V-\frac{\partial V}{\partial x}\,\delta x-\frac{\partial V}{\partial y}\,\delta y\right)dx\,dy$$

$$=\int V\left(\delta x\frac{dy}{d\sigma}-\delta y\frac{dx}{d\sigma}\right)d\sigma, \text{ round the perimeter,}$$

$$+\iint(Z\omega+P\omega_x+Q\omega_y+R\omega_{xx}+S\omega_{xy}+T\omega_{yy}+\dots)\,dx\,dy,$$

over the area.

1551. In proceeding further it will be sufficient for our purposes to limit the discussion to the case where

$$V=\phi(x, y, z\,;\, p, q\,;\, r, s, t),$$

containing no partial differential coefficients of z of higher order than the second. For this will include all cases likely to be useful, and in any case if higher order differential coefficients should occur the process to be followed would be the same.

Now, by Arts. 471 and 472, writing ω for U,

$$\iint (P\omega_x+Q\omega_y)\,dx\,dy=-\iint \omega\left(\frac{\partial . P}{\partial x}+\frac{\partial . Q}{\partial y}\right)dx\,dy+\int \omega\left(P\frac{dy}{d\sigma}-Q\frac{dx}{d\sigma}\right)d\sigma$$

and $$\iint (R\omega_{xx}+S\omega_{xy}+T\omega_{yy})\,dx\,dy=\iint \omega\left(\frac{\partial^2 . R}{\partial x^2}+\frac{\partial^2 . S}{\partial x\,\partial y}+\frac{\partial^2 . T}{\partial y^2}\right)dx\,dy$$

$$+\int\left[\left\{\begin{vmatrix}\omega, & \omega_y\\ T, & T_y\end{vmatrix}+S_x\omega\right\}\frac{dx}{d\sigma}+\left\{\begin{vmatrix}R, & R_x\\ \omega, & \omega_x\end{vmatrix}+S\omega_y\right\}\frac{dy}{d\sigma}\right]d\sigma,$$

where in each case the line integral is taken in the positive direction round the contour of the region.

Thus we have $\delta\iint V\,dx\,dy=[H]+\iint K\omega\,dx\,dy,$

where $$K=Z-\frac{\partial . P}{\partial x}-\frac{\partial . Q}{\partial y}+\frac{\partial^2 . R}{\partial x^2}+\frac{\partial^2 . S}{\partial x\,\partial y}+\frac{\partial^2 . T}{\partial y^2},$$

and $$H=\int V\left(\delta x\frac{dy}{d\sigma}-\delta y\frac{dx}{d\sigma}\right)d\sigma+\int \omega\left(P\frac{dy}{d\sigma}-Q\frac{dx}{d\sigma}\right)d\sigma$$

$$+\int\left[\left\{\begin{vmatrix}\omega, & \omega_y\\ T, & T_y\end{vmatrix}+S_x\omega\right\}\frac{dx}{d\sigma}+\left\{\begin{vmatrix}R, & R_x\\ \omega, & \omega_x\end{vmatrix}+S\omega_y\right\}\frac{dy}{d\sigma}\right]d\sigma.$$

The terms of the group H depend solely upon the variations at the boundary of the contour. The terms in the surface integral are multiplied by the variation ω, *i.e.* by $\delta z-p\,\delta x-q\,\delta y$, which varies arbitrarily from point to point of the area bounded by the contour.

1552. Conditions for a Stationary Value.

As in the case of one independent variable, if the functional relation of z with x and y is to be determined so that $\iint V\,dx\,dy$ is to have a stationary value, *i.e.* so that $\delta\iint V\,dx\,dy=0$, we must have in the first place $K=0$, viz. a differential equation between z, x and y; and in addition the coefficients of the several independent variations in the limit terms $[H]$ must also vanish.

1553. **The Differential Equation.**

For the case considered, viz. $V \equiv \phi(x, y, z; p, q; r, s, t)$, the equation $K=0$ is a partial differential equation, in general of the fourth order.

Forsyth (*Diff. Eq.*, Ch. X.) discusses the solution of some forms of Partial Differential Equations of the second and higher order, but so far, even in the case of partial differential equations of the second order, it is only possible to perform the integration in special cases.

The chief methods available are in the cases in which the equation takes the form

$$\left.\begin{array}{ll}(\alpha) & Ar+Bs+Ct=U \\ \text{or } (\beta) & Ar+Bs+Ct+D(rt-s^2)=U,\end{array}\right\} \text{where } A, B, C, D, U \text{ are functions of } x, y, z, p \text{ and } q,$$

for which we have the methods of Monge and of Ampère (Forsyth, Arts. 232, 265).

These methods, however, are purely tentative and may fail.

(γ) We have also an important method known as the Principle of Duality, which amounts to reciprocation with regard to a quadric, usually taken as an elliptic paraboloid (Forsyth, Arts. 197 and 242).

(δ) For equations of form $A=(rt-s^2)^n B$, where A is a function of p, q, r, s, t, homogeneous with regard to r, s and t; and B a function of x, y, z, p, q, remaining finite when $rt=s^2$, we have Poisson's method, which begins with the assumption of a functional relation between p and q, and which thereby limits any solution to be found in that way to developable surfaces satisfying the equation.

(ϵ) We have the case where the differential equation is of the class "linear with constant coefficients."

(ζ) There are also various miscellaneous methods applicable in particular cases.

The solution of the equation $K=0$ is therefore in any but very simple cases, in the present state of knowledge of the mode of treatment of Partial Differential Equations, an insuperable barrier.

When r, s, t are absent and $V \equiv \phi(x, y, z, p, q)$, we have $K \equiv Z - \dfrac{\partial . P}{\partial x} - \dfrac{\partial . Q}{\partial y}$, and $K=0$ is in general an equation of the second order, and if it be of one of the forms enumerated a solution may perhaps be effected.

Ex. *It is required to discover the class of surfaces for which* $\iint (p^2+q^2)dx\,dy$ *has a stationary value.*

Here $V=p^2+q^2$, $Z=0$, $P=2p$, $Q=2q$; and $K=0$ becomes $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}=0$, whence $z=F_1(x+\iota y)+F_2(x-\iota y)$.

1554. It will be seen, however, that in some cases, even when the solution of the equation $K=0$ in general terms is impossible, important geometrical properties of the class of surfaces satisfying it may nevertheless be deduced.

1555. If V be of form $V \equiv A+Br+2Cs+Dt+E(rt-s^2)$, the capitals A, B, C, D, E being functions of x, y, z, p, q, it will be found by ordinary differentiation that the function K is an expression of the same type. Thus $K=0$ becomes in this case an equation of the nature to which the tentative processes of Monge or Ampère may be applied.

1556. **The Boundary Conditions.**

Taking the case $V \equiv \phi(x, y, z;\ p, q;\ r, s, t)$, we have

$$[H]=\int\Bigg[V\left(\delta x\frac{dy}{d\sigma}-\delta y\frac{dx}{d\sigma}\right)+\omega\left(P\frac{dy}{d\sigma}-Q\frac{dx}{d\sigma}\right)$$

$$+\left\{\begin{vmatrix}\omega, & \omega_y\\ T, & T_y\end{vmatrix}+S_x\omega\right\}\frac{dx}{d\sigma}+\left\{\begin{vmatrix}R, & R_x\\ \omega, & \omega_x\end{vmatrix}+S\omega_y\right\}\frac{dy}{d\sigma}\Bigg]d\sigma,$$

which is to vanish when taken round the contour of the region.

There will be as many equations resulting from this as there are independent boundary variations amongst the three δx, δy, δz, and this will depend upon the nature of the boundary.

Take the case r, s, t absent, *i.e.* $V \equiv \phi(x, y, z;\ p, q)$.

Then $\quad [H]=\int\left[(V\,\delta x+\omega P)\dfrac{dy}{d\sigma}-(V\,\delta y+\omega Q)\dfrac{dx}{d\sigma}\right]d\sigma,$

where $\omega=\delta z-p\,\delta x-q\,\delta y$.

1557. The ordinary cases occurring in geometrical applications are:

(i) When the boundary is altogether unspecified.

(ii) When the surface to be discovered is to pass through a given plane curve fixed in space.

(iii) When the surface is to be bounded by a curve which lies on a given surface but is otherwise unspecified.

(iv) When in the latter case that given surface is a plane, to which the z-plane may be taken as parallel.

Take the case $V \equiv \phi(x, y, z; p, q)$ and consider these cases.

(i) Boundary unspecified. Here δx, δy, δz are all independent at the boundary. Hence

$$P\frac{dy}{d\sigma} - Q\frac{dx}{d\sigma} = 0, \quad V\frac{dy}{d\sigma} - p\left(P\frac{dy}{d\sigma} - Q\frac{dx}{d\sigma}\right) = 0,$$

$$V\frac{dx}{d\sigma} + q\left(P\frac{dy}{d\sigma} - Q\frac{dy}{d\sigma}\right) = 0,$$

that is, $P\dfrac{dy}{d\sigma} - Q\dfrac{dx}{d\sigma} = 0$ and $V = 0$ are to hold at all points of the boundary for which all conditions are unassigned.

(ii) Boundary a given fixed curve in a plane parallel to the x-y plane.

Here z is incapable of variation at all points of the boundary, *i.e.* $\delta z = 0$. Also at all points of the boundary,

$$\frac{\delta y}{\delta x} = \frac{dy}{dx}, \quad \textit{i.e.}\ \delta x\frac{dy}{d\sigma} = \delta y\frac{dx}{d\sigma}.$$

Hence $P\dfrac{dy}{d\sigma} - Q\dfrac{dx}{d\sigma} = 0$ for all points of the fixed boundary.

(iii) If the boundary of the surface sought is to be on a fixed surface, $\phi(x, y, z) = 0$, but to be otherwise unspecified, we have $\phi_x\delta x + \phi_y\delta y + \phi_z\delta z = 0$, *i.e.* $\delta z = -\dfrac{\phi_x}{\phi_z}\delta x - \dfrac{\phi_y}{\phi_z}\delta y$; δx, δy being independent variations.

Hence

$$\left[V\delta x - P\left(p + \frac{\phi_x}{\phi_z}\right)\delta x - P\left(q + \frac{\phi_y}{\phi_z}\right)\delta y\right]\frac{dy}{d\sigma}$$

$$-\left[V\delta y - Q\left(p + \frac{\phi_x}{\phi_z}\right)\delta x - Q\left(q + \frac{\phi_y}{\phi_z}\right)\delta y\right]\frac{dx}{d\sigma} = 0,$$

and therefore

$$\left.\begin{aligned} V\frac{dy}{d\sigma} &= \left(P\frac{dy}{d\sigma}-Q\frac{dx}{d\sigma}\right)\left(p+\frac{\phi_x}{\phi_z}\right), \\ V\frac{dx}{d\sigma} &= -\left(P\frac{dy}{d\sigma}-Q\frac{dx}{d\sigma}\right)\left(q+\frac{\phi_y}{\phi_z}\right). \end{aligned}\right\}$$

Remembering also that $dz=p\,dx+q\,dy$ at all points of the surface to be discovered, and that $\phi_x dx+\phi_y dy+\phi_z dz=0$ along the boundary, we have $(\phi_x+p\phi_z)\,dx+(\phi_y+q\phi_z)\,dy=0$ along the boundary, *i.e.* $dx/(\phi_y+q\phi_z)=-dy/(\phi_x+p\phi_z)$.

Hence the equations obtained above become

$$\{P(\phi_x+p\phi_z)+Q(\phi_y+q\phi_z)\}(\phi_x+p\phi_z)-V(\phi_x+p\phi_z)\,\phi_z=0$$

and $\{P(\phi_x+p\phi_z)+Q(\phi_y+q\phi_z)\}(\phi_y+q\phi_z)-V(\phi_y+q\phi_z)\,\phi_z=0,$

i.e. they each reduce to $V\phi_z=P(\phi_x+p\phi_z)+Q(\phi_y+q\phi_z)$, or $(V-Pp-Qq)\,\phi_z=P\phi_x+Q\phi_y$, which is to hold at all points of the bounding line upon the given surface.

(iv) When the surface is merely a plane $z=$const.,

$$\phi_x=0, \quad \phi_y=0, \quad \phi_z=1,$$

and the condition becomes $V-Pp-Qq=0$, which is to hold at all points of the bounding line which lies on the given plane.

1558. **Relative Maxima and Minima.**

In the case where a maximum or minimum value of $u\equiv\iint V\,dx\,dy$ is sought conditionally upon a second surface integral $V\equiv\iint W\,dx\,dy$ retaining a definite value a, the same process applies as already employed in the case of a single independent variable (Art. 1504), viz. to make

$$\iint(V+\lambda W)\,dx\,dy$$

an unconditional maximum or minimum. For it is obvious that if u is to be a maximum or minimum, $u+\lambda a$ is a maximum or minimum, *i.e.* $\iint(V+\lambda W)\,dx\,dy$ is so also.

1559. **Surfaces of Maximum or Minimum Area; Bubbles.**

Apply the theorems now established to obtain the condition that $\iint\sqrt{1+p^2+q^2}\,dx\,dy$ shall have a stationary value. That is, we are to find the nature of a surface which, whilst satisfying certain bounding conditions which may be subsequently imposed, is to have a maximum or minimum curved area.

Here $V=\sqrt{1+p^2+q^2}$, $X=Y=Z=0$, $P=\dfrac{p}{\sqrt{1+p^2+q^2}}$, $Q=\dfrac{q}{\sqrt{1+p^2+q^2}}$.

The equation $K=0$ gives $\dfrac{\partial\,.\,P}{\partial x}+\dfrac{\partial\,.\,Q}{\partial y}=0$, *i.e.*

$$\frac{r}{(1+p^2+q^2)^{\frac{1}{2}}}-\frac{p(pr+qs)}{(1+p^2+q^2)^{\frac{3}{2}}}+\frac{t}{(1+p^2+q^2)^{\frac{1}{2}}}-\frac{q(ps+qt)}{(1+p^2+q^2)^{\frac{3}{2}}}=0,$$

i.e. $$(1+p^2+q^2)(r+t)=p^2r+2pqs+q^2t,$$

or $$(1+p^2)t-2pqs+(1+q^2)r=0.$$

This is a second order partial differential equation to determine z as a function of x and y. Without proceeding to its solution, it will be noticed that since the equation giving the principal radii of curvature at any point of a surface $z=f(x, y)$ is

$$(rt-s^2)\rho^2-\sqrt{1+p^2+q^2}\{(1+p^2)t-2pqs+(1+q^2)r\}\rho+(1+p^2+q^2)^2=0,$$

this equation reduces for such surfaces as we are searching for to

$$\rho^2=(1+p^2+q^2)^2/(s^2-rt).$$

The roots are equal and of opposite sign. And if ρ_1, ρ_2 be the roots, $\rho_1+\rho_2=0$, or what is the same thing, $\dfrac{1}{\rho_1}+\dfrac{1}{\rho_2}=0$, *i.e.* the sum of the principal curvatures is zero, and the surface is an anticlastic one with this peculiarity. Moreover, this is the condition of equilibrium (stable or unstable) of possible shapes of soap-bubble films with equal pressures on opposite sides of the film. For the hydrostatic equation for that difference of pressure is $p=\dfrac{\tau}{\rho}+\dfrac{\tau}{\rho'}$, where τ is the surface tension. And it will be recalled that a number of known surfaces satisfy this condition and are possible forms for soap-bubble films, *e.g.* the catenoid formed by the revolution of a catenary about its directrix; and this is the only possible form if it is to be also a surface of revolution. The helicoidal surface and the surfaces $e^z=\cos y\sec x$, $\sin z=\sinh x\sinh y$ are shown by Catalan to satisfy the same differential equation (*Journal de l'École Polytechnique*, 1856). See Besant, *Hydromech.*, p. 217, who refers to Darboux, *Théorie Générale de Surfaces*, T. i., Liv. iii., for a full discussion of minima surfaces.

Since the Potential Energy of a soap-bubble film is $\int\tau\,dS$, where τ is the surface tension and a constant, it will be evident that if the potential energy is to be a minimum the surface is to be a minimum.

If the pressure on opposite sides of the film be not the same, we have $\frac{1}{\rho}+\frac{1}{\rho'}=\frac{p}{\tau}$, and the mean curvature is constant but not in this case zero.

1560. If the boundary is to be on the surface $\phi(x, y, z)=0$, the equation $(V-Pp-Qq)\,\phi_z=P\phi_x+Q\phi_y$ of Art. 1557 (iii) gives $\phi_z=p\phi_x+q\phi_y$, indicating that the minimum surface is to cut $\phi(x, y, z)=0$ orthogonally at all points of the bounding curve.

1561. Let us next find the conditions that must hold when, *for a given volume* expressed by $\iint z\,dx\,dy$, we have a surface of maximum or minimum area.

We are then to make $\iint(\sqrt{1+p^2+q^2}+\lambda z)\,dx\,dy$ an unconditional maximum or minimum. Here

$$V=\sqrt{1+p^2+q^2}+\lambda z,\quad Z=\lambda,\quad X=Y=0,\quad P=\frac{p}{\sqrt{1+p^2+q^2}},\quad Q=\frac{q}{\sqrt{1+p^2+q^2}};$$

and $K\equiv Z-\frac{\partial . P}{\partial x}-\frac{\partial . Q}{\partial y}=0$ gives, similarly to the work in the last case,

$$\lambda-\frac{(1+p^2)\,t-2pqs+(1+q^2)\,r}{(1+p^2+q^2)^{\frac{3}{2}}}=0,$$

so that in this case we have $\frac{1}{\rho_1}+\frac{1}{\rho_2}=\lambda$, a constant, which is the case of soap-bubble films in equilibrium, with a constant difference of pressure on opposite sides, such as might be maintained by closing the ends in the case of a film in the form of a surface of revolution and maintaining a constant air pressure in the interior, so that, provided the temperature remains constant, the volume also remains constant.

It may be noted that a sphere and a right circular cylinder are surfaces which satisfy this differential equation, but that neither of them satisfy that of Art. 1559.

1562. **Case of a Surface of Revolution.**

This case may be discussed in an elementary way by making $\int 2\pi y\,ds$ a minimum whilst $\int \pi y^2\,dx$ is constant; *i.e.* $\delta\int(y\sqrt{1+y'^2}+\lambda y^2)\,dx=0$.

Here $\qquad V=y\sqrt{1+y'^2}+\lambda y^2,\quad X=0,\quad Y_{\prime}=yy'/\sqrt{1+y'^2}$,

whence $y\sqrt{1+y'^2}+\lambda y^2=yy'^2/\sqrt{1+y'^2}+C$ or $y/\sqrt{1+y'^2}=C-\lambda y^2$.

One of the radii of curvature (ρ') of the surface is equal (in magnitude) to the normal $(n)=y\sqrt{1+y'^2}$. Thus, $\frac{1}{n}=\frac{C}{y^2}-\lambda$.

For the other, we have

$$\frac{dx}{ds}=\frac{C}{y}-\lambda y,\quad \frac{d^2x}{ds^2}=-\left(\frac{C}{y^2}+\lambda\right)\frac{dy}{ds},$$

and
$$\frac{1}{\rho}=-\frac{d^2x}{ds^2}\Big/\frac{dy}{ds}=\frac{C}{y^2}+\lambda,$$

whence $\frac{1}{\rho}-\frac{1}{n}=2\lambda$; and if ρ' be measured in the same direction as ρ, $\rho'=-n$, so that $\frac{1}{\rho}+\frac{1}{\rho'}=2\lambda=\text{const.}$; the same result as before.

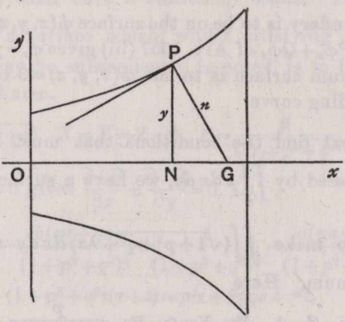

Fig. 450.

1563. It is convenient in many cases to choose a less general variation.

Let us take δx and δy both zero, but vary z and the partial differential coefficients of z. We shall then have

$$\omega=\delta z, \quad \omega_x=\delta p, \quad \omega_y=\delta q, \quad \omega_{xx}=\delta r, \quad \omega_{xy}=\delta s, \quad \omega_{yy}=\delta t.$$

With this variation the limiting terms $[H]$, when r, s, t are absent, reduce to

$$[H]=\left[\int \delta z\left(P\frac{dy}{d\sigma}-Q\frac{dx}{d\sigma}\right)d\sigma\right] \quad \text{(Art. 1556)};$$

and for the very important case frequently occurring in geometrical applications, in which the region to be considered is bounded by a fixed closed curve in the plane of x-y, we have $\delta z=0$ at every point of the bounding curve, so that $[H]$ vanishes identically.

The partial differential equation $K=0$ will, when solved, usually give z as a functional form containing x and y, and, in the case cited of a fixed boundary, the functional form occurring in the solution will have to be so chosen that the surface obtained passes through the bounding curve.

1564. Ex. *Find whether a developable surface can be found which passes through the circle $z=0$, $x^2+y^2=a^2$, and for which $\iint\sqrt{1+p^2+q^2}\,dx\,dy$ has a stationary value.*

The partial differential equation to be satisfied is

$$(1+p^2)t-2pqs+(1+q^2)r=0.$$

If the surface is to be developable, we must take $q=f(p)$.

This will give $[1+\{f(p)\}^2]-2pf(p)f'(p)+(1+p^2)\{f'(p)\}^2=0$,

i.e. $\{f(p)-pf'(p)\}^2=-1-\{f'(p)\}^2$ or $f(p)=pf'(p)+\sqrt{-1-\{f'(p)\}^2}$, which is of Clairaut's form (see *I.C. for Beginners*, p. 230), with a solution $f(p)=Ap+\sqrt{-1-A^2}$, *i.e.* $Ap-q=-\sqrt{-1-A^2}$.

Applying Lagrange's method to this (Forsyth, *D. Eq.*, Art. 184),

$$\frac{dx}{A}=\frac{dy}{-1}=\frac{dz}{-\sqrt{-1-A^2}};$$

whence $\quad x+Ay=B, \quad z-y\sqrt{-1-A^2}=\phi(B)$,

i.e. $z=y\sqrt{-1-A^2}+\phi(x+Ay)$ is the functional solution sought.

If we take ϕ to be zero and A to be $\sqrt{-1}$, we have a solution of our problem, viz. $z=0$. The circular disc bounded by $x^2+y^2=a^2$ is the developable surface which has a minimum area, and the principal curvatures of the plane surface are both zero, so that all the conditions are satisfied.

1565. *Consider the stationary value of $\iint U\,dS$, where dS is an element of the surface represented by a supposititious relation between x, y and z, and suppose that there is an accompanying condition that $\iint W\,dx\,dy=a$, taking U and W to be functions of x, y, z alone.*

Here $\quad V=U\sqrt{1+p^2+q^2}+\lambda W, \quad Z=\frac{\partial U}{\partial z}\sqrt{1+p^2+q^2}+\lambda\frac{\partial W}{\partial z}$,

$$P=U\frac{p}{\sqrt{1+p^2+q^2}}, \quad Q=U\frac{q}{\sqrt{1+p^2+q^2}},$$

$$\frac{\partial.P}{\partial x}=\left(\frac{\partial U}{\partial x}+\frac{\partial U}{\partial z}p\right)\frac{p}{(1+p^2+q^2)^{\frac{1}{2}}}+U\frac{r}{(1+p^2+q^2)^{\frac{1}{2}}}-U\frac{p(pr+qs)}{(1+p^2+q^2)^{\frac{3}{2}}},$$

$$\frac{\partial.Q}{\partial y}=\left(\frac{\partial U}{\partial y}+\frac{\partial U}{\partial z}q\right)\frac{q}{(1+p^2+q^2)^{\frac{1}{2}}}+U\frac{t}{(1+p^2+q^2)^{\frac{1}{2}}}-U\frac{q(ps+qt)}{(1+p^2+q^2)^{\frac{3}{2}}}.$$

Hence $K\equiv Z-\frac{\partial.P}{\partial x}-\frac{\partial.Q}{\partial y}=0$ becomes

$$\frac{\partial U}{\partial z}(1+p^2+q^2)^2+\lambda\frac{\partial W}{\partial z}(1+p^2+q^2)^{\frac{3}{2}}-\left(\frac{\partial U}{\partial x}+\frac{\partial U}{\partial z}p\right)p(1+p^2+q^2)$$
$$-\left(\frac{\partial U}{\partial y}+\frac{\partial U}{\partial z}q\right)q(1+p^2+q^2)-U\{(1+p^2)t-2pqs+(1+q^2)r\}=0,$$

i.e.
$$\lambda\frac{\partial W}{\partial z}(1+p^2+q^2)^{\frac{3}{2}}+\left(\frac{\partial U}{\partial z}-p\frac{\partial U}{\partial x}-q\frac{\partial U}{\partial y}\right)(1+p^2+q^2)$$
$$=U[(1+p^2)t-2pqs+(1+q^2)r];$$

$$\therefore\ \lambda\frac{\partial W}{\partial z}(1+p^2+q^2)^{\frac{1}{2}}+\frac{\partial U}{\partial z}-p\frac{\partial U}{\partial x}-q\frac{\partial U}{\partial y}=U(1+p^2+q^2)^{\frac{1}{2}}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right).$$

If l, m, n be the direction cosines of the normal to the supposititious surface $z=\phi(x, y)$, say, viz. $(\xi-x)/(-p)=(\eta-y)/(-q)=\zeta-z$,

$$l=\frac{-p}{\sqrt{1+p^2+q^2}},\quad m=\frac{-q}{\sqrt{1+p^2+q^2}},\quad n=\frac{1}{\sqrt{1+p^2+q^2}},$$

and
$$\frac{1}{\rho_1}+\frac{1}{\rho_2}=\frac{1}{U}\left(l\frac{\partial U}{\partial x}+m\frac{\partial U}{\partial y}+n\frac{\partial U}{\partial z}\right)+\frac{\lambda}{U}\frac{\partial W}{\partial z},$$

and when $\iint U\,dS$ is unconditionally stationary,

$$\frac{1}{\rho_1}+\frac{1}{\rho_2}=\frac{1}{U}\left(l\frac{\partial U}{\partial x}+m\frac{\partial U}{\partial y}+n\frac{\partial U}{\partial z}\right).$$

If the surface in either case is to terminate in a line on any surface $\psi(x, y, z)=0$, the bounding condition $(V-Pp-Qq)\psi_z=P\psi_x+Q\psi_y$ becomes

$$(U+\lambda W\sqrt{1+p^2+q^2})\psi_z=U(p\psi_x+q\psi_y)\quad\text{or}\quad p\psi_x+q\psi_y-\psi_z=\frac{\lambda}{n}\frac{W}{U}\psi_z,$$

and in the unconditional case $p\psi_x+q\psi_y-\psi_z=0$, and the surfaces then cut orthogonally at each point of such bounding line or lines.

1566. A Method of Discrimination when the Limits are fixed.

If we consider the case of fixed limits of integration for such an integral as $v=\iint\sqrt{1+p^2+q^2}\,dx\,dy$, say from $y=y_0$ to $y=y_1$, and from $x=x_0$ to $x=x_1$, the discrimination between maxima and minima may be conducted as follows, taking such a variation as described in Art. 1563.

Suppose z becomes $z+\delta z$ and p, q respectively $p+\delta p$ and $q+\delta q$. Then V becomes $\sqrt{1+(p+\delta p)^2+(q+\delta q)^2}$. This we must expand to terms of the second order, and we have

$$V+\delta V=\sqrt{1+p^2+q^2}\left[1+\frac{1}{2}\frac{2p\,\delta p+2q\,\delta q+\delta p^2+\delta q^2}{1+p^2+q^2}-\frac{1}{8}\frac{(2p\,\delta p+2q\,\delta q)^2}{(1+p^2+q^2)^2}+\dots\right];$$

$$\therefore\ \delta V=\frac{p\,\delta p+q\,\delta q}{(1+p^2+q^2)^{\frac{1}{2}}}+\frac{\delta p^2+\delta q^2+(p\,\delta q-q\,\delta p)^2}{2(1+p^2+q^2)^{\frac{3}{2}}}.$$

Hence the second order variation in δv is

$$\frac{1}{2}\iint\frac{\delta p^2+\delta q^2+(p\,\delta q-q\,\delta p)^2}{(1+p^2+q^2)^{\frac{3}{2}}}\,dx\,dy,$$

which being essentially positive for all variations, the solution of Art. 1559 gives a true *minimum* solution.

1567. Taking the case of Art. 1561, the second order terms in δV are those in $\sqrt{1+(p+\delta p)^2+(q+\delta q)^2}+\lambda(z+\delta z)$, *i.e.* the same as the above, and are essentially positive. We therefore find a true *minimum* in this case also. We turn, however, to a more detailed consideration of the second order terms in the general case.

1568. **Culverwell's Method of Discrimination between Maxima and Minima Values. Reconsideration of the Variations to be given.**

In estimating the variation of

$$u \equiv \int_{x_0}^{x_1} V\,dx, \quad \text{where } V \equiv \phi\{x, y, y', y'', \ldots y^{(n)}\},$$

we have so far given to each letter, inclusive of x, an arbitrary change, so that the point x, y is displaced to $x+\delta x$, $y+\delta y$; and the direction of the path, its curvature and higher order peculiarities, indicated by y', y'' and higher order differential coefficients, have also undergone arbitrary variations and become $y'+\delta y'$, $y''+\delta y''$, etc.

Many writers prefer to keep x unaltered, and to vary y and its differential coefficients alone (see Art. 1563).

Considerable simplification results in taking δx to be zero. For then we have $\omega=\delta y$, $\omega'=\delta y'$, $\omega''=\delta y''$, etc., instead of the more cumbrous expressions $\delta y - y'\,\delta x$, $\delta y' - y''\,\delta x$, $\delta y'' - y'''\delta x$, etc., for which they respectively stand. But there is this disadvantage, that when in an investigation δx has once been taken to be zero it cannot be restored at a later stage, whilst if we retain the variation of x from the beginning we can at any time make it zero. And in dealing with the terminal conditions, these terminals are not in general compelled to move upon lines parallel to the y-axis, but may lie on specific curves in which δx necessarily varies with δy, and it has therefore been so far convenient to retain command of the variation of x as well as over those of the other letters.

1569. To make $\delta x=0$ throughout clearly means that the deformation chosen of the hypothetical curve which represents a relation between y and x, is one which is obtained by an arbitrary point to point variation of each ordinate. That is, each point is displaced parallel to the y-axis, through an arbitrary small distance *with consequent alterations* in the values of the differential coefficients of y, which depend upon the particular variations arbitrarily assigned from point to point to the ordinates. That is, taking $y=\chi(x)$ to be a supposititious relation between x and y, which we are to test as to the possibility of its giving a stationary value to $\int V\,dx$ between the limits $x=x_0$ and $x=x_1$, then $y=\chi(x)+\epsilon\theta(x)$, where ϵ is an infinitesimal constant not containing x, and $\theta(x)$ is an arbitrary function of x understood to be finite for the whole range of integration, would be the equation of a contiguous curve to $y=\chi(x)$, and such that the variation of y at any point is $\delta y \equiv \epsilon\theta(x)$. We shall write χ and θ for $\chi(x)$ and $\theta(x)$ respectively for short; and we shall take θ to have been chosen so that neither it nor any of its differential coefficients up to the $(n-1)^{\text{th}}$ becomes infinite or discontinuous, but that they each remain either zero or finite throughout the whole range of integration. Then as ϵ is taken independent of x, $\delta y'=\epsilon\theta'$, $\delta y''=\epsilon\theta''$, $\delta y'''=\epsilon\theta'''$, ... $\delta y^{(n-1)}=\epsilon\theta^{(n-1)}$ and $\delta y^{(n)}=\epsilon\theta^{(n)}$.

But with regard to the last of these, viz. $\epsilon\theta^{(n)}$, we reserve to ourselves the right to make an abrupt change in the value we choose for it, provided such change be from one finite value to another finite value. With this supposition all the differentiations performed are valid operations, all the functions *differentiated* being finite and continuous real functions of x between the limits of the integration.

1570. With such a system of increments, V is changed to

$$V+\delta V=\phi\{x,\ y+\epsilon\theta,\ y'+\epsilon\theta',\ y''+\epsilon\theta'',\ \dots\ y^{(n)}+\epsilon\theta^{(n)}\};$$

and assuming V to be such that we may use Taylor's Theorem, we have

$$V+\delta V=V+\epsilon\Delta V+\frac{\epsilon^2}{2!}\Delta^2 V+\frac{\epsilon^3}{3!}R,$$

where $\Delta\equiv\theta\dfrac{\partial}{\partial y}+\theta'\dfrac{\partial}{\partial y'}+\dots+\theta^{(n)}\dfrac{\partial}{\partial y^{(n)}}$, and $\dfrac{\epsilon^3}{3!}R$ is the "Remainder" after three terms. This expansion involves the assumption that all the Partial Differential Coefficients of V of the first and second orders with regard to y, y', y'', ... $y^{(n)}$ are finite and continuous functions for values of y, y', etc., within the ranges from y, y', etc., respectively to $y+\epsilon\theta$, $y'+\epsilon\theta'$, etc., for all values of x which lie within the limits of integration of the integral $\int V\,dx$, *i.e.* from x_0 to x_1.

Now x being taken as not subject to variation, we have

$$\delta\int V\,dx=\int\delta V\,dx=\epsilon\int(\Delta V)\,dx+\frac{\epsilon^2}{2!}\int(\Delta^2 V)\,dx+\frac{\epsilon^3}{3!}\int R\,dx,$$

and by taking ϵ sufficiently small each of the terms on the right-hand side may be made greater than the sum of all that follow it. Hence, so long as $\int(\Delta V)dx$ does not vanish, the sign of $\delta\int V dx$ can be made to change by changing the sign of ϵ. Therefore the primary condition for a maximum or a minimum value is that $\int(\Delta V)dx$ should vanish, the limits being the same as those of the integral $\int V dx$.

Now $$\Delta V \equiv \left(\theta\frac{\partial V}{\partial y}+\theta'\frac{\partial V}{\partial y'}+\theta''\frac{\partial V}{\partial y''}+\ldots+\theta^{(n)}\frac{\partial V}{\partial y^{(n)}}\right),$$

where θ itself is arbitrary. And this will be recognised as what the expression $Y\omega+Y_{,}\omega'+Y_{,,}\omega''+\ldots$ of Art. 1495 becomes upon putting $\delta x=0$ therein.

By integration by parts, as in Art. 1496,

$$\int(\Delta V)dx=[\overline{Y}_{,}\theta+\overline{Y}_{,,}\theta'+\ldots+\overline{Y}_{(n)}\theta^{(n-1)}]+\int\overline{Y}\theta dx,$$

the term $V\delta x$ not now appearing in the limit terms, as $\delta x=0$.

Now let us take one variation between the two points (x_0, y_0) and (x_1, y_1) to be such that *at each terminal the values of x, y, y', y'', ... $y^{(n-1)}$ are the same for the varied curve* $y=\chi+\epsilon\theta$ as for the supposititious curve $y=\chi$ *itself*. That is, suppose the two curves to have contact of the $(n-1)^{\text{th}}$ order at the terminals. Then δy, $\delta y'$, ... $\delta y^{(n-1)}$ all vanish at the terminals, and therefore also $\theta, \theta', \theta'', \ldots \theta^{(n-1)}$ all vanish at the terminals.

Therefore, with this variation $\int(\Delta V)dx=\int\overline{Y}\theta\, dx$, and θ being arbitrary from point to point along the path of integration, we must have $\overline{Y}=0$ as a necessary condition that $\int(\Delta V)dx$ should vanish. This is the differential equation before obtained, and its solution has been seen to be of the form

$$y=F(x, c_1, c_2, \ldots c_{2n}), \quad \text{or shortly, } y=F, \text{ say,}$$

in which we may suppose that the several constants occurring have been found as heretofore explained by aid of the terminal conditions existing, and their values inserted. This relation is that for which the integral $\int V dx$ assumes a stationary value, and the graph is called a stationary curve. This value of y

and those of its differential coefficients may now be substituted in V.

1571. The variation of the integral now reduces to

$$\delta\int_{x_0}^{x_1} V\,dx = \frac{\epsilon^2}{2!}\int_{x_0}^{x_1}(\Delta^2 V)\,dx + \frac{\epsilon^3}{3!}\int_{x_0}^{x_1} R\,dx,$$

in which we are to consider a variation *from the stationary curve*, the supposititious curve $y=\chi(x)$ having been discovered to be of the now known form $y=F$.

As before, if we take ϵ sufficiently small the sign of $\frac{\epsilon^2}{2!}\int_{x_0}^{x_1}(\Delta^2 V)\,dx$ governs the sign of the right-hand side of the equation, so that the variation $\delta\int_{x_0}^{x_1} V\,dx$ is positive or negative according as $\int_{x_0}^{x_1}(\Delta^2 V)\,dx$ is positive or negative for all sufficiently small values of ϵ of whatever sign.

Therefore if $\int_{x_0}^{x_1}(\Delta^2 V)\,dx$ be positive, $\int_{x_0}^{x_1} V\,dx$ is increased by such a variation from the stationary curve, and if negative, decreased. It follows, therefore, that the stationary curve $y=F$ gives a maximum or a minimum value to $\int_{x_0}^{x_1} V\,dx$ according as $\int_{x_0}^{x_1}(\Delta^2 V)\,dx$ is negative or positive. We therefore have to examine the second order terms $\int_{x_0}^{x_1}(\Delta^2 V)\,dx$.

1572. In the following examination of the second order terms, we shall follow the method given by Mr. E. P. Culverwell in Vol. XXIII. of the *Proc. of the Lond. Math. Soc.*, 1892. It is only possible to give here a very abridged account of the results arrived at in Mr. Culverwell's researches, and his paper should be read carefully by the advanced student. Various modifications of his notation and procedure are necessarily adopted here to bring the discussion into line with previous work, but the main course of his work is adhered to.

1573. Such a variation of a path $y=\chi$ between two specific terminals P and Q, as has been described in Art. 1570, having contact of the $(n-1)^{\text{th}}$ order with $y=\chi$ at the terminals, so that $\theta=\theta'=\theta''=\ldots=\theta^{(n-1)}=0$ at P, and at Q, is said to be a "fixed limit" variation, and is a legitimate variation, provided the conditions for the existence and continuity of the several differential coefficients and the validity of Taylor's Theorem are not violated.

1574. **"Short Range" Variation.**

Let $APCQB$ be any path $y=\chi$, and let $PC'Q$ be a "fixed limit" variation of the portion PCQ. Let the abscissae of P and Q be ξ_0 and ξ_1 respectively ($\xi_1 > \xi_0$), and let ξ be the abscissa of an intermediate point C on the arc PCQ. Then

$$\int_{\xi_0}^{\xi} \theta^{(p)}(x)\,dx = \left[\theta^{(p-1)}(x)\right]_{\xi_0}^{\xi} = \theta^{(p-1)}(\xi) - \theta^{(p-1)}(\xi_0) = \theta^{(p-1)}(\xi),$$

where $n \nless p > 0$, for by the condition of Art. 1573, $\theta^{(p-1)}(\xi_0)=0$.

If then the greatest numerical value of $\theta^{(p)}(x)$ in the range ξ_0 to ξ be called ρ, which is by supposition finite, we have $\theta^{(p-1)}(\xi) \ngtr (\xi-\xi_0)\rho$, and therefore $\ngtr (\xi_1-\xi_0)\rho$, and *if we take a very short range* from P to Q, $\xi_1-\xi_0$ may be made as small as we please. Hence the numerical value of each of the quantities $\theta, \theta', \theta'', \ldots \theta^{(n-1)}, \theta^{(n)}$, may in such short range be regarded as indefinitely small in comparison with the next in order. Therefore $\theta, \theta', \theta'', \ldots \theta^{(n-1)}$ are all negligible in comparison with the last variation $\theta^{(n)}$ for a "short fixed limit" variation.

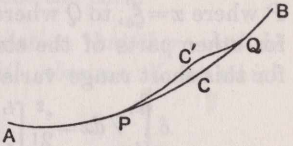

Fig. 451.

Now $\Delta^2 V \equiv \left(\theta\frac{\partial}{\partial y} + \theta'\frac{\partial}{\partial y'} + \ldots + \theta^{(n)}\frac{\partial}{\partial y^{(n)}}\right)^2 V$, and for such a variation reduces to $(\theta^{(n)})^2 \frac{\partial^2 V}{\partial (y^{(n)})^2}$.

Hence for this short variation,

$$\delta\int V\,dx = \frac{\epsilon^2}{2!}\int (\theta^{(n)})^2 \frac{\partial^2 V}{\partial (y^{(n)})^2}dx + \frac{\epsilon^3}{3!}\int R\,dx,$$

and $\theta^{(n)}$ occurs with an *even* power, so that if $\frac{\partial^2 V}{\partial (y^{(n)})^2}dx$ retains one sign within these short limits from P to Q, $\delta\int V dx$ is positive or negative according as $\frac{\partial^2 V}{\partial (y^{(n)})^2}dx$ is positive or negative throughout that range when ϵ is taken sufficiently small.

Now, considering the *finite* range from $x=x_0$ to $x=x_1$, the integral $\int_{x_0}^{x_1} V\,dx$ could not have a maximum for this range unless $\frac{\partial^2 V}{\partial(y^{(n)})^2}\,dx$ remained negative *throughout the whole range* from $x=x_0$ to $x=x_1$, nor a minimum unless $\frac{\partial^2 V}{\partial(y^{(n)})^2}\,dx$ remained positive *throughout* the same range. For suppose that there be a small portion of the range from x_0 to x_1, say from ξ_0 to ξ_1, in which $\frac{\partial^2 V}{\partial(y^{(n)})^2}\,dx$ has ceased to be negative and become positive. We could then take a "short range fixed limit" variation from P where $x=\xi_0$, to Q where $x=\xi_1$, without any variation at all for other parts of the stationary curve from x_0 to x_1. Then for this short range variation,

$$\delta\int_{\xi_0}^{\xi_1} V\,dx=\frac{\epsilon^2}{2!}\int_{\xi_0}^{\xi_1}(\theta^{(n)})^2\frac{\partial^2 V}{\partial(y^{(n)})^2}\,dx+\frac{\epsilon^3}{3!}\int_{\xi_0}^{\xi_1} R\,dx,$$

and for the rest of the range from x_0 to x_1 there is no variation; therefore $\delta\int_{x_0}^{x_1} V\,dx$ for the whole range is positive *for such a variation*, and the condition for a maximum is that it shall be negative. Hence, unless $\frac{\partial^2 V}{\partial(y^{(n)})^2}\,dx$ retains a negative sign *for the whole range* from x_0 to x_1, a maximum value of $\int_{x_0}^{x_1} V\,dx$ cannot occur. Similarly a minimum could not occur if $\frac{\partial^2 V}{\partial(y^{(n)})^2}\,dx$, starting with a positive value, became negative for part of the range.

Hence, *supposing that in the whole range from* $A(x=x_0)$ *to* $B(x=x_1)$, x *increasing throughout, there is no point at which* $\int_{x_0}^{x}(\Delta^2 V)\,dx$ *vanishes*, small short range variations such as that just described from the point P to the point Q upon it can be supposed to be made, and if in each of these $\frac{\partial^2 V}{\partial(y^{(n)})^2}\,dx$ retains the same sign, $\int_{x_0}^{x_1} V\,dx$ will have a maximum or a minimum value according as that sign is negative or positive, remaining so throughout the whole range of integration.

1575. It will be noted that in the above statement we have written $\frac{\partial^2 V}{\partial(y^{(n)})^2}dx$, including the dx as a factor, because if in the case when in travelling from A to B we pass a point C at which the tangent to the path is parallel to the y-axis, and x increases up to a certain amount, viz. the abscissa of C, and then decreases on approaching B, dx itself in such cases changes sign. Hence also in such cases $\frac{\partial^2 V}{\partial(y^{(n)})^2}$ must for a maximum or minimum also change sign at C in order to preserve an invariable sign in $\frac{\partial^2 V}{\partial(y^{(n)})^2}dx$ throughout the path.

We have now to consider the stipulation that *there shall be no point between A and B, say with abscissa X, at which* $\int_{x_0}^{X} \Delta^2 V\, dx$ *vanishes.*

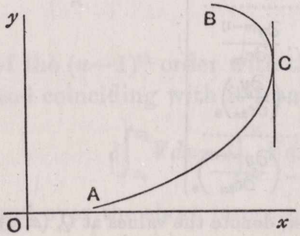

Fig. 452.

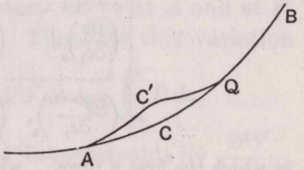

Fig. 453.

1576. **Conjugate Points on a Stationary Curve.**

Let A, Q be two points on a stationary path $ACQB$.

Then, if Q be the *first* point along the arc for which it is possible to draw a contiguous fixed limit variation $AC'Q$, *which is itself also stationary,* the points A, Q are said to be 'conjugate' to each other.

If both paths be stationary, we must have $\delta\int V\,dx=0$ to the first order along each, and therefore each must be a solution of the same differential equation $\overline{Y}=0$. Therefore, if the curve ACQ have the equation $y=F(x, c_1, c_2, \dots c_{2n})$, the variation $AC'Q$ must have an equation of the same form, and the corresponding ordinate may be written

$$y+\delta y=F(x,\, c_1+\delta c_1,\, c_2+\delta c_2, \dots c_{2n}+\delta c_{2n}),$$

so that $$\delta y=\frac{\partial y}{\partial c_1}\delta c_1+\frac{\partial y}{\partial c_2}\delta c_2+\dots+\frac{\partial y}{\partial c_{2n}}\delta c_{2n}.$$

Differentiating this $(n-1)$ times with regard to n,

$$\delta y'=\frac{\partial y'}{\partial c_1}\delta c_1+\frac{\partial y'}{\partial c_2}\delta c_2+\dots+\frac{\partial y'}{\partial c_{2n}}\delta c_{2n},$$

etc.,

$$\delta y^{(n-1)}=\frac{\partial y^{(n-1}}{\partial c_1}\delta c_1+\frac{\partial y^{(n-1)}}{\partial c_2}\delta c_2+\dots+\frac{\partial y^{(n-1)}}{\partial c^{2n}}\delta c_{2n}.$$

Now $\delta y, \delta y', \dots \delta y^{(n-1)}$ are to vanish at $A(x_0, y_0)$ and also at $Q(x, y)$. Hence we obtain by elimination of $\delta c_1, \delta c_2, \dots \delta c_{2n}$ between the $2n$ equations arising, a determinant with $2n$ rows and columns, viz.

$$\begin{vmatrix} \dfrac{\partial y}{\partial c_1}, & \dfrac{\partial y}{\partial c_2}, & \dots\dots & \dfrac{\partial y}{\partial c_{2n}} \\ \dots & \dots & \dots & \dots \\ \dfrac{\partial y^{(n-1)}}{\partial c_1}, & \dfrac{\partial y^{(n-1)}}{\partial c_2}, & \dots\dots & \dfrac{\partial y^{(n-1)}}{\partial c_{2n}} \\ \left(\dfrac{\partial y}{\partial c_1}\right)_0, & \left(\dfrac{\partial y}{\partial c_2}\right)_0, & \dots\dots & \left(\dfrac{\partial y}{\partial c_{2n}}\right)_0 \\ \dots & \dots & \dots & \dots \\ \left(\dfrac{\partial y^{(n-1)}}{\partial c_1}\right)_0, & \left(\dfrac{\partial y^{(n-1)}}{\partial c_2}\right)_0, & \dots & \left(\dfrac{\partial y^{\,n-1)}}{\partial c_{2n}}\right)_0 \end{vmatrix}=0,$$

in which the first n rows, without suffix, denote the values at Q, (x, y), and the second n rows, with suffix $_0$, denote the values at A, (x_0, y_0).

This equation determines x in terms of x_0. That is, it gives the various points Q on the first stationary curve $ACQB$, starting from A, to which it is possible to draw a contiguous fixed limit curve $AC'Q$, which is also stationary. And the first of the points Q which satisfies this condition is the point conjugate to A.

1577. Now let a point P (abscissa X) travel along the curve AB from $A(x_0, y_0)$ towards $B(x_1, y_1)$, the curve being a stationary one for $\int V\,dx$. Then we have seen that for this curve to give a *maximum value* to the integral, it is a primary *necessary* condition that $\frac{\partial^2 V}{\partial (y^{(n)})^2}dx$ should be negative for all values of x from A to B.

We shall show that as P travels along AB, the point conjugate to A is also the first position of P for which

$$\int_{x_0}^{X} \Delta^2 V\,dx=0.$$

Take a position of P very near A and connect AB by a "short range fixed limit" variation $AQPDB$ having contact

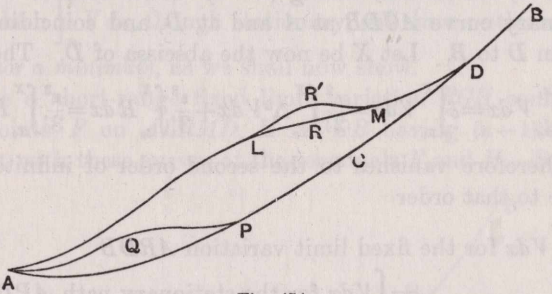

Fig. 454.

of the $(n-1)^{\text{th}}$ order with the stationary curve at A and at P, and coinciding with it from P to B. Then, for this variation

$$\delta\int_{x_0}^{x_1} V\,dx=\delta\int_{x_0}^{X} V\,dx=\frac{\epsilon^2}{2!}\int_{x_0}^{X}\Delta^2 V\,dx+\frac{\epsilon^3}{3!}\int_{x_0}^{X} R\,dx,$$

and over the short range x_0 to X, $\Delta^2 V$ is replaceable by $\frac{\partial^2 V}{\partial (y^{(n)})^2}$, which is of necessity negative, and therefore within this short range $\int_{x_0}^{x_1} V dx$ is decreased by the variation whatever be the sign of ϵ when sufficiently small. Therefore $\frac{\partial^2 V}{\partial (y^{(n)})^2}dx$ negative is a *sufficient* condition that the stationary path should yield a *maximum* value to $\int V dx$ *for this short range.*

Now let P travel onwards towards B. Then, $\Delta^2 V$ being by supposition a *finite and continuous* function of x, it cannot change sign except by passing through a zero value. Suppose that $\Delta^2 V$, which started from A as a negative quantity, retains that sign until P arrives at a point C on the stationary curve

AB, and that at C, $\Delta^2 V=0$, and beyond C that $\Delta^2 V$ becomes positive. Then $\int \Delta^2 V dx$ from A to C is a negative quantity. Suppose now that P travels beyond C to a point D such that $\int \Delta^2 V=0$ when the integration is from A to D, the positive values of the integrand which accrue beyond C having cancelled the aggregate of the negative values occurring before arrival at C. Take a "fixed limit" variation connecting A and D, viz. $ARDB$, having $(n-1)^{\text{th}}$ order contact with the stationary curve $ACDB$ at A and at D, and coinciding with it from D to B. Let X be now the abscissa of D. Then

$$\delta\int_{x_0}^{x_1} V dx=\delta\int_{x_0}^{X} V dx=\frac{\epsilon^2}{2!}\int_{x_0}^{X}\Delta^2 V dx+\frac{\epsilon^3}{3!}\int_{x_0}^{X} R\,dx=\frac{\epsilon^3}{3!}\int_{x_0}^{X} R\,dx,$$

and therefore vanishes to the second order of infinitesimals. Hence to that order

$$\int V dx \text{ for the fixed limit variation } ARDB$$
$$=\int V dx \text{ for the stationary path } APDB.$$

It will follow that $ARDB$ is itself also a stationary path from A to D.

For if any short portion of it, say LRM, were not of stationary character, we could connect RM by a stationary short-range fixed limit path $LR'M$, and therefore

$$\int V dx \text{ (for } LR'M) > \int V dx \text{ (for } LRM);$$

$$\therefore \int V dx \text{ (for } ALR'MDB) > \int V dx \text{ (for } ALRMDB),$$

and $$\therefore > \int V dx \text{ (for } APDB),$$

and this would necessitate $\int \Delta^2 V dx$ becoming positive between A and D, which is contrary to the hypothesis that D is the first point for which the integral ceases to be negative. Therefore the variation $ALRMD$ must itself be a stationary curve between A and D, and D is itself the point conjugate to A.

Since $\int_{x_0}^{x} \Delta^2 V\,dx$ is negative so long as $x < X$, viz. the abscissa of D, $\int_{x_0}^{x} V\,dx$ has a maximum value along APD for all values of x which are less than X.

In the same way $\int_{x_0}^{x} V\,dx$ has a minimum value for all values of x which are $< X$ if $\Delta^2 V$ be positive at starting from A.

1578. If, however, the conjugate point of A occurs before B is reached, $\int_{x_0}^{x} V\,dx$, though stationary, *will have neither a maximum nor a minimum*, as we shall now show.

Take a short-range fixed limit variation FGH connecting two points, F on $ALRMD$, H on DB having $(n-1)^{\text{th}}$ order contact with these curves at the terminals F and H. Suppose

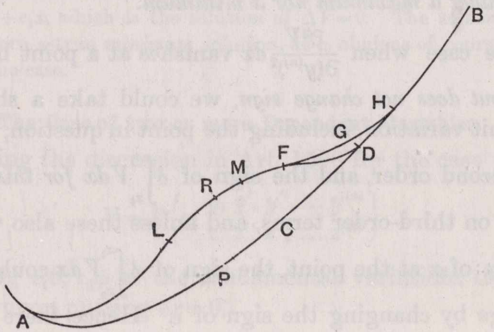

Fig. 455.

this variation to have been selected a stationary curve. Then, since by hypothesis $\dfrac{\partial^2 V}{\partial (y^{(n)})^2}dx$ is negative, this variation gives a maximum value for $\int V\,dx$ for that range, and therefore

$$\int V\,dx \text{ (for } FGH) > \int V\,dx \text{ (for } FDH).$$

Hence $\int V\,dx$ (for $ARFGHB$) $> \int V\,dx$ (for $ARFDB$),

and therefore $\qquad\qquad > \int V\,dx$ (for $APDB$).

Hence $\int V\,dx$ along $APDB$ would not have a *maximum* value; and it could not have a minimum value, for $\frac{\partial^2 V}{\partial (y^{(n)})^2}dx$ is negative.

Therefore, if the conjugate point to A lies between A and B the stationary path AB gives neither a maximum value nor a minimum value for $\int V\,dx$ for that range.

We therefore have the following test:

The stationary path AB having been determined, it will yield a maximum or a minimum value for $\int V\,dx$, according as $\frac{\partial^2 V}{\partial (y^{(n)})^2}dx$ is negative or positive from A to B, provided there be no point conjugate to A lying between A and B. But in case of such point being existent between A and B the stationary curve from A to B yields neither a maximum nor a minimum.

In the case when $\frac{\partial^2 V}{\partial (y^{(n)})^2}dx$ vanishes at a point between A and B, *but does not change sign*, we could take a short-range fixed limit variation, including the point in question, vanishing to the second order, and the sign of $\delta\int_{x_0}^{x_1} V\,dx$ *for this variation* depends on third-order terms, and unless these also vanish for the value of x at the point, the sign of $\delta\int_{x_0}^{x_1} V\,dx$ could be made to change by changing the sign of ϵ. Hence there would be neither a maximum nor a minimum for such a variation. But for other variations $\int_{x_0}^{x_1} V\,dx$ has a maximum or a minimum as before.

1579. Illustrative Examples.

(i) *Take the case of the integral* $\int (y'')^2\,dx$ *of Art.* 1502 (3). *To find the point conjugate to the point* x_0, y_0 *on the stationary curve.*

The stationary curve is $y = c_0 + c_1 x + \frac{1}{2!}c_2 x^2 + \frac{1}{3!}c_3 x^3$.

Here $\delta y = \delta c_0 + x\,\delta c_1 + \frac{1}{2!}x^2\,\delta c_2 + \frac{1}{3!}x^3\,\delta c_3$, $\delta y' = \delta c_1 + x\,\delta c_2 + \frac{1}{2!}x^2\,\delta c_3$, and these are to vanish at (x_0, y_0) and at (x, y). Hence the point conjugate to (x_0, y_0) is given by

$$\begin{vmatrix} 1, & x, & \frac{1}{2!}x^2, & \frac{1}{3!}x^3 \\ 0, & 1, & x, & \frac{1}{2!}x^2 \\ 1, & x_0, & \frac{1}{2!}x_0^2, & \frac{1}{3!}x_0^3 \\ 0, & 1, & x_0, & \frac{1}{2!}x_0^2 \end{vmatrix} = 0, \text{ that is } \tfrac{1}{12}(x-x_0)^4=0,$$

and $x=x_0$ is the only solution.

Hence, in this case, there is *no* point on the stationary curve which is conjugate to any other.

We also have $V=y''^2$ and $\dfrac{\partial^2 V}{\partial y''^2}=2$, which, being positive, the stationary curve gives a true *minimum* value to $\int y''^2 dx$ for any selected portion of the curve.

(ii) *In Ex.* 1 *of Art.* 1502, *viz. the shortest distance between two points,* $V=\sqrt{1+y'^2}$, $\Delta\equiv\theta'\dfrac{\partial}{\partial y'}$, $\dfrac{\partial^2 V}{\partial y'^2}=\dfrac{\partial}{\partial y'}\dfrac{y'}{\sqrt{1+y'^2}}=\dfrac{1}{(1+y'^2)^{\frac{3}{2}}}$, and is essentially positive. And there is obviously no point conjugate to any other on the locus $y=c_0+c_1x$, which is the solution of $\Delta V=0$. The solution arrived at is therefore a true *minimum* solution, as is obvious of course from the nature of the case.

1580. **The Case of two or more Dependent Variables.**

Resuming the discussion in Art. 1508 for the case

$$V\equiv F\left\{x, \begin{matrix} y, y', y'', \dots y^{(n)} \\ z, z', z'', \dots z^{(m)} \end{matrix}\right\},$$

and taking $\epsilon_1\theta$, $\epsilon_2\phi$ as the fundamental variations of y and z, we have, upon putting $\delta x=0$,

$$\eta=\delta y=\epsilon_1\theta,\quad \eta'=\epsilon_1\theta',\quad \eta''=\epsilon_1\theta'' \text{ etc.},$$
$$\zeta=\delta z=\epsilon_2\phi,\quad \zeta'=\epsilon_2\phi',\quad \zeta''=\epsilon_2\phi'' \text{ etc.},$$

and taking $$\Delta_1\equiv\theta\frac{\partial}{\partial y}+\theta'\frac{\partial}{\partial y'}+\dots+\theta^{(n)}\frac{\partial}{\partial y^{(n)}},$$

$$\Delta_2=\phi\frac{\partial}{\partial z}+\phi'\frac{\partial}{\partial z'}+\dots+\phi^{(m)}\frac{\partial}{\partial z^{(m)}},$$

$$\delta\int V\,dx=[H]+\int(\overline{Y}\epsilon_1\theta+\overline{Z}\epsilon_2\phi)\,dx+\frac{1}{2!}\int(\epsilon_1\Delta_1+\epsilon_2\Delta_2)^2 V dx+\frac{1}{3!}\int R\,dx,$$

and the general forms of y and z are determinable from the differential equations $\overline{Y}=0$ and $\overline{Z}=0$, and the constants involved obtainable from $[H]=0$ as before explained. And the same theorems hold as in the case of one independent variable. But the second-order variation will in its highest differential coefficients become

$$\frac{1}{2!}\int\left\{\epsilon_1{}^2(\theta^{(n)})^2\frac{\partial^2 V}{\partial(y^{(n)})^2}+2\epsilon_1\epsilon_2\theta^{(n)}\phi^{(m)}\frac{\partial^2 V}{\partial y^{(n)}\,\partial z^{(m)}}+\epsilon_2{}^2(\phi^{(m)})^2\frac{\partial^2 V}{\partial(z^{(m)})^2}\right\}dx,$$

in which the integrand is of the form

$$r\epsilon_1{}^2(\theta^{(n)})^2+2s\epsilon_1\epsilon_2\theta^{(n)}\phi^{(m)}+t\epsilon_2{}^2(\phi^{(m)})^2;$$

and, as in *D. C.*, Art. 497, the condition for an invariable sign is that $rt-s^2$ shall be positive, and the sign in question will be that of r or of t, for since $rt-s^2$ is to be positive, r and t must have the same sign.

Thus it will be essential that $\dfrac{\partial^2 V}{\partial(y^{(n)})^2}\cdot\dfrac{\partial^2 V}{\partial(z^{(m)})^2}-\left\{\dfrac{\partial^2 V}{\partial(y^{(n)})\,\partial z^{(m)}}\right\}^2$ shall be positive, and for a maximum we must have $\dfrac{\partial^2 V}{\partial(y^{(n)})^2}$ negative, and for a minimum, positive.

1581. The case $rt=s^2$ in general necessitates an examination of the terms of $(\epsilon_1\Delta_1+\epsilon_2\Delta_2)^2V$, which contain lower order differentials. This case is discussed by Mr. Culverwell in the paper cited above, to which the reader is referred.

The method employed in the last article is clearly applicable if there be more dependent variables than two. Following the same method as before, the second-order variation takes a form similar to that discussed in Art. 502, *Diff. Calc.*, with an exactly similar result.

1582. **Relative Maxima and Minima.**

It has been explained that when we are to search for the maximum or minimum value of $v\equiv\int V\,dx$, with condition $w\equiv\int W\,dx=$ a given constant, say a, we are to treat $\int(V+\lambda W)\,dx$ as an unconditional maximum or minimum, and we get

$$\begin{aligned}\delta(v+\lambda w)&\equiv\delta\int(V+\lambda W)\,dx=\int(\delta V+\lambda\,\delta W)\,dx\\&=\epsilon\int(\Delta V+\lambda\Delta W)\,dx+\frac{\epsilon^2}{2!}\int(\Delta^2 V+\lambda\Delta^2 W)\,dx+\frac{\epsilon^3}{3!}\int R\,dx,\end{aligned}$$

and with the same precautions as before with regard to choice of legitimate variations which will not violate conditions of continuity in the several differential coefficients, and which will ensure the validity of Taylor's expansion, the terms of first order having been made to vanish as a primary condition for a maximum or minimum, we have $\int(\Delta V+\lambda\,\Delta W)\,dx=0$, an equation already arrived at in Art. 1504; and then

$$\delta(v+\lambda w)=\frac{\epsilon^2}{2!}\int(\Delta^2 V+\lambda\Delta^2 W)\,dx+\frac{\epsilon^3}{3!}\int R\,dx,$$

and the terms of the highest order in the integrand $\Delta^2 V+\lambda\Delta^2 W$ are all we require in the discrimination between maxima and minima. These terms are $\frac{\partial^2 V}{\partial(y^{(n)})^2}+\lambda\frac{\partial^2 W}{\partial(y^{(n)})^2}$, and for a maximum this expression must be negative throughout the whole range of integration, and for a minimum, positive. In case of the existence of a point conjugate to (x_0, y_0), such as D of Art. 1577 on the stationary path, with abscissa X, lying between the limits of integration, the variations chosen must be such as to make $\delta\int_{x_0}^{X} W\,dx$ zero. For (see Fig. 455) beyond the point D the variation $\delta\int_{X}^{x_1} W\,dx$ has been taken as zero. Therefore X must be such that $\int_{x_0}^{X} W\,dx$ along the stationary fixed limit variation $ALRD$ has the same value as $\int_{x_0}^{x_1} W\,dx$ along the original stationary curve $APCDB$, for which in general the value of λ is different.

The equation to find the position of the conjugate point is therefore modified by the introduction of λ.

The equation of the stationary path is now of the form $y=\chi(x, \lambda, c_1, c_2, \ldots c_{2n})$. If, upon substitution of this value of y and its several differential coefficients we get

$$w\equiv\int_{x_0}^{x_1} W\,dx\equiv F(x_0, x_1, \lambda, c_1, c_2, \ldots c_{2n})=a,$$

upon variation of the constants we get the additional equation

$$\frac{\partial F}{\partial\lambda}\delta\lambda+\frac{\partial F}{\partial c_1}\delta c_1+\frac{\partial F}{\partial c_2}\delta c_2+\ldots+\frac{\partial F}{\partial c_{2n}}\delta c_{2n}=0,$$

and the equations arising from the vanishing of δy, $\delta y'$,

$\delta y'', \dots \delta y^{(n-1)}$ at (x_0, y_0) and at its conjugate, which are now altered by the presence of λ to

$$\left.\begin{array}{l}
\dfrac{\partial y}{\partial \lambda}\delta\lambda + \dfrac{\partial y}{\partial c_1}\delta c_1 + \dfrac{\partial y}{\partial c_2}\delta c_2 + \dots + \dfrac{\partial y}{\partial c_{2n}}\delta c_{2n} = 0, \\
\dfrac{\partial y'}{\partial \lambda}\delta\lambda + \dfrac{\partial y'}{\partial c_1}\delta c_1 + \dfrac{\partial y'}{\partial c_2}\delta c_2 + \dots + \dfrac{\partial y'}{\partial c_{2n}}\delta c_{2n} = 0, \\
\qquad\qquad \text{etc.,} \\
\dfrac{\partial y^{(n-1)}}{\partial \lambda}\delta\lambda + \dfrac{\partial y^{(n-1)}}{\partial c_1}\delta c_1 + \dfrac{\partial y^{(n-1)}}{\partial c_2}\delta c_2 + \dots + \dfrac{\partial y^{(n-1)}}{\partial c_{2n}}\delta c_{2n} = 0,
\end{array}\right\} \begin{array}{l}\text{true at} \\ (x_0, y_0) \\ \text{and its} \\ \text{conjugate} \\ (x, y).\end{array}$$

These $2n+1$ equations give, upon the elimination of $\delta\lambda, \delta c_1, \delta c_2, \dots \delta c_{2n}$,

$$\begin{vmatrix}
\dfrac{\partial y}{\partial \lambda}, & \dfrac{\partial y}{\partial c_1}, & \dfrac{\partial y}{\partial c_2}, & \dots\dots\dots & \dfrac{\partial y}{\partial c_{2n}} \\
\dfrac{\partial y'}{\partial \lambda}, & \dfrac{\partial y'}{\partial c_1}, & \dfrac{\partial y'}{\partial c_2}, & \dots\dots\dots & \dfrac{\partial y'}{\partial c_{2n}} \\
\dots & \dots & \dots & \dots & \dots \\
\dfrac{\partial y^{(n-1)}}{\partial \lambda}, & \dfrac{\partial y^{(n-1)}}{\partial c_1}, & \dfrac{\partial y^{(n-1)}}{\partial c_2}, & \dots\dots & \dfrac{\partial y^{(n-1)}}{\partial c_{2n}}, \\
\left(\dfrac{\partial y}{\partial \lambda}\right)_0, & \left(\dfrac{\partial y}{\partial c_1}\right)_0, & \left(\dfrac{\partial y}{\partial c_2}\right)_0, & \dots\dots & \left(\dfrac{\partial y}{\partial c_{2n}}\right)_0, \\
\dots & \dots & \dots & \dots & \dots \\
\left(\dfrac{\partial y^{(n-1)}}{\partial \lambda}\right)_0, & \left(\dfrac{\partial y^{(n-1)}}{\partial c_1}\right)_0, & \left(\dfrac{\partial y^{(n-1)}}{\partial c_2}\right)_0, & \dots & \left(\dfrac{\partial y^{(n-1)}}{\partial c_{2n}}\right)_0 \\
\dfrac{\partial F}{\partial \lambda}, & \dfrac{\partial F}{\partial c_1} & \dfrac{\partial F}{\partial c_2} & \dots\dots\dots & \dfrac{\partial F}{\partial c_{2n}},
\end{vmatrix} = 0,$$

to determine the position of a point (x, y) on the stationary path conjugate to (x_0, y_0).

If such a point occurs between the limits $x=x_0$ and $x=x_1$ on the stationary path, this path will give neither a maximum nor a minimum.

1583. When V contains more than one dependent variable, and these dependent variables are connected by an equation $L=0$, viz. the case discussed in Art. 1513, we proceed as there explained with the first-order variation to obtain the stationary solution. In passing to the second-order variation, we have

$$\frac{1}{2!}\int \Delta^2(V+\lambda L)dx, \quad \text{where } \Delta \equiv \epsilon_1\Delta_1 + \epsilon_2\Delta_2 \quad \text{(Art. 1580)},$$

where $\epsilon_1\theta$ and $\epsilon_2\phi$ are the fundamental variations of y and z, and $\epsilon_1\theta^{(n)}$, $\epsilon_2\phi^{(n)}$ those of $y^{(n)}$ and $z^{(n)}$. We shall suppose that the orders of the highest differentials occurring in V and L are the same. Then taking as before a short-range variation, the variations $\theta, \theta', \theta'', \ldots \theta^{(n-1)}$ may be all neglected in comparison with $\theta^{(n)}$, and $\phi, \phi', \phi'', \ldots \phi^{(n-1)}$ in comparison with $\phi^{(n)}$. The only terms of $\Delta^2(V+\lambda L)$ which need be retained are therefore

$$\frac{\partial^2(V+\lambda L)}{\partial(y^{(n)})^2}\epsilon_1{}^2(\theta^{(n)})^2+2\frac{\partial^2(V+\lambda L)}{\partial y^{(n)}\,\partial z^{(n)}}\epsilon_1\epsilon_2\theta^{(n)}\phi^{(n)}+\frac{\partial^2(V+\lambda L)}{\partial(z^{(n)})^2}\epsilon_2{}^2(\phi^{(n)})^2,$$

where $\theta^{(n)}, \phi^{(n)}$ are not independent but connected by the equation

$$\frac{\partial L}{\partial y^{(n)}}\epsilon_1\theta^{(n)}+\frac{\partial L}{\partial z^{(n)}}\epsilon_2\phi^{(n)}=0,$$

so that $$\left\{\frac{\partial^2(V+\lambda L)}{\partial(y^{(n)})^2}\left(\frac{\partial L}{\partial z^{(n)}}\right)^2-2\frac{\partial^2(V+\lambda L)}{\partial y^{(n)}\,\partial z^{(n)}}\frac{\partial L}{\partial z^{(n)}}\frac{\partial L}{\partial y^{(n)}}+\frac{\partial^2(V+\lambda L)}{\partial(z^{(n)})^2}\left(\frac{\partial L}{\partial y^{(n)}}\right)^2\right\}dx$$

must retain the same sign throughout the integration if a maximum or a minimum is to occur; and that sign must be negative for a maximum, positive for a minimum.

For details of the case in which the orders of the highest degree differentials in V and L are not the same, the reader is referred to Mr. Culverwell's paper [p. 252, *L. Math. Soc. Proc.*, Vol. XXIII.].

1584. **Bibliography.**

Readers wishing to pursue the subject of the Calculus of Variations further are referred to Todhunter's ***History of the Progress of the Calculus of Variations*** during the nineteenth century and ***Researches in the Calculus of Variations***, and to the treatises on the subject by Jellett and Strauch. Professor Williamson, in Chapter XV. of his ***Integral Calculus***, gives an account of the "Sign of Substitution" used by Sarrus in his Essay, ***Recherches sur le Calcul des Variations***, and makes much use of the same. In his Chapter XVII. the student will find much useful information with regard to the bounding variations in the case of a double integral and a discussion of some cases which arise in the treatment of the partial differential equation as well as several other interesting matters. The papers by Culverwell, of which considerable use has been made, should be referred to in ***R.S. Trans.***, 1887, and in ***Proc. of the Lond. Math. Soc.***, 1891-2. Other writers are Moigno and Lindelöf referred to by Dr. Williamson (*I.C.*, p. 465), Lagrange (*Th. des Fonct.*), Lacroix (*Calc. Int.*, pp. 655-724), Jacobi, Legendre (***Mém. de l'Acad. des Sc.***, 1783), De Morgan (*D. and I. Calc.*, pp. 446-474), Poisson (***Mém. de l'Institut***, T. XII.), Abbott (*Calc. of Var.*), Airy (***Math. Tracts***), Woodhouse (***Isoperimetrical Problems***).

PROBLEMS.

1. Find the stationary value of $\int V dx$, taken between definitely fixed limits, where $V = y'^2 + 2myy' + ny^2$, and discuss its nature.

[LACROIX, *C.I.*, II., p. 721.]

2. Mark out the range of limits on the parabola $(x+a)^2 = 4cy$ between which the integral $\int_{x_0}^{x_1} y\left(\frac{dy}{dx}\right)^{-2} dx$ is a maximum, the range between which it is a minimum, and the range between which it is neither.

[MATH. TRIP., 1890.]

3. The integral $\iint f(x, y, z, p, q)\,dx\,dy$ is found to be stationary when taken over the surface $z = \phi(x, y)$; show, by confining the actual variation of z to a small area on this surface, that the variation of the integral cannot always have the same sign within limits specified by a given curve through which the surface must pass, unless $\frac{\partial^2 f}{\partial p^2}\delta p^2 + 2\frac{\partial^2 f}{\partial p\,\partial q}\delta p\,\delta q + \frac{\partial^2 f}{\partial q^2}\delta q^2$ always retains the same sign within these limits, and deduce a criterion for discriminating maxima and minima. Show further that, for a true maximum or minimum, it must not be possible to draw a consecutive surface of stationary character which meets the original one in a closed curve within the given limits. Are these conditions sufficient as well as necessary?

[MATH. TRIP., 1890.]