LVIII

ICOSIAN CYCLES

[Note-book 139, 13 November 1863.]

1. Let τ denote any triad of successive points, on my Icosian Board, for example the initial triad, *bcd*, using here italic letters. I refer here to the adapted diagram* published by Mr Jacques. For the moment, it may be sufficient to note here, that we have the six positive subcycles of five,

$$bcdfg,\quad gfkjh,\quad fdmlk,\quad dcpnm,\quad cbzqp,\quad bghxz;$$

and the six negative subcycles of five, respectively opposite to the foregoing,

$$tvwrs,\quad srqpn,\quad rwxzq,\quad wvjhx,\quad vtlkj,\quad tsnml,$$

because we have the ten pairs of opposite points on the dodecahedron,

$$bt,\quad cv,\quad dw,\quad fr,\quad gs,\quad hn,\quad jp,\quad kq,\quad lz,\quad mx;$$

we have also ten pairs of adjacent points,

$$bg,\quad cp\quad dm,\quad fk,\quad hx,\quad jv,\quad lt,\quad ns,\quad qz,\quad rw,$$

besides the twenty other pairs of adjacent points given by the twenty consonants of the English Alphabet, in their order,

$$b\,c\,d\,f\,g\,h\,j\,k\,l\,m\,n\,p\,q\,r\,s\,t\,v\,w\,x\,z,\quad b\,\ldots.$$

2. Let $\alpha\tau$ denote the triad formed from τ, by advancing a step within the plane, or within the pentagon, or subcycle of five, to which the triad τ belongs, for example, if $\tau = bcd$ as above, then $\alpha\tau = cdf$, $\alpha^2\tau = dfg$, $\alpha^3\tau = fgb$, $\alpha^4\tau = gbc$, $\alpha^5\tau = bcd = \tau$ and therefore

$$\alpha^5 = 1.$$

3. Let the triad $\beta\tau$, on the other hand, be formed from τ, by leaving the plane, or the pentagon, which is determined by τ, but still entering the second and third points of the given triad τ, as the first and second points of the new triad, $\beta\tau$ for example, if $\tau = bcd$, as before, then $\beta\tau = cdm$, $\beta^2\tau = dml$, $\beta^3\tau = mlt$, $\beta^4\tau = ltv$, $\beta^5\tau = tvw$, $\beta^6\tau = vwx$, $\beta^7\tau = wxz$, $\beta^8\tau = xrb$, $\beta^9\tau = rbc$, $\beta^{10}\tau = bcd = \tau$ so that

$$\beta^{10} = 1.$$

We see at the same time that $\quad \beta^5 = \omega, \quad$ and $\quad \omega^2 = 1,$

if ω denote the operation of passing from a given triad *bcd* to the opposite triad,

$$tvw = \omega\,.\,bcd.$$

4. Compounding successive operations of these two kinds, α and β, we have, if

$$\tau = bcd,\quad \alpha\tau = cdf,\quad \beta\alpha\tau = dfk,\quad \alpha\beta\alpha\tau = fkl,\quad \beta\alpha\beta\alpha\tau = klt,$$

and $\quad \alpha\beta\alpha\beta\alpha\tau = ltv = \beta^4\tau, \quad$ by No. 3;

hence

$$\alpha\beta\alpha\beta\alpha = \beta^4.$$

* [For the diagram referred to, see Appendix 2. In place of the twenty consonants used by Hamilton the numerals 1 to 20 are used in the diagram.]

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5. Again, $\alpha^2\beta\alpha\tau = \alpha . fkl = klm, \quad \beta\alpha^2\beta\alpha\tau = lmn, \quad \alpha\beta\alpha^2\beta\alpha\tau = mns;$

and $\alpha\beta\tau = \alpha . cdm = dmn, \quad \beta\alpha\beta\tau = mns = \alpha\beta\alpha^2\beta\alpha\tau,$

therefore $\alpha\beta\alpha^2\beta\alpha = \beta\alpha\beta.$

6. If we had started with *pcd*, instead of *bcd*, as the first subject of the operations, we should have had successively,

$$\alpha . pcd = cdm = \beta\tau; \quad \beta\alpha . pcd = \beta^2\tau = dml; \quad \beta^2\alpha . pcd = \beta^3\tau = mlt;$$

$$\beta^3\alpha . pcd = \beta^4\tau = ltv; \quad \beta . pcd = cdf = \alpha\tau; \quad \beta^2 . pcd = \beta\alpha\tau = dfk;$$

$$\alpha\beta\alpha\beta^2 . pcd = \alpha\beta\alpha\beta\alpha\tau = ltv,$$

abstracting then from the new operand, *pcd*, we have now,

$$\alpha\beta\alpha\beta^2 = \beta^3\alpha;$$

and we see that, for the same reason, in any equation between two symbolic products, of α and β or their powers, taken in any two determined orders, it is permitted to interchange the two initial (or right-hand) factors.

7. Again, if we thus pass by the successive processions, β, β, α, β, α from *pcd* to *cdf*, *dfk*, *fkl*, *klt*, and *ltv*, we can then return, by the processions α, β, α, β, β from *vtl*, to *tlk*, *lkf*, *kfd*, *fdc*, to *dcp*; and in like manner if we have passed by the processions α, β, β, β in the order thus written, from *pcd* to *cdm*, *dml*, *mlt*, and *ltv*, we can then return by the processions β, β, β, α from *vtl* to *tlm*, *lmd*, *mdc*, and *dcp*; we may, therefore, write the equation,

$$\beta^2\alpha\beta\alpha = \alpha\beta^3.$$

Generally, we may invert the order of the factors, on both sides, of any such symbolic equation.

8. Interchanging, by No. 6, the right-hand factors α and β in the equation last written (No. 7), we obtain this other equation,

$$\beta^2\alpha\beta^2 = \alpha\beta^2\alpha,$$

which might also have been at once obtained from the equation of No. 6,

$$\alpha\beta\alpha\beta^2 = \beta^3\alpha$$

by interchanging the left-hand factors. In general, a combination of recent results conducts to the rule, that it is permitted, in any symbolic equation of the form here considered, to interchange the final (or left-hand) factors.

9. With the principles of transformation (Nos. 6, 7, 8), and with the four particular equations (of Nos. 2, 3, 4, 5),

$$\alpha^5 = 1, \quad \beta^{10} = 1, \quad \alpha\beta\alpha\beta\alpha = \beta^4; \quad \alpha\beta\alpha^2\beta\alpha = \beta\alpha\beta, \quad (\text{or } (\alpha\beta\alpha)^2 = \beta\alpha\beta),$$

it is possible to derive a large number of other equations of the same sort, which are useful in the study of the cycles, or subcycles, of points on the Icosian Board, or corners of the pentagonal dodecahedron.

($\beta^{10} = 1$ can be deduced from the others. In fact $\beta^8 = \alpha\beta\alpha\beta\alpha^2\beta\alpha\beta\alpha = \alpha\beta^2\alpha\beta^2\alpha = \alpha^2\beta^2\alpha^2$, $\beta^{10} = (\alpha^2\beta^2)^2 = \alpha^5 = 1$, see below.)

10. For example, from the last written equation, we can derive at sight these others,

$$\alpha\beta\alpha^2\beta^2 = \beta\alpha^2, \quad \beta^2\alpha^2\beta\alpha = \alpha^2\beta, \quad \beta^2\alpha^2\beta^2 = \alpha^3;$$

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and therefore (because $\alpha^5=1$)

$$\alpha^2\beta^2\alpha^2\beta^2=1, \quad \beta^2\alpha^2\beta^2\alpha^2=1, \quad \text{or} \quad (\alpha^2\beta^2)^2=1, \quad (\beta^2\alpha^2)^2=1.$$

11. Again, the equation $\alpha^2\beta^2\alpha^2\beta^2=1$ gives (because $\alpha^6=\alpha$), $\alpha\beta^2\alpha^2\beta^2=\alpha^4$, and therefore $\alpha\beta^2\alpha^2\beta^2\alpha=\alpha^5=1$ and generally, when one member of a symbolical equation of this sort is unity, it is permitted to make any cyclical permutation of the factors of the other member, and also (as is easily seen), to reverse their order. We may then write,

$$(\alpha\beta^2\alpha)^2=1; \quad (\beta\alpha^2\beta)^2=\beta\alpha^2\beta^2\alpha^2\beta=1.$$

12. But from $\alpha\beta\alpha\beta\alpha=\beta^4$, we derive

$$\beta^2\alpha\beta\alpha=\alpha\beta^3, \quad \alpha\beta\alpha\beta^2=\beta^3\alpha, \quad \beta^2\alpha\beta^2=\alpha\beta^2\alpha;$$

hence $$1=(\alpha\beta^2\alpha)^2=\beta^2\alpha\beta^2.\alpha\beta^2\alpha=(\beta^2\alpha)^3;$$

thus $$(\beta^2\alpha)^3=1, \quad (\beta\alpha\beta)^3=1, \quad (\alpha\beta^2)^3=1.$$

13. Again $(\beta^2\alpha\beta^2)^2=(\alpha\beta^2\alpha)^2=1$; therefore

$$\beta^2\alpha\beta^4\alpha\beta^2=1, \quad \beta\alpha\beta^4\alpha\beta^3=1, \quad (\alpha\beta^4)^2=1, \quad (\beta^4\alpha)^2=1, \quad (\beta^3\alpha\beta)^2=1, \quad (\beta\alpha\beta^3)^2=1.$$

Hence also $$\beta^4\alpha\beta^4\alpha=1=\beta^{10}, \quad \alpha\beta^4\alpha=\beta^6, \quad \alpha\beta^5=\beta^5\alpha.$$

In fact, by No. 3, this last equation is evidently true under the form,

$$\alpha\omega=\omega\alpha.$$

14. Since $\alpha\beta\alpha\beta\alpha=\beta^4$ and $\alpha\beta^4\alpha=\beta^6$, while $\beta^{10}=1$, we have

$$\alpha^2\beta\alpha\beta\alpha^2=\beta^6, \quad \alpha^2\beta\alpha\beta\alpha^2\beta^4=1.$$

But $$\alpha\beta\alpha\beta\alpha\beta^6=1.$$

Also $$(\alpha\beta\alpha\beta\alpha)^2\beta^2=1.$$

Also $$(\alpha\beta)^3=\beta^5=\omega; \quad (\alpha\beta)^6=\beta^{10}=\omega^2=1.$$

Hence $$(\beta\alpha)^3=\beta^5=\omega; \quad (\beta\alpha)^6=1.$$

Also $$\alpha\beta\alpha\beta^3=\beta^3\alpha\beta;$$

therefore $$(\alpha\beta\alpha\beta^3)^2=1.$$

15. So far, then, we have the ten cycles, or subcycles, following:

I, 5, $\alpha^5=1$,

II, 8, $(\alpha^2\beta^2)^2=1$,

III, 9, $(\alpha\beta^2)^3=1$,

IV, 10, $(\alpha\beta^4)^2=1$,

V, 10, $\beta^{10}=1$,

VI, 11, $\alpha^2\beta\alpha\beta\alpha^2\beta^4=1$,

VII, 11, $\alpha\beta\alpha\beta\alpha\beta^6=1$,

VIII, 12, $(\alpha\beta\alpha\beta\alpha)^2\beta^2=1$,

IX, 12, $(\alpha\beta\alpha\beta^2)^2=1$,

X, 12, $(\alpha\beta)^6=1$.

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16. Since $(\alpha\beta\alpha)^2=\beta\alpha\beta$ and $\alpha\beta\alpha\beta\alpha=\beta^4$, we have

$$(\alpha\beta\alpha)^3=\beta^5=\omega;\quad (\alpha\beta\alpha)^6=1;\quad (\alpha^2\beta)^6=1;\quad \text{(cycle of 18)}.$$

Also $(\alpha\beta\alpha)^3\beta^5=1$; cycle of 14.

Hence $(\alpha\beta\alpha)^2\beta^5\alpha\beta\alpha=1$,

but $\beta^2\alpha\beta\alpha=\alpha\beta^3$,

therefore $(\alpha\beta\alpha)^2\beta^3\alpha\beta^3=\alpha\beta\alpha^2\beta(\alpha\beta^3)^2=1$; cycle of 13.

Also, $\beta^4=\alpha(\alpha\beta\alpha)^2\alpha=\alpha^2\beta\alpha^2\beta\alpha^2$,

therefore $(\alpha^2\beta)^3=\beta^5=\omega$

and, as before, $(\alpha^2\beta)^6=1$, $(\alpha^2\beta)^2\alpha^2\beta^6=1$; cycle of 14.

Also, since $\alpha\beta\alpha\beta^2=\beta^3\alpha$ and $(\alpha\beta^3\alpha\beta)^2=1$, therefore

$(\alpha^2\beta\alpha\beta^3)^2=1$; cycle of 14.

We have, therefore, (reversing the cycle $(\alpha^2\beta)^6=1$) the four additional cycles:

XI, 13, $(\alpha\beta\alpha)^2\beta^3\alpha\beta^3=1$,

XII, 14, $(\alpha^2\beta)^3\beta^5=1$,

XIII, 14, $(\alpha^2\beta\alpha\beta^3)^2=1$,

XIV, 14, $(\alpha\beta\alpha)^3\beta^5=1$.

17. Besides cyclical formulae, (equation = 1), we have at this stage the equations:

$$\omega=\beta^5=(\alpha\beta)^3=(\beta\alpha)^3=(\alpha\beta\alpha)^3=(\alpha^2\beta)^3=(\beta\alpha^2)^3;$$

$$\alpha\beta\alpha\beta\alpha=\beta^4,\quad \beta^2\alpha\beta\alpha=\alpha\beta^3,\quad \beta^3\alpha=\alpha\beta\alpha\beta^2,\quad \alpha\beta^2\alpha=\beta^2\alpha\beta^2;$$

$$(\alpha\beta\alpha)^2=\alpha\beta\alpha^2\beta\alpha=\beta\alpha\beta;\quad \beta^2\alpha^2\beta\alpha=\alpha^2\beta;\quad \alpha\beta\alpha^2\beta^2=\beta\alpha^2;\quad \beta^2\alpha^2\beta^2=\alpha^3.$$

18. The twenty-four cycles, or subcycles, which I have found to exist and to be the only ones, may be expressed by the twenty-four equations:

$1=\alpha^5$	I, 5
$=(\alpha^2\beta^2)^2$	II, 8
$=(\alpha\beta^2)^3$	III, 9
$=(\alpha\beta^4)^2$	IV, 10
$=\beta^{10}$	V, 10
$=\alpha^2\beta\alpha\beta\alpha^2\beta^4$	VI, 11
$=\alpha\beta\alpha\beta\alpha\beta^6$	VII, 11
$=(\alpha\beta\alpha\beta\alpha)^2\beta^2$	VIII, 12
$=(\alpha\beta\alpha\beta^3)^2$	IX, 12
$=(\alpha\beta)^6$	X, 12
$=\alpha\beta\alpha^2\beta(\alpha\beta^3)^2$	XI, 13
$=(\alpha^2\beta)^3\beta^5$	XII, 14
$=(\alpha^2\beta\alpha\beta^3)^2$	XIII, 14
$=\alpha^2\beta\alpha\beta^5\alpha\beta\alpha^2\beta$	XIV, 14

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$=(\alpha^2\beta)^2\alpha\beta^2\alpha\beta\alpha\beta^3$ XV, 15

$=(\alpha^2\beta)^3(\alpha\beta)^3$ XVI, 15

$=(\alpha^2\beta)^3\beta^2(\alpha\beta^3)^2$ XVII, 16

$=(\alpha^2\beta\alpha\beta\alpha^2\beta)^2$ XVIII, 16

$=\alpha^2\beta\alpha^2\beta^3\alpha^2\beta\alpha\beta^5$ XIX, 17

$=\alpha^2(\beta\alpha\beta)^2\alpha(\alpha\beta^3)^2$ XX, 17

$=(\alpha^2\beta)^2(\alpha\beta)^2(\alpha^2\beta)^2\beta^2$ XXI, 18

$=\alpha^2(\beta\alpha\beta)\alpha\beta^2\alpha\beta\alpha^2\beta^3$ XXII, 18

$=(\alpha^2\beta)^6$ XXIII, 18

$=(\alpha^2\beta\alpha^2\beta^5)^2$ XXIV, 20

19. We may also write VII as $(\alpha\beta)^3\beta^5=1$; IX as $((\alpha\beta)^2\beta^2)^2=1$; XI as $(\alpha\beta\alpha)^2\beta^3\alpha\beta^3=1$; XII as $\beta^3\alpha^2\beta\alpha^2\beta\alpha^2\beta^3=1$; XIV as $(\alpha\beta\alpha)^3\beta^5=1$; XVII as $\alpha^2\beta\alpha(\alpha\beta^3)^3=1$; XVIII as $((\beta\alpha\beta)^2\alpha^2)^2=1$; and XIX as $(\alpha^2\beta)^2\beta^2\alpha^2\beta\beta^5=1$.

20. If we operate on the initial triad *bcd* by these twenty symbolic products read from left to right, the results are:

I. *bcdfgbcd*,
II. *bcdfghxzbcd*,
III. *bcdfjkhxzbcd*,
IV. *bcdfjkvwxzbcd*,
V. *bcdmltvwxzbcd*,
VI. *bcdfghjvwxzbcd*,
VII. *bcdfkltvwxzbcd*,
VIII. *bcdfkltvjhxzbcd*,
IX. *bcdfkltsrwxzbcd*,
X. *bcdfkltvwrqzbcd*,
XI. *bcdfklmnsrwxzbcd*,
XII. *bcdfghjkltv wxrbcd*,
XIII. *bcdfghjvtsrwxzbcd*,
XIV. *bcdfghjvtsnpqzbcd*,
XV. *bcdfghjkltsrwxzbcd*,
XVI. *bcdfghjkltvwrqzbcd*,
XVII. *bcdfghjklmnsrwxzbcd*,
XVIII. *bcdfkjhxwvtsnpqzbcd*,
XIX. *bcdfghjklmnstvwxzbcd*,
XX. *bcdfghjvtlmnsrwxzbcd*,
XXI. *bcdfghjkltsnpqrwxzbcd*,
XXII. *bcdfghjvtlmnpqrwxzbcd*,
XXIII. *bcdfghjkltvwrsnpqzbcd*,
XXIV. *bcdfghjklmnpqrstvwxzbcd*.

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21. Of these,*

XXIV. *bcdfghjklmnpqrstvwxz* is a finished cycle of 20,

XXI. *bcdfghjkltsnpqrwxz* a finished cycle of 18,

XXII. *bcdfghjvtlmnpqrwxz* a finished cycle of 18,

XVIII. *bcdfkjhxwvtsnpqz* a finished cycle of 16.

22. Starting with the cycles I and V, or with the equations $\alpha^5 = 1$, $\beta^{10} = 1$, and with the two other equations, $\alpha\beta\alpha\beta\alpha = \beta^4$, $(\alpha\beta\alpha)^2 = \beta\alpha\beta$ and admitting that an equation may be read backwards and that extreme factors may be interchanged, we have proved symbolically the cycles II, III, IV, VI, VII, VIII, IX, X, XI, XII, XIII, XIV, XXIII.

23. The cycle XII gives $(\alpha^2\beta)^2\alpha . \alpha\beta^3 . \beta^3 = 1$; but $\alpha\beta^3 = \beta^2\alpha\beta\alpha$; $(\alpha^2\beta)^2\alpha\beta^2\alpha\beta\alpha\beta^3 = 1$, as in XV. Multiply $(\alpha^2\beta)^3 = \beta^5$ and $(\alpha\beta)^3 = \beta^5$ to get XVI. Changing $\alpha\beta^2\alpha$ in XV to $\beta^2\alpha\beta^2$, we get XVII, which may be written thus, $1 = \alpha^2\beta\alpha . \alpha\beta^3 . \alpha\beta . \beta^2\alpha\beta^2 . \beta$, but $\alpha\beta^3 = \beta^2\alpha\beta\alpha$ and $\beta^2\alpha\beta^2 = \alpha\beta^2\alpha$, therefore $1 = \alpha^2\beta\alpha . \beta^2\alpha\beta\alpha . \alpha\beta . \alpha\beta^2\alpha . \beta = (\alpha^2(\beta\alpha\beta)^2)^2$; therefore XVIII.

24. Again, because $(\alpha\beta\alpha)^2 = \beta\alpha\beta$, we have $\beta^2\alpha^2\beta\alpha = \alpha^2\beta$; therefore $(\alpha^2\beta)^2\beta^2\alpha^2\beta\alpha = (\alpha^2\beta)^3 = \beta^5$; therefore $(\alpha^2\beta)^2\beta^2\alpha^2\beta\alpha\beta^5 = 1$; therefore XIX. Or thus, by XIV, $1 = (\alpha^2\beta)^3\beta^5$ and $\alpha^2\beta = \beta^2\alpha^2\beta\alpha$ and therefore XIX.

It may have been observed that XIII gives

$$1 = \alpha^2\beta\alpha . \beta^3\alpha^2 . \beta\alpha\beta^3;$$

but $\beta^3\alpha = \alpha\beta\alpha\beta^2$ and $\alpha\beta^2\alpha = \beta^2\alpha\beta^2$; $\beta^3\alpha^2 = \alpha\beta^3\alpha\beta^2$, therefore

$$1 = \alpha^2\beta\alpha^2\beta\alpha\beta^3\alpha\beta^3 = \alpha^2\beta\alpha(\alpha\beta^3)^3 = (\alpha^2\beta)^2\beta^2(\alpha\beta^3)^2,$$

therefore XVIII.

Write $\alpha\beta^3 = \beta^2\alpha\beta\alpha$; therefore

$$1 = \alpha^2\beta\alpha\beta^2\alpha\beta\alpha(\alpha\beta^3)^2 = \alpha^2(\beta\alpha\beta)^2\alpha(\alpha\beta^3)^2; \quad \text{XX}.$$

It only remains then to prove XXI, XXII, XXIV.

25. In XVII, $\beta^2\alpha\beta^3 = \beta^2 . \beta^2\alpha\beta\alpha = \beta^4\alpha\beta\alpha = \alpha\beta\alpha\beta\alpha^2\beta\alpha$; therefore

$$1 = (\alpha^2\beta)^2\alpha\beta\alpha\beta\alpha^2\beta\alpha\alpha\beta^3 = (\alpha^2\beta)^2(\alpha\beta)^2(\alpha^2\beta)^2\beta^2; \quad \text{XXI}.$$

And this equation may also be thus written:

$$1 = (\alpha^2\beta)^2\alpha\beta(\alpha\beta\alpha)^2\alpha\beta^3; \quad \text{but} \quad (\alpha\beta\alpha)^2 = \beta\alpha\beta;$$

therefore $$1 = (\alpha^2\beta)^2\alpha\beta^2\alpha\beta\alpha\beta^3,$$

which is XV. Conversely, in XV, we may change $\beta\alpha\beta$ to $(\alpha\beta\alpha)^2$, and so obtain

$$1 = (\alpha^2\beta)^2\alpha\beta(\alpha\beta\alpha)^2\alpha\beta^3,$$

namely XXI. Again in XIX, we may change β^5 to $(\alpha\beta)^3$ and so obtain

$$1 = \alpha^2\beta\alpha^2\beta^3(\alpha^2\beta)^2(\alpha\beta)^2,$$

that is $1 = (\alpha^2\beta)^2(\alpha\beta)^2(\alpha^2\beta)^2\beta^2$ as in XXI, which may thus be obtained at pleasure from XV, or from XVII, or from XIX. But we have still to deduce the cycle XXII of 18 and the complete cycle XXIV of 20.

* [A reference to a former catalogue is here omitted.]

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26. The equation XV may be thus written,

$$1 = \alpha^2\beta\alpha^2(\beta\alpha\beta)^2\alpha\beta^3.$$

Now $$(\beta\alpha\beta)^2 = \beta\alpha\beta(\alpha\beta\alpha)^2 = \beta\,.\,\alpha\beta\alpha\beta\alpha\,.\,\alpha\beta\alpha = \beta^5\alpha\beta\alpha = \beta^3\,.\,\beta^2\alpha\beta\alpha;$$

also $$\alpha\beta^3 = \beta^2\alpha\beta\alpha;$$

therefore we may either ascend from XV to a next higher cycle of the form

$$1 = \alpha^2\beta\alpha^2\beta^3\alpha\beta^3\alpha\beta^3 = \alpha^2\beta\alpha(\alpha\beta^3)^3,$$

namely to XVII, or to a still higher cycle, with equation,

$$1 = \alpha^2\beta\alpha(\beta^2\alpha\beta\alpha)^2\alpha\beta^3 = \alpha^2\beta\alpha\beta^2\alpha\beta\alpha\beta^2\alpha\beta\alpha^2\beta^3 = \alpha^2(\beta\alpha\beta)^2\alpha\beta^2\alpha\beta\alpha^2\beta^3,$$

namely XXII. At the same time, we have proved that

$$\alpha(\beta\alpha\beta)^2 = (\alpha\beta^3)^2 = (\beta^2\alpha\beta\alpha)^2.$$

The cycle XXII is therefore deducible from the lower cycle, XV. It is simpler now to change the first factor $\alpha\beta^3$, in XX, to $\beta^2\alpha\beta\alpha$, for then we get at once

$$1 = \alpha^2(\beta\alpha\beta)^2\alpha\,.\,\beta^2\alpha\beta\alpha\,.\,\alpha\beta^3,$$

that is XXII.

27. Finally, as regards the complete and cyclical succession, XXIII, or to show that

$$\alpha^2\beta\alpha^2\beta^5\alpha^2\beta\alpha^2 = \beta^5 = (\alpha^2\beta)^3,$$

or that $$\beta^4\alpha^2\beta\alpha^2 = \alpha^2\beta = \beta^2\alpha^2\beta\alpha$$

we have only to change $\alpha^2\beta$ in $\beta^2\,.\,\alpha^2\beta\,.\,\alpha$ to $\beta^2\alpha^2\beta\alpha$. Thus, returning, from $\beta\alpha\beta = (\alpha\beta\alpha)^2 = \alpha\beta\alpha^2\beta\alpha$ we infer, $\alpha^2\beta = \beta^2\,.\,\alpha^2\beta\,.\,\alpha = \beta^2\,.\,\beta^2\alpha^2\beta\alpha\,.\,\alpha = \beta^4\alpha^2\beta\alpha^2$; therefore,

$$\beta^5 = (\alpha^2\beta)^3 = (\alpha^2\beta)^2\beta^4\alpha^2\beta\alpha^2 = \alpha^2\beta\alpha^2\beta^5\,.\,\alpha^2\beta\alpha^2;$$

therefore, $1 = \beta^5\,.\,\beta^5 = (\alpha^2\beta\alpha^2\beta^5)^2$, that is the cycle XXIV.

28. We have therefore symbolically proved, from the principles stated in No. 22, or in No. 9, the correctness of the twenty-two mixed cycles, or of those which involve both α and β as factors in their equations; but we have not yet proved that there are no other mixed cycles of the kind sought.

29. Nor has it yet been symbolically proved, that all these twenty-four cycles are passages without repetition of a letter; or that no letter occurs more than once, in any of the cycles above assigned. This has however been proved by actual construction, in No. 20, of the twenty-four literal formulae, with *bcd* for the initial triad in each.

30.

No.	$\alpha\beta$	$\alpha\beta$	$\alpha\beta$	$\alpha\beta$	$\alpha\beta$	$\alpha\beta$	No. of Points	No.	$\alpha\beta$	$\alpha\beta$	$\alpha\beta$	$\alpha\beta$	$\alpha\beta$	$\alpha\beta$	No. of Points
1	21	21	21	21	22	—	19	10	21	21	11	20	—	—	13
2	21	21	21	22	—	—	16	11	21	21	11	11	21	20	18
3	21	21	21	11	20	—	16	12	21	21	11	11	22	—	17
4	21	21	21	12	20	—	17	13	21	21	11	11	12	20	18
5	21	21	21	12	11	—	17	14	21	21	11	11	12	11	18
6	21	21	23	21	20	—	19	15	21	21	12	20	—	—	14
7	21	21	23	21	11	—	19	16	21	21	12	11	23	—	19
8	21	21	23	12	20	—	19	17	21	21	13	20	—	—	16
9	21	21	23	13	—	—	18	18	21	21	14	21	—	—	17

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No.	$\alpha\beta$	$\alpha\beta$	$\alpha\beta$	$\alpha\beta$	$\alpha\beta$	$\alpha\beta$	No. of Points
19	21	23	21	21	20	—	19
20	21	23	21	22	—	—	18
21	21	23	21	12	20	—	19
22	21	23	21	12	11	—	19
23	21	23	11	20	—	—	15
24	21	23	11	11	22	—	19
25	21	23	12	20	—	—	16
26	21	23	13	21	—	—	17
27	21	25	21	22	—	—	20
28	21	25	11	20	—	—	17
29	21	25	12	20	—	—	18
30	21	27	21	20	—	—	20
31	21	27	12	20	—	—	20
32	21	11	21	21	20	—	16
33	21	11	21	21	11	20	18
34	21	11	21	21	11	11	18
35	21	11	21	23	21	—	19
36	21	11	21	24	—	—	17
37	21	11	21	12	20	—	16
38	21	11	21	12	11	20	18
39	21	11	21	12	11	11	18
40	21	11	21	14	21	11	19
41	21	11	21	15	21	11	17
42	21	11	11	21	24	—	19
43	21	11	11	22	24	—	14
44	21	11	11	11	22	—	16
45	21	11	11	12	20	—	15
46	21	11	11	12	11	—	15
47	21	11	12	20	—	—	13
48	21	11	13	21	20	—	17
49	21	11	13	21	11	—	17
50	21	11	13	12	20	—	17
51	21	11	13	13	20	—	16
52	21	12	20	—	—	—	11
53	21	12	11	21	20	—	16
54	21	12	11	21	11	20	18
55	21	12	11	23	21	20	19
56	21	12	11	12	20	—	16
57	21	12	11	12	11	20	18
58	21	12	11	14	21	—	19
59	21	13	21	25	—	—	20
60	21	13	11	23	—	—	17
61	21	13	12	20	—	—	15
62	21	14	21	—	—	—	14
63	21	15	21	20	—	—	17
64	21	15	12	20	—	—	17
65	22	11	21	21	22	—	19
66	22	11	21	22	—	—	16
67	22	11	21	11	20	—	16
68	22	11	21	12	20	—	17
69	22	11	21	12	11	—	17
70	22	11	23	21	20	—	19
71	22	11	23	21	11	—	19
72	22	11	23	12	20	—	19
73	22	11	23	13	—	—	18
74	22	11	11	20	—	—	13
75	22	11	11	11	21	20	18
76	22	11	11	11	22	—	17
77	22	11	11	11	12	20	18
78	22	11	11	11	12	11	18
79	22	11	12	20	—	—	14
80	22	11	12	11	23	—	19
81	22	11	13	—	—	—	16
82	22	11	14	21	—	—	17
83	22	13	21	21	20	—	19
84	22	13	21	22	—	—	18
85	22	13	21	12	20	—	19
86	22	13	21	12	11	—	19
87	22	13	11	20	—	—	15
88	22	13	11	11	22	—	19
89	22	13	12	20	—	—	16
90	22	13	13	21	—	—	18
91	22	15	21	22	—	—	20
92	22	15	11	20	—	—	17
93	22	15	12	20	—	—	18
94	22	17	21	20	—	—	20
95	22	17	12	20	—	—	20
96	23	21	21	20	—	—	16
97	23	21	21	11	20	—	18
98	23	21	21	11	11	—	18
99	23	21	23	21	—	—	19
100	23	21	24	—	—	—	17
101	23	21	12	20	—	—	16
102	23	21	12	11	20	—	18
103	23	21	12	11	11	—	18
104	23	21	14	21	—	—	19
105	23	21	15	—	—	—	17
106	23	11	21	24	—	—	19
107	23	11	22	—	—	—	14
108	23	11	11	22	—	—	16
109	23	11	12	20	—	—	15
110	23	11	12	11	—	—	15
111	23	12	20	—	—	—	13
112	23	13	21	20	—	—	17
113	23	13	21	11	—	—	17
114	23	13	12	20	—	—	17
115	23	13	13	—	—	—	16
116	24	20	—	—	—	—	11
117	24	11	21	20	—	—	16
118	24	11	21	11	20	—	18
119	24	11	23	21	20	—	19
120	24	11	12	20	—	—	16
121	24	11	12	11	20	—	18
122	24	11	14	21	—	—	19
123	25	21	25	—	—	—	20
124	25	11	23	—	—	—	17
125	25	12	20	—	—	—	15
126	26	21	—	—	—	—	14
127	27	21	20	—	—	—	17
128	27	12	20	—	—	—	17
129	11	21	21	21	21	10	18
130	11	21	21	21	22	—	18
131	11	21	21	22	—	—	15
132	11	21	21	11	20	—	15

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No.	$\alpha\beta$	$\alpha\beta$	$\alpha\beta$	$\alpha\beta$	$\alpha\beta$	$\alpha\beta$	No. of Points	No.	$\alpha\beta$	$\alpha\beta$	$\alpha\beta$	$\alpha\beta$	$\alpha\beta$	$\alpha\beta$	No. of Points
133	11	21	21	11	11	—	15	190	11	11	12	11	—	—	12
134	11	21	21	12	20	—	16	191	11	11	13	21	20	—	16
135	11	21	21	12	11	—	16	192	11	11	13	21	11	—	16
136	11	21	23	21	20	—	18	193	11	11	13	12	20	—	16
137	11	21	23	21	11	—	18	194	11	11	13	13	—	—	15
138	11	21	23	12	20	—	18	195	11	12	20	—	—	—	10
139	11	21	23	13	—	—	17	196	11	12	11	21	20	—	15
140	11	21	24	—	—	—	14	197	11	12	11	21	11	20	17
141	11	21	11	20	—	—	12	198	11	12	11	21	12	—	16
142	11	21	11	11	21	21	18	199	11	12	11	23	21	—	18
143	11	21	11	11	22	—	16	200	11	12	11	23	11	—	17
144	11	21	11	11	11	10	15	201	11	12	11	12	20	—	15
145	11	21	11	11	12	20	17	202	11	12	11	12	11	21	18
146	11	21	11	11	12	11	17	203	11	12	11	13	10	—	15
147	11	21	12	20	—	—	13	204	11	12	11	14	21	—	18
148	11	21	12	11	23	—	18	205	11	13	21	25	—	—	19
149	11	21	12	11	11	—	15	206	11	13	21	11	—	—	14
150	11	21	13	21	—	—	15	207	11	13	11	23	—	—	16
151	11	21	14	21	—	—	16	208	11	13	12	20	—	—	14
152	11	21	15	—	—	—	14	209	11	13	13	—	—	—	13
153	11	23	21	21	20	—	18	210	11	14	21	—	—	—	13
154	11	23	21	21	11	—	18	211	11	15	21	21	—	—	17
155	11	23	21	22	—	—	17	212	11	15	11	10	—	—	14
156	11	23	21	12	20	—	18	213	11	15	12	20	—	—	16
157	11	23	21	12	11	—	18	214	12	20	—	—	—	—	8
158	11	23	21	13	—	—	17	215	12	11	21	21	22	—	18
159	11	23	11	20	—	—	14	216	12	11	21	22	—	—	15
160	11	23	11	11	22	—	18	217	12	11	21	11	20	—	15
161	11	23	11	12	—	—	15	218	12	11	21	12	20	—	16
162	11	23	12	20	—	—	15	219	12	11	21	12	11	—	16
163	11	23	13	21	—	—	17	220	12	11	23	21	20	—	18
164	11	23	13	11	—	—	16	221	12	11	23	21	11	—	18
165	11	25	21	23	—	—	20	222	12	11	23	12	20	—	18
166	11	25	11	21	—	—	17	223	12	11	23	13	—	—	17
167	11	25	12	20	—	—	17	224	12	11	11	20	—	—	12
168	11	26	10	—	—	—	14	225	12	11	11	11	21	20	17
169	11	27	21	20	—	—	19	226	12	11	11	11	22	—	16
170	11	27	12	20	—	—	19	227	12	11	11	11	12	20	17
171	11	11	21	21	20	—	15	228	12	11	11	11	12	11	17
172	11	11	21	21	11	20	17	229	12	11	12	20	—	—	13
173	11	11	21	21	11	11	17	230	12	11	12	11	23	—	18
174	11	11	21	23	21	—	18	231	12	11	13	21	—	—	15
175	11	11	21	24	—	—	16	232	12	11	14	21	—	—	16
176	11	11	21	12	20	—	15	233	12	13	21	21	20	—	18
177	11	11	21	12	11	20	17	234	12	13	21	22	—	—	17
178	11	11	21	12	11	11	17	235	12	13	21	12	20	—	18
179	11	11	21	14	21	—	18	236	12	13	21	12	11	—	18
180	11	11	21	15	—	—	16	237	12	13	11	20	—	—	14
181	11	11	22	—	—	—	11	238	12	13	11	11	22	—	18
182	11	11	11	21	23	10	18	239	12	13	12	20	—	—	15
183	11	11	11	21	24	—	18	240	12	13	13	21	—	—	17
184	11	11	11	22	—	—	13	241	12	15	21	22	—	—	19
185	11	11	11	11	21	10	15	242	12	15	11	20	—	—	16
186	11	11	11	11	22	—	15	243	12	15	12	20	—	—	17
187	11	11	11	12	20	—	14	244	12	17	21	20	—	—	19
188	11	11	11	12	11	—	14	245	12	17	12	20	—	—	19
189	11	11	12	20	—	—	12	246	13	21	21	20	—	—	15

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No.	αβ	αβ	αβ	αβ	αβ	αβ	No. of Points
247	13	21	21	11	20	—	17
248	13	21	21	11	11	10	18
249	13	21	23	21	—	—	18
250	13	21	25	—	—	—	17
251	13	21	12	20	—	—	15
252	13	21	12	11	20	—	17
253	13	21	12	11	12	—	18
254	13	21	14	21	—	—	18
255	13	21	15	10	—	—	17
256	13	11	21	24	—	—	18
257	13	11	23	—	—	—	14
258	13	11	11	22	—	—	15
259	13	11	12	20	—	—	14
260	13	11	12	11	10	—	15
261	13	12	20	—	—	—	12
262	13	13	21	20	—	—	16
263	13	13	21	12	—	—	17
264	13	13	12	20	—	—	16
265	13	13	13	10	—	—	16
266	14	21	—	—	—	—	11
267	14	11	21	20	—	—	15
268	14	11	21	11	20	—	17
269	14	11	23	21	—	—	18
270	14	11	12	20	—	—	15
271	14	11	12	11	20	—	17
272	14	11	14	21	—	—	18
273	15	21	25	10	—	—	20
274	15	11	23	10	—	—	17
275	15	12	20	—	—	—	14
276	16	21	10	—	—	—	14
277	17	21	20	—	—	—	16
278	17	12	20	—	—	—	16
	βαβ	αβ	αβ	αβ	αβ	αβ	
279	121	21	21	21	20	—	18
280	121	21	21	22	—	—	17
281	121	21	21	12	20	—	18
282	121	21	21	12	11	—	18
283	121	21	22	—	—	—	14
284	121	21	11	20	—	—	14
285	121	21	11	11	21	10	18
286	121	21	11	11	22	—	18
287	121	21	12	20	—	—	15
288	121	21	12	11	—	—	15
289	121	21	13	21	—	—	17
290	121	23	21	22	—	—	19
291	121	23	21	11	—	—	17
292	121	23	11	20	—	—	16
293	121	23	12	20	—	—	17
294	121	23	13	—	—	—	16
295	121	25	21	21	—	—	20
296	121	25	11	10	—	—	17
297	121	25	12	20	—	—	19
298	121	11	20	—	—	—	11
299	121	11	11	21	22	—	18
300	121	11	11	22	—	—	15

No.	βαβ	αβ	αβ	αβ	αβ	αβ	No. of Points
301	121	11	11	11	20	—	15
302	121	11	11	12	20	—	16
303	121	11	11	12	11	—	16
304	121	11	13	21	20	—	18
305	121	11	13	21	11	—	18
306	121	11	13	12	20	—	18
307	121	11	13	13	—	—	17
308	121	12	20	—	—	—	12
309	121	12	11	21	20	—	17
310	121	12	11	23	—	—	17
311	121	12	11	12	20	—	17
312	121	12	11	12	11	10	18
313	121	13	21	—	—	—	14
314	121	13	11	23	—	—	18
315	121	14	21	—	—	—	15
316	121	15	21	10	—	—	17
317	123	21	21	20	—	—	17
318	123	21	21	11	20	—	19
319	123	21	23	21	—	—	20
320	123	21	12	20	—	—	17
321	123	21	12	11	20	—	19
322	123	21	14	21	—	—	20
323	123	11	21	24	—	—	20
324	123	11	11	22	—	—	17
325	123	11	12	20	—	—	16
326	123	12	20	—	—	—	14
327	123	13	21	20	—	—	18
328	123	13	12	20	—	—	18
329	124	11	21	20	—	—	17
330	124	11	21	11	20	—	19
331	124	11	23	21	—	—	20
332	124	11	12	20	—	—	17
333	124	11	12	11	20	—	19
334	124	11	14	21	—	—	20
335	125	21	24	—	—	—	20
336	125	11	22	—	—	—	17
337	125	12	20	—	—	—	16
338	126	20	—	—	—	—	14
339	127	21	20	—	—	—	18
340	127	12	20	—	—	—	18
341	111	21	21	22	—	—	16
342	111	21	21	11	20	—	16
343	111	21	21	11	11	—	16
344	111	21	23	21	11	—	19
345	111	21	24	—	—	—	15
346	111	21	11	20	—	—	13
347	111	21	12	20	—	—	14
348	111	21	12	11	20	—	16
349	111	21	12	11	11	—	16
350	111	21	14	21	—	—	17
351	111	21	15	—	—	—	15
352	111	23	21	21	20	—	19
353	111	23	21	21	11	—	19
354	111	23	21	12	20	—	19
355	111	23	21	13	—	—	18
356	111	23	11	11	22	—	19
357	111	23	11	12	—	—	16

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No.	$\beta\alpha\beta$	$\alpha\beta$	$\alpha\beta$	$\alpha\beta$	$\alpha\beta$	$\alpha\beta$	No. of Points	No.	$\beta\alpha\beta$	$\alpha\beta$	$\alpha\beta$	$\alpha\beta$	$\alpha\beta$	$\alpha\beta$	No. of Points
358	111	23	12	20	—	—	16	415	221	21	11	11	22	—	19
359	111	23	13	20	—	—	17	416	221	21	12	20	—	—	16
360	111	23	13	11	—	—	17	417	221	21	13	21	—	—	18
361	111	11	21	21	11	20	18	418	221	23	21	22	—	—	20
362	111	11	21	21	11	11	18	419	221	23	11	20	—	—	17
363	111	11	21	23	20	—	18	420	221	23	12	20	—	—	18
364	111	11	21	24	—	—	17	421	221	25	21	20	—	—	20
365	111	11	22	—	—	—	12	422	221	25	12	20	—	—	20
366	111	11	11	21	20	—	15	423	221	11	11	21	22	—	19
367	111	11	11	22	—	—	14	424	221	11	11	22	—	—	16
368	111	11	11	12	20	—	15	425	221	11	11	11	20	—	16
369	111	11	11	12	11	—	15	426	221	11	11	12	20	—	17
370	111	11	12	20	—	—	13	427	221	11	11	12	11	—	17
371	111	11	12	11	—	—	13	428	221	11	13	21	20	—	19
372	111	12	20	—	—	—	11	429	221	11	13	21	11	—	19
373	111	12	11	21	11	20	18	430	221	11	13	12	20	—	19
374	111	12	11	21	12	—	17	431	221	11	13	13	—	—	18
375	111	12	11	23	20	—	18	432	221	12	20	—	—	—	13
376	111	12	11	23	11	—	18	433	221	12	11	21	20	—	18
377	111	13	21	20	—	—	15	434	221	12	11	22	—	—	17
378	111	13	21	11	—	—	15	435	221	12	11	12	20	—	18
379	111	13	12	20	—	—	15	436	221	12	11	12	11	—	18
380	111	13	13	—	—	—	14	437	221	13	20	—	—	—	14
381	111	14	21	—	—	—	14	438	221	13	11	23	—	—	19
382	112	20	—	—	—	—	9	439	221	14	21	—	—	—	16
383	112	11	21	21	22	—	19	440	221	15	21	—	—	—	17
384	112	11	21	11	20	—	16	441	211	21	21	22	—	—	17
385	112	11	21	12	20	—	17	442	211	21	24	—	—	—	16
386	112	11	23	21	20	—	19	443	211	21	11	20	—	—	14
387	112	11	23	12	20	—	19	444	211	21	12	20	—	—	15
388	112	11	11	11	21	20	18	445	211	21	12	11	11	—	17
389	112	11	11	11	22	—	17	446	211	23	21	20	—	—	17
390	112	11	11	11	12	20	19	447	211	23	21	13	—	—	19
391	112	11	11	11	12	11	19	448	211	23	12	20	—	—	17
392	112	11	12	20	—	—	14	449	211	23	13	11	—	—	18
393	112	11	12	11	22	—	18	450	211	11	22	—	—	—	13
394	112	11	13	20	—	—	15	451	211	11	11	21	20	—	16
395	112	11	14	21	—	—	17	452	211	11	11	22	—	—	15
396	113	21	21	20	—	—	16	453	211	11	11	12	20	—	16
397	113	21	25	—	—	—	18	454	211	11	11	12	11	—	16
398	113	21	12	20	—	—	16	455	211	12	20	—	—	—	12
399	113	21	12	11	12	—	19	456	211	12	11	23	11	—	19
400	113	11	23	—	—	—	15	457	211	13	21	11	—	—	16
401	113	11	11	22	—	—	16	458	211	14	21	—	—	—	15
402	113	12	20	—	—	—	13	459	212	20	—	—	—	—	10
403	113	13	21	12	—	—	18	460	213	21	21	20	—	—	17
404	114	21	—	—	—	—	12	461	213	21	22	—	—	—	16
405	115	21	22	—	—	—	17	462	213	21	12	20	—	—	17
406	115	11	20	—	—	—	14	463	213	21	12	11	—	—	17
407	115	12	20	—	—	—	15	464	213	11	20	—	—	—	13
408	117	21	20	—	—	—	17	465	213	11	11	22	—	—	17
409	117	12	20	—	—	—	17	466	213	12	20	—	—	—	14
410	221	21	21	21	20	—	19	467	213	13	21	—	—	—	16
411	221	21	21	22	—	—	18	468	215	21	22	—	—	—	18
412	221	21	21	12	20	—	19	469	215	11	20	—	—	—	15
413	221	21	21	12	11	—	19	470	215	12	20	—	—	—	16
414	221	21	11	20	—	—	15	471	217	21	20	—	—	—	18

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No.	$\beta\alpha\beta$	$\alpha\beta$	$\alpha\beta$	$\alpha\beta$	$\alpha\beta$	$\alpha\beta$	No. of Points	No.	$\beta\alpha\beta$	$\alpha\beta$	$\alpha\beta$	$\alpha\beta$	$\alpha\beta$	$\alpha\beta$	No. of Points
472	217	12	20	—	—	—	18	497	313	12	20	—	—	—	15
473	321	21	20	—	—	—	14	498	313	13	20	—	—	—	16
474	321	21	11	20	—	—	16	499	421	21	11	20	—	—	17
475	321	21	11	11	20	—	18	500	421	21	11	11	20	—	19
476	321	21	13	21	—	—	19	501	421	21	13	21	—	—	20
477	321	23	21	—	—	—	17	502	421	23	20	—	—	—	17
478	321	23	11	20	—	—	18	503	421	23	11	20	—	—	19
479	321	25	21	10	—	—	20	504	421	25	21	—	—	—	20
480	321	12	20	—	—	—	14	505	411	21	20	—	—	—	14
481	321	12	11	21	20	—	19	506	411	21	11	20	—	—	16
482	321	12	11	12	20	—	19	507	411	23	21	—	—	—	17
483	321	13	11	23	—	—	20	508	411	12	20	—	—	—	14
484	321	14	21	—	—	—	17	509	411	12	11	20	—	—	16
485	321	15	20	—	—	—	17	510	411	14	21	—	—	—	17
486	311	21	24	—	—	—	17	511	412	20	—	—	—	—	12
487	311	21	11	20	—	—	15	512	521	23	11	20	—	—	20
488	311	23	21	13	—	—	20	513	521	25	20	—	—	—	20
489	311	11	22	—	—	—	14	514	511	21	11	20	—	—	17
490	311	12	20	—	—	—	13	515	511	23	20	—	—	—	17
491	311	12	11	20	—	—	15	516	512	20	—	—	—	—	13
492	311	14	21	—	—	—	16	517	621	20	—	—	—	—	14
493	312	20	—	—	—	—	11	518	612	20	—	—	—	—	14
494	313	21	21	20	—	—	18	519	721	20	—	—	—	—	15
495	313	21	12	20	—	—	18	520	712	20	—	—	—	—	15
496	313	11	11	22	—	—	18								

31. [This section has been omitted. It is a table referring to a former catalogue called 'Old Numbers', to 30, called 'New Numbers'.]

32. Every finished non-cyclical succession must include two cyclical successions, of which one is finished; and every finished succession must include an unfinished cyclical succession.

33. Thus, No. 1 of the old or alphabetical catalogue is the finished cyclical succession, indeed it is a complete one, its literal formula is *b c d f g h j k l m n p q r s t v w x z* and its numeral formula is 21, 25, 21, 22; its symbol is therefore $\alpha^2, \beta, \alpha^2, \beta^5, \alpha^2, \beta, \alpha^2, \beta^2$ or briefly $\alpha^2\beta\alpha^2\beta^5\alpha^2\beta\alpha^2\beta^2$ if we agree to read from left to right. Also *wxz–bcd* gives the additional factors β, β, β or β^3 and *wxz–qrs* gives $\alpha^2\beta$; we have, therefore, the additional equations

$$(\alpha^2\beta\alpha^2\beta^5)^2 = 1 \text{ (XXIV)} \quad \text{and} \quad \beta\alpha^2\beta^2\alpha^2\beta = 1 \text{ (II)},$$

striking off the 12 left-hand factors in order to obtain the intended lesser cycle (No. 32), these two symbolic equations are true by former results.

34. Treating these two last equations as known, we may infer that the operator

$$\alpha^2\beta\alpha^2\beta^5\alpha^2\beta\alpha^2\beta^2$$

must produce a complete cyclical succession. First, it produces a cyclical succession, because when we follow it by β^3 we get an operator which is equal to 1; it is complete because it is of the seventeenth dimension in α and β. It may also be seen to be finished, because we can follow it neither by α nor by β, without the inadmissible repetition of a letter. In fact, if we follow the whole by β, we repeat the initial letter, because we get an operator equal to β^{-2}, by XXIV; and if we follow the final part $\beta\alpha^2\beta^2$ by α and then by $\alpha\beta$ we get an operator equal

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to 1, by II, so that $\beta\alpha^2\beta^2\alpha$ is an inadmissible operator in the game, as well as $\alpha^2\beta\alpha^2\beta^5\alpha^2\beta\alpha^2\beta^3$. Indeed, no admissible operator, in the game, can exceed the seventeenth dimension.

35. In fact, an operator which produces a finished cyclical succession must be of the form $\gamma\delta$, and both $\gamma\delta\alpha$ and $\gamma\delta\beta$ must be inadmissible: also $\gamma\delta\alpha$ may either be inadmissible as a whole, or because $\delta\alpha$ is inadmissible. We must therefore have one or other of two alternatives, either (i) $\delta\alpha$ and $\gamma\delta\beta$ are inadmissible; or (ii) $\delta\beta$ and $\gamma\delta\alpha$ are inadmissible. And an operator is inadmissible, when being followed by any two moves, that is by any one of the four binary operators α^2, $\alpha\beta$, $\beta\alpha$, β^2, it restores the original triad.

36. In the recent example, taken from No. 27, we have $\gamma=\alpha^2\beta\alpha^2\beta^5\alpha^2=21,25,2$ and $\delta=\beta\alpha^2\beta^2=01,22$; $\delta\alpha$ is inadmissible because $\delta\alpha^2\beta=1$, by II, and $\gamma\delta\beta$ is inadmissible because $\gamma\delta\beta^3=1$, by XX; we are therefore in the case (ii) of No. 35, and

$$\gamma\delta=\alpha^2\beta\alpha^2\beta^5\alpha^2\beta\alpha^2\beta^2=21,25,21,22$$

is an operation which produces a finished cyclical succession (indeed a complete one), as before.

37. Old No. 2 (New 29) is a finished non-cyclical succession, but by suppressing nine left-hand factors we get the finished cyclical succession *npqrwvts*, or the operator $\alpha\beta^2\alpha^2=12,2=\gamma\delta$ with $\gamma=\alpha\beta^2$ and $\delta=\alpha^2$. Here $vts \rightarrow npq$ and $vts \rightarrow rwv$, which show that $\delta\alpha^3=1$ and $\gamma\delta\beta^2\alpha=1$: in fact $\alpha^5=1$ and $\alpha\beta^2\alpha^2\beta^2\alpha=1$, by II; therefore $\delta\alpha$ and $\gamma\delta\beta$ are inadmissible and we are still in case (i).

38. In general, δ and $\gamma\delta$ both generate cycles, the latter alone being finished and the former being included in it.*

39. [A further table of successions.]

40. [A continuation of a previous table 31.]

41. Type I evidently gives no finished succession of five because we can always go out from a given pentagon.

42. Type II gives 12, 2, or No. 214,† as one finished cyclical succession of eight.

43. Type III gives no finished cyclical succession of nine, because the nine points surround a vacant point, and we can either go inwards or outwards from any one of the nine.

44. Type IV gives no finished cyclical succession of ten, because the ten points surround two adjacent vacant points, or an edge, and it is always possible to go outwards or inwards.

45. Type V gives no finished cyclical succession of ten, because the zone leaves two pentagons free.

46. Type VI gives four finished cyclical successions of eleven, because the three successive pentagons leave no interior point free. Nos. 116, 181, 266, 298.

47. Type VII gives no other finished cyclical succession of eleven, because in circulating around three successive vacant points, it is always possible to go outwards or inwards.

48. Type VIII gives two successive cyclical successions of twelve, Nos. 190 and 224.

49. Types IX and X give no other finished cyclical succession of twelve.

50. Type XI gives two finished cyclical successions of thirteen, Nos. 209, 464.

* [The remainder of this section is omitted.]

† [The Nos. here refer to article 30.]

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51. Type XII gives eight finished cyclical successions of fourteen, Nos. 126, 140, 168, 276, 283, 338, 473, 517.

52. Type XIII gives six finished cyclical successions of thirteen, Nos. 107, 159, 205, 257, 313, 437.

53. Type XIV gives four finished cyclical successions of fourteen, Nos. 152, 212, 406, 505.

54. Type XV gives twelve finished cyclical successions of fifteen, Nos. 110, 149, 161, 196, 203, 216, 231, 260, 288, 377, 394, 491.

55. Type XVI gives six finished cyclical successions of fifteen, Nos. 133, 144, 171, 185, 301, 366.

56. Type XVII gives eight finished cyclical successions of sixteen, Nos. 115, 164, 262, 265, 294, 461, 467, 498.

57. Type XVIII gives three finished cyclical successions of sixteen, Nos. 198, 219, 348.

58. Type XIX gives twenty-four finished cyclical successions of seventeen, Nos. 28, 63, 100, 105, 124, 155, 158, 166, 211, 250, 255, 274, 291, 296, 316, 336, 405, 440, 446, 477, 485, 502, 507, 515.

59. Type XX gives ten finished cyclical successions of seventeen, Nos. 113, 163, 200, 223, 252, 263, 310, 359, 434, 463.

60. Type XXI gives twelve finished cyclical successions of eighteen, Nos. 11, 98, 136, 142, 154, 174, 182, 248, 285, 299, 363, 475.

61. Type XXII gives twelve others of eighteen, Nos. 57, 103, 157, 199, 202, 221, 230, 253, 312, 375, 393, 436.

62. Type XXIII gives two others of eighteen, Nos. 129, 279.

63. Type XXIV gives ten cyclical successions of twenty, Nos. 20, 123, 165, 273, 295, 335, 421, 479, 504, 513.

64. There are in all $1+4+2+2+18+18+11+34+26+10=126$ finished cyclical successions.

65. And the $520-126=394$ finished, but non-cyclical, successions must all be dependent on these.

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