LVII

LETTER TO JOHN T. GRAVES ON THE ICOSIAN

[Note-book 139.]

Observatory, 17 *October* 1856.

My dear John Graves,

The constant friendship to which you have admitted me, during a period of more than thirty years, may cause you to remember with some interest, that you were the first person to whom I communicated my invention of the Quaternions in a letter of 17 October 1843, which has since, by your permission, been printed.*

I had some hope that you might have anticipated me, in a more recent mathematical conception, the result of a train of thought for the suggestion of which I am indebted to yourself. With your usual candour you have informed me that you have no claim to make in the matter. Let me at least have the pleasure of in some degree associating your name therewith, if this letter shall be thought worth preserving, for I feel sure that if you had not lately pressed on my attention the geometrical interest of the polyhedra, although the feeling of such an interest is among my very earliest mathematical recollections, I should not have been conducted to that novel† system of symbols, respecting which I have now the pleasure of giving you some enlarged particulars, without pretending to attach more than a very moderate degree of importance to the results.

[1]. As in the little paper‡ which I lately sent you, let me continue to assume three symbols, ι, κ, λ, which shall satisfy the four following equations:

$$(\mathrm{A}) \quad \begin{cases} \iota^2 = 1, & (1) \\ \kappa^3 = 1, & (2) \\ \lambda^5 = 1, & (3) \\ \lambda = \iota\kappa. & (4) \end{cases}$$

What I have first to show, by one or two examples, is that the symbols so defined have curious but determinate properties, making them the legitimate instrument of a calculus: every symbolic result of which, so far as I can judge, and I have examined a great number of them, admits of easy and often interesting interpretation, with reference to the passage from face to face, or from corner to corner, of one or other of the solids considered in the ancient geometry.

[2]. For the purpose of such calculation, I assume an associative property of multiplication

* In the Supplementary Number of the *Phil. Mag.* for December 1844. [See IV.]

† Of course, when I say 'novel', I can only mean that it is such to myself, at present.

‡ See the 'Memorandum' dated 7 October 1856, the receipt of which has been acknowledged by you. [See LVI.]

of symbols; and (as in the calculus of quaternions) I reject the commutative property. For instance, I admit that $\iota . \iota\kappa = \iota^2\kappa = \kappa$ and therefore have

$$\iota\lambda = \kappa, \tag{5}$$

but I find it necessary to reject the formula $\iota\lambda = \lambda\iota$. Indeed it is evident from the mere inspection of my assumed system of equations (A), that the symbols here discussed are not the ordinary roots of unity, and that therefore they are not to be expected to follow all the laws of such roots. The distributive property of multiplication is not requisite for my present purpose, which is not concerned with addition.

[3]. It is clear, from what precedes, that in this system,

$$1 = (\iota\lambda)^3 = (\iota\kappa)^5, \tag{6}$$

also that
$$\iota\kappa = (\iota\kappa)^{-4} = (\kappa^2\iota)^4, \tag{7}$$

$$1 = \iota . \iota\kappa . \kappa^2 = (\iota\kappa^2)^5, \tag{8}$$

and therefore that
$$\mu^5 = 1 \tag{9}$$

if
$$\mu = \iota\kappa^2 = \lambda\kappa. \tag{10}$$

So that this symbol μ denotes a new fifth root of unity, which is found to be connected with the former fifth root λ, by some simple relations of reciprocity. For example,

$$\mu = \lambda\iota\lambda, \quad \lambda = \mu\iota\mu, \tag{11}$$

$$\iota = \lambda\iota\mu = \mu\iota\lambda, \tag{12}$$

whence
$$1 = (\lambda\iota\lambda)^5 = (\mu\iota\mu)^5 = (\lambda\iota\mu)^2 = (\mu\iota\lambda)^2. \tag{13}$$

Many other equations of the same general character are found,

$$1 = (\iota\kappa)^{-5} = (\kappa^2\iota)^5 = (\iota\lambda\iota\lambda\iota)^5, \tag{14}$$

$$1 = \kappa\mu^5\kappa^{-1} = \kappa(\iota\kappa^2)^5\kappa^{-1} = (\kappa\iota\kappa)^5 = (\kappa\lambda)^5 = (\iota\lambda^2)^5, \tag{15}$$

$$1 = (\kappa\iota)^5, \quad 1 = (\lambda^2\iota)^5, \quad 1 = (\iota\lambda^3)^5, \quad 1 = (\lambda^3\iota)^5, \tag{16}$$

but, with a change of the exponent,

$$1 = (\lambda\iota)^3, \quad 1 = (\lambda^4\iota)^3, \quad 1 = (\iota\lambda^4)^3. \tag{17}$$

A few of these new symbolic roots of unity might deserve to be denoted by special signs, if the system were to be fully developed. Meantime I shall remark that I have found the following formulae of reduction useful:

$$\lambda\mu^2\lambda = \mu\lambda\mu; \quad \mu\lambda^2\mu = \lambda\mu\lambda, \tag{18}$$

$$\lambda\mu^3\lambda = \mu^2; \quad \mu\lambda^3\mu = \lambda^2, \tag{19}$$

and that if we make, for abridgment,
$$\omega = \lambda\mu\lambda\mu\lambda, \tag{20}$$

we shall have also,
$$\omega = \mu\lambda\mu\lambda\mu \tag{21}$$

and
$$\omega^2 = 1. \tag{22}$$

[4]. To give an example of the use, in calculation, of some of the foregoing formulae of reduction, let us take the equation which, with a slightly different notation, I lately communicated* to you: namely the following,

$$(\lambda^3\mu^3(\lambda\mu)^2)^2 = 1. \tag{23}$$

What I am to shew is that this equation is true, as consequence of the definitional laws of the symbols λ and μ. I change $\lambda^3\mu^3(\lambda\mu)^2$ to $\lambda^2\mu^3\lambda\mu$, and this to $\lambda\mu^3$, in virtue of the first equation (19); and then the equation is reduced to deciding whether it be true that

$$1 = (\lambda\mu^3)^2 = \lambda\mu^3\lambda\mu^3. \tag{24}$$

But this is obvious, in virtue of the same first equation (19), combined with the property (9) of μ, as being a symbolic fifth root of unity.

[5]. Now for a sketch of the geometrical interpretation of such symbolic results as these. Let a, b, c, d, e denote five successive faces of the icosahedron, arranged about one common corner of the body. Let the five respectively adjacent faces, which are not about that corner, be called α, β, γ, δ, ϵ and let the ten other faces which are respectively opposite to these, be marked as a', b', c', d', e' and α', β', γ', δ', ϵ', a' being the face opposite to the face a, and similarly in other cases.

Then these twenty faces will arrange themselves in twelve quines, round the 12 corners of the solid, which quines, by a suitable selection of the letters, may be thus described, one order of rotation being preserved throughout: namely,

$$\left.\begin{array}{llllll}
abcde & \text{round} & F; & a'e'd'c'b' & \text{round} & F'; \\
\alpha'\delta dc\gamma & \text{round} & A; & \alpha\gamma'c'd'\delta' & \text{round} & A'; \\
\beta'\epsilon ed\delta & \text{round} & B; & \beta\delta'd'e'\epsilon' & \text{round} & B'; \\
\gamma'\alpha ae\epsilon & \text{round} & C; & \gamma\epsilon'e'a'\alpha' & \text{round} & C'; \\
\delta'\beta ba\alpha & \text{round} & D; & \delta\alpha'a'b'\beta' & \text{round} & D'; \\
\epsilon'\gamma cb\beta & \text{round} & E; & \epsilon\beta'b'c'\gamma' & \text{round} & E'.
\end{array}\right\} \tag{25}$$

I interpret the symbol λ as denoting the operation of passing, in any one of these twelve quines, and in the order of rotation here adopted as direct, from one pair of adjacent faces to a consecutive one, the second face of the first pair being the first face of the second pair: for example, in the quine F, from ab to bc; or, in the same quine, from bc to cd; or from cd to de; or from de to ea; or finally, from ea to ab again: the character of a fifth root of positive unity being evidently thus presented. And I interpret the connected symbol μ as denoting the operation of passing, in the contrary order of rotation, from pair to pair of faces: for instance, in the quine D, from ab to $b\beta$; thence to $\beta\delta'$; thence to $\delta'\alpha$; thence to αa; and thence to ab again: the character of a fifth root of unity being thus anew presented to our notice. As to the symbol ι, I interpret it as signifying, that we invert the order of the two faces of one pair; passing by its operation (for example) from ab to ba, without introducing the consideration of any third face of the body. That ι is thus a square root of positive unity is evident: but that the symbolic relations,

$$\mu = \lambda\iota\lambda, \quad \lambda = \mu\iota\mu \tag{11}$$

are also satisfied, may be verified in the following manner.

* [See LVI, equation (B).]

[6]. From the general symmetry of the solid, not here considered as implying any equality of angles, but only the existence of one common law of construction throughout—and specially this, that five triangles meet at every corner—we may select any pair of faces as the subject of our first operation: and I shall take the pair ab. Operating on this pair, successively, by λ, ι, λ we get, on the plan already explained,

$$\left.\begin{aligned} \lambda(ab) &= (bc), \\ \iota\lambda(ab) &= \iota(bc) = (cb), \\ \lambda\iota\lambda(ab) &= \lambda(cb) = (b\beta). \end{aligned}\right\} \tag{26}$$

But also, $$\mu(ab) = (b\beta), \tag{27}$$

therefore $\mu = \lambda\iota\lambda$, as had been asserted. In like manner it may be proved that, with the recent interpretations, we have $\mu\iota\mu = \lambda$, as in (11).

[7]. As regards the interpretation of the symbol κ, which is perhaps less interesting, though useful as an auxiliary, for our present purpose, I may just remark that

$$\left.\begin{aligned} \kappa(ab) &= \iota\lambda(ab) = \iota(bc) = (cb), \\ \kappa^2(ab) &= \iota\lambda(cb) = \iota(b\beta) = (\beta b), \\ \kappa^3(ab) &= \iota\lambda(\beta b) = \iota(ba) = (ab), \end{aligned}\right\} \tag{28}$$

and that thus the symbolical property of the operator κ, as a cube root* of positive unity, is verified.

[8]. I may also remark that the symbol ω receives an extremely simple geometrical interpretation, connected with the property of being a square root of positive unity. It implies the passage from one pair of adjacent faces of the icosahedron to the pair which is opposite thereto. In fact, if we take the pair ab to operate on, and proceed by means of the expression (20), we find successively,

$$\lambda(ab) = (bc); \quad \mu(bc) = c\gamma; \quad \lambda(c\gamma) = (\gamma\alpha'); \quad \mu(\gamma\alpha') = (\alpha'a'); \quad \lambda(\alpha'a') = (a'b'); \tag{29}$$

and therefore finally $$\omega(ab) = (a'b'). \tag{30}$$

The same conclusion would be reached through a different set of steps, if we used the expression (21) for ω.

[9]. Let us now explain what relation the formula numbered (23) in this letter, or the equivalent formula (B) of my memorandum of 7 October,† bears to the theory of the icosahedron. It represents, as I conceive that I have determined, the only method of cyclic succession, whereby all the faces of that body can be passed over, one after another, so as to end in immediate proximity to the face with which we had begun: allowance being made for the circumstance, that such a cycle may be begun at any stage, and may be traversed backwards: and also that all the directions of rotation may be reversed together. Availing ourselves of these permissions, and operating, as an example, on the initial pair of faces ab, by the twenty successive operators included in the symbol $((\mu\lambda)^2\mu^2\lambda^3)^2$, namely by

$$\lambda, \lambda, \lambda, \mu, \mu, \mu, \lambda, \mu, \lambda, \mu; \quad \lambda, \lambda, \lambda, \mu, \mu, \mu, \lambda, \mu, \lambda, \mu; \tag{31}$$

* The operator κ is seen to have the effect of causing the second face of the pair to turn round the first, in an order of rotation opposite to that which has been assumed as the direct one, in forming the quines (25); or in conceiving a pair of consecutive faces to turn together round a common corner.

† [See LVI.]

we obtain the following succession of twenty derived pairs of faces, whereof the last is (as it ought to be) the same with that first operated on:

$$\left.\begin{array}{l} bc,\, cd,\, de,\, e\epsilon,\, \epsilon\beta',\, \beta'\delta,\, \delta\alpha',\, \alpha'\gamma,\, \gamma\epsilon',\, \epsilon'\beta; \\ \beta\delta',\, \delta'\alpha',\, \alpha'e',\, e'a',\, a'b',\, b'c',\, c'\gamma',\, \gamma'\alpha,\, \alpha a,\, ab. \end{array}\right\} \tag{32}$$

More briefly, we thus obtain the cyclical and complete succession of the twenty faces,

$$a\,b\,c\,d\,e\,\epsilon\,\beta'\,\delta\,\alpha'\,\gamma\,\epsilon'\,\beta\,\delta'\,d'\,e'\,a'\,b'\,c'\,\gamma'\,\alpha. \tag{33}$$

On the same plan I have found nine other complete and cyclical successions, commencing with the three faces here called $a\,b\,c$, and do not believe that any others, under the same conditions, remain to be discovered: but speak, of course, with a due consciousness of the difficulty of being sure that any subject, of even moderate complexity, has been exhausted; or any one department hereof. Yet the intellect of man recognizes an irrepressible instinct to seek for such a completion: and as I deeply enjoyed, when almost a child, the proof that (in the ancient sense of the words) only five regular solids are possible, I shall hazard here what I suppose to be an exhaustive statement, for your examination, of the 9 ways in which, besides the way marked (33)—thus making on the whole ten ways—the twenty faces of the icosahedron can be cyclically traversed, if three successive faces ($a\,b\,c$) be given, or assumed, as the initial ones. You will find that they are all represented by my formula (23), with very easy symbolical transformations.* They are these:

$$a\,b\,c\,d\,e\,\epsilon\,\gamma'\,c'\,d'\,e'\,a'\,b'\,\beta'\,\delta\,\alpha'\,\gamma\,\epsilon'\,\beta\,\delta'\,\alpha; \tag{34}$$

$$a\,b\,c\,d\,\delta\,\beta'\,b'\,c'\,d'\,e'\,a'\,\alpha'\,\gamma\,\epsilon'\,\beta\,\delta'\,\alpha\,\gamma'\,\epsilon\,e; \tag{35}$$

$$a\,b\,c\,d\,\delta\,\alpha'\,\gamma\,\epsilon'\,\beta\,\delta'\,\alpha\,\gamma'\,c'\,d'\,e'\,a'\,b'\,\beta'\,\epsilon\,e; \tag{36}$$

$$a\,b\,c\,\gamma\,\alpha'\,\delta\,d\,e\,\epsilon\,\beta'\,b'\,a'\,e'\,\epsilon'\,\beta\,\delta'\,d'\,c'\,\gamma'\,\alpha; \tag{37}$$

* The formula for the cyclical succession (33) has

$$((\mu\lambda)^2\mu^3\lambda^3)^2 = 1. \tag*{(33)$'$}$$

The corresponding formulae for the nine other successions of the same sort, beginning with the same three faces $a\ b\ c$, may be thus written:

$$(\mu^3(\lambda\mu)^2\lambda^3)^2 = 1, \tag*{(34)$'$}$$

$$(\lambda\mu^3(\lambda\mu)^2\lambda^2)^2 = 1, \tag*{(35)$'$}$$

$$(\lambda(\mu\lambda)^2\mu^3\lambda^2)^2 = 1, \tag*{(36)$'$}$$

$$(\mu\lambda\mu^3\lambda^3\mu\lambda)^2 = 1, \tag*{(37)$'$}$$

$$(\mu^3\lambda^3(\mu\lambda)^2)^2 = 1, \tag*{(38)$'$}$$

$$(\lambda^2\mu^3(\lambda\mu)^2\lambda)^2 = 1, \tag*{(39)$'$}$$

$$(\mu\lambda^3\mu^3\lambda\mu\lambda)^2 = 1, \tag*{(40)$'$}$$

$$(\mu\lambda\mu\lambda^3\mu^3\lambda)^2 = 1, \tag*{(41)$'$}$$

$$(\lambda^2(\mu\lambda)^2\mu^3\lambda)^2 = 1, \tag*{(42)$'$}$$

of these, (34)$'$, (35)$'$, (39)$'$, (40)$'$ and (41)$'$ can be reduced to the form (23), by cyclical transpositions of the factors; and (36)$'$, (37)$'$, (38)$'$ and (42)$'$ can be reduced to the form (33)$'$, which, after such transpositions, differs only by the interchange of the two symbols λ and μ, from the same earlier equation (23). No other formula of the same sort, with λ for the right-hand factor, can be deduced from that equation by any such transformation or interchange. But if we permit ourselves to begin with the three faces $a\ b\ \beta$, and therefore to perform first the operation μ, then ten other complete and cyclical successions result, the formulae of which may be deduced from the ten marked above as (33)$'$ to (42)$'$, by interchanging λ and μ.

$$a\,b\,c\,\gamma\,\alpha'\,a'\,b'\,\beta'\,\delta\,d\,e\,\epsilon\,\gamma'\,c'\,d'\,e'\,\epsilon'\,\beta\,\delta'\,\alpha; \tag{38}$$

$$a\,b\,c\,\gamma\,\alpha'\,a'\,b'\,c'\,d'\,e'\,\epsilon'\,\beta\,\delta'\,\alpha\,\gamma'\,\epsilon\,\beta'\,\delta\,d\,e; \tag{39}$$

$$a\,b\,c\,\gamma\,\alpha'\,a'\,e'\,\epsilon'\,\beta\,\delta'\,d'\,c'\,b'\,\beta'\,\delta\,d\,e\,\epsilon\,\gamma'\,\alpha; \tag{40}$$

$$a\,b\,c\,\gamma\,\epsilon'\,\beta\,\delta'\,d'\,e'\,a'\,\alpha'\,\delta\,d\,e\,\epsilon\,\beta'\,b'\,c'\,\gamma'\,\alpha; \tag{41}$$

$$a\,b\,c\,\gamma\,\epsilon'\,\beta\,\delta'\,\alpha\,\gamma'\,\epsilon\,\beta'\,b'\,c'\,d'\,e'\,a'\,\alpha'\,\delta\,d\,e. \tag{42}$$

If you can point out any eleventh way of solving the same problem (of assigning a complete and cyclical succession with the same set of 3 initial faces given), I shall be thankful for the instruction, and correction, thereby conveyed.

[10]. There exist several subcycles, of faces of the icosahedron, expressed by formulae of the same general character as the recently illustrated formula (23), namely by equations asserting that certain products of powers of the symbols λ and μ, taken in certain orders, and with fewer than twenty factors, admit of being equated to unity. Thus besides the assumed equation (3), to which the derived equation (9) is an exact counterpart, and which answers to the passage, in one order or in its opposite, round one of the twelve corners of the solid, in one of the twelve quines (25), we have also the equation (24), or the other which is equivalent thereto, being obtained from it by merely interchanging λ and μ,

$$(\mu\lambda^3)^2 = 1, \tag{43}$$

and which may be interpreted as corresponding to a subcycle of eight faces, such as the following:

$$a\,b\,c\,d\,e\,\epsilon\,\gamma'\,\alpha. \tag{44}$$

The equation $$(\mu\lambda^2)^3 = 1, \tag{45}$$

which results easily from those marked (18) and (19), represents the subcycle of nine faces,

$$a\,b\,c\,d\,\delta\,\beta'\,\epsilon\,\gamma'\,\alpha. \tag{46}$$

Two subcycles, of ten faces each, namely

$$a\,b\,c\,d\,\delta\,\beta'\,b'\,c'\,\gamma'\,\alpha, \tag{47}$$

and $$a\,b\,c\,\gamma\,\alpha'\,a'\,b'\,c'\,\gamma'\,\alpha, \tag{48}$$

are represented by the equations, $$(\mu\lambda\mu\lambda^2)^2 = 1 \tag{49}$$

and $$(\mu\lambda)^5 = 1, \tag{50}$$

the latter answering to the passage along a zone of ten faces, which are each remote from both of two opposite corners of the body, as B and B' in the succession (48), or F and F' in the following,

$$\alpha\,\delta'\,\beta\,\epsilon'\,\gamma\,\alpha'\,\delta\,\beta'\,\epsilon\,\gamma'. \tag{51}$$

I have found also two such cycles of eleven faces; three of twelve; one of thirteen; three of fourteen; two of fifteen; two of sixteen; two of seventeen; and three of eighteen faces: and I think that no other subcycles exist essentially different from these to which I thus refer. The formulae of some are very simple, as for instance these two,

$$(\mu^2\lambda^2)^3 = 1, \tag{52}$$

$$(\mu^3\lambda^3)^3 = 1, \tag{53}$$

which correspond to subcycles of twelve and eighteen faces, such as the following,

$$a\,b\,c\,d\,\delta\,\alpha'\,a'\,b'\,c'\,d'\,\delta'\,\alpha, \tag{54}$$

$$a\,b\,c\,d\,e\,\epsilon\,\beta'\,\delta\,\alpha'\,a'\,b'\,c'\,d'\,e'\,\epsilon'\,\beta\,\delta'\,\alpha. \tag{55}$$

In general, the reductions of the symbolic equations which occur, are easy and entertaining, as are also several of the geometrical interpretations, conducted on the same plan as before. It may be useful to remark that some of these reductions are connected with properties of the symbol ω, which besides the former (20) and (21), may also receive several others, among which these seem to deserve attention,

$$\omega = \lambda^2\mu^2\lambda^2, \quad \omega = \lambda^3\mu^3\lambda^3. \tag{56}$$

It may be noticed also that

$$\lambda\omega = \omega\mu = (\lambda\mu)^3. \tag{57}$$

In all such formulae, involving only the symbols ι, λ, μ, ω, it is permitted to interchange λ with μ, while leaving ι and ω unaltered, but if the symbol κ enter, we must then at the same time change this symbol to its square, or to its reciprocal.

[11]. Besides cyclical and subcyclical successions of the faces of the icosahedron, I have found several complete but non-cyclical successions of those twenty faces, and have expressed them all by formulae. Of these I have found only twelve types essentially distinct, remembering that here, as in the case of the cyclic, we may read backwards, and may interchange λ and μ. Preserving only the exponents of the powers of those two symbols, so that the left-hand member of the formula (23), for example, shall be written thus,

$$3\,3\,1\,1\,1\,1\,3\,3\,1\,1\,1\,1, \tag{58}$$

or by an additional, but natural, abridgment thus,

$$(3^2\,1^4); \tag{59}$$

the following are the twelve types to which I have referred:

$$\left.\begin{array}{c} 3\,1\,2\,1^6\,2\,1\,3; \quad 3^2\,1^6\,3^2; \quad 1\,3\,1^3\,2^2\,1^3\,3\,1; \quad 1\,3\,1^2\,3^2\,1^2\,3\,1; \\ 1^2\,3^2\,1^2\,3^2\,1^2; \quad 1^3\,2\,3\,1^2\,3\,2\,1^3; \quad 1^3\,3\,2\,1^2\,2\,3\,1^3; \end{array}\right\} \tag{60}$$

$$3\,1\,2\,1^6\,3^2; \quad 3\,1\,2\,1^4\,3^2\,1^2; \quad 3\,2\,1^2\,3^2\,1^5; \quad 1\,3\,1^3\,2\,3\,1^2\,3\,1; \quad 1\,3\,1^2\,2\,3^2\,1^4. \tag{61}$$

For instance, the second of the seven types (60) corresponds to the complete succession,

$$a\,b\,c\,d\,e\,\epsilon\,\beta'\,\delta\,\alpha'\,\gamma\,\epsilon'\,\beta\,\delta'\,\alpha\,\gamma'\,c'\,d'\,e'\,a'\,b', \tag{62}$$

where the twentieth face is not in immediate proximity to the first. In this case, we traverse first the quine round F, then the ten faces of the zone between F and F', though not precisely in the order (51), and finally the quine round F'. The formula of this complete (but non-cyclical) succession, more fully written, is

$$\mu^3\lambda^3(\mu\lambda)^3\mu^3\lambda^3, \tag{63}$$

and in this, as in all similar cases, we may observe, by contrast to the formula for all cyclical (or subcyclical) successions, that we must not equate the product of the powers of λ and μ to unity, and secondly, that the sum of the exponents is always equal to eighteen: whereas in the formula for a complete cycle, such as (23), it is equal to twenty, the two steps by which the original pair is recovered being thus counted. In the case (63), the product in the formula is easily reduced to the value ω, and accordingly, on inspection of the succession (62), we see that the last pair of faces, $a'\,b'$, is opposite to the first pair, $a\,b$: but this is far from always happening. It is also necessary, for the completeness of the succession, that no intermediate product, nor the total one, formed by taking any or all of the eighteen symbolical factors of the formula, in their order, and without interruption or hiatus, should be equal either to λ^4, or to μ^4. For example, if in any product of eighteen factors (λ's and μ's) the succession of exponents 3 1 3 should occur, or the intermediate or initial or final factor $\lambda^3\mu\lambda^3$, or $\mu^3\lambda\mu^3$, that succession must be rejected as incomplete, because

$$\lambda^3\mu\lambda^3 = \mu^4, \quad \mu^3\lambda\mu^3 = \lambda^4 \tag{64}$$

in virtue of the equations (3), (9), (24), (43), and because the operation of λ^4, or of μ^4, reproduces the first face of a quine as the second face of a fourth pair. Conversely, every product of eighteen factors (λ's and μ's), in which no fourth power, directly or indirectly, occurs, represents a complete succession. It represents also a cyclical one, if it happen that the total product has any one of the four following values:

$$\lambda^3; \quad \mu^3; \quad \lambda^4\mu^4; \quad \mu^4\lambda^4. \tag{65}$$

Since the five formulae (61) are varied by inverting the order of the exponents, there are, in all, seventeen ways of completing, non-cyclically, a succession commencing with any three adjacent faces, such as abc.

[12]. The subject of numbers has interested you, and you know much more about it than I do. Has it ever happened to you to meet, or to invent, such formulae of geometrical congruity as the following, which (after what has been already said) seem to require no special explanation, it being understood that they all relate to the successive and cyclical passage over some, or all, of the faces of the icosahedron, as a species of geometrical modulus: and to the employment of the two symbolical operators, λ and μ?

		*	
$\lambda^5 = 1$	$5 \equiv 0$	5	(66)
$\mu\lambda^3\mu\lambda^3 = \lambda^2\lambda^3 = 1$	$(3\,1)^2 \equiv 0$ or $(1\,3)^2 \equiv 0$	8	(67)
$\mu\lambda^2\mu\lambda^2\mu\lambda^2 = \mu\lambda^3\mu\lambda^3 = 1$	$(1\,2)^3 \equiv 0$	9	(68)
$\mu\lambda\mu\lambda^2\mu\lambda\mu\lambda^2 = \mu\lambda^2\mu\lambda^2\mu\lambda^2 = 1$	$(1^3\,2)^2 \equiv 0$	10	(69)
$\mu\lambda\mu\lambda\mu\,.\,\lambda\mu\lambda\mu\lambda = \omega\,.\,\omega = 1$	$1^{10} \equiv 0$	10	(70)
$\mu\lambda\mu\lambda^3\mu^2\lambda^3 = \mu\lambda^3\mu\lambda^3 = 1$	$1^3\,3\,2\,3 \equiv 0$	11	(71)
$\mu\lambda\mu\lambda\mu\,.\,\lambda^2\mu^2\lambda^2 = \omega^2 = 1$	$1^5\,2^3 \equiv 0$	11	(72)
$\mu\lambda^2\mu^2\lambda^3\mu^2\lambda^2 = \mu\lambda^2\mu\lambda^2\mu\lambda^2 = 1$	$1\,2^2\,3\,2^2 \equiv 0$	12	(73)
$(\mu\lambda\mu^2\lambda^2)^2 = (\mu^2\lambda\mu\lambda)^2 = 1$	$(1^2\,2^2)^2 \equiv 0$	12	(74)
$\mu^2\lambda^2\mu^2\,.\,\lambda^2\mu^2\lambda^2 = \omega^2 = 1$	$2^6 \equiv 0$	12	(75)
$\mu\lambda\mu^2\lambda\mu\lambda^2\mu^3\lambda^2 = \mu\lambda\mu^2\lambda\mu\lambda\mu^2\lambda = 1$	$1^2\,2\,1^2\,2\,3\,2 \equiv 0$	13	(76)
$\mu\lambda\mu\lambda\mu\lambda^3\mu^3\lambda^3 = \omega^2 = 1$	$1^5\,3^3 \equiv 0$	14	(77)
$(\mu\lambda\mu^2\lambda^3)^2 = (\mu^2\lambda\mu\lambda^2)^2 = 1$	$(1^2\,2\,3)^2 \equiv 0$	14	(78)
$\mu^2\lambda^3\mu^3\lambda^2\mu\lambda\mu\lambda = \mu^2\lambda^2\mu^2\lambda\mu\lambda\mu\lambda = 1$	$2\,3^2\,2\,1^4 \equiv 0$	14	(79)
$\mu\lambda\mu^2\lambda^2\mu\lambda^2\mu^3\lambda^3 = \mu\lambda\mu^2\lambda^2\mu\lambda\mu^2\lambda^2 = 1$	$1^2\,2^2\,1\,2\,3^2 \equiv 0$	15	(80)
$\mu^2\lambda^2\mu^2\,.\,\lambda^3\mu^3\lambda^3 = \omega^2 = 1$	$2^3\,3^3 \equiv 0$	15	(81)
$\mu\lambda\mu^2\lambda\mu\lambda^2\mu\lambda\mu^3\lambda^3 = \mu\lambda\mu^2\lambda\mu\lambda^2\mu^3\lambda^2 = 1$	$(1^2\,2)^2\,1^2\,3^2 \equiv 0$	16	(82)

* This column is headed 'Number of symbols in cycle or subcycle'.

		*	
$(\lambda^2\mu^3\lambda^2\mu)^2 = (\lambda\mu^2\lambda\mu)^2 = 1$	$(3\,2\,1\,2)^2 \equiv 0$	16	(83)
$\mu\lambda\mu\lambda\mu^2\lambda^3\mu\lambda\mu^3\lambda^3 = \mu\lambda\mu\lambda\mu\lambda^3\mu^3\lambda^3 = 1$	$1^6\,2\,3\,1^2\,3^2 \equiv 0$	17	(84)
$\mu\lambda\mu^2\lambda\mu\lambda^3\mu^2\lambda\mu^2\lambda^3 = \mu\lambda\mu^2\lambda^3\mu\lambda\mu^2\lambda^3 = 1$	$1^2\,2\,1^2\,3\,2\,1\,2\,3 \equiv 0$	17	(85)
$\mu\lambda\mu^3\lambda^3\mu^2\lambda^2\mu^3\lambda^3 = \mu^3\lambda^2\mu^2\lambda^2\mu^3\lambda^3 = 1$	$1^2\,3^2\,2^2\,3^2 \equiv 0$	18	(86)
$\mu\lambda\mu^3\lambda^2\mu\lambda^2\mu^2\lambda\mu^2\lambda^3 = \mu^3\lambda\mu\lambda^2\mu^2\lambda\mu^2\lambda^3 = 1$	$3\,2\,1\,2^2\,1\,2\,3 \equiv 0$	18	(87)
$\mu^3\lambda^3\mu^3 . \lambda^3\mu^3\lambda^3 = \omega^2 = 1$	$3^6 \equiv 0$	18	(88)

and finally

$(\lambda^3\mu^3\lambda\mu\lambda\mu)^2 = 1$	$(3^2\,1^4)^2 \equiv 0$	20	(89)

the last being a concise expression of the only complete and cyclical succession possible: including all the cases of such successions which have been heretofore explained. My present impression is, that the foregoing table of congruities is an exhaustive one, in the general sense of this letter, and with such easy modifications as the rules of this calculus allow: and which, if called on, I shall be happy to explain more fully: though I wish to dismiss this subject from my thoughts, and to turn, or return, to other things. You see, of course, meanwhile, that the congruity (66) represents a quine, and that (70) denotes a zone, of the icosahedron; and generally that all the congruities have natural geometrical interpretations. If we take, for instance, the congruity (75), we see that it is merely a shorter expression for the formula (52), and corresponds to the subcycle of twelve faces (54).

[13]. Instead of passing from face to face, we may propose to pass from corner to corner, of the celebrated solid which we have been considering: and thus the triad of symbols,

$$\iota, \kappa, \lambda, \tag{A}$$

to which this letter relates, will receive a new application. Our object must now be, retaining the fundamental laws of this new calculus, to interpret them anew, as bearing on this new geometrical question. On this point I wish to be brief. There are twelve quines of corners, corresponding to so many pentagonal sections of the body, which with one common order of rotation may be thus arranged.

$$\left.\begin{array}{rll rll}
ABCDE & \text{round} & F; & A'E'D'C'B' & \text{round} & F'; \\
FEC'D'B & \text{round} & A; & F'B'DCE' & \text{round} & A'; \\
FAD'E'C & \text{round} & B; & F'C'EDA' & \text{round} & B'; \\
FBE'A'D & \text{round} & C; & F'D'AEB' & \text{round} & C'; \\
FCA'B'E & \text{round} & D; & F'E'BAC' & \text{round} & D'; \\
FDB'C'A & \text{round} & E; & F'A'CBD' & \text{round} & E';
\end{array}\right\} \tag{90}$$

I propose to interpret λ, when regarded as thus operating on any pair of adjacent corners (and not now on a pair of adjacent faces as before), as signifying that we make the first corner turn round the second, through one step, or stage of the rotation, in the order here adopted as direct, in one or other of the twelve quines (90). Thus,

$$\lambda(AF) = (BF); \quad \lambda^2(AF) = \lambda(BF) = (CF); \quad \lambda^5(AF) = (AF); \tag{91}$$

and the symbolic property of λ as a fifth root of positive unity, is seen to be preserved. It is

* This column is headed 'Number of symbols in cycle or subcycle'.

still more obvious to interpret the square root of unity, ι, anew, as denoting the simple inversion of the order of a pair of adjacent corners, so that, for instance,

$$\iota(AB)=(BA). \tag{92}$$

But to know whether the little calculus, to which the foregoing portion of this letter relates, can be applied, without any alteration of its rules, to the new geometrical subject of discussion now proposed, we must inquire whether it be true that the product,

$$\kappa=\iota\lambda, \quad (5)$$

is still, as it had been when we were treating of the passage from face to face, a symbolical cube root of unity. There is, however, no difficulty in establishing this agreement, between the laws of calculation, for faces and corners of the icosahedron. In fact we have,

$$\left.\begin{aligned}\kappa(AF)=\iota\lambda(AF)=\iota(BF)=(FB);\\ \kappa^2(AF)=\iota\lambda(FB)=\iota(AB)=(BA);\\ \kappa^3(AF)=\iota\lambda(BA)=\iota(FA)=(AF);\end{aligned}\right\} \tag{93}$$

or $\kappa^3=1$, as in (2). We may, therefore, adopt anew the system of equations (A), or the system (1) (2) (3) (4) of this letter, with this varied system of interpretations: and shall be sure that every thing, which was legitimately deduced before, will be still legitimate, and applicable. For example, if we still define that

$$\mu=\iota\kappa^2, \quad \omega=\lambda\mu\lambda\mu\lambda,$$

as in the equations (10) and (20), we shall indeed have new interpretations of these two symbols, namely those expressed, or exemplified, by the equations,

$$\mu(AF)=(AB); \quad \omega(AF)=(F'A'); \tag{94}$$

so that μ has now the effect of causing the second corner of a pair to turn round the first, through one inverse step, or stage, of rotation, and ω now changes a pair of corners, not simply to the opposite pair, but to that pair taken in an inverted order. But still the properties of these two symbols, μ and ω, as denoting, respectively, a fifth root and a square root of unity, remain; we have also still the relations (11) and (12), and it may be remarked that although the commutative property of multiplication does not generally hold good in this calculus, yet with both systems of interpretation we have the very simple formula,

$$\iota\omega=\omega\iota. \tag{95}$$

It will now be useful for one new object to introduce three new symbols, (compare [3]). We shall denote three roots of unity, not hitherto specially considered. Let us, therefore, write,

$$\kappa'=\iota\lambda^4=\iota\lambda^{-1}; \tag{96}$$

$$\nu=\iota\lambda^2; \quad \nu'=\iota\lambda^3; \tag{97}$$

retaining the symbol $\kappa=\iota\lambda$, as before. We shall thus have, by (15) (16) (17), the values,

$$\kappa'^3=1, \tag{98}$$

$$\nu^5=1, \tag{99}$$

$$\nu'^5=1. \tag{100}$$

Then, whereas we have seen, in (93), that

$$\kappa(AF)=(FB),$$

we may establish also the formulae $\kappa'(AF) = (FE),$ (101)

$$\nu(AF) = (FC), \tag{102}$$

$$\nu'(AF) = (FD). \tag{103}$$

Thus all modes of successively passing from corner to corner of the icosahedron come to be symbolically expressed by products of powers of the four roots of unity

$$\kappa, \kappa', \nu, \nu', \tag{B}$$

as all passages from face to face were symbolically expressed by products of the two roots λ and μ. A single example of a complete and cyclical succession of corners may suffice, after the length to which the remarks have extended on the other chief problem of this letter. The formula,

$$(\kappa\kappa'\nu'^2\nu^2)^2 = 1, \tag{104}$$

is found to be symbolically correct and to correspond to the cyclical succession of the twelve corners of the solid:

$$F\,C\,E'\,D'\,B'\,F'\,A'\,D\,E\,A\,B. \tag{105}$$

The halves of the succession, answering to the square root of the expression (104), possess an interesting property, whereto I may perhaps return. Besides the subcycles of three corners, denoted by the equations (2) (98) and those of five corners, denoted by (99) (100), there exist many other subcycles: for example, we have the following formula for a certain subcycle of four points,

$$(\kappa\nu)^2 = 1, \tag{106}$$

which may be exemplified by the succession,

$$A\,F\,C\,B. \tag{107}$$

But unless a complete treatise on the subject of the Icosian Calculus were to be written, it would be tedious to pursue this matter.

[14]. Yet you may think it not irrelevant, if a short additional paragraph be devoted to the more perfect proof that such a calculus exists; or that such equations, as those lately given, can be shewn to be symbolically true, without the slightest reference to the geometry, which (as you see) suggested them. Take then the last equation, (106). We have by the definition (97) of ν, combined with the equations (1) and (4), the value,

$$\nu = \kappa\iota\kappa, \tag{108}$$

which expression has accordingly been seen, in (15), to be one of the fifth roots of unity, as it ought, by (99) to be. Hence

$$\kappa\nu = \kappa^2\iota\kappa, \tag{109}$$

and therefore

$$\begin{aligned}(\kappa\nu)^2 &= \kappa^2\iota\kappa\,.\,\kappa^2\iota\kappa\\ &= \kappa^2\iota\,.\,\iota\kappa, \quad \text{by (2)},\\ &= \kappa^2\,.\,\kappa, \quad \text{by (1)},\\ &= \kappa^3 = 1, \quad \text{by (2) again.}\end{aligned}$$

For you, or indeed for any reader of this letter, it would be useless to give other examples.

[15]. Of course, whatever has been proved respecting the faces and corners of the icosahedron applies, without any change whatever in the rules of calculation, to questions respecting the corners and faces of the dodecahedron: provided merely that we now denote the corners by the twenty symbols $a\,b\,c$ etc., and the faces by the twelve symbols, $A\,B\,C$ etc. Thus a suffi-

cient type for the only cyclical method of passing from corner to corner of the dodecahedron, (inversions etc. being understood) is contained in the equation (23), or in the congruity (89), or in the succession of letters (33), whereof (34) to (42) are merely modifications. There is, however, the slight advantage gained, by this new view of the subject, that it is perhaps a little easier to conceive geometrically, the passage, along successive edges, from corner to corner of a body, than that from face to face. We may, for instance, ask, what closed wire, or what closed though twisted polygon, would fit along some suitably selected twenty edges (out of the thirty) of a dodecahedron, so that every one of its twenty bends should adapt itself to some one of the twenty corners of the solid, no one such corner being omitted, and none being repeated. Or, to make the question, if possible, more clear and definite, we may ask, what shall be the plane development of such a twisted or gauche polygon. Without presuming to deny that I am likely to have been anticipated in the proposal and solution of this problem, I can give you, in a very few words, my own mode of resolving it. We have merely to take the first (or the last) half of the expression (105); for example, this half considered as denoting a succession of six pentagonal faces:

$$F\,C\,E'\,D'\,C'\,B' \tag{110}$$

and to suppress the five common sides, whereby each face is connected with the one that precedes or follows it. The closed figure thus obtained, or its development, inversion being allowed, is what we have been seeking for: and its formula is simply,

$$\nu'^2\nu^2, \tag{111}$$

or

$$\nu^2\nu'^2. \tag{112}$$

[16]. It is time to bring this long letter to a close. You may remember that, in the Memorandum of 7 October I remarked that 'for other solids I use other exponents.' Let it here suffice to observe, that for the octahedron and hexahedron I employ a certain system of square-roots, cube-roots, and fourth-roots of positive unity; fifth-roots being not for these surfaces required. I assume and interpret such a system of equations as

$$\iota^2 = \eta^3 = \theta^4 = 1, \quad \theta = \iota\eta, \tag{113}$$

instead of the system (A) of this letter; and then by making also

$$\theta' = \iota\eta^2, \quad \eta' = \iota\theta^2, \quad \eta'' = \iota\theta^3 \tag{114}$$

and

$$\omega = \iota\eta'^2 = \theta^2\iota\theta^2, \tag{115}$$

I reproduce some old results, such as

$$\omega^2 = 1, \quad \iota\omega = \omega\iota, \qquad (22), (95)$$

and arrive at several new ones, for example,

$$1 = \eta'^4 = \eta''^3 = \theta'^4; \tag{116}$$

$$\omega = \theta\theta'\theta = \theta'\theta\theta' = \eta\eta'\eta''; \quad \text{etc.} \tag{117}$$

all of which have geometrical meanings and assist in expressing by equations certain cyclical or subcyclical successions, of faces or of corners, of the two solids just now mentioned. As regards the tetrahedron, which is almost too simple to be interesting in the present theory, it may be enough to observe that if we assume the system of equations,

$$\iota^2 = \omega^3 = (\iota\omega)^3 = 1, \quad \omega' = \omega\iota\omega, \tag{118}$$

we shall have also $$\omega'^3 = 1; \tag{119}$$

and the formula $$(\omega'\omega)^2 = 1 \tag{120}$$

represents a passage over a cycle of faces of the pyramid.

[17]. In conclusion, let me ask you to accept a diagram,* of a certain system of twelve plane pentagons, to which I am disposed to give the name of the Icosion, because it has been suggested by the consideration of the twenty corners of the dodecahedron, or of the twenty faces of the icosahedron; and is designed to assist the conception of the passage from one such corner, or face, to another. In counting the pentagons of the figure as twelve I include the total or outer pentagon, whereof the eleven others appear to the eye as parts, but which is to be conceived as situated at the back of the paper, as is also the twelfth point F'; so that the order of the rotation $a'\,b'\,c'\,d'\,e'$ in this large pentagon round F', must, as in the table of the twelve quines (25), be regarded as opposite to the order of the rotation round F, though these two orders seem to be the same. All the other rotations are as they appear, in comparison with the standard order $a\,b\,c\,d\,e$.

I have found that some young persons have been much amused by trying a new mathematical game which the Icosion furnishes, one person sticking five pins in any five consecutive points, such as $a\,b\,c\,d\,e$, or $a\,b\,c\,d\,e'$, and the other player then aiming to insert, which by the theory in this letter can always be done, fifteen other pins, in cyclical succession, so as to cover all the other points, and to end in immediate proximity to the pin wherewith his antagonist had begun. Whatever then may be thought of the utility of these new systems of roots of unity, suggested to me by the study of the ancient solids, which we know to have interested Plato, not only as geometrical forms, but as having (supposed) analogies in the speculation in which Kepler too indulged, it may be hoped, at least, that by this mode of representation, they will be found to have supplied (a new and innocent) pleasure, not only to algebraists, but even to children: and I am willing to suppose that, on the whole, you will derive some satisfaction from examining them. At all events, I remain

your faithful friend,

William Rowan Hamilton.

John T. Graves Esq.

Postscript of 25 *October to the letter of* 17 *October* 1856

Let $$1 = \iota^2 = (\iota\lambda)^3 = \lambda^n, \tag{1}$$

then all the results of my letter, so far as they are symbolical, are included in the theory of this system of roots of unity: the exponent n being equal to five for two of the solids, equal to four for two others, and equal to three for the fifth: for instance

$$1 = (\lambda\iota\lambda)^n = (\iota\lambda^2)^n = (\lambda^2\iota)^n = (\iota\lambda^{n-2})^n = (\lambda^m\iota\lambda^{n-2-m})^n; \tag{2}$$

where m is any integer.

If we make $$\mu = \lambda\iota\lambda, \tag{3}$$

* [See Appendix 2.]

we shall have generally, (that is, for all values of the exponent n,)

$$\lambda=\mu\iota\mu, \quad (\iota\mu)^3=1, \quad \mu^n=1; \tag{4}$$

so that λ and μ are connected by relations of perfect reciprocity. Making also,

$$\kappa=\iota\lambda, \tag{5}$$

we have

$$\lambda=\iota\kappa, \quad \mu=\iota\kappa^2, \quad \kappa^3=1, \tag{6}$$

and in all formulae involving ι, κ, λ, μ, we may interchange λ and μ if we at the same time change κ to κ^2, or to κ^{-1}. This specimen of an extension of the theory may be sufficient for the present.

W. R. H.