XV

ON THE EXISTENCE OF A SYMBOLIC AND BIQUADRATIC EQUATION, WHICH IS SATISFIED BY THE SYMBOL OF LINEAR OPERATION IN QUATERNIONS*

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1. In a recent communication (of 9 June 1862),† I showed how the general Linear and Quaternion Function of a Quaternion could be expressed, under a standard quadrinomial form; and how that function, when so expressed, could be inverted.

2. I have since perceived, that whatever *form* be adopted, to represent the *Linear Symbol of Quaternion Operation* thus referred to, that *symbol* always satisfies a certain *Biquadratic Equation*, with *Scalar Coefficients*, of which the *values* depend upon the particular *constants* of the *Function* above referred to.

3. This result, with the properties of the *Auxiliary Linear* and *Quaternion Functions* with which it is connected, appears to me to constitute the most remarkable accession to the *Theory of Quaternions proper*, as distinguished from their *separation* into *scalar* and *vector parts*, and from their *application to Geometry* and *Physics*, which has been made since I had first the honour of addressing the Royal Irish Academy on the subject, in the year 1843.

4. The following is an outline of one of the proofs of the existence of the biquadratic equation, above referred to. Let

$$fq = r \tag{1}$$

be a given linear equation in quaternions; r being a given quaternion, q a sought one, and f the symbol of a linear or distributive operation: so that

$$f(q+q') = fq + fq', \tag{2}$$

whatever two quaternions may be denoted by q and q'.

5. I have found that the *formula of solution* of this equation (1), or the formula of *inversion* of the *function*, f, may be thus stated:

$$nq = nf^{-1}r = Fr; \tag{3}$$

where n is a *scalar constant* depending for its *value*, and F is an auxiliary and *linear symbol* of operation depending for its *form* (or rather for the *constants* which it involves), on the *particular form* of f; or on the special values of the *constants*, which enter into the composition of the *particular function*, fq.

6. We have thus, independently of the particular quaternions, q and r, the equations,

$$Ffq = nq, \quad fFr = nr; \tag{4}$$

or, briefly and symbolically,

$$Ff = fF = n. \tag{5}$$

* [See *Elements*, Book III, chapter II, sections 350 and 365. See also Introduction re Cayley–Hamilton Theorem.]

† [See XIV.]

7. Changing next f to $f_c = f + c$, that is to say, proposing next to resolve the *new linear equation*,

$$f_c q = fq + cq = r, \tag{6}$$

where c is an *arbitrary scalar*, I find that the *new* formula of solution, or of inversion, may be thus written:

$$f_c F_c = n_c; \tag{7}$$

where

$$F_c = F + cG + c^2 H + c^3, \tag{8}$$

and

$$n_c = n + n'c + n''c^2 + n'''c^3 + c^4; \tag{9}$$

G and H being the symbols (or characteristics) of *two new linear operations*, and n', n'', n''' denoting *three new scalar constants*.

8. Expanding then the symbolical product $f_c F_c$, and comparing powers of c, we arrive at *three new symbolical equations*, namely, the following:

$$fG + F = n'; \quad fH + G = n''; \quad f + H = n'''; \tag{10}$$

by elimination of the symbols, F, G, H, between which and the equation (5), the *symbolical biquadratic*,

$$0 = n - n'f + n''f^2 - n'''f^3 + f^4, \tag{A}$$

is obtained.

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1. As early as the year 1846, I was led to perceive the existence of a certain *symbolic* and *cubic equation*, of the form

$$0 = m - m'\phi + m''\phi^2 - \phi^3, \tag{1}$$

in which ϕ is used as a symbol of *linear* and *vector operation* on a *vector*, so that $\phi\rho$ denotes a vector depending on ρ, such that

$$\phi(\rho + \rho') = \phi\rho + \phi\rho', \tag{2}$$

if ρ and ρ' be any two vectors; while m, m', and m'' are *three scalar constants*, depending on the *particular* form of the linear and vector function $\phi\rho$, or on the (scalar or vector) constants which enter into the composition of that function. And I saw, of course, that the problem of *inversion* of such a *function* was at once given by the formula

$$m\phi^{-1} = m' - m''\phi + \phi^2, \tag{3}$$

—the required assignment of the inverse function, $\phi^{-1}\rho$, being thus reduced to the performance of a limited number of *direct operations*.

2. Quite recently I have discovered that the far more general *linear* (or distributive) and *quaternion function of a quaternion* can be *inverted*, by an analogous process, or that there always exists, for any *such* function fq, satisfying the condition

$$f(q + q') = fq + fq', \tag{4}$$

where q and q' are any two quaternions, a *symbolic* and *biquadratic* equation, of the form

$$0 = n - n'f + n''f^2 - n'''f^3 + f^4, \tag{5}$$

in which n, n', n'', and n''' are *four scalar constants*, depending on the particular composition of the linear function fq; and that therefore we may write generally this *Formula of Quaternion Inversion*,

$$nf^{-1} = n' - n''f + n'''f^2 - f^3. \tag{6}$$

3. As it was in the Number of the *Philosophical Magazine* for July 1844 that the first *printed* publication of the Quaternions occurred (though a paper on them had been previously read to the Royal Irish Academy in November 1843), I have thought that the Editors of the Magazine might perhaps allow me thus to put on record what seems to myself an important addition to the theory, and that they may even allow me to add, in a Postscript to this communication, so much as may convey a distinct conception of the *Method* which I have pursued.