XIV

ON A NEW AND GENERAL METHOD OF INVERTING A LINEAR AND QUATERNION FUNCTION OF A QUATERNION*

Communicated 9 June 1862.

[*Proc. Roy. Irish Acad.* vol. VIII (1864), pp. 182–3.]

Let a, b, c, d, e represent any five quaternions, and let the following notations be admitted, at least as temporary ones:

$$ab - ba = [ab]; \quad S[ab]c = (abc);$$

$$(abc) + [cb]Sa + [ac]Sb + [ba]Sc = [abc]; \quad Sa[bcd] = (abcd);$$

then it is easily seen that

$$[ab] = -[ba]; \quad (abc) = -(bac) = (bca) = \&c.; \quad [abc] = -[bac] = [bca] = \&c.;$$

$$(abcd) = -(bacd) = (bcad) = \&c.; \quad 0 = [aa] = (aac) = [aac] = (aacd), \ \&c.$$

We have then these two Lemmas respecting Quaternions, which answer to two of the most continually occurring transformations of vector expressions:

I. $$0 = a(bcde) + b(cdea) + c(deab) + d(eabc) + e(abcd),$$

or I′. $$e(abcd) = a(ebcd) + b(aecd) + c(abed) + d(abce);$$

and II. $$e(abcd) = [bcd]Sae - [cda]Sbe + [dab]Sce - [abc]Sde;$$

as may be proved in various ways.

Assuming therefore *any four* quaternions a, b, c, d, which are *not* connected by the relation,

$$(abcd) = 0,$$

we can *deduce* from them four others, a', b', c', d', by the expressions,

$$a'(abcd) = f[bcd], \quad b'[abcd] = -f[cda], \ \&c.,$$

where f is used as the characteristic of a linear or *distributive quaternion function* of a quaternion, of which the form is supposed to be given; and thus the *general form* of *such* a function comes to be represented by the expression,

V. $$r = fq = a'Saq + b'Sbq + c'Scq + d'Sdq;$$

involving *sixteen scalar constants*, namely those contained in $a'b'c'd'$.

* [See *Elements*, Chapter II, Section 6.]

The *Problem* is to *invert* this *function* f; and the *solution* of that problem is easily found, with the help of the new Lemmas I and II, to be the following:

VI. $$q(abcd)(a'b'c'd') = (abcd)(a'b'c'd')f^{-1}r$$
$$= [bcd](rb'c'd') + [cda](rc'd'a') + [dab](rd'a'b') + [abc](ra'b'c');$$

of which solution the correctness can be verified, *à posteriori*, with the help of the same Lemmas.

Although the foregoing problem of *Inversion* has been *virtually* resolved by Sir W. R. H. many years ago, through a reduction of it to the corresponding problem respecting *vectors*, yet he hopes that, as regards the Calculus of *Quaternions*, the new solution will be considered to be an important step. He is, however, in possession of a general *method* for treating questions of this class, on which he may perhaps offer some remarks at the next meeting of the Academy.*

* [See XV.]