VII

RESEARCHES RESPECTING QUATERNIONS FIRST SERIES

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The researches respecting Quaternions, of the first series of which an account is submitted in the following pages, are to be considered as being, at least in their first aspect and conception, a continuation of those speculations concerning algebraic Couples, and respecting Algebra itself, regarded as the science of Pure Time, which were first communicated to the Royal Irish Academy in November 1833, and were published in the year 1835 in the seventeenth Volume of its Transactions.* The author has thus endeavoured to fulfil, at least in part, the intention which he expressed in the concluding sentence of his former Essay, in the volume just referred to, of publishing, at a time then future, some applications of the same view of algebra to a theory of *sets* of moments, steps, and numbers, which should include that former theory of *couples*. Some general remarks on this whole train of speculation, and on its application to geometrical and physical questions, will be offered at the end of this paper. And the author indulges a hope that the papers containing an account of those subsequent investigations respecting Quaternions, which he has made, and (in part) communicated to the Academy, since the date prefixed to this First Series of Researches, will tend to place the subject in a still clearer point of view: and, by exhibiting more fully to mathematicians its interest and its importance, increase the likelihood of their contributing their aid to its development.

General Conception and Notation of a System or Set of Moments

1. When we have in any manner been led to form successively the separate conceptions of any number of moments of time, we may afterwards form the *new* conception of a *system*, or MOMENTAL SET, to which all these separate moments belong; and may say that this set is of the second, third, fourth, or n^{th} *order*, according as the number of the moments which compose it is 2, 3, 4, or n: we may also call those moments the *constituent moments* of the set. A *symbol* for such a set may be formed by enclosing in parentheses, with commas interposed between them, the separate symbols of the moments which compose the set; thus the symbol of a *momental quaternion*, or set of the fourth order, will be of the form

$$(A_0, A_1, A_2, A_3),$$

if A_0, A_1, A_2, A_3 be employed as symbols to denote the four separate moments of the quaternion. If we employ any other symbol, such as the letter Q, to denote the same quaternion, or set, we may then write an *equation* between the two equisignificant symbols, as follows:

$$Q = (A_0, A_1, A_2, A_3); \tag{1}$$

* [See I.]

and, in like manner, if Q' denote another quaternion, of which the four separate moments are denoted by A'_0, A'_1, A'_2, A'_3, we shall have this other similar equation,

$$Q' = (A'_0, A'_1, A'_2, A'_3). \tag{2}$$

An equation of this sort, between two symbols of equinumerous momental sets, is to be understood as expressing that the *several* moments of the one set coincide respectively with the *homologous* moments of the other set, primary with primary, secondary with secondary, and so on: thus if, with the recent significations of the symbols, we write the *quaternion equation*,

$$Q' = Q, \tag{3}$$

or more fully,

$$(A'_0, A'_1, A'_2, A'_3) = (A_0, A_1, A_2, A_3), \tag{4}$$

we indicate concisely, thereby, the system of the *four* following *momental equations*, or expressions of four coincidences between moments of time denoted by different symbols:

$$A'_0 = A_0, \quad A'_1 = A_1, \quad A'_2 = A_2, \quad A'_3 = A_3. \tag{5}$$

The same complex equation, or system of equations, may also be thus written:

$$(A'_0, A'_1, A'_2, A'_3)\,(=,=,=,=)\,(A_0, A_1, A_2, A_3); \tag{6}$$

or more concisely thus:

$$Q'(=,=,=,=)\,Q. \tag{7}$$

Characteristics of momental Separation, Recombination, and Transposition

2. In the foregoing article, *parentheses* have been used as *characteristics of systematic combination*, in order to combine the symbols of separate moments into the symbol of a common set. If we now agree to prefix, conversely, *characteristics of momental separation*, such as $M_0, M_1, \ldots$ to the symbol of a momental set, in order to form separate symbols for the *separate moments* of that set, we may resolve the equation (1) into the four following:

$$M_0 Q = A_0; \quad M_1 Q = A_1; \quad M_2 Q = A_2; \quad M_3 Q = A_3; \tag{8}$$

and an equation, such as (3), between two momental quaternions or other sets, Q and Q', may, in like manner, be resolved into equations between moments as follows:

$$M_0 Q' = M_0 Q; \quad M_1 Q' = M_1 Q; \quad \&c. \tag{9}$$

With these characteristics of combination and separation of moments we may write, *for any four moments*, A, B, C, D, the *identical* equations,

$$A = M_0(A, B, C, D); \quad B = M_1(A, B, C, D); \quad \&c. \tag{10}$$

and for *any momental quaternion* Q, the identity,

$$Q = (M_0 Q, M_1 Q, M_2 Q, M_3 Q); \tag{11}$$

with other similar expressions for other sets of moments.

The identical expression (11) may also conveniently be written thus:

$$1Q = (M_0, M_1, M_2, M_3)\,Q = M_{0,1,2,3}\,Q; \tag{12}$$

$1\,Q$ being regarded as a symbol equivalent to Q, and the third member of the formula being an abridgment of the second; and then, by omitting the symbol Q of that quaternion of moments which is here the *common operand*, we may write, more concisely,

$$1 = (M_0, M_1, M_2, M_3) = M_{0,1,2,3}; \tag{13}$$

and may call the second or the third member of this last symbolical equation a *characteristic of recombination* (of a momental set). The same analogy of notation enables us easily to form *characteristics of momental transposition*, which shall serve to express the effect of changing the places or ranks, as primary, secondary, &c., of the moments of any set, with reference merely to that conceived and written arrangement on which the set itself depends for its subjective or symbolic existence, and without any regard being *here* had to the objective or phenomenal succession of the moments in the actual progression of time. Thus, from the proposed or assumed quaternion (1), we may, in general, derive twenty-three other quaternions, which shall be all different from it, and from each other, in consequence of their involving different mental and symbolic arrangements of the same four moments of time; and these new quaternions may be denoted by the following expressions:

$$\left.\begin{array}{l}(A_0, A_1, A_3, A_2) = M_{0,1,3,2}\,Q;\\ \cdots\cdots\cdots\cdots\\ (A_3, A_2, A_1, A_0) = M_{3,2,1,0}\,Q.\end{array}\right\} \qquad (14)$$

In this notation we may write the symbolical equations,

$$M^4_{3,0,1,2} = 1; \quad M_{3,0,1,2} = 1^{\frac{1}{4}}; \qquad (15)$$

to imply that *four successive transpositions*, which are each of the kind directed by the characteristic $M_{3,0,1,2}$, will *reproduce* any proposed momental quaternion (A, B, C, D), as the last of the four successive results:

$$(D, A, B, C), \quad (C, D, A, B), \quad (B, C, D, A), \quad (A, B, C, D). \qquad (16)$$

And generally, for any set of moments, we may write, by an analogous use of exponents, the formula

$$M^n_{n-1,0,1,\ldots,n-2} = 1; \qquad (17)$$

which allows us to establish also this other symbolical equation:

$$M^{rn}_{sn-1,0,1,\ldots sn-2} = 1^{\frac{r}{s}}. \qquad (18)$$

For example, if we take, in this last expression, the values $n = 4$, $r = 1$, $s = 2$, we are conducted to the following characteristic of a certain transposition of the moments of an *octad*, which transposition, if it be once repeated, will *restore* those eight moments to their original arrangement, and which is therefore to be regarded as being a *symbolical square root of unity*; namely,

$$\omega = 1^{\frac{1}{2}}, \qquad (19)$$

if

$$\omega = M_{4,5,6,7,0,1,2,3}. \qquad (20)$$

It may also be here observed, as another example of the notation of the present article, that if, in addition to this last characteristic ω, we introduce three other signs of the same sort, which we shall call (for a reason that will afterwards appear) *three coordinate characteristics of octadic transposition*, and shall define as follows:

$$\left.\begin{array}{l}\omega_1 = M_{5,0,7,2,1,4,3,6};\\ \omega_2 = M_{6,3,0,5,2,7,4,1};\\ \omega_3 = M_{7,6,1,0,3,2,5,4};\end{array}\right\} \qquad (21)$$

then these four symbols ω, ω_1, ω_2, ω_3, will be found to be connected by the relations,

$$\omega_1^2 = \omega_2^2 = \omega_3^2 = \omega_1\omega_2\omega_3 = \omega; \qquad (22)$$

$$\omega\omega_1 = \omega_1\omega; \quad \omega\omega_2 = \omega_2\omega; \quad \omega\omega_3 = \omega_3\omega; \qquad (23)$$

from which, when combined with the equation

$$\omega^2 = 1, \tag{24}$$

these other symbolic equations may be deduced:

$$\left.\begin{array}{lll} \omega_1\omega_2 = \omega_3; & \omega_2\omega_3 = \omega_1; & \omega_3\omega_1 = \omega_2; \\ \omega_2\omega_1 = \omega\omega_3; & \omega_3\omega_2 = \omega\omega_1; & \omega_1\omega_3 = \omega\omega_2; \end{array}\right\} \tag{25}$$

$$\left.\begin{array}{l} \omega_1\omega_2\omega_3 = \omega_2\omega_3\omega_1 = \omega_3\omega_1\omega_2 = \omega; \\ \omega_3\omega_2\omega_1 = \omega_1\omega_3\omega_2 = \omega_2\omega_1\omega_3 = 1; \end{array}\right\} \tag{26}$$

$$\left.\begin{array}{c} (\omega\omega_1)^2 = (\omega\omega_2)^2 = (\omega\omega_3)^2 = \omega; \\ (\omega\omega_1)^4 = (\omega\omega_2)^4 = (\omega\omega_3)^4 = 1; \\ \omega_1^4 = \omega_2^4 = \omega_3^4 = 1. \end{array}\right\} \tag{27}$$

Forms of ordinal Relations between Moments, or Sets of Moments; and Comparisons of Pairs of Moments, or Pairs of Sets, with respect to Analogy or Non-analogy

3. If the moment denoted by the symbol A′ be supposed to be essentially, as well as symbolically, distinct from the moment denoted by A, so that these two symbols denote two *different moments* in the progression of time, and that therefore the momental equation A′ = A does *not* hold good; then it is an immediate and necessary result of our notion or intuition of *time*, that the moment A′, since it is *not coincident with* A, must be *either later or earlier* than it. Using, therefore, as in a former Essay,* the signs > <, which are commonly employed as marks of inequality of magnitude, to denote these *two modes of ordinal diversity*, and thus employing the formula

$$A' > A, \tag{28}$$

to express, *without any reference to magnitude*, that the moment A′ is *later* than A; but, on the contrary, using this other formula, in like manner without reference to magnitude,

$$A' < A, \tag{29}$$

to express that A′ is *earlier* than A; so that the character > is here used as a *sign of subsequence*, whereas the mark < is, on the contrary, in this notation, a *sign of precedence*; while the formula, or equation,

$$A' = A, \tag{30}$$

still expresses that the moment A′ is *coincident* (or simultaneous) with A, so that the mark = is at once an expression of *symbolic equivalence* and also a *sign of simultaneity*; we see that the comparison of any *sought* moment A′, regarded as an *ordinand*, with any *given* moment A regarded as an *ordinator*, must conduct to *one or other* of these *three forms of ordinal relation*, (28), (29), (30); and that no such comparison of two moments can conduct to two of these three forms, or modes of relation, at once. In like manner, if we compare any *set* of *n* moments $(A'_0, A'_1, \ldots, A'_{n-1})$, regarded as an *ordinand set*, with any other equinumerous momental set $(A_0, A_1, \ldots, A_{n-1})$, regarded as an *ordinator set*, by comparing *each* moment of the one set with the homologous moment of the other set, primary with primary, secondary with secondary, and so forth, we shall obtain in general *n* different ordinal relations, which may, however, be combined, in thought and in expression, into *one system*, or ORDINAL SET; and this set, which

* On Algebra as the Science of Pure Time.—*Transactions of the Royal Irish Academy*, vol. xvii. Dublin, 1835. [See I.]

may be said to be of the n^{th} *order*, will admit of 3^n different *forms*, obtained by attributing separately to each of its *n constituent ordinal relations* each of the 3 forms $>$ $<$ $=$. For example, the complex ordinal relation which a sought momental quaternion Q′, regarded as an ordinand, bears to a given momental quaternion Q, regarded as an ordinator, is composed of four ordinal relations between the homologous moments of these two momental sets, of which four relations each separately may be one of subsequence ($>$), or of precedence ($<$), or of simultaneity ($=$): and hence this complex ordinal relation of Q′ to Q may receive any one of $3^4 = 81$ different forms, of which one, namely, the case of *quadruple momental coincidence*, has been considered in the first article, and of which the others may be denoted on a similar plan. Thus to write the formula

$$Q'(>,=,<,=)Q, \tag{31}$$

if Q and Q′ denote the quaternions (1) and (2), may be regarded as a mode of concisely expressing the following system of four separate ordinal relations between moments,

$$A'_0 > A_0; \quad A'_1 = A_1; \quad A'_2 < A_2; \quad A'_3 = A_3; \tag{32}$$

or, in the notation of the second article,

$$M_0 Q' > M_0 Q; \quad M_1 Q' = M_1 Q; \quad M_2 Q' < M_2 Q; \quad M_3 Q' = M_3 Q; \tag{33}$$

and similarly in other cases.

4. Again, as we have compared two moments, or two sets of moments, or have conceived them to be compared with each other, with a view to discover the (simple or complex) *ordinal relations* existing between them, so we may now compare, or conceive to be compared, *two pairs* of moments, or of momental sets, with respect to their (simple or complex) *analogy* or *non-analogy*; that is, with respect to the *similarity or dissimilarity of the two simple or complex ordinal relations*, which are discovered by the two separate comparisons of the moments or sets belonging to each separate pair. Representing (as in the former Essay) by the notation

$$D - C = B - A, \tag{34}$$

the existence of an *analogy* of this sort between the two pairs of moments, A, B, and C, D, or the supposition of an *exact similarity* between the two ordinal relations of D to C, and of B to A; we may, in like manner, denote by the formula,

$$Q''' - Q'' = Q' - Q, \tag{35}$$

the *complex analogy* which may be conceived to exist between the two pairs of quaternions, or other momental sets, Q, Q′, and Q″, Q‴, belonging all to any one determined order n, that is, containing each n moments. This analogy (35) requires, for its existence, in the view here taken, that the n constituent ordinal relations between moments which compose, by their mental and symbolic combination into one system, the complex ordinal relation of the set Q‴ to the set Q″, should, *separately* and respectively, be exactly similar to those n other constituent ordinal relations between moments, which collectively compose the other complex ordinal relation of the set Q′ to the set Q; for then, but not otherwise, do we regard the one *complex* ordinal relation as being in *all* respects similar to the other. In symbolical language, the complex *set-analogy* (or analogy between pairs of sets) *of the* n^{th} *order* (35) may be resolved into *n momental analogies* (or analogies between pairs of moments), namely, the following:

$$\left.\begin{array}{l} M_0 Q''' - M_0 Q'' = M_0 Q' - M_0 Q; \\ \cdots\cdots\cdots\cdots \\ M_{n-1} Q''' - M_{n-1} Q'' = M_{n-1} Q' - M_{n-1} Q; \end{array}\right\} \tag{36}$$

of which each separately is to be interpreted on the same plan as the analogy (34). The two formulae of *momental non-analogies*, or of *dissimilar ordinal relations* between pairs of moments,

$$\left.\begin{aligned} D-C>B-A,\\ D-C<B-A,\end{aligned}\right\} \qquad (37)$$

may still be interpreted as in the former Essay; the first formula (37) denoting that the relation of the moment D to C is, as compared with the relation of B to A, a relation of *comparative lateness*; and the second formula (37) denoting, on the contrary, that the former ordinal relation, as compared with the latter, is one of *comparative earliness*: and because, in the first case the moment D is *too late*, while in the second case this moment is *too early*, to satisfy the analogy (34), we may still call the first formula (37) a momental *non-analogy of subsequence*, and may call the second formula (37) a *non-analogy of precedence.* By compounding several such momental non-analogies, or even one such, with any number of momental analogies, into one system, we shall compose a *complex non-analogy* between two pairs of momental sets, which may easily be denoted on the plan of recent notations; thus, if we make, for abridgment,

$$\left.\begin{aligned} Q''=(A_0'',A_1'',A_2'',A_3''),\\ Q'''=(A_0''',A_1''',A_2''',A_3'''),\end{aligned}\right\} \qquad (38)$$

retaining for Q and Q′ the same meanings as in the equations (1), (2), and then write the formula

$$Q'''-Q''(>,=,<,=)\,Q'-Q, \qquad (39)$$

we are to be considered as expressing concisely hereby a complex non-analogy between two pairs of momental quaternions, Q, Q′, and Q″, Q‴, which may be resolved into the following system of mixed analogies and non-analogies between four pairs of moments:

$$\left.\begin{aligned} &M_0Q'''-M_0Q''>M_0Q'-M_0Q; \quad \text{or,} \quad &A_0'''-A_0''>A_0'-A_0;\\ &M_1Q'''-M_1Q''=M_1Q'-M_1Q; &A_1'''-A_1''=A_1'-A_1;\\ &M_2Q'''-M_2Q''<M_2Q'-M_2Q; &A_2'''-A_2''<A_2'-A_2;\\ &M_3Q'''-M_3Q''=M_3Q'-M_3Q; &A_3'''-A_3''=A_3'-A_3.\end{aligned}\right\} \qquad (40)$$

A little consideration suffices to show, by the aid of the fundamental notion of TIME, which enters essentially into this whole theory (at least as the subject is here viewed), that every simple or complex analogy or non-analogy of the kind considered in the present article admits of *alternation*; that is to say, if we call the moments B and C, or the sets Q′ and Q″, the *means*, and call the moments A and D, or the sets Q and Q‴, the *extremes*, of the analogy or non-analogy, it is allowed to *interchange the means* or to *interchange the extremes* among themselves, without destroying the truth or changing the character of the formula. For example, under the conditions (40), we may write, instead of (39), either of the two following forms:

$$\left.\begin{aligned} Q'''-Q'(>,=,<,=)\,Q''-Q;\\ Q-Q''(>,=,<,=)\,Q'-Q'''.\end{aligned}\right\} \qquad (41)$$

We may also employ *inversion*, that is, we may substitute extremes for means, and means for extremes, provided that we, at the same time, change each of the two signs of ordinal diversity between moments, and every complex sign of ordinal non-analogy between momental pairs, to the contrary or *opposite sign*, by changing > to <, and < to >; thus we may write the complex non-analogy (39) under this other or *inverse* form:

$$Q''-Q'''(<,=,>,=)\,Q-Q'. \qquad (42)$$

And with the same conceptions, and the same plan of notation, we are led to regard the following formula of *quadruple momental analogy*,

$$Q''' - Q'' (=, =, =, =) Q' - Q, \tag{43}$$

as being only a fuller expression of that complex analogy between the two pairs of quaternions Q, Q′, and Q″, Q‴, which is more briefly denoted by the formula (35).

5. Consistently with the same modes of interpreting formulae for the expression of any simple or complex analogy or non-analogy between pairs of moments or of sets, or of any similarity or dissimilarity between simple or complex ordinal relations, if we agree that the symbol 0, *when it occurs as one member of any such formula*, shall be regarded as a *symbol of the relation of ordinal identity*, writing thus for any two identical moments, or identical sets,

$$A - A = 0, \quad Q - Q = 0; \tag{44}$$

we may then not only write

$$A' - A = 0, \quad Q' - Q = 0, \tag{45}$$

as transformations of the equations (30) and (3); but also

$$A' - A > 0, \quad A' - A < 0, \tag{46}$$

as transformations respectively of the two formulae of ordinal diversity, (28) and (29); and may write

$$Q' - Q(>, =, <, =) 0, \tag{47}$$

instead of the formula (31). And if we employ small Roman letters, with or without accents or indices, such as a, a_0, &c., to denote generally *any ordinal relations* between moments, which may or may not be relations of identity, and which may otherwise be denoted by such symbols as B − A, $A_0' - A_0$, &c., which have been already used as members of formulae expressing analogies or non-analogies; writing, for example,

$$\left.\begin{array}{ll} A_0' - A_0 = a_0, & A_1' - A_1 = a_1, \ \dots \\ A_0''' - A_0'' = a_0', & A_1''' - A_1'' = a_1', \ \dots \end{array}\right\} \tag{48}$$

and extending this notation so as to introduce the corresponding abridgments,

$$Q' - Q = q, \quad Q''' - Q'' = q'; \tag{49}$$

then we may not only transform the formula (31), or the system of the formulae (32), by writing

$$q(>, =, <, =) 0; \tag{50}$$

but also, on the same plan, may substitute for the expression of the complex non-analogy (39) this more concise expression,

$$q'(>, =, <, =) q. \tag{51}$$

For in this notation (as in that of the former Essay), the first, second, and third of the three formulae,

$$a > 0, \quad a < 0, \quad a = 0, \tag{52}$$

express, respectively, that the ordinal relation between moments, denoted by the letter a, is one of lateness, or of earliness, or of simultaneity; and in like manner, the three written assertions,

$$b > a, \quad b < a, \quad b = a, \tag{53}$$

express, respectively, that the ordinal relation between the two moments of one pair, denoted by b, as *compared with the relation* between the two moments of another pair, denoted by a, is one of *comparative* lateness, comparative earliness, or comparative coincidence, that is, analogy. And to mark generally the *unity of the conception of an ordinal set*, or system of ordinal relations, such as was considered in the foregoing article, we may agree to denote such a system or set of relations by writing in parentheses, with commas interposed, the symbols of those separate relations; and thus may write the formula,

$$Q' - Q = (A'_0 - A_0, A'_1 - A_1, \ldots, A'_{n-1} - A_{n-1}); \tag{54}$$

or, more concisely, by the abridgments (48) and (49), if we confine ourselves to the case of an ordinal quaternion,

$$q = (a_0, a_1, a_2, a_3). \tag{55}$$

Operations on an Ordinal Set; Coordinate Characteristics of Quaternion-Derivation

6. We may now treat this last expression for an ordinal quaternion in the same way as the expression for a momental quaternion was treated in the second article. Let R_0, R_1, &c., be *characteristics of ordinal separation*, analogous to the characteristics of momental separation, M_0, M_1, &c.; we may then, with their help, decompose the equation (55) into four others, as follows:

$$R_0 q = a_0; \quad R_1 q = a_1; \quad R_2 q = a_2; \quad R_3 q = a_3; \tag{56}$$

we may therefore write, *for any four ordinal relations*, a, b, c, d, between moments, the identical equations,

$$a = R_0(a, b, c, d); \quad b = R_1(a, b, c, d); \quad \&c.; \tag{57}$$

and, for *any ordinal quaternion*, we may write the corresponding identity,

$$q = (R_0 q, R_1 q, R_2 q, R_3 q); \tag{58}$$

or more concisely, by abridgments analogous to those marked (13),

$$1 = (R_0, R_1, R_2, R_3) = R_{0,1,2,3}; \tag{59}$$

with formulae of the same kind for ordinal sets of higher orders. *Characteristics of ordinal transposition* are easily formed on the same plan; and we may write, for example, as the expression of one such transposition performed on the ordinal quaternion (55),

$$R_{3,0,1,2}\, q = (a_3, a_0, a_1, a_2); \tag{60}$$

and may hence deduce this symbolic equation, analogous to (15),

$$R^4_{3,0,1,2} = 1. \tag{61}$$

If, instead of thus *transposing the ordinal relations*, we transpose, in the expression of any one relation, *the two related moments*, or momental sets, we then obtain, in general, a new ordinal relation, which is the *inverse* or *opposite* of the old relation, or is that old one with its *sign* (*or signs*) *changed*, each constituent relation of *earliness* being altered to a relation of *lateness* (in the same degree), and *vice versâ*: a change which may be expressed, according to known analogies of notation, by prefixing the sign $-$ to the symbol of the simple or complex relation which has thus been altered: for example, the equations (48), (49) give, by this change of signs,

$$A_0 - A'_0 = -a_0, \quad A_1 - A'_1 = -a_1, \quad \&c.; \tag{62}$$

and

$$Q - Q' = -q, \quad \&c. \tag{63}$$

Hence we may write, as a consequence of the formula (55), the following:

$$-q = (-a_0, -a_1, -a_2, -a_3); \tag{64}$$

that is, for any ordinal quaternion, we have

$$-1=(-R_0, -R_1, -R_2, -R_3), \tag{65}$$

with similar results for other ordinal sets. The notation may be abridged if we agree to write, for the present, such formulae as the following:

$$\left.\begin{array}{l} -R_0=R_{-0}; \quad -R_1=R_{-1}; \ \ldots \\ (R_{-1}, R_{-0}, \ldots)=R_{-1,-0,\ldots}; \ \&c. \end{array}\right\} \tag{66}$$

for then we can not only express the symbolical equation (65) under the shorter form,

$$-1=R_{-0,-1,-2,-3}, \tag{67}$$

but can compose, generally, *characteristics of ordinal derivation*, which shall express the joint or combined performance of several simultaneous or successive acts of separation, inversion, transposition, and recombination of the constituent relations of any ordinal set. Thus if we operate twice successively on an *ordinal couple* (a_0, a_1), by the characteristic of derivation $R_{-1,0}$, we obtain thereby the two new or derived couples:

$$\left.\begin{array}{l} R_{-1,0}(a_0, a_1)=(-a_1, a_0); \\ R^2_{-1,0}(a_0, a_1)=R_{-1,0}(-a_1, a_0) \\ \qquad =(-a_0, -a_1)=-(a_0, a_1); \end{array}\right\} \tag{68}$$

of which the last is merely the original couple (a_0, a_1) with its sign changed; so that we have the symbolic equation,

$$R^2_{-1,0}=-1. \tag{69}$$

This symbolic result, presented under a slightly different form, was made the foundation of the theory of algebraic couples, and of the use of the symbol $\sqrt{-1}$ in algebra, proposed by the present writer, in that Essay, already several times referred to, which was published in a former volume of the *Transactions* of this Academy; for the symbolic equation (vol. XVII, page 417, equation 157)

$$\sqrt{(-1)}=(0, 1),$$

was there given, in which the essential character of the *number-couple* $(0, 1)$ was that, when used as a multiplier, it transformed one *step-couple* (a_1, a_2), that is to say, one couple of steps, a_1, a_2, in the progression of time, or one couple of ordinal relations between moments, into another couple of steps or of relations in the same progression of time, according to the law,

$$(0, 1)\,(a_1, a_2)=(-a_2, a_1);$$

which agrees with the process directed by the recent characteristic of derivation, $R_{-1,0}$, and was included in the equation (37), page 401, of the volume lately cited. Again, if we now regard i, j, k as three characteristics of operation on an ordinal quaternion, defined as follows:

$$\left.\begin{array}{r} i=R_{-1,0,-3,2}; \\ j=R_{-2,3,0,-1}; \\ k=R_{-3,-2,1,0}; \end{array}\right\} \tag{70}$$

we shall have the four following symbolic equations, which will be found to be of essential importance in the present theory of quaternions:

$$\left.\begin{array}{r} i^2=-1; \\ j^2=-1; \\ k^2=-1; \\ ijk=-1; \end{array}\right\} \tag{71}$$

and which may be concisely expressed under the form of a single but continued equation, as follows:

$$i^2 = j^2 = k^2 = ijk = -1. \qquad (72) = (\text{A})$$

7. To leave no doubt respecting the truth or meaning of these important symbolical relations, (72) or (A), between the *three coordinate characteristics of quaternion-derivation*, i, j, k, defined by the equations (70), we shall here exhibit distinctly the successive steps or stages of the transformations which are indicated by those characteristics. Suppose then that any ordinal quaternion q, or any set of four ordinal relations, a, b, c, d, between moments of time, is proposed as the subject of the operations.

For the purpose of operating on this quaternion by the characteristic of derivation i, we may first write the following definitional equation between its two symbols,

$$\mathrm{q} = (\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}), \qquad (73)$$

and then resolve this complex equation into its four components, or constituents, with the help of the signs of ordinal separation, R_0, &c., as follows:

$$\mathrm{R}_0\mathrm{q} = \mathrm{a}; \quad \mathrm{R}_1\mathrm{q} = \mathrm{b}; \quad \mathrm{R}_2\mathrm{q} = \mathrm{c}; \quad \mathrm{R}_3\mathrm{q} = \mathrm{d}. \qquad (74)$$

In the next place, the definition (70) of i, combined with the notation (66), directs us to change the signs of the second and fourth of these equations (74), and then to make the first and second equations change places with each other, interchanging also, at the same time, the places of the third and fourth, so as to form this new system of four equations:

$$\mathrm{R}_{-1}\mathrm{q} = -\mathrm{b}; \quad \mathrm{R}_0\mathrm{q} = \mathrm{a}; \quad \mathrm{R}_{-3}\mathrm{q} = -\mathrm{d}; \quad \mathrm{R}_2\mathrm{q} = \mathrm{c}. \qquad (75)$$

We are then to combine these four constituent ordinal relations, thus partially inverted and transposed, namely, −b, a, −d, and c, into a new ordinal quaternion; and this will be, by definition, the *first coordinate derivative*, iq, of the proposed quaternion q; so that we may now write, as derived from the equation (73), by the *first coordinate mode of quaternion derivation*, the equation,

$$i\mathrm{q} = (-\mathrm{b}, \mathrm{a}, -\mathrm{d}, \mathrm{c}). \qquad (76)$$

If now we repeat this process of derivation, we get successively the two following systems of four equations:

$$\mathrm{R}_0.i\mathrm{q} = -\mathrm{b}; \quad \mathrm{R}_1.i\mathrm{q} = \mathrm{a}; \quad \mathrm{R}_2.i\mathrm{q} = -\mathrm{d}; \quad \mathrm{R}_3.i\mathrm{q} = \mathrm{c}; \qquad (77)$$

$$\mathrm{R}_{-1}.i\mathrm{q} = -\mathrm{a}; \quad \mathrm{R}_0.i\mathrm{q} = -\mathrm{b}; \quad \mathrm{R}_{-3}.i\mathrm{q} = -\mathrm{c}; \quad \mathrm{R}_2.i\mathrm{q} = -\mathrm{d}; \qquad (78)$$

and, finally, by a new combination of these four last ordinal relations into one ordinal quaternion, which is *the derivative of the derivative of* q *in the first coordinate mode*, we find

$$i^2\mathrm{q} = i.i\mathrm{q} = (-\mathrm{a}, -\mathrm{b} \;\; -\mathrm{c}, -\mathrm{d}) = -\mathrm{q}; \qquad (79)$$

so that this *repeated process of derivation* by the characteristic i has *changed the sign of the quaternion*, q, by changing the sign of *each* of its four constituent ordinal relations, a, b, c, d; which is the property expressed by the first equation (71), namely, by the formula,

$$i^2 = -1. \qquad (71, 1)$$

By exactly similar operations, except so far as the second symbolic equation (70) differs from the first, we find, for the *second coordinate derivative*, jq, of the same proposed quaternion, q, the expression,

$$j\mathrm{q} = (-\mathrm{c}, \mathrm{d}, \mathrm{a}, -\mathrm{b}); \qquad (80)$$

and for the *derivative of the derivative in the second mode*,

$$j^2\mathrm{q} = j.j\mathrm{q} = (-\mathrm{a}, -\mathrm{b}, -\mathrm{c}, -\mathrm{d}) = -\mathrm{q} = -1\mathrm{q}; \qquad (81)$$

the symbols 1q and q (like 1Q and Q) being regarded as equivalent: which result (81) justifies the second equation (71), by giving the symbolic equation,

$$j^2 = -1. \tag{71,2}$$

And in like manner the *third coordinate derivative*, kq, is, by the third equation (70), expressed as follows:

$$k\mathrm{q} = (-\mathrm{d}, -\mathrm{c}, \mathrm{b}, \mathrm{a}); \tag{82}$$

so that, by repeating this process of derivation, we find that the *derivative of the second order, in the third mode*, as well as in each of the two other modes, is *the original quaternion with its sign changed*,

$$k^2\mathrm{q} = k\,.\,k\mathrm{q} = (-\mathrm{a}, -\mathrm{b}, -\mathrm{c}, -\mathrm{d}) = -1\mathrm{q}; \tag{83}$$

or, by detaching the symbols of operation from those of the common operand,

$$k^2 = -1. \tag{71,3}$$

Finally, if we operate on the expression (82) for kq, by the characteristic j, we find

$$j\,.\,k\mathrm{q} = \mathrm{R}_{-2,\,3,\,0,\,-1}(-\mathrm{d}, -\mathrm{c}, \mathrm{b}, \mathrm{a})$$

$$= (-\mathrm{b}, \mathrm{a}, -\mathrm{d}, \mathrm{c}) = i\mathrm{q}; \tag{84}$$

and, therefore, operating on this result by i, we obtain,

$$i\,.\,j\,.\,k\mathrm{q} = i\,.\,i\mathrm{q} = -1\mathrm{q}, \tag{85}$$

that is,

$$ijk = -1; \tag{71,4}$$

so that *the first coordinate derivative, of the second coordinate derivative, of the third coordinate derivative of any ordinal quaternion, is equal to that quaternion with its sign changed*; and all the parts of the compound assertion (72), or (A), are justified.

8. We see, at the same time, by (84), that

$$jk = i; \tag{86}$$

or that a derivation in the third mode, followed by a derivation in the second mode, is equivalent to a derivation in the first mode. If, on the contrary, we had effected the two successive derivations in the *opposite order*, operating first in the second mode, and afterwards in the third mode, we should have obtained an *opposite result*, that is, a result which might be formed from the previous result by changing the sign of the final ordinal quaternion: for if we operate on the expression (80) by k, we get

$$kj\mathrm{q} = (\mathrm{b}, -\mathrm{a}, \mathrm{d}, -\mathrm{c}) = -i\mathrm{q}, \tag{87}$$

giving the symbolic equation,

$$kj = -i, \tag{88}$$

of which the contrast to the equation (86) is highly worthy of attention. Another contrast of the same sort presents itself, between the results of operating on the expression (80) by the characteristic i, and on the expression (76) by the characteristic j; for these two processes give,

$$\left.\begin{aligned} ij\mathrm{q} &= (-\mathrm{d}, -\mathrm{c}, \mathrm{b}, \mathrm{a}) = k\mathrm{q}; \\ ji\mathrm{q} &= (\mathrm{d}, \mathrm{c}, -\mathrm{b}, -\mathrm{a}) = -k\mathrm{q}; \end{aligned}\right\} \tag{89}$$

or, more concisely,

$$ij = k; \quad ji = -k. \tag{90}$$

And, finally, we find, in like manner, by operating on (76) by k, and on (82) by i, the two contrasted results,

$$\left.\begin{aligned} ki\mathrm{q} &= (-\mathrm{c}, \mathrm{d}, \mathrm{a}, -\mathrm{b}) = j\mathrm{q}; \\ ik\mathrm{q} &= (\mathrm{c}, -\mathrm{d}, -\mathrm{a}, \mathrm{b}) = -j\mathrm{q}; \end{aligned}\right\} \tag{91}$$

giving $$ki = j; \quad ik = -j. \tag{92}$$

The importance and singularity of these results (86) (88) (90) (92) induce us to collect them here into one view, as follows:

$$\left.\begin{aligned} ij = k; \quad ji = -k; \\ jk = i; \quad kj = -i; \\ ki = j; \quad ik = -j. \end{aligned}\right\} \tag{93)=(B}$$

9. It ought, however, to be observed, that when once the fundamental formula, or continued equation (A), has been established, no new operations of *actual* derivation of quaternions, by inversions and transpositions of ordinal relations between moments, such as have been performed in the foregoing article, are *necessary*, for the deduction of these equations (B). Thus if we knew, by any process independent of the actual derivations (84), that $i^2 = ijk = -1$, or that $i^2\mathrm{q} = ijk\mathrm{q} = -\mathrm{q}$, whatever ordinal quaternion q may be, we could infer immediately that

$$jk\mathrm{q} = -i^2.jk\mathrm{q} = -i.ijk\mathrm{q} = -i(-\mathrm{q}) = i\mathrm{q}, \tag{94}$$

and thus could return to the symbolic equation (86), or to the essential part of the relation (84), from the equations (A). Again, from those equations (A) we can infer that

$$ij.k\mathrm{q} = ijk\mathrm{q} = -\mathrm{q} = k^2\mathrm{q} = k.k\mathrm{q}, \tag{95}$$

and, therefore, suppressing the symbol kq of the common operand, which may represent any ordinal quaternion, we obtain the first equation (90), namely, $ij = k$. Operating on this by i, and changing i^2 to -1, we find the second equation (92), $ik = -j$. Operating with this on $-k$q, we obtain again $i = jk$. Operating on this by j, we get $ji = -k$; that is, we are conducted to the second equation (90). Operating with this on $-i$q, we find the first equation (92), namely, $ki = j$. And, finally, operating on this equation by k, we are brought to the equation (88), namely, $kj = -i$, which completes the symbolic deduction of (B) from (A).

Either by a deduction of this sort, or by actually performing the operations indicated, we find also that

$$kji = 1; \tag{96}$$

that is to say, if we operate successively on any ordinal quaternion q by the three modes of coordinate derivation, i, j, k, in their order (first by i, then by j, and finally by k), the result will be the original quaternion itself. And if we make, for abridgment, in the notation of the sixth article,

$$\left.\begin{aligned} i' = \mathrm{R}_{1,-0,3,-2}; \\ j' = \mathrm{R}_{2,-3,-0,1}; \\ k' = \mathrm{R}_{3,2,-1,-0}; \end{aligned}\right\} \tag{97}$$

so that the results of the operation of these three new characteristics, i', j', k', on the quaternion (73), are, respectively,

$$\left.\begin{aligned} i'\mathrm{q} = (\mathrm{b}, -\mathrm{a}, \mathrm{d}, -\mathrm{c}); \\ j'\mathrm{q} = (\mathrm{c}, -\mathrm{d}, -\mathrm{a}, \mathrm{b}); \\ k'\mathrm{q} = (\mathrm{d}, \mathrm{c}, -\mathrm{b}, -\mathrm{a}); \end{aligned}\right\} \tag{98}$$

we shall then have not only the relations,

$$i' = -i, \quad j' = -j, \quad k' = -k, \tag{99}$$

but also these others,

$$\left.\begin{aligned} i'i = ii' = 1; \\ j'j = jj' = 1; \\ k'k = kk' = 1; \end{aligned}\right\} \tag{100}$$

on which account we may call these three new signs, i', j', k', as compared with the signs i, j, k, *coordinate characteristics of contra-derivation*, performed on an ordinal quaternion.

Connexions between the coordinate Characteristics of Quaternion-Derivation and those of Octadic Transposition, introduced in the foregoing Articles

10. It may serve to throw some additional light on the foregoing relations between the coordinate characteristics, i, j, k, of quaternion-derivation, if we point out a connexion which exists between (1st) the system of these three signs and the sign $-$, which enters with them into the formula (A), on the one hand, and (2nd) the system of the four characteristics of octadic transposition, $\omega_1, \omega_2, \omega_3$, and ω, which were considered in the second article, on the other hand. In general, an *ordinal set* of the n^{th} order, since it involves n constituent ordinal relations, which are each between two moments, or because it is a complex ordinal relation between two momental sets, which are each of the n^{th} order, may be regarded as containing, in its first conception, a reference to $2n$ moments; and these moments may always be supposed to be collected, in thought and in expression, into *a new momental set, of twice as high an order* as the ordinal set which was proposed. In symbols, the ordinal set (54), which may be thus denoted:

$$Q' - Q = (A'_0, A'_1, \ldots, A'_{n-1}) - (A_0, A_1, \ldots, A_{n-1}), \tag{101}$$

may naturally suggest the consideration of the following momental set, with which it is *connected*:

$$(A'_0, A'_1, \ldots, A'_{n-1}, A_0, A_1, \ldots, A_{n-1}); \tag{102}$$

and if the latter set be given, the former can be deduced from it. Hence every operation of transposition performed on the $2n$ moments of the set (102), is connected with, and determines, a certain *corresponding* change of the n ordinal relations of the set (101). For example, if in the formula of momental transposition (18) we make $s = 2, r = 1$, then, with reference to a certain operation on the momental set (102), which consists here in exchanging the places of each moment A with the corresponding moment A$'$, we obtain the symbolic equation,

$$M^n_{2n-1, 0, 1, \ldots, 2n-2} = 1^{\frac{1}{2}}; \tag{103}$$

which implies that a repetition of this process of transposition would restore the set (102) to its original state. But the same operation on this momental set corresponds to, and determines, a certain other operation, performed on the ordinal set (101), which consists in changing the sign of each constituent ordinal relation, and in therefore changing, by the sixth article, the sign of the ordinal set itself, or in operating on that ordinal set by the characteristic $-$, or -1; we might therefore, in this way, be conducted to the known result, or principle, that the sign $-$, or the coefficient -1, is a symbolic square root of unity. And we might be led to express in words the corresponding conception, by saying that as two successive interchanges of the places of two moments, or of two momental sets, regarded respectively as ordinand and as ordinator, do not finally affect their ordinal relation to each other; the *second transposition* of these two moments or sets having *destroyed the effect of the first*: so too, and for a similar reason, the character (as well as the degree) of an ordinal *relation* is not changed, or is *restored*, when it undergoes *two successive inversions*: the *opposite of the opposite* of a relation being the same with that *original relation* itself. Thus, in particular, for the case $n=4$, the characteristic of octadic transposition, ω, of which the symbolic square was unity, is connected with the sign $-$, or -1, prefixed, as a characteristic of inversion, to the symbol of an ordinal quaternion.

11. Again, with respect to the *sign of* SEMI-INVERSION, $\sqrt{(-1)}$, we may observe that *if the exponent* n *of the order of the ordinal set be an even number*, $=2m$, then we shall have in general, as a symbolic *fourth root of unity*, the following characteristic of momental transposition, which may be obtained by changing r to 1, s to 4, and n to m, in the formula (18):

$$\mathrm{M}^m_{4m-1,0,1,\ldots,4m-2} = 1^{\frac{1}{4}}; \tag{104}$$

and which takes the particular form (15), when m is changed to 1. And because the symbolic square of the first member of (104) acquires the form (103) by restoring n in the place of $2m$, we see that *an ordinal set, if it be of an even order*, such as is an ordinal couple or quaternion, may always be *semi-inverted, and therefore operated on by the sign* $\sqrt{(-1)}$, in, *at least, one way*, through the medium of that momental transposition, performed *on a momental set of an evenly even order*, which is indicated by this first member. For example, when we operate on a *momental quaternion* $(\mathrm{A}'_0, \mathrm{A}'_1, \mathrm{A}_0, \mathrm{A}_1)$ by the characteristic $\mathrm{M}_{3,0,1,2}$ we obtain the new momental quaternion,

$$(\mathrm{A}_1, \mathrm{A}'_0, \mathrm{A}'_1, \mathrm{A}_0) = \mathrm{M}_{3,0,1,2}(\mathrm{A}'_0, \mathrm{A}'_1, \mathrm{A}_0, \mathrm{A}_1); \tag{105}$$

and it is evident that, as was remarked in the second article, and as is included in the more general assertion (104), four successive transpositions of this sort *reproduce* the momental quaternion which was originally proposed to be operated on. But we now see, further, that if, on the plan of the article immediately preceding the present, we *connect*, in thought, this *momental quaternion* with the *ordinal couple*,

$$(\mathrm{A}'_0, \mathrm{A}'_1) - (\mathrm{A}_0, \mathrm{A}_1) = (\mathrm{A}'_0 - \mathrm{A}_0, \mathrm{A}'_1 - \mathrm{A}_1), \tag{106}$$

we shall thereby *connect* the foregoing operation of *momental transposition* with an operation of *ordinal derivation*, which must admit of being symbolically represented by the sign $\sqrt{(-1)}$, and which here consists in passing from the couple (106) to this other ordinal couple:

$$(\mathrm{A}_1, \mathrm{A}'_0) - (\mathrm{A}'_1, \mathrm{A}_0) = (\mathrm{A}_1 - \mathrm{A}'_1, \mathrm{A}'_0 - \mathrm{A}_0). \tag{107}$$

In fact, if we examine the changes of ordinal relation which have been made, in passing from the form (106) to the form (107), we shall perceive that they may be said to consist in first inverting the second constituent relation of the couple, namely, $\mathrm{A}'_1 - \mathrm{A}_1$, which thus becomes $\mathrm{A}_1 - \mathrm{A}'_1$, and in then transposing the two constituent relations. But this is precisely the process of ordinal derivation which was indicated in the sixth article by the characteristic $\mathrm{R}_{-1,0}$, and which we saw to be a symbolic square root of -1. Indeed, as was noticed in that sixth article, it was on this property of this mode of derivation, that the present writer proposed, in a former Essay, to found a theory of algebraic couples, and of the use of the symbol $\sqrt{(-1)}$ in algebra.

12. Proceeding on a similar plan, though not precisely by the formula (104), to illustrate those *new symbolic fourth roots of unity* which enter into the present theory of algebraic quaternions, by regarding those roots as certain characteristics of ordinal derivation, which are connected with certain other characteristics of momental transposition, we are now to consider a *momental octad*, which we shall denote as follows:

$$\Omega = (\mathrm{A}'_0, \mathrm{A}'_1, \mathrm{A}'_2, \mathrm{A}'_3, \mathrm{A}_0, \mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3); \tag{108}$$

and shall regard as being *connected*, on the plan of the tenth article, with the *ordinal quaternion*,

$$q=(A_0', A_1', A_2', A_3')-(A_0, A_1, A_2, A_3); \tag{109}$$

that is, by (48) and (49), with the ordinal quaternion (55). If we operate on the octad Ω by the characteristic of transposition ω, defined by the symbolic equation (20) of the second article, then, according to a remark lately made, the resulting octad $\omega\Omega$ *corresponds* to, or is (on the present plan) connected with, the quaternion $-q$; and thus the two signs ω and $-$, as here used, have a certain correspondence, or connexion, though not an identity, with each other. Again, if we operate on the same octad Ω by the three coordinate characteristics of transposition ω_1, ω_2, ω_3, defined by the equations (21), we obtain these three new octads:

$$\left.\begin{aligned} \omega_1\Omega &= (A_1, A_0', A_3, A_2', A_1', A_0, A_3', A_2);\\ \omega_2\Omega &= (A_2, A_3', A_0', A_1, A_2', A_3, A_0, A_1');\\ \omega_3\Omega &= (A_3, A_2, A_1', A_0', A_3', A_2', A_1, A_0);\end{aligned}\right\} \tag{110}$$

to which *correspond* these three derived quaternions:

$$\left.\begin{aligned} iq &= (A_1-A_1', A_0'-A_0, A_3-A_3', A_2'-A_2);\\ jq &= (A_2-A_2', A_3'-A_3, A_0'-A_0, A_1-A_1');\\ kq &= (A_3-A_3', A_2-A_2', A_1'-A_1, A_0'-A_0);\end{aligned}\right\} \tag{111}$$

the characteristics of derivation ijk being easily seen to have the same effect and significance here as in the recent articles. Thus the three coordinate characteristics of quaternion-derivation, i, j, k, *correspond* respectively to the three coordinate characteristics of octadic transposition, ω_1, ω_2, ω_3; and since the sign $-$ has been seen to correspond in like manner, as a sign of ordinal inversion performed on the quaternion q, to the other octadic characteristic ω, we see that a correspondence is at once established between the symbolic equations (22), respecting transpositions of the moments of an octad, and the formulae (72) or (A), respecting derivations of an ordinal quaternion. The equations (25) correspond in like manner to the formulae (93) or (B); the octadic characteristics, $\omega\omega_1$, $\omega\omega_2$, $\omega\omega_3$, correspond to the characteristics of contraderivation of a quaternion, i', j', k'; the equation (27) might remind us that i, j, k, i', j', k' are, all of them, symbolic fourth roots of unity; and, finally, the equations (26) show, by the same kind of correspondence of relations, that we may write the following formulae, which include the results (71, 4) and (96):

$$\left.\begin{aligned} ijk=jki=kij&=-1;\\ kji=ikj=jik&=1.\end{aligned}\right\} \tag{112}$$

Addition and Subtraction, or Composition and Decomposition of Ordinal Relations between any Sets of Moments

13. The usual correlation between the signs $+$ and $-$ may be extended by definition to expressions involving those signs in conjunction with symbols for momental and ordinal sets; and thus, by the use already mentioned of *zero*, the following equations,

$$\left.\begin{aligned} (Q'-Q)+Q&=Q',\\ (Q''-Q')+(Q'-Q)&=Q''-Q,\\ 0+Q&=Q,\end{aligned}\right\} \tag{113}$$

together with those others which are formed from them by changing each Q to q, may here, as elsewhere, be regarded as *identically* true. At the same time, the two symbols $0-q$ and $-q$ will thus be equisignificant, each denoting the inverse or *opposite* of that complex ordinal relation between two sets of moments, which is denoted by the symbol q; because the symbol $-q$ has been already defined to denote that inverse relation, and therefore we have now the two equations, $(-q)+q=0$, $(0-q)+q=0$; and the other isolated, but *affected* symbol, $+q$, may in like manner be interpreted as being equivalent in signification to $0+q$, and therefore to q. With the conceptions of *addition* and *subtraction*, or of *composition and decomposition of ordinal relations*, which correspond to these notations, we may write:

$$(a', b', \ldots) \pm (a, b, \ldots) = (a' \pm a, b' \pm b, \ldots); \tag{114}$$

$$\left.\begin{aligned} R_0(q' \pm q) &= R_0 q' \pm R_0 q; \\ R_1(q' \pm q) &= R_1 q' \pm R_1 q; \ldots \end{aligned}\right\} \tag{115}$$

or, using Σ and Δ as characteristics of *sum* and *difference*, we may establish the important identities:

$$R_m \Sigma q = \Sigma R_m q; \quad R_m \Delta q = \Delta R_m q. \tag{116}$$

Addition of ordinal sets is a *commutative* and also an *associative* operation; that is, we have the formulae,

$$q' + q = q + q'; \tag{117}$$

$$(q'' + q') + q = q'' + (q' + q); \tag{118}$$

the former of these two properties of addition being connected with the principle of *alternation of an analogy*, which was mentioned in the fourth article. An ordinal set, of any order n, may always be regarded as the *sum* of n other sets of the same order, in each of which only *one* constituent ordinal relation (at most) shall be a relation of diversity; for we may write, generally,

$$q = (R_0 q, 0, \ldots) + (0, R_1 q, \ldots) + \&c. \tag{119}$$

Thus, for example, the ordinal quaternion (73) may be expressed as the *sum of four others*, which may be called respectively a *pure primary* (ordinal quaternion), a *pure secondary*, *pure tertiary*, and *pure quaternary*, as follows:

$$(a, b, c, d) = (a, 0, 0, 0) + (0, b, 0, 0) + (0, 0, c, 0) + (0, 0, 0, d). \tag{120}$$

Multiplication of an ordinal Set by a Number

14. With these preparations it is easy to attach a perfectly clear conception to the act or process of *multiplying* any single ordinal relation, a, or any ordinal set, q, *by any positive or negative number*, m. For having already agreed to regard 1q and q, as well as 1a and a, as being symbols equivalent to each other, so that we have identically, or by definition,

$$a = 1a, \quad q = 1q; \tag{121}$$

and adopting also from common Arithmetic, which may itself be regarded as a branch of the *Science of Pure Time*, since it involves the conception of *succession* between things or thoughts as *counted*, the abbreviations 2, 3, &c., for the symbols $1+1$, $1+1+1$, &c., we shall have an analogous *system of abbreviated symbols* to denote the *composition of* any number of *similar ordinal relations*, whether those components be simple, as a, or complex, as q; namely, the following:

$$\left.\begin{aligned} a + a = 2a, \quad a + a + a = 3a, \ \&c.; \\ q + q = 2q, \quad q + q + q = 3q, \ \&c. \end{aligned}\right\} \tag{122}$$

We may also agree to write, at pleasure, $2 \times \mathrm{a}$, $3 \times \mathrm{q}$, &c., instead of 2a, 3q, &c.; and with this use of elementary notations, the *distributive* and *associative* properties of multiplication offer themselves in the present theory, under the well-known and elementary forms,

$$m(\mathrm{a}' \pm \mathrm{a}) = m\mathrm{a}' \pm m\mathrm{a}; \quad (m' \pm m)\mathrm{a} = m'\mathrm{a} \pm m\mathrm{a}; \tag{123}$$

$$(m'm) \times \mathrm{a} = m' \times (m\mathrm{a}); \quad (m' \div m) \times m\mathrm{a} = m'\mathrm{a}; \tag{124}$$

in each of which each symbol a or a′ of a simple ordinal relation may be changed to the corresponding symbol q or q′ of an ordinal set, and in which we may, *at first*, suppose that m, m', $m'-m$, and $m' \div m$, denote positive whole numbers. Then writing (as usual),

$$0 \times \mathrm{a} = 0, \quad 0 \times \mathrm{q} = 0, \tag{125}$$

we shall be able, with the help of the interpretations in the last article, to *remove* the last mentioned restriction, and to suppose that m, m', $m'+m$, $m'-m$, $m' \times m \,(=m'm)$, and $m' \div m \left(=\frac{m'}{m}\right)$, denote *any* numbers, whole or fractional, and positive or negative, or null, from $-\infty$ to $+\infty$, without violating any of the usual rules for operating on such numbers, by addition, subtraction, multiplication, and division; or rather we might *deduce anew* all those known rules for those fundamental operations on what are usually called *real* numbers, as consequences of the foregoing formulae, or as necessary conditions for their generalization; observing, indeed, that for the case of *incommensurable* (but still real) multipliers, whether operating on a simple ordinal relation a, or on an ordinal set q, we are to use also an equation of *limits*, of the form,

$$(\lim m) \times \mathrm{a} = \lim (m \times \mathrm{a}). \tag{126}$$

It is a consequence of these conceptions and notations that *an ordinal set* q *is multiplied by a number* m, *when each of its constituent ordinal relations*, $\mathrm{R}_0\mathrm{q}$, $\mathrm{R}_1\mathrm{q}$, &c., *is separately multiplied thereby*; so that we may establish the formula,

$$m(\mathrm{a}, \mathrm{b}, \mathrm{c}, \ldots) = (m\mathrm{a}, m\mathrm{b}, m\mathrm{c}, \ldots); \tag{127}$$

and therefore also, $$\mathrm{R}_0 . m\mathrm{q} = m\mathrm{R}_0\mathrm{q}; \quad \mathrm{R}_1 . m\mathrm{q} = m\mathrm{R}_1\mathrm{q}; \quad \&\mathrm{c}. \tag{128}$$

And any ordinal relations, such as ma, mb, &c., or any ordinal sets, such as mq, mq′, &c., which are thus obtained from others, such as a, b, &c., or q, q′, &c., by multiplying them respectively by any common number m, may be said to be *proportional* to those others.

We may also say that any ordinal relations, such as ma, m'a, &c., and that any ordinal sets, such as mq, m'q, &c., are *proportional to the multiplying numbers* m, m', &c., by which they are generated from any common relation a, or set q, as from a common multiplicand, when such generation is possible.

Case of Existence of a simple numeral Quotient, obtained by a particular Division of one ordinal Set by another

15. The recent theory of the *multiplication* of an ordinal set by a number, enables us to assign, in one extensive case, an expression for the result of the *division* of one ordinal set by another; for if we regard the equations

$$(\mathrm{a}' \div \mathrm{a}) \times \mathrm{a} = \mathrm{a}', \quad (\mathrm{q}' \div \mathrm{q}) \times \mathrm{q} = \mathrm{q}', \tag{129}$$

as being identically or definitionally true by the general symbolical correlation of the signs $\times$ and $\div$, we may then write, in virtue of the formula (127), this other and correlative formula,

$$(\mathrm{a}', \mathrm{b}', \mathrm{c}', \ldots) \div (\mathrm{a}, \mathrm{b}, \mathrm{c}, \ldots) = m, \tag{130}$$

whenever the following conditions are satisfied:

$$\mathrm{a}' \div \mathrm{a} = \mathrm{b}' \div \mathrm{b} = \mathrm{c}' \div \mathrm{c} = \ldots = m. \tag{131}$$

In other words, we know how to *interpret the quotient* $\mathrm{q}' \div \mathrm{q}$, of *one ordinal set* q' *divided by another* q, namely, as being another expression for a simple or single number m, in the case when the n constituent ordinal relations of the one set are *proportional* (in the sense lately defined) to the *n homologous constituents* of the other set; and we have, *in that case*, the continued equation,

$$\mathrm{q}' \div \mathrm{q} = \mathrm{R}_0 \mathrm{q}' \div \mathrm{R}_0 \mathrm{q} = \mathrm{R}_1 \mathrm{q}' \div \mathrm{R}_1 \mathrm{q} = \&\mathrm{c}. \tag{132}$$

But in the infinitely many *other* cases in which this condition of proportionality is *not* satisfied, the n numerical quotients, $\mathrm{R}_0 \mathrm{q}' \div \mathrm{R}_0 \mathrm{q}$, $\mathrm{R}_1 \mathrm{q}' \div \mathrm{R}_1 \mathrm{q}$, &c., being at least partially different among themselves, and therefore being not each equal to one common number m (whether commensurable or incommensurable, and whether positive or negative or null), it is, for the same reason, *impossible to find any* ONE *such number*, m, *which shall be correctly equated to the quotient* $\mathrm{q}' \div \mathrm{q}$ *of the two proposed ordinal sets*, in consistency with the foregoing principles. It is, however, *not impossible to find a* SYSTEM *of numbers*, which may, consistently with those principles, be regarded as representing this *quotient of the division of one ordinal set by another*; and we proceed to give an outline of a process by which such a numeral system, or *complex quotient*, may be found.

Investigation of a complex numeral Quotient, resulting from the general symbolical Division of one ordinal Set by another

16. Conceive that from any proposed expression of the form,

$$\mathrm{q} = (\mathrm{a}_0, \mathrm{a}_1, \ldots, \mathrm{a}_t, \ldots, \mathrm{a}_{n-1}), \tag{133}$$

for an ordinal set q of the n^{th} order, we form n other expressions of *coordinate derivative sets*, $\mathrm{q}_0, \mathrm{q}_1, \ldots, \mathrm{q}_{n-1}$, according to the type,

$$1 \mathrm{X}_r \mathrm{q} = \mathrm{X}_r \mathrm{q} = \mathrm{q}_r = (\mathrm{a}_{r,0}, \mathrm{a}_{r,1}, \ldots, \mathrm{a}_{r,s}, \ldots, \mathrm{a}_{r,n-1}); \tag{134}$$

in which it is supposed that the constituent ordinal relation $\mathrm{a}_{r,s}$, of the derivative set q_r, has a determinate and known dependence on the n constituents, such as a_t, of the proposed set q; and let us conceive that this dependence is expressed by a formula such as the following:

$$\mathrm{a}_{r,s} = c_{r,s,0} \mathrm{a}_0 + \ldots + c_{r,s,t} \mathrm{a}_t + \ldots + c_{r,s,n-1} \mathrm{a}_{n-1}; \tag{135}$$

the n^3 *coefficients of coordinate derivation*, $c_{r,s,t}$, being all regarded as constant and known numbers, whether positive or negative or null. It will then be possible, without altering the constant numerical values thus supposed to belong to these n^3 coefficients, $c_{r,s,t}$, to form a *complex* and variable *derivative* q' of the set q, by multiplying each of the *n simple* or *elementary derivatives* already obtained, such as q_r, by a *variable number* m_r, and adding the n products together; and the resulting set may be denoted thus:

$$\left.\begin{aligned} &(m_0 \mathrm{X}_0 + m_1 \mathrm{X}_1 + \ldots + m_r \mathrm{X}_r + \ldots + m_{n-1} \mathrm{X}_{n-1})\, \mathrm{q} \\ &\quad = m_0 \mathrm{q}_0 + m_1 \mathrm{q}_1 + \ldots + m_r \mathrm{q}_r + \ldots + m_{n-1} \mathrm{q}_{n-1} = \mathrm{q}'; \end{aligned}\right\} \tag{136}$$

where we shall have $$q' = (a'_0, a'_1, \ldots, a'_s, \ldots, a'_{n-1}), \tag{137}$$
if we make, for abridgment,

$$a'_s = m_0 a_{0,s} + m_1 a_{1,s} + \ldots + m_r a_{r,s} + \ldots + m_{n-1} a_{n-1,s}; \tag{138}$$

and the entire collection of signs of operation, $m_0 \times_0 + \&c.$, which is prefixed between parentheses to the symbol q in the first line of the formula (136), may be said to be a *characteristic of complex derivation*, or a *complex symbolic multiplier*. But instead of thus conceiving the set q′ to be *deduced* from q by this mode of complex derivation, or *symbolical multiplication* (136), with the assistance of the constant coefficients of derivation c, and of n *given values* for the variable multiplying numbers m, we may inquire, conversely, *what system of numerical multipliers*, $m_0, \ldots m_r, \ldots m_{n-1}$, *must be assumed*, in order to produce or generate a *given ordinal set* q′, as the *symbolical product* of this sort of multiplication; the *multiplicand set* q, and the *constant coefficients* c, being still supposed to be *given*. This inverse or reciprocal process may be called the *symbolical division of one ordinal set by another*, namely, of the set q′ by the set q; and it may be denoted by the following formula, which is the reciprocal or inverse of the formula (136):

$$q' \div q = m_0 \times_0 + m_1 \times_1 + \ldots + m_{n-1} \times_{n-1}. \tag{139}$$

To describe more fully the process which is thus briefly indicated, we may observe that, besides the n^3 constant coefficients c, there are now given, or supposed to be known, $2n$ ordinal relations of the forms a_t and a'_s (or numbers proportional to these $2n$ relations), as the constituents of the two given ordinal sets of the n^{th} order, q and q′; which sets are here regarded as the *divisor set* and the *dividend set* respectively. Thus the n^2 ordinal relations of the form $a_{r,s}$ are conceived to be known, as depending in a known manner on the n given relations a_t, by the n^2 expressions of the form (135); and on substituting for these n^2 ordinal relations, and for the n other given relations of the form a'_s, in the n formulae (138), any system of numerical values which shall be (in the sense of the 14th article) *proportional* to these different ordinal relations, we shall thereby obtain n *linear equations*, of an ordinary algebraical kind, between the n sought *numbers*, m_r: from which these latter numbers may then in general be deduced, by any of the usual processes of solution of such ordinary and linear equations.

For example, after fixing upon any standard ordinal relation, or relation between two selected moments of time, and calling it a, we may first prepare the equation (138) by putting it under the form,

$$a'_s \div a = \Sigma_r . m_r (a_{r,s} \div a); \tag{140}$$

in which Σ_r is the characteristic of a summation performed with respect to r, and the quotients in both members are numerical. And then, by suitable combinations of the numerical quotients in the second member of this last equation, which combinations are determined by the given expressions (135), we may find a system of n^2 numerical *coefficients of elimination*, $l_{r,s}$, of which the values depend on the constant coefficients c, and on the n given numerical quotients of the form $a_t \div a$, but are independent of the n other quotients $a'_s \div a$, and satisfy the n^2 conditions included in the formula,

$$\Sigma_s . l_{r,s}(a_{r',s} \div a) = 0, \quad \text{or} \quad = l, \quad \text{according as} \quad r' \gtrless \quad \text{or} \quad = r; \tag{141}$$

l being here another number, namely, the *common denominator* of the elimination. For in this manner we shall have n final expressions of the form,

$$m_r = l^{-1} \Sigma_s . l_{r,s} (a'_s \div a); \tag{142}$$

by which the n sought *coefficients of the symbolical quotient* (139) can be, in general, determined.

Successive complex Derivation: Conception of a numeral Set

17. Suppose that, after deducing q′ from q, by the complex derivation or symbolical multiplication (136), we again derive another ordinal set q″ from q′ by another multiplication of the same sort, with the *same constant coefficients of derivation*, c, but with a *new system of variable numerical multipliers*, m; which supposition we shall, on the same plan as before, express as follows:

$$(m'_0 \times_0 + \ldots + m'_{r'} \times_{r'} + \ldots + m'_{n-1} \times_{n-1})\, q' = q''. \tag{143}$$

Making now, in imitation of the expression (137),

$$q'' = (a''_0, \ldots, a''_{s'}, \ldots, a''_{n-1}), \tag{144}$$

we shall have, as expressions analogous to (138) and (135), the following:

$$a''_{s'} = \Sigma_{r'} . m'_{r'} a'_{r', s'}; \tag{145}$$

$$a'_{r', s'} = \Sigma_s . c_{r', s', s} a'_s; \tag{146}$$

and thus the result of this *successive multiplication* will be a determined and known set, q″. In the next place, let this resulting set, or *successive symbolical product*, q″, be *divided by the original set* q, which was at first proposed as a multiplicand; we shall then obtain, by the method described in the foregoing article, a symbolical quotient of the form,

$$q'' \div q = m''_0 \times_0 + \ldots + m''_{r''} \times_{r''} + \ldots + m''_{n-1} \times_{n-1}; \tag{147}$$

in which, on the same plan as in the formula (142), and with the *same system of eliminational coefficients* of the form *l*, determined by (141), we have,

$$m''_{r''} = l^{-1} \Sigma_{s'} . l_{r'', s'} (a''_{s'} \div a). \tag{148}$$

Substituting for $a''_{s'}$ its value, given by (145), (146), and by (138) or (140), and eliminating the numerical denominator *l* by (141), we find that we may write:

$$m''_{r''} = \Sigma_{r, r'} . m_r m'_{r'} n_{r, r', r''}; \tag{149}$$

if we establish, for conciseness, the following formula, including n^3 separate expressions for so many separate numbers:

$$n_{r,r',r''} = (\Sigma_{s, s'} . l_{r'', s'} c_{r', s', s} a_{r, s}) \div (\Sigma_s . l_{r, s} a_{r, s}); \tag{150}$$

in which it is to be observed that the sum which enters as a divisor is the same for all the n^3 quotients. The value of each of these numerical quotients (150) will, *in general*, depend on the $n-1$ ratios of the constituents $a_0, a_1, \ldots, a_{n-1}$ of the first proposed ordinal set q, or the ratios of the numbers to which these *n* ordinal constituents are proportional; but it may be possible to *assign* (at the outset) *such values to the constant but arbitrary coefficients of derivation c*, or to subject those n^3 coefficients to such restrictions, that *these $n-1$ arbitrary ratios of the n constituents* a_t, in the expression (133), *shall have no influence on the value of any one of the n^3 numbers* included in the expression (150). When this last condition, or system of conditions, is satisfied, we are allowed to *detach the characteristics of the successive symbolical multiplications of an ordinal set from the symbol of the original multiplicand*; and as the result of the comparison of the formulae (136) and (143), and of (147) under the form,

$$q'' = (m''_0 \times_0 + \ldots + m''_{n-1} \times_{n-1})\, q, \tag{151}$$

we may write,

$$m''_0 \times_0 + \ldots + m''_{n-1} \times_{n-1} = (m'_0 \times_0 + \ldots + m'_{n-1} \times_{n-1})(m_0 \times_0 + \ldots + m_{n-1} \times_{n-1}); \tag{152}$$

which will denote the *reduction of a system of two successive and complex derivations*, or symbolic multiplications of the kind (136), *to one complex derivation of the same kind.* Under the same conditions, the *successive performance of two simple or elementary derivations*, of the kind (134), will be equivalent to the performance of *one complex derivation*, of the kind (136), *with numerical coefficients independent of the original derivand*, as follows:

$$\times_{r'} \times_r = \Sigma_{r''} . n_{r,r',r''} \times_{r''}. \tag{153}$$

We may also regard the n variable numerical coefficients m_r, in the quotient (139), obtained by the symbolical division of one ordinal set by another, as composing, under the same conditions, a NUMERAL SET; and this new sort of *set* may be *detached*, in thought and in expression, from the two ordinal sets which have served, by their mutual comparison, to *suggest* it. The quotient (139), when thus regarded as a *numeral set*, may be denoted as follows:

$$\mathrm{q}' \div \mathrm{q} = q = (m_0, m_1, \ldots, m_{n-1}); \tag{154}$$

the letter q, when used as a symbol of *such* a set, being written in the Italic character: and then the n numerical relations, which are included in the formula (149), may be supposed to be otherwise summed up in the one equation:

$$(m''_0, \ldots, m''_{r''}, \ldots, m''_{n-1}) = (m'_0, \ldots, m'_{r'}, \ldots, m'_{n-1})\,(m_0, \ldots, m_r, \ldots, m_{n-1}). \tag{155}$$

And conversely, this last equation, which asserts that *the numeral set in its first member is equal to the symbolical product of the two numeral sets in its second member*, may be considered to receive its *interpretation* from the formula (149); in which the n^3 numbers $n_{r,r',r''}$ may be called the *coefficients of multiplication of a numeral set.* But it is necessary to consider more closely what are the *forms* of those *conditions of detachment* which have been above alluded to, and which (according to the view here taken) are required for the (separate) *existence* of such a numeral set; it will also be proper to give, at least, some examples of the possibility of satisfying the conditions thus determined.

Conditions of Detachment

18. The following appears to be a sufficiently simple mode of discovering the *conditions of detachment*, under which the values of the numerical coefficients, $n_{r,r',r''}$, in (149) or (150), shall be independent of the ratios of the ordinal constituents of the set q, which is originally operated upon. Employing the characteristics of ordinal separation, as explained in a former article, we may now regard it as being the *definition* of the sign of derivation $\times_r$, that this sign satisfies the symbolic equation,

$$\mathrm{R}_s \times_r = \Sigma_t . c_{r,s,t} \mathrm{R}_t; \tag{156}$$

which gives

$$\mathrm{R}_s \times_{r'} \times_r = \Sigma_s . c_{r',s',s} \mathrm{R}_s \times_r$$
$$= \Sigma_{s,t} . c_{r',s',s} c_{r,s,t} \mathrm{R}_t. \tag{157}$$

On the other hand, the equation (153), when operated on by the characteristic of separation $\mathrm{R}_{s'}$, gives, by changing r'' to s, and by afterwards changing r, s in (156) to s, s':

$$\mathrm{R}_{s'} \times_{r'} \times_r = \Sigma_s . n_{r,r',s} \mathrm{R}_{s'} \times_s$$
$$= \Sigma_{s,t} . n_{r,r',s} c_{s,s',t} \mathrm{R}_t. \tag{158}$$

We are then to satisfy the equation,

$$0 = \Sigma_s (n_{r,r',s} \mathrm{R}_{s'} \times_s - c_{r',s',s} \mathrm{R}_s \times_r)$$
$$= \Sigma_{s,t} (n_{r,r',s} c_{s,s',t} - c_{r',s',s} c_{r,s,t}) \mathrm{R}_t; \tag{159}$$

and because we are to do this independently of the ratios of the n constituent ordinal relations a_t, which are obtained from the ordinal set q by the n operations of separation R_t, we must endeavour to satisfy all the numerical conditions which are included in the form,

$$0=\Sigma_s(n_{r,r',s}c_{s,s',t}-c_{r',s',s}c_{r,s,t}). \tag{160}$$

The *number* of these *conditions of detachment* (160) is n^4, because each of the four indices, r, r', s', t, may receive any one of the n values $0, 1, \ldots, n-1$; and they involve only $2n^3$ numerical coefficients, or rather their ratios, which are fewer by one, to be determined; from which it may at first sight seem to be impossible to satisfy all these conditions of detachment, except by making all the coefficients of derivation vanish. Yet we shall see that when $n=2$, namely, for the case of *numeral couples*, the conditions admit of an *indeterminate* form of solution: and for the case $n=4$, it will be shown that they can also be satisfied by that system of coefficients on which is founded our theory of *numeral quaternions*, and even by a system of coefficients somewhat more general. A more complete discussion of the important formula (160) will not be needed for the purposes of the present Essay.

Case of Couples

19. If we suppose $n=2$, then the index s, with respect to which the summation is to be performed, can be only 0 or 1; the formula (160) becomes, therefore, in this case,

$$n_{r,r',0}c_{0,s',t}+n_{r,r',1}c_{1,s',t}=c_{r',s',0}c_{r,0,t}+c_{r',s',1}c_{r,1,t}. \tag{161}$$

If we suppose also that the two simple or elementary derivations of one ordinal couple from another are denoted thus:

$$\left.\begin{aligned}\mathrm{X}_0(\mathrm{a}_0,\mathrm{a}_1)=(\mathrm{a}_{0,0},\mathrm{a}_{0,1})=(a\mathrm{a}_0+a'\mathrm{a}_1, b\mathrm{a}_0+b'\mathrm{a}_1);\\ \mathrm{X}_1(\mathrm{a}_0,\mathrm{a}_1)=(\mathrm{a}_{1,0},\mathrm{a}_{1,1})=(c\mathrm{a}_0+c'\mathrm{a}_1, d\mathrm{a}_0+d'\mathrm{a}_1);\end{aligned}\right\} \tag{162}$$

we shall have, by (135), for the $2^3=8$ coefficients of derivation of the form $c_{r,s,t}$, the abridged symbols:

$$\left.\begin{aligned}c_{0,0,0}=a;\quad c_{0,0,1}=a';\quad c_{0,1,0}=b;\quad c_{0,1,1}=b';\\ c_{1,0,0}=c;\quad c_{1,0,1}=c';\quad c_{1,1,0}=d;\quad c_{1,1,1}=d'.\end{aligned}\right\} \tag{163}$$

And if we employ in like manner these other temporary abridgments, for the eight coefficients of multiplication of one numeral couple by another,

$$\left.\begin{aligned}n_{0,0,0}=e;\quad n_{0,0,1}=e';\quad n_{0,1,0}=f;\quad n_{0,1,1}=f';\\ n_{1,0,0}=g;\quad n_{1,0,1}=g';\quad n_{1,1,0}=h;\quad n_{1,1,1}=h';\end{aligned}\right\} \tag{164}$$

the *equations of detachment*, included in the general formula (160), will then, by (161), be the sixteen following:

$$\begin{array}{lll} & (t=0) & (t=1) \end{array}$$

$$\left.\begin{array}{lll}(s'=0) & ea+e'c=aa+a'b; & ea'+e'c'=aa'+a'b';\\ (s'=1) & eb+e'd=ba+b'b; & eb'+e'd'=ba'+b'b';\end{array}\right\}\ (r=0, r'=0) \tag{165}$$

$$\left.\begin{array}{lll}(s'=0) & fa+f'c=ca+c'b; & fa'+f'c'=ca'+c'b';\\ (s'=1) & fb+f'd=da+d'b; & fb'+f'd'=da'+d'b';\end{array}\right\}\ (r=0, r'=1) \tag{166}$$

$$\left.\begin{array}{lll}(s'=0) & ga+g'c=ac+a'd; & ga'+g'c'=ac'+a'd';\\ (s'=1) & gb+g'd=bc+b'd; & gb'+g'd'=bc'+b'd';\end{array}\right\}\ (r=1, r'=0) \tag{167}$$

$$\left.\begin{array}{lll}(s'=0) & ha+h'c=cc+c'd; & ha'+h'c'=cc'+c'd';\\ (s'=1) & hb+h'd=dc+d'd; & hb'+h'd'=dc'+d'd'.\end{array}\right\}\ (r=1, r'=1) \tag{168}$$

Now the twelve equations (165) (166) (167) are all satisfied, independently of c, c', d, d', if we suppose

$$a=b'=e=f'=g'; \quad a'=b=e'=f=g=0; \tag{169}$$

and then the four remaining equations (168) take the forms,

$$\left.\begin{aligned} ha+(h'-c)c=c'd; \quad (h'-c-d')c'=0; \\ (h'-c-d')d=0; \quad ha+(h'-d')d'=c'd; \end{aligned}\right\} \tag{170}$$

which are satisfied by supposing

$$h'=c+d'; \quad ha=c'd-cd'. \tag{171}$$

Accordingly, with the values (169), the sign of derivation $\times_0$ reduces itself to the ordinary numeric multiplier a, so that we may write simply,

$$\times_0=a; \tag{172}$$

and while the other sign of linear derivation $\times_1$ retains its greatest degree of generality, consistent with the *order* of the sets, namely, *couples*, which are at present under consideration, so that the four numerical constants $c\,c'\,d\,d'$ remain entirely unrestricted, the symbolic equations of the form (153) become now, by (164), (169), and (171):

$$\left.\begin{aligned} \times_0\times_0&=e\times_0+e'\times_1=a\times_0; \\ \times_1\times_0&=f\times_0+f'\times_1=a\times_1; \\ \times_0\times_1&=g\times_0+g'\times_1=a\times_1; \\ \times_1\times_1&=h\times_0+h'\times_1=a^{-1}(c'd-cd')\times_0+(c+d')\times_1; \end{aligned}\right\} \tag{173}$$

and these equations are, as we aimed that they should be, independent of the original derivand, that is, here, of the ordinal couple $(\mathrm{a}_0, \mathrm{a}_1)$. In fact, the three first equations (173) are evidently true, by (172), whatever the constant coefficients of derivation included in the sign $\times_1$ may be; and if, by the definition (162) of that sign of derivation, we form the *successive derivative*,

$$\begin{aligned} \times_1\times_1(\mathrm{a}_0, \mathrm{a}_1)&=\times_1(\mathrm{a}_{1,0}, \mathrm{a}_{1,1}) \\ &=(c\mathrm{a}_{1,0}+c'\mathrm{a}_{1,1}, d\mathrm{a}_{1,0}+d'\mathrm{a}_{1,1}) \\ &=(c(c\mathrm{a}_0+c'\mathrm{a}_1)+c'(d\mathrm{a}_0+d'\mathrm{a}_1), d(c\mathrm{a}_0+c'\mathrm{a}_1)+d'(d\mathrm{a}_0+d'\mathrm{a}_1)), \end{aligned} \tag{174}$$

we are conducted, whatever the two original constituent ordinal relations a_0 and a_1 may be, to the same final ordinal couple, as if we add together the two partial results, which are obtained by the two derivations represented by the two terms of the last member of the fourth equation (173), namely, the two following couples:

$$\left.\begin{aligned} a^{-1}(c'd-cd')\times_0(\mathrm{a}_0, \mathrm{a}_1)&=((c'd-cd')\mathrm{a}_0, (c'd-cd')\mathrm{a}_1); \\ (c+d')\times_1(\mathrm{a}_0, \mathrm{a}_1)&=((c+d')(c\mathrm{a}_0+c'\mathrm{a}_1), (c+d')(d\mathrm{a}_0+d'\mathrm{a}_1)). \end{aligned}\right\} \tag{175}$$

We may therefore express the result of two successive and complex derivations of this sort, performed on an ordinal couple $(\mathrm{a}_0, \mathrm{a}_1)$, by a *symbolical equation independent of that original derivand*, or operand couple, namely, by the following:

$$(m_0'\times_0+m_1'\times_1)(m_0\times_0+m_1\times_1)=m_0''\times_0+m_1''\times_1, \tag{176}$$

which is included in the form (152), and in which we have now these two relations, of the form (149), between the numerical coefficients:

$$\left.\begin{aligned} m_0''&=am_0'm_0+a^{-1}(c'd-cd')m_1'm_1; \\ m_1''&=am_1'm_0+am_0'm_1+(c+d')m_1'm_1. \end{aligned}\right\} \tag{177}$$

Under the same conditions we may also write, more briefly,

$$(m''_0, m''_1) = (m'_0, m'_1)(m_0, m_1), \tag{178}$$

as in the general form (155); and may regard the one *numeral couple* (m''_0, m''_1) as the *symbolical product* of the other two. If we simplify the formulae by assuming the five constant coefficients of derivation which still remain disposable, namely, a, c, c', d, d', as follows:

$$a=1, \quad c=0, \quad c'=-1, \quad d=1, \quad d'=0, \tag{179}$$

we shall then have $\qquad \mathrm{X}_0(\mathrm{a}_0, \mathrm{a}_1) = (\mathrm{a}_0, \mathrm{a}_1); \quad \mathrm{X}_1(\mathrm{a}_0, \mathrm{a}_1) = (-\mathrm{a}_1, \mathrm{a}_0);$ (180)

or more concisely, $\qquad \mathrm{X}_0 = 1; \quad \mathrm{X}_1 = \mathrm{R}_{-1,0};$ (181)

this last symbol being here the same characteristic of derivation of an ordinal couple which was considered in former articles of this paper. And the equation for the *multiplication of two numeral couples* will then reduce itself to the following form:

$$(m'_0, m'_1)(m_0, m_1) = (m'_0 m_0 - m'_1 m_1, \; m'_1 m_0 + m'_0 m_1); \tag{182}$$

which agrees with that assigned in the earlier Essay*. With the same values of the coefficients of derivation, and consequently with the same values of the coefficients of multiplication likewise, we may write also, as in that Essay (compare the page just cited), a *formula for the division of one numeral couple by another*, namely:

$$\frac{(m''_0, m''_1)}{(m_0, m_1)} = (m'_0, m'_1) = \left(\frac{m_0 m''_0 + m_1 m''_1}{m_0^2 + m_1^2}, \; \frac{m_0 m''_1 - m_1 m''_0}{m_0^2 + m_1^2}\right). \tag{183}$$

It is not necessary, and it would detain us too long from the main subject of this memoir, to consider here any other and less simple formulae of the same sort, which may be obtained for the same case of couples, by any other systems of coefficients of derivation and multiplication, which satisfy the same conditions of detachment, assigned in the present article.

20. It may be instructive, however, to consider here the same case of couples, as an exemplification of some other general formulae which have been already given in this Essay. Writing, for abridgment, $\qquad \mathrm{a}_t \div \mathrm{a} = a_t; \quad \mathrm{a}'_s \div \mathrm{a} = a'_s; \quad \mathrm{a}''_{s'} \div \mathrm{a} = a''_{s'};$ (184)

and in like manner, $\qquad \mathrm{a}_{r,s} \div \mathrm{a} = a_{r,s}; \quad \mathrm{a}'_{r',s'} \div \mathrm{a} = a'_{r',s'};$ (185)

the quotients thus denoted being numerical; we have, by article 16, for the case $n=2$, the commas in the compound indices being here omitted for the sake of conciseness:

$$\left.\begin{aligned} a_{00} &= c_{000} a_0 + c_{001} a_1; \quad a_{01} = c_{010} a_0 + c_{011} a_1; \\ a_{10} &= c_{100} a_0 + c_{101} a_1; \quad a_{11} = c_{110} a_0 + c_{111} a_1; \end{aligned}\right\} \tag{186}$$

$$a'_0 = m_0 a_{00} + m_1 a_{10}; \qquad a'_1 = m_0 a_{01} + m_1 a_{11}; \tag{187}$$

$$\left.\begin{aligned} l &= l_{00} a_{00} + l_{01} a_{01} = l_{10} a_{10} + l_{11} a_{11}; \\ 0 &= l_{00} a_{10} + l_{01} a_{11} = l_{10} a_{00} + l_{11} a_{01}; \end{aligned}\right\} \tag{188}$$

and, consequently, $\qquad l m_0 = l_{00} a'_0 + l_{01} a'_1; \quad l m_1 = l_{10} a'_0 + l_{11} a'_1.$ (189)

Again, by article 17, for the same case $n=2$, we have the analogous formulae:

$$\left.\begin{aligned} a'_{00} &= c_{000} a'_0 + c_{001} a'_1; \quad a'_{01} = c_{010} a'_0 + c_{011} a'_1; \\ a'_{10} &= c_{100} a'_0 + c_{101} a'_1; \quad a'_{11} = c_{110} a'_0 + c_{111} a'_1; \end{aligned}\right\} \tag{190}$$

$$a''_0 = m'_0 a'_{00} + m'_1 a'_{10}; \qquad a''_1 = m'_0 a'_{01} + m'_1 a'_{11}; \tag{191}$$

* [See I, p. 83.]

and then, assuming these other expressions,

$$a_0''=m_0''a_{00}+m_1''a_{10};\quad a_1''=m_0''a_{01}+m_1''a_{11}, \tag{192}$$

we find, by (188), two equations of the same forms as (189), namely,

$$\left.\begin{aligned} lm_0''&=l_{00}a_0''+l_{01}a_1'';\\ lm_1''&=l_{10}a_0''+l_{11}a_1''.\end{aligned}\right\} \tag{193}$$

Making, therefore, according to the general rule contained in the formula (150),

$$\begin{aligned} ln_{rr'r''}&=\Sigma_{s,s'}\,.\,l_{r''s'}c_{r's's}a_{rs}\\ &=(l_{r''0}c_{r'00}+l_{r''1}c_{r'10})a_{r0}+(l_{r''0}c_{r'01}+l_{r''1}c_{r'11})a_{r1},\end{aligned} \tag{194}$$

we have results included in the formula (149), namely,

$$m_0''=\Sigma_{r,r'}\,.\,m_r m_{r'}' n_{rr'0};\quad m_1''=\Sigma_{r,r'}\,.\,m_r m_{r'}' n_{rr'1}; \tag{195}$$

that is, more fully,

$$\left.\begin{aligned} m_0''&=m_0m_0'n_{000}+m_0m_1'n_{010}+m_1m_0'n_{100}+m_1m_1'n_{110};\\ m_1''&=m_0m_0'n_{001}+m_0m_1'n_{011}+m_1m_0'n_{101}+m_1m_1'n_{111}.\end{aligned}\right\} \tag{196}$$

Thus, in particular, the coefficient of the product m_0m_0', in the expression thus obtained for m_0'', is,

$$\begin{aligned} n_{000}=l^{-1}l_{00}(c_{000}a_{00}+c_{001}a_{01})\\ +l^{-1}l_{01}(c_{010}a_{00}+c_{011}a_{01}).\end{aligned} \tag{197}$$

The equations (188) permit us to write

$$l_{00}=a_{11};\quad l_{01}=-a_{10};\quad l_{10}=-a_{01};\quad l_{11}=a_{00}; \tag{198}$$

provided that we assign to l the value

$$l=a_{00}a_{11}-a_{10}a_{01}. \tag{199}$$

Hence

$$n_{000}=\frac{a_{11}(c_{000}a_{00}+c_{001}a_{01})-a_{10}(c_{010}a_{00}+c_{011}a_{01})}{a_{00}a_{11}-a_{10}a_{01}}. \tag{200}$$

If we substitute, in this expression for n_{000}, the values (186) for $a_{00}, a_{01}, a_{10}, a_{11}$, we shall thereby obtain, in general, a certain function of a_0, a_1, which will be homogeneous of the dimension zero, because it will present itself under the form of a fraction, of which the numerator and the denominator will be homogeneous and quadractic functions of the same a_0, a_1. In order that this quotient of two quadratic functions of the number expressing the ratio of a_1 to a_0, or of a_1 to a_0, may be itself independent of that ratio, we must have certain relations between the coefficients c_{000}, &c., and the fraction itself must take a particular value connected with those coefficients; which relations and value may be determined by the three equations:

$$\begin{aligned} n_{000}(c_{000}c_{110}-c_{100}c_{010})=c_{110}(c_{000}^2+c_{001}c_{010})\\ -c_{100}c_{010}(c_{000}+c_{011});\end{aligned} \tag{201}$$

$$\begin{aligned} n_{000}(c_{000}c_{111}-c_{100}c_{011}+c_{001}c_{110}-c_{101}c_{010})\\ =c_{111}(c_{000}^2+c_{001}c_{010})-c_{101}c_{010}(c_{000}+c_{011})\\ +c_{110}c_{001}(c_{000}+c_{011})-c_{100}(c_{010}c_{001}+c_{011}^2);\end{aligned} \tag{202}$$

$$\begin{aligned} n_{000}(c_{001}c_{111}-c_{101}c_{011})=c_{111}c_{001}(c_{000}+c_{011})\\ -c_{101}(c_{010}c_{001}+c_{011}^2).\end{aligned} \tag{203}$$

In like manner, each of the seven other coefficients, n_{010}, &c. in the expressions (196), will furnish three other equations of condition, which must all be satisfied, in order that the values of these coefficients of multiplication of couples may be independent of the original ratio of a_1 to a_0, or of a_1 to a_0; and each of the twenty-four equations thus furnished, of which the equations (201), (202), (203), are three, is an equation of the third dimension, with respect to the coefficients of derivation and multiplication, c_{000}, &c., n_{000}, &c. We should, therefore, by this method, have obtained equations more numerous and less simple than those which were given by the method of the eighteenth article: which method there is, therefore, an advantage in introducing, even for the case of couples, and much more for the case of quaternions, or other ordinal and numeral sets; although the method above exemplified appears to offer itself more immediately from the principles of the seventeenth article.

But to exhibit by an example the agreement of the two methods in their results, let the symbols defined by the equations (163), (164), be employed to abridge the expression of the equations (201), (202), (203); the latter will then become:

$$\left.\begin{aligned} &e(ad-cb)=d(a^2+a'b)-cb(a+b'); \\ &e(ad'-cb'+a'd-c'b)=d'(a^2+a'b)-c'b(a+b') \\ &\qquad\qquad\qquad\qquad +da'(a+b')-c(ba'+b'^2); \\ &e(a'd'-c'b')=d'a'(a+b')-c'(ba'+b'^2); \end{aligned}\right\} \tag{204}$$

and it is evident, upon inspection, that these three equations (204) may be deduced by elimination of e' from the four equations of detachment (165), which were obtained by the simplified method; and which, in that method, formed part of a system of only sixteen (instead of twenty-four) equations, each rising no higher than the second (instead of the third) dimension.

Associative Principle of the Multiplication of numeral Sets: Characteristics of numeral Separation

21. Whenever, for any value of the exponent n of the order of a set, we have succeeded in satisfying the n^4 simplified *equations of detachment*, included in the formula (160) of the eighteenth article, and have thereby found a system of n^3 coefficients of derivation, and a connected system of n^3 coefficients of multiplication, with reference to which two systems of coefficients an equation, or rather a system of equations, of the form (153) can be established, independently of the $n-1$ ratios of the constituents of that ordinal set q, on which the two successive derivations are performed; it is evident that we can then proceed, in like manner, to perform on the resulting set a third successive derivation; and that, with respect to such successive operations of derivation, the following simple but important formula holds good:

$$\mathsf{X}_{r'}.\mathsf{X}_r\mathsf{X}_t=\mathsf{X}_{r'}\mathsf{X}_r.\mathsf{X}_t. \tag{205}$$

To develop this symbolical equation, which may be said to contain the *associative principle of the multiplication of numeral sets*, we may conveniently employ a *characteristic of numeral separation*, N, analogous to those two characteristics, M and R, which we have already introduced in this paper, for the purpose of expressing separately the different moments of a momental set, and of separating, in like manner, those constituent ordinal relations between moments which compose an ordinal set. Let us, therefore, agree to regard the n equations,

$$m_0=\mathrm{N}_0q;\quad m_1=\mathrm{N}_1q;\quad \ldots,\ m_{n-1}=\mathrm{N}_{n-1}q, \tag{206}$$

as jointly equivalent to the one complex equation or expression (154), for a numeral set q, of any proposed order n; in such a manner that we shall have, identically, for numeral constituents and numeral sets, the equations

$$\left.\begin{aligned} m_0 &= \mathrm{N}_0(m_0, m_1, \ldots, m_{n-1}), \\ m_1 &= \mathrm{N}_1(m_0, m_1, \ldots, m_{n-1}), \ldots \end{aligned}\right\} \tag{207}$$

and
$$q = (\mathrm{N}_0 q, \mathrm{N}_1 q, \ldots, \mathrm{N}_{n-1} q); \tag{208}$$

which are analogous to those marked (10) and (11), for moments and momental sets, and also to the formulae (57), (58), for constituent ordinal relations, and for the ordinal sets to which they belong. We may then substitute for the formula (153) of symbolic multiplication, or of successive derivation, the following:

$$\mathrm{N}_s . \mathsf{X}_{r'} \mathsf{X}_r = n_{r, r', s}; \tag{209}$$

which will give, also, by suitably changing the letters,

$$\mathrm{N}_{s'} . \mathsf{X}_s \mathsf{X}_t = n_{t, s, s'}; \tag{210}$$

the commas in the indices being here, for the sake of greater clearness, restored. In this manner we find that

$$\mathrm{N}_{s'}(\mathsf{X}_{r'} \mathsf{X}_r . \mathsf{X}_t) = \Sigma_s . n_{r, r', s'} n_{t, s, s'}. \tag{211}$$

But, also,
$$\mathrm{N}_s . \mathsf{X}_r \mathsf{X}_t = n_{t, r, s}; \quad \mathrm{N}_{s'} . \mathsf{X}_{r'} \mathsf{X}_s = n_{s, r', s'}; \tag{212}$$

and, therefore,
$$\mathrm{N}_{s'}(\mathsf{X}_{r'} . \mathsf{X}_r \mathsf{X}_t) = \Sigma_s . n_{s, r', s'} n_{t, r, s}; \tag{213}$$

consequently, by operating with the characteristic $\mathrm{N}_{s'}$ on the symbolical equation (205), we obtain this other form for the expression of the associative principle, considered as establishing a certain system of relations between the coefficients of multiplication:

$$0 = \Sigma_s(n_{r, r', s} n_{t, s, s'} - n_{s, r', s'} n_{t, r, s}). \tag{214}$$

We are, therefore, entitled to regard this last formula, or the system of numerical equations of condition which it includes, as being a consequence of the analogous system of conditions included in the formula (160), because the associative property of multiplication is a consequence of the principle of detachment. And on comparing the two formulae, we perceive that as soon as the one last deduced, namely, (214), has been satisfied by a suitable system of coefficients of multiplication, then the one previously established, namely, (160), can be immediately satisfied also, by connecting with this latter system a system of coefficients of derivation, according to the rule expressed by the following very simple equation:

$$c_{r, s, t} = n_{t, r, s}. \tag{215}$$

For example, in the case of couples, with the abridged symbols (163), (164), for the two systems of coefficients, this rule (215) would have shewn that if we had in any manner succeeded in satisfying the sixteen equations of detachment (165)...(168) between a, b, c, d a', b', c', d' and $e, f, g, h\, e', f', g', h'$, we could then satisfy the same equations of detachment with the same values of the eight latter symbols, and with the following values for the eight former:

$$\left.\begin{aligned} a&=e; \quad b=e'; \quad c=f; \quad d=f'; \\ a'&=g; \quad b'=g'; \quad c'=h; \quad d'=h'; \end{aligned}\right\} \tag{216}$$

which, in fact, will be found to agree with the values of the nineteenth article.

Connexion between the Coefficients of Derivation and of Multiplication; simplified Conception of a numeral Set, regarded as expressing the complex Ratio of an ordinal Set to a single ordinal Relation

22. The rule (215), for connecting together the two systems of coefficients, of derivation and of multiplication, admits of being interpreted or accounted for in a very simple manner.

The coefficient $c_{r,s,t}$, introduced in the sixteenth article, may be regarded as having been generated, or, at least, brought under our view as follows. We first supposed an ordinal set, q, to be operated on by the elementary characteristic of derivation $\times_r$, so as to produce thereby a derivative set, q_r. We then operated on this derived set, in a way which may be indicated by the characteristic of ordinal separation, R_s, and so obtained a result of the form

$$\mathrm{R}_s \times_r \mathrm{q} = \mathrm{a}_{r,s}. \tag{217}$$

And, lastly, we analyzed this result, so as to find the part of it which depended on, and arose from, the constituent a_t or $\mathrm{R}_t\mathrm{q}$ of the original operand set; and the coefficient of this constituent a_t, in the part obtained by this analysis, was denoted by $c_{r,s,t}$, and was regarded as a coefficient of derivation. On the other hand, the coefficient of multiplication, $n_{t,r,s}$, may be said to arise thus: an elementary derivation, denoted by $\times_t$, is succeeded by another, denoted by $\times_r$; the compound operation, $\times_r \times_t$, is detached from the operand, and regarded as equivalent to a single complex derivation, of which the characteristic may be symbolically equated to a certain numeral set; this last set is subjected to the characteristic of numeral separation, N_s, or to an analysis equivalent thereto; and the result is, by (212), the coefficient of multiplication in question.

Now the agreement of the results of these two processes, which is expressed by the equation (215), becomes quite intelligible and natural, if we conceive that the constituent a_t of the operand set q, on which constituent alone we really operate in the former process, the others being, in fact, set aside, as contributing nothing to the result here sought for, has been *itself produced* or generated *by an earlier operation of the form* $a_t \times_t$ (where a_t has the same signification as in (184)), from some one primary or original ordinal relation, such as that which was denoted in some recent articles by the letter a. In this manner we may be led to look upon *any ordinal set*, such as the set q in the equation (133), as being *generated by a certain complex derivation*, which is expressed by a certain numeral set *q*, *from a single standard ordinal relation*, a, or from the relation between some two standard or selected moments of time, according to either of the two reciprocal formulae:

$$\mathrm{q} = q\mathrm{a} = \Sigma_t . a_t \times_t \mathrm{a}; \quad \text{or,} \quad q = \mathrm{q} \div \mathrm{a} = \Sigma_t . a_t \times_t; \tag{218}$$

in which last equation the members are symbols for a numeral set. And thus a *numeral set* (*q*) may come to be conceived as being *a system or set of numbers, serving to mark or to express the complex ratio which an ordinal set* (q) *bears to a simple or single ordinal relation* (a), regarded as a standard of comparison.

Case of Quaternions; Coefficients of Multiplication

23. In the case of quaternions, the formula (214) gives a system of $4^4 = 256$ equations of condition, included in the following type (in which u has been written instead of r', and the accent common to all the indices s' has been omitted as unnecessary in the result):

$$n_{r,u,0}n_{t,0,s} + n_{r,u,1}n_{t,1,s} + n_{r,u,2}n_{t,2,s} + n_{r,u,3}n_{t,3,s}$$
$$= n_{0,u,s}n_{t,r,0} + n_{1,u,s}n_{t,r,1} + n_{2,u,s}n_{t,r,2} + n_{3,u,s}n_{t,r,3}; \tag{219}$$

each of the four indices, r, s, t, u, in this last formula, being allowed to receive any one of the four values, $0, 1, 2, 3$. And all these two hundred and fifty-six equations are satisfied when we establish the following system of numerical values of the sixty-four coefficients of multiplication (in which the commas between the indices are again omitted for conciseness):

$$\left.\begin{array}{llll} n_{000}=1; & n_{001}=0; & n_{002}=0; & n_{003}=0; \\ n_{010}=0; & n_{011}=1; & n_{012}=0; & n_{013}=0; \\ n_{020}=0; & n_{021}=0; & n_{022}=1; & n_{023}=0; \\ n_{030}=0; & n_{031}=0; & n_{032}=0; & n_{033}=1; \end{array}\right\} \quad (220)$$

$$\left.\begin{array}{llll} n_{100}=0; & n_{101}=1; & n_{102}=0; & n_{103}=0; \\ n_{110}=-1; & n_{111}=0; & n_{112}=0; & n_{113}=0; \\ n_{120}=0; & n_{121}=0; & n_{122}=0; & n_{123}=-1; \\ n_{130}=0; & n_{131}=0; & n_{132}=1; & n_{133}=0; \end{array}\right\} \quad (221)$$

$$\left.\begin{array}{llll} n_{200}=0; & n_{201}=0; & n_{202}=1; & n_{203}=0; \\ n_{210}=0; & n_{211}=0; & n_{212}=0; & n_{213}=1; \\ n_{220}=-1; & n_{221}=0; & n_{222}=0; & n_{223}=0; \\ n_{230}=0; & n_{231}=-1; & n_{232}=0; & n_{233}=0; \end{array}\right\} \quad (222)$$

$$\left.\begin{array}{llll} n_{300}=0; & n_{301}=0; & n_{302}=0; & n_{303}=1; \\ n_{310}=0; & n_{311}=0; & n_{312}=-1; & n_{313}=0; \\ n_{320}=0; & n_{321}=1; & n_{322}=0; & n_{323}=0; \\ n_{330}=-1; & n_{331}=0; & n_{332}=0; & n_{333}=0. \end{array}\right\} \quad (223)$$

We might content ourselves with proving the truth of this assertion by actual arithmetical substitution of these sixty-four values in the two hundred and fifty-six equations; but the following method, if less elementary, will probably be considered to be more elegant, or less tedious. It will have, also, the advantage of conducting to a somewhat more general system of expressions, by which the same equations can be satisfied; and will serve to exemplify the application of the fundamental relations, (A), (B), which were assigned in the sixth and eighth articles, between the important symbols $i\,j\,k$, and on which the present Theory of Quaternions may be regarded as essentially depending.

24. Let us, then, first form, from the type (219), by changing the index r to the value 0, the following less general type, which, however, contains under it sixty-four out of the two hundred and fifty-six equations of condition to be satisfied:

$$n_{0u0}n_{t0s}+n_{0u1}n_{t1s}+n_{0u2}n_{t2s}+n_{0u3}n_{t3s}$$
$$=n_{0us}n_{t00}+n_{1us}n_{t01}+n_{2us}n_{t02}+n_{3us}n_{t03}. \quad (224)$$

Make, for abridgment, $$q_{tu}=n_{tu0}+in_{tu1}+jn_{tu2}+kn_{tu3}; \quad (225)$$

$i\,j\,k$ being the three symbols just now referred to; we may then substitute for (224) the following formula, deduced from it, but not involving the index s:

$$n_{0u0}q_{t0}+n_{0u1}q_{t1}+n_{0u2}q_{t2}+n_{0u3}q_{t3}$$
$$=q_{0u}n_{t00}+q_{1u}n_{t01}+q_{2u}n_{t02}+q_{3u}n_{t03}. \quad (226)$$

This, again, will reduce itself, by the same definition (225) of the symbol q_{tu}, to the identity,

$$q_{0u}q_{t0} = q_{0u}q_{t0}, \tag{227}$$

and therefore will be satisfied, if we satisfy the six conditions:

$$\left.\begin{array}{lll} q_{t1} = iq_{t0}; & q_{t2} = jq_{t0}; & q_{t3} = kq_{t0}; \\ q_{1u} = q_{0u}i; & q_{2u} = q_{0u}j; & q_{3u} = q_{0u}k. \end{array}\right\} \tag{228}$$

If, instead of making $r = 0$, we make $r = 1$, in (219), we then obtain, instead of (224), the formula:

$$\begin{aligned} &n_{1u0}n_{t0s} + n_{1u1}n_{t1s} + n_{1u2}n_{t2s} + n_{1u3}n_{t3s} \\ &\quad = n_{0us}n_{t10} + n_{1us}n_{t11} + n_{2us}n_{t12} + n_{3us}n_{t13}; \end{aligned} \tag{229}$$

and the symbolic equation (226) is replaced by the following:

$$\begin{aligned} &n_{1u0}q_{t0} + n_{1u1}q_{t1} + n_{1u2}q_{t2} + n_{1u3}q_{t3} \\ &\quad = q_{0u}n_{t10} + q_{1u}n_{t11} + q_{2u}n_{t12} + q_{3u}n_{t13}; \end{aligned} \tag{230}$$

which, under the conditions (228), becomes first, by the definition (225),

$$q_{1u}q_{t0} = q_{0u}q_{t1}; \tag{231}$$

and then is seen to be satisfied, in virtue of the same conditions.

In like manner by making $r = 2$, in (219), we find

$$\begin{aligned} &n_{2u0}n_{t0s} + n_{2u1}n_{t1s} + n_{2u2}n_{t2s} + n_{2u3}n_{t3s} \\ &\quad = n_{0us}n_{t20} + n_{1us}n_{t21} + n_{2us}n_{t22} + n_{3us}n_{t23}; \end{aligned} \tag{232}$$

and this, under the form

$$\begin{aligned} &n_{2u0}q_{t0} + n_{2u1}q_{t1} + n_{2u2}q_{t2} + n_{2u3}q_{t3} \\ &\quad = q_{0u}n_{t20} + q_{1u}n_{t21} + q_{2u}n_{t22} + q_{3u}n_{t23}, \end{aligned} \tag{233}$$

is satisfied by the same conditions (228), since they give

$$q_{2u}q_{t0} = q_{0u}q_{t2}. \tag{234}$$

Finally, the formula obtained from (219) by making $r = 3$, namely,

$$\begin{aligned} &n_{3u0}n_{t0s} + n_{3u1}n_{t1s} + n_{3u2}n_{t2s} + n_{3u3}n_{t3s} \\ &\quad = n_{0us}n_{t30} + n_{1us}n_{t31} + n_{2us}n_{t32} + n_{3us}n_{t33}, \end{aligned} \tag{235}$$

or this other, deduced from it by the help of (225),

$$\begin{aligned} &n_{3u0}q_{t0} + n_{3u1}q_{t1} + n_{3u2}q_{t2} + n_{3u3}q_{t3} \\ &\quad = q_{0u}n_{t30} + q_{1u}n_{t31} + q_{2u}n_{t32} + q_{3u}n_{t33}, \end{aligned} \tag{236}$$

is satisfied by the same conditions (228), which give

$$q_{3u}q_{t0} = q_{0u}q_{t3}. \tag{237}$$

We shall therefore satisfy not only the sixty-four arithmetical conditions included in the type (224), but also the sixty-four others included in the type (229), the sixty-four included in (232), and the sixty-four included in (235); that is to say, we shall satisfy the whole system of the *two hundred and fifty-six arithmetical* (or ordinary algebraical) *conditions* included in the formula (219), if we satisfy the system of the *six symbolical equations* (228), which involve the three symbols $i\,j\,k$ in their composition; provided that we do so *without establishing any linear relation between those three symbols and unity.* This last restriction is necessary, in order that each of the four symbolical formulae, (226), (230), (233), (236), not involving the index s, may be, as we have supposed, equivalent to the corresponding one of the four arithmetical formulae, (224), (229), (232), (235), in which that index s, occurs, and is permitted to receive any one of the four values, 0, 1, 2, 3.

25. If we write, for conciseness,

$$q_0 = n_{000} + in_{001} + jn_{002} + kn_{003}, \tag{238}$$

the conditions of the preceding article give the sixteen symbolical equations:

$$\left.\begin{array}{llll} q_{00} = q_0; & q_{01} = iq_0; & q_{02} = jq_0; & q_{03} = kq_0; \\ q_{10} = q_0 i; & q_{11} = iq_0 i; & q_{12} = jq_0 i; & q_{13} = kq_0 i; \\ q_{20} = q_0 j; & q_{21} = iq_0 j; & q_{22} = jq_0 j; & q_{23} = kq_0 j; \\ q_{30} = q_0 k; & q_{31} = iq_0 k; & q_{32} = jq_0 k; & q_{33} = kq_0 k; \end{array}\right\} \tag{239}$$

in which, while still retaining the *linear independence* lately assumed to exist between i, j, k, and 1, we may now suppose that the *squares and products* of the three symbols, i, j, k, are determined, or eliminated, by the help of the fundamental formula (A), assigned in the sixth article, namely,

$$i^2 = j^2 = k^2 = ijk = -1; \tag{A}$$

together with those others which this may be considered as including, especially the following:

$$ij = k, \quad ji = -k; \qquad jk = i, \quad kj = -i; \qquad ki = j, \quad ik = -j. \tag{B}$$

In this manner, by (225) and (238), while the first of the sixteen symbolical equations (239) is identically satisfied, each of the other fifteen will resolve itself into four ordinary equations, independent of the three symbols, i, j, k; and thus, if we denote, for conciseness, four of the numerical coefficients of quaternion multiplication as follows,

$$n_{000} = a; \quad n_{001} = b; \quad n_{002} = c; \quad n_{003} = d, \tag{240}$$

the other sixty coefficients of such multiplication may be expressed in terms of these: and the values so obtained will satisfy the two hundred and fifty-six conditions included in the formula (219); whatever four numbers may be chosen for a, b, c, d.

And if we farther simplify the formulae by supposing

$$a = 1, \quad b = 0, \quad c = 0, \quad d = 0, \tag{241}$$

which will be found in the applications to involve no *essential* loss of generality, we then obtain, from this last-mentioned system of expressions, that system of sixty-four numerical values for the sixty-four coefficients of multiplication of quaternions, which was assigned in the equations (220) ... (223), of the twenty-third article.

Coefficients of Quaternion-Derivation; Comparison of Characteristics

26. Adopting, then, those values, (220) ... (223), for the sixty-four coefficients of multiplication, let us, at the same time, in accordance with the rule (215), adopt also such a connected system of values for the sixty-four connected coefficients of derivation, $c_{r,s,t}$, as shall give the continued equation,

$$\begin{aligned} 1 = c_{000} = c_{011} = c_{022} = c_{033} = -c_{101} = c_{110} = -c_{123} = c_{132} \\ = -c_{202} = c_{213} = c_{220} = -c_{231} = -c_{303} = -c_{312} = c_{321} = c_{330}; \end{aligned} \tag{242}$$

ten of these coefficients c being thus equal to $+1$, and six others being each equal to -1, while the other forty-eight coefficients of derivation shall, by the same rule, vanish.

The formula (135) will thus give the sixteen following equations:

$$\left.\begin{array}{llll} a_{00}= \ \ a_0; & a_{01}= \ \ a_1; & a_{02}= \ \ a_2; & a_{03}= \ \ a_3; \\ a_{10}=-a_1; & a_{11}= \ \ a_0; & a_{12}=-a_3; & a_{13}= \ \ a_2; \\ a_{20}=-a_2; & a_{21}= \ \ a_3; & a_{22}= \ \ a_0; & a_{23}=-a_1; \\ a_{30}=-a_3; & a_{31}=-a_2; & a_{32}= \ \ a_1; & a_{33}= \ \ a_0; \end{array}\right\} \quad (243)$$

and, therefore, by comparing the definitions (134) and (70), we shall have the four expressions:

$$\left.\begin{array}{l} X_0 q = (a_0, \ \ a_1, \ \ a_2, \ \ a_3) = 1q; \\ X_1 q = (-a_1, \ \ a_0, \ \ -a_3, \ \ a_2) = iq; \\ X_2 q = (-a_2, \ \ a_3, \ \ a_0, \ \ -a_1) = jq; \\ X_3 q = (-a_3, \ \ -a_2, \ \ a_1, \ \ a_0) = kq; \end{array}\right\} \quad (244)$$

for the results of operating, by the four elementary characteristics of derivation, X_0, X_1, X_2, X_3, which are thus seen to be equivalent to $1, i, j, k$, on the ordinal quaternion,

$$q = (a_0, a_1, a_2, a_3). \quad (55)$$

Whatever the constituents of this original operand may be, since the equations of detachment have been satisfied by the choice of the constant coefficients, we shall have, by the formula (153), and by the values (220) ... (223), sixteen expressions for the symbolic squares and products of these elementary characteristics of derivation, which are independent of the quaternion first operated on; namely, the sixteen expressions following:

$$\left.\begin{array}{llll} X_0 X_0 = X_0; & X_1 X_0 = \ \ X_1; & X_2 X_0 = \ \ X_2; & X_3 X_0 = \ \ X_3; \\ X_0 X_1 = X_1; & X_1 X_1 = -X_0; & X_2 X_1 = -X_3; & X_3 X_1 = \ \ X_2; \\ X_0 X_2 = X_2; & X_1 X_2 = \ \ X_3; & X_2 X_2 = -X_0; & X_3 X_2 = -X_1; \\ X_0 X_3 = X_3; & X_1 X_3 = -X_2; & X_2 X_3 = \ \ X_1; & X_3 X_3 = -X_0; \end{array}\right\} \quad (245)$$

which might also be deduced from the equations,

$$X_0 = 1; \quad X_1 = i; \quad X_2 = j; \quad X_3 = k. \quad (246)$$

Product and Quotient of two numeral Quaternions; Law of the Modulus

27. We may also write, by (155),

$$(m''_0, m''_1, m''_2, m''_3) = (m'_0, m'_1, m'_2, m'_3)\,(m_0, m_1, m_2, m_3), \quad (247)$$

and may say that the *numeral quaternion* $(m''_0, m''_1, m''_2, m''_3)$ is equal to the *product* obtained when the numeral quaternion (m_0, m_1, m_2, m_3) is *multiplied*, as a *multiplicand*, by the numeral quaternion (m'_0, m'_1, m'_2, m'_3) as a *multiplier*; provided that, by the formula (149), with the same values of the coefficients of multiplication, we establish the four following equations between the twelve numerical constituents of these three numeral quaternions:

$$\left.\begin{array}{l} m''_0 = m'_0 m_0 - m'_1 m_1 - m'_2 m_2 - m'_3 m_3; \\ m''_1 = m'_0 m_1 + m'_1 m_0 + m'_2 m_3 - m'_3 m_2; \\ m''_2 = m'_0 m_2 - m'_1 m_3 + m'_2 m_0 + m'_3 m_1; \\ m''_3 = m'_0 m_3 + m'_1 m_2 - m'_2 m_1 + m'_3 m_0. \end{array}\right\} \quad (248)$$

Under the same conditions we may say that the *multiplier* quaternion (or the *left-hand factor* in the expression for a product) is the *quotient* obtained by *dividing* the product by the multiplicand; and may write the formula,

$$(m'_0, m'_1, m'_2, m'_3) = \frac{(m''_0, m''_1, m''_2, m''_3)}{(m_0, m_1, m_2, m_3)}. \tag{249}$$

It is easy to see that if we make, for abridgment,

$$\left.\begin{aligned} \mu^2 &= m_0^2 + m_1^2 + m_2^2 + m_3^2, \\ \mu'^2 &= m_0'^2 + m_1'^2 + m_2'^2 + m_3'^2, \\ \mu''^2 &= m_0''^2 + m_1''^2 + m_2''^2 + m_3''^2, \end{aligned}\right\} \tag{250}$$

and regard μ, μ', μ'' as positive (or absolute) numbers, the equations (248) give the following very simple but important relation:

$$\mu'' = \mu'\mu. \tag{251}$$

If then we give the name of *modulus* to the (positive or absolute) square-root of the sum of the squares of the four (positive or negative or null) numbers, which enter as *constituents* into the expression of a numeral quaternion, we see that it is allowed to say, for such quaternions (as well as for couples and their analogous moduli), that *the modulus of the product is equal to the product of the moduli*. The equations (248) give also, for the numerical constituents of the quotient (249), the expressions:

$$\left.\begin{aligned} m'_0 &= \mu^{-2}(+m''_0 m_0 + m''_1 m_1 + m''_2 m_2 + m''_3 m_3); \\ m'_1 &= \mu^{-2}(-m''_0 m_1 + m''_1 m_0 - m''_2 m_3 + m''_3 m_2); \\ m'_2 &= \mu^{-2}(-m''_0 m_2 + m''_1 m_3 + m''_2 m_0 - m''_3 m_1); \\ m'_3 &= \mu^{-2}(-m''_0 m_3 - m''_1 m_2 + m''_2 m_1 + m''_3 m_0); \end{aligned}\right\} \tag{252}$$

which may be compared with the expression (183) for the quotient that results from the division of one couple by another. As a verification, we may observe that they give, as it is not difficult to see that they ought to do,

$$\frac{(m_0, m_1, m_2, m_3)}{(m_0, m_1, m_2, m_3)} = (1, 0, 0, 0). \tag{253}$$

And these results respecting products and quotients of two numeral quaternions may easily be remembered, or reproduced, if we observe that we have the following *general expression for a numeral quaternion*:

$$q = (m_0, m_1, m_2, m_3) = m_0 + im_1 + jm_2 + km_3; \qquad (254) = (\text{C})$$

where i, j, k are still those three coordinate symbols, or new fourth roots of unity, already introduced in this Essay, of which the squares and products are subject to the fundamental formula:

$$i^2 = j^2 = k^2 = ijk = -1; \tag{A}$$

and to the relations which are consequences of this formula, especially the following:

$$ij = -ji = k; \quad jk = -kj = i; \quad ki = -ik = j. \tag{B}$$

These equations, (A) and (B), had indeed occurred before in this paper; but on account of their great importance in the present theory, they have been written once more in this place, in connexion with the general expression (C), which may represent any numeral quaternion.

On the more general System of Coefficients, obtained by a recent Investigation

28. If we had not adopted the particular numerical values (241), but had allowed the four letters a, b, c, d, in the equations (240), to denote any four constant numbers, which numbers, or their symbols, should thus enter as *arbitrary constants* into the expressions for the coefficients of multiplication, and into those for the connected coefficients of derivation of quaternions; then it is not difficult to see that, with the same fundamental system of expressions for the squares and products of i, j, k, contained in the formula (A), the results of the investigation in the twenty-fourth and twenty-fifth articles might be concisely presented as follows:

$$m_0 X_0 + m_1 X_1 + m_2 X_2 + m_3 X_3 = (m_0 + m_1 i + m_2 j + m_3 k)(a + bi + cj + dk). \qquad (255)$$

And then the formula of symbolic multiplication of one numeral quaternion by another, which is included in (152), namely,

$$\begin{aligned} &m_0'' X_0 + m_1'' X_1 + m_2'' X_2 + m_3'' X_3 \\ &\quad = (m_0' X_0 + m_1' X_1 + m_2' X_2 + m_3' X_3)(m_0 X_0 + m_1 X_1 + m_2 X_2 + m_3 X_3), \end{aligned} \qquad (256)$$

would become, with the same system of non-linear relations between the same three symbols i, j, k:

$$\begin{aligned} &m_0'' + m_1'' i + m_2'' j + m_3'' k \\ &\quad = (m_0' + m_1' i + m_2' j + m_3' k)(a + bi + cj + dk)(m_0 + m_1 i + m_2 j + m_3 k). \end{aligned} \qquad (257)$$

This formula resolves itself, by those relations, and by the linear independence of i, j, k, and 1, into four separate equations, which may be obtained from the four equations (248), by changing m_0, m_1, m_2, m_3, respectively, to

$$\left.\begin{aligned} m_0^{\backprime} &= am_0 - bm_1 - cm_2 - dm_3; \\ m_1^{\backprime} &= am_1 + bm_0 + cm_3 - dm_2; \\ m_2^{\backprime} &= am_2 - bm_3 + cm_0 + dm_1; \\ m_3^{\backprime} &= am_3 + bm_2 - cm_1 + dm_0; \end{aligned}\right\} \qquad (258)$$

so that, with these abridgments, the four equations included in the formula (257) may be thus written:

$$\left.\begin{aligned} m_0'' &= m_0' m_0^{\backprime} - m_1' m_1^{\backprime} - m_2' m_2^{\backprime} - m_3' m_3^{\backprime}; \\ m_1'' &= m_0' m_1^{\backprime} + m_1' m_0^{\backprime} + m_2' m_3^{\backprime} - m_3' m_2^{\backprime}; \\ m_2'' &= m_0' m_2^{\backprime} - m_1' m_3^{\backprime} + m_2' m_0^{\backprime} + m_3' m_1^{\backprime}; \\ m_3'' &= m_0' m_3^{\backprime} + m_1' m_2^{\backprime} - m_2' m_1^{\backprime} + m_3' m_0^{\backprime}. \end{aligned}\right\} \qquad (259)$$

In this manner we should obtain the four expressions:

$$\left.\begin{aligned} m_0'' &= aA_0 + bB_0 + cC_0 + dD_0; \\ m_1'' &= aA_1 + bB_1 + cC_1 + dD_1; \\ m_2'' &= aA_2 + bB_2 + cC_2 + dD_2; \\ m_3'' &= aA_3 + bB_3 + cC_3 + dD_3; \end{aligned}\right\} \qquad (260)$$

where

$$\left.\begin{aligned} A_0 &= m_0' m_0 - m_1' m_1 - m_2' m_2 - m_3' m_3; \\ A_1 &= m_0' m_1 + m_1' m_0 + m_2' m_3 - m_3' m_2; \\ A_2 &= m_0' m_2 - m_1' m_3 + m_2' m_0 + m_3' m_1; \\ A_3 &= m_0' m_3 + m_1' m_2 - m_2' m_1 + m_3' m_0; \end{aligned}\right\} \qquad (261)$$

$$\left.\begin{aligned}B_0&=-m'_0m_1-m'_1m_0+m'_2m_3-m'_3m_2;\\B_1&=+m'_0m_0-m'_1m_1+m'_2m_2+m'_3m_3;\\B_2&=-m'_0m_3-m'_1m_2-m'_2m_1+m'_3m_0;\\B_3&=+m'_0m_2-m'_1m_3-m'_3m_0-m'_3m_1;\end{aligned}\right\}\quad(262)$$

$$\left.\begin{aligned}C_0&=-m'_0m_2-m'_1m_3-m'_2m_0+m'_3m_1;\\C_1&=+m'_0m_3-m'_1m_2-m'_2m_1-m'_3m_0;\\C_2&=+m'_0m_0+m'_1m_1-m'_2m_2+m'_3m_3;\\C_3&=-m'_0m_1+m'_1m_0-m'_2m_3-m'_3m_2;\end{aligned}\right\}\quad(263)$$

$$\left.\begin{aligned}D_0&=-m'_0m_3+m'_1m_2-m'_2m_1-m'_3m_0;\\D_1&=-m'_0m_2-m'_1m_3+m'_2m_0-m'_3m_1;\\D_2&=+m'_0m_1-m'_1m_0-m'_2m_3-m'_3m_2;\\D_3&=+m'_0m_0+m'_1m_1+m'_2m_2-m'_3m_3.\end{aligned}\right\}\quad(264)$$

And thus may the problem of the multiplication of numeral quaternions be resolved, without any restriction being laid on the numerical values of the four arbitrary constants, a, b, c, d. The *modular equation* (251), namely, $\mu''=\mu'\mu$, will extend to this more general system, if we define the modulus μ of the quaternion (m_0, m_1, m_2, m_3) by the formula:

$$\mu^2=(a^2+b^2+c^2+d^2)\,(m_0^2+m_1^2+m_2^2+m_3^2). \quad (265)$$

Thus, with the recently established forms (261) ... (264), of the sixteen functions $A_0 \dots D_3$, we must have, as an *identity*, independent of the values of the *twelve* numbers denoted by the symbols $a\,b\,c\,d\,m_0\,m_1\,m_2\,m_3\,m'_0\,m'_1\,m'_2\,m'_3$, the following equation:

$$\begin{aligned}&(aA_0+bB_0+cC_0+dD_0)^2+(aA_1+bB_1+cC_1+dD_1)^2\\&\quad+(aA_2+bB_2+cC_2+dD_2)^2+(aA_3+bB_3+cC_3+dD_3)^2\\&\quad=(a^2+b^2+c^2+d^2)\,(m_0'^2+m_1'^2+m_2'^2+m_3'^2)\,(m_0^2+m_1^2+m_2^2+m_3^2);\end{aligned}\quad(266)$$

and therefore, independently of the values of the *eight* numbers $m_0 \dots m'_3$, we must have these *ten* other equations:

$$\left.\begin{aligned}&(m_0'^2+m_1'^2+m_2'^2+m_3'^2)\,(m_0^2+m_1^2+m_2^2+m_3^2)\\&=A_0^2+A_1^2+A_2^2+A_3^2=B_0^2+B_1^2+B_2^2+B_3^2\\&=C_0^2+C_1^2+C_2^2+C_3^2=D_0^2+D_1^2+D_2^2+D_3^2;\end{aligned}\right\}\quad(267)$$

$$\left.\begin{aligned}&0=A_0B_0+A_1B_1+A_2B_2+A_3B_3;\quad 0=A_0C_0+A_1C_1+A_2C_2+A_3C_3;\\&0=A_0D_0+A_1D_1+A_2D_2+A_3D_3;\quad 0=B_0C_0+B_1C_1+B_2C_2+B_3C_3;\\&0=B_0D_0+B_1D_1+B_2D_2+B_3D_3;\quad 0=C_0D_0+C_1D_1+C_2D_2+C_3D_3.\end{aligned}\right\}\quad(268)$$

Although these identities admit of being established in a more elementary way, yet it has been thought worth while to point out the foregoing method of arriving at them, because that method follows easily from the principles of the present theory.

On the Extension of the Theory of Multiplication of Quaternions to other numeral Sets

29. This seems to be a proper place for offering a few remarks on the treatment of the general equation (214), which may assist in the future extension of the present theory of

multiplication of quaternions to other numeral sets; and may serve, in the meanwhile, to throw some fresh light on the process which has been employed in the twenty-fourth and twenty-fifth articles, for discovering a mode of satisfying that general equation, in the case when the exponent n of the order of the set is 4.

Let $i_0, i_1, \ldots, i_{n-1}$ be a system of n symbolical multipliers, which we shall assume to be unconnected with each other by any *linear* relation; and let us establish the following formula, analogous to (225),

$$q_{t,u} = i_0 n_{t,u,0} + \cdots + i_{n-1} n_{t,u,n-1} = \Sigma_v \cdot i_v n_{t,u,v}. \qquad (269)$$

Then, operating by the characteristic $\Sigma . i_{s'}$ on the equation (214), we shall transform that equation into the following:

$$0 = \Sigma_s (n_{r,r',s}\, q_{t,s} - q_{s,r'}\, n_{t,r,s}); \qquad (270)$$

and may satisfy it by supposing

$$q_{t,u} = i_u q_0 i_t; \quad q_0 = i_0^{-1} q_{0,0} i_0^{-1}; \qquad (271)$$

for we shall then have

$$\Sigma_s \cdot n_{r,r',s}\, q_{t,s} = q_{r,r'} q_0 i_t = i_{r'} q_0 i_r q_0 i_t = i_{r'} q_0 q_{t,r} = \Sigma_s \cdot q_{s,r'}\, n_{t,r,s}. \qquad (272)$$

We are therefore to endeavour to satisfy the symbolical condition,

$$\Sigma_v \cdot i_u^{-1} i_v i_t^{-1} n_{t,u,v} = \text{const.} = q_0; \qquad (273)$$

this constant q_0 being independent of t and u, and the n symbols i_0, i_1, &c., being still unconnected by any linear relation. When this shall have been accomplished, we may then employ the formula,

$$\mathrm{X}_t = i_t q_0; \qquad (274)$$

which will give

$$\mathrm{X}_u \mathrm{X}_t = i_u q_0 i_t q_0 = q_{t,u} q_0 = \Sigma_v \cdot n_{t,u,v} \mathrm{X}_v; \qquad (275)$$

and therefore will agree with the formula (153). And thus the equations of detachment will have been satisfied, and a numeral *set*, of the kind above supposed, will be found under the form,

$$q = \Sigma_t \cdot m_t \mathrm{X}_t = \Sigma_t \cdot m_t i_t q_0. \qquad (276)$$

For the case of *couples*, we may make

$$i_0 = 1; \quad i_1 = \sqrt{(-1)}; \quad q_0 = 1; \qquad (277)$$

and then the condition (273) will be satisfied by the values of the coefficients of multiplication assigned in the nineteenth article; and the numeral couple will present itself under the well-known form, $m_0 + m_1 \sqrt{(-1)}$.

For the case of *quaternions*, if we suppose

$$i_0 = 1; \quad i_1 = i; \quad i_2 = j; \quad i_3 = k; \qquad (278)$$

the symbols i, j, k being still connected by the fundamental relations (A); the six symbolical equations (228), and the sixteen symbolical equations (239), will then be included, by (269), in the formula (273), in which we may write, by (240), and by (271), or (238),

$$q_0 = a + bi + cj + dk; \qquad (279)$$

and the expression (255) will be included in the more general expression (276). And if we farther particularize, and at the same time simplify, by adopting, as we propose henceforth to do, the values (241), which reduce q_0 to 1, we shall then obtain from (276), by (278), the same expression (254), or (C), which has already been assigned in the twenty-seventh article, as the representation of a numeral quaternion.

Successive Multiplication of Quaternions: Application of the associative Principle

30. It has been stated that we design to adopt, in our theory of numeral quaternions, the simplifications contained in the equations (241). We shall therefore regard, henceforth, the constituents of any product of *two* numeral quaternions as being given by the simpler formulae (248), and not by the more complex formulae (260), in which $A_0 \dots D_3$ are abridged representatives of the sixteen quadrinomials (261) ... (264). Yet the trouble of investigating these latter expressions will not have been thrown away: for we may see, by (257), that they will serve, hereafter, to express the result of a successive multiplication, or the *continued product of three numeral quaternions*. And by applying the *associative principle*, already considered in the twenty-first article, to such *successive multiplication*, we see that, instead of developing the formula (257) by a process which was equivalent to the development of the system of the two equations,

$$m_0^{\backprime} + m_1^{\backprime} i + m_2^{\backprime} j + m_3^{\backprime} k = (a + bi + cj + dk)(m_0 + m_1 i + m_2 j + m_3 k), \tag{280}$$

and

$$m_0'' + m_1'' i + m_2'' j + m_3'' k = (m_0' + m_1' i + m_2' j + m_3' k)(m_0^{\backprime} + m_1^{\backprime} i + m_2^{\backprime} j + m_3^{\backprime} k), \tag{281}$$

we might have developed the same formula (257) by a different, but analogous process, founded on a different mode of grouping or *associating* the three quaternions which enter as symbolic factors. For we might have introduced this other quaternion,

$$m_0^{\backprime\backprime} + m_1^{\backprime\backprime} i + m_2^{\backprime\backprime} j + m_3^{\backprime\backprime} k = (m_0' + m_1' i + m_2' j + m_3' k)(a + bi + cj + dk); \tag{282}$$

which would have given the expression,

$$m_0'' + m_1'' i + m_2'' j + m_3'' k = (m_0^{\backprime\backprime} + m_1^{\backprime\backprime} i + m_2^{\backprime\backprime} j + m_3^{\backprime\backprime} k)(m_0 + m_1 i + m_2 j + m_3 k); \tag{283}$$

and then the four values (260), for the four constituents of the final product of the three quaternion factors which enter into the second member of the formula (257), would have presented themselves as the result of the elimination of the four constituents of the intermediate quaternion product (282), between the eight following equations:

$$\left.\begin{aligned} m_0^{\backprime\backprime} &= m_0' a - m_1' b - m_2' c - m_3' d; \\ m_1^{\backprime\backprime} &= m_0' b + m_1' a + m_2' d - m_3' c; \\ m_2^{\backprime\backprime} &= m_0' c - m_1' d + m_2' a + m_3' b; \\ m_3^{\backprime\backprime} &= m_0' d + m_1' c - m_2' b + m_3' a; \end{aligned}\right\} \tag{284}$$

$$\left.\begin{aligned} m_0'' &= m_0^{\backprime\backprime} m_0 - m_1^{\backprime\backprime} m_1 - m_2^{\backprime\backprime} m_2 - m_3^{\backprime\backprime} m_3; \\ m_1'' &= m_0^{\backprime\backprime} m_1 + m_1^{\backprime\backprime} m_0 + m_2^{\backprime\backprime} m_3 - m_3^{\backprime\backprime} m_2; \\ m_2'' &= m_0^{\backprime\backprime} m_2 - m_1^{\backprime\backprime} m_3 + m_2^{\backprime\backprime} m_0 + m_3^{\backprime\backprime} m_1; \\ m_3'' &= m_0^{\backprime\backprime} m_3 + m_1^{\backprime\backprime} m_2 - m_2^{\backprime\backprime} m_1 + m_3^{\backprime\backprime} m_0. \end{aligned}\right\} \tag{285}$$

And accordingly, on comparing these eight equations with the four expressions (260), we arrive at the same quadrinomial values for the sixteen coefficients $A_0 \dots D_3$, which have been already given in the equations (261) ... (264). We may perceive that they would conduct also to the relations (267), (268) between those coefficients, and to the formula (266) for the *decomposition of a product of three sums, containing each four squares*, by eliminating the modulus $\mu^{\backprime\backprime}$ of the quaternion (282) between two equations analogous to (251), namely, the two following:

$$\mu^{\backprime\backprime} = \mu' e, \quad \mu'' = \mu^{\backprime\backprime} \mu; \tag{286}$$

where μ, μ', μ'' have the significations (250), and where

$$\mu^{\backprime\backprime 2} = m_0^{\backprime\backprime 2} + m_1^{\backprime\backprime 2} + m_2^{\backprime\backprime 2} + m_3^{\backprime\backprime 2}; \quad e^2 = a^2 + b^2 + c^2 + d^2. \tag{287}$$

13-2

Addition and Subtraction of Numeral Sets; Non-commutative Character of Quaternion Multiplication

31. Any two numeral sets may be *added* to each other, by adding their respective constituent numbers, primary to primary, secondary to secondary, and so forth; and on a similar plan may *subtraction* of such sets be performed; thus, for any two numeral quaternions we may write,

$$(m'_0, m'_1, m'_2, m'_3) \pm (m_0, m_1, m_2, m_3) = (m'_0 \pm m_0, m'_1 \pm m_1, m'_2 \pm m_2, m'_3 \pm m_3); \qquad (288)$$

and generally, by using Σ and Δ as the characteristics of sum and difference, and employing those signs of numeral separation which were proposed in the twenty-first article, we may write formulae for sums and differences of numeral sets, which are analogous to, and may be considered as depending upon those marked (116), for the addition and subtraction of ordinal sets; namely, the following:

$$\mathrm{N}_r \Sigma q = \Sigma \mathrm{N}_r q; \quad \mathrm{N}_r \Delta q = \Delta \mathrm{N}_r q. \qquad (289)$$

For the *multiplication* of numeral sets, we have already established principles and formulae which involve, generally, the *distributive* and the *associative* properties of the operation of the same name, as performed on single numbers; but which *do not retain*, in general, *the commutative property* of that ordinary operation upon numbers. Thus we may write,

$$\Sigma q' \times \Sigma q = \Sigma(q' \times q), \qquad (290)$$

and also,

$$q'' \times q'q = q''q' \times q = q''q'q, \qquad (291)$$

the mark of multiplication being allowed to be omitted, because its place is unimportant to the result, in the successive multiplication of any three or more numeral sets. But we are *not* at liberty to write, *generally*, for *any two* such sets, as factors, the commutative formula,

$$q'q = qq';$$

since, although, by the equation (182), this last formula of commutation of factors holds good, not only for single *numbers*, but also when the factors are numeral *couples*, of the kind considered in the nineteenth article of the present paper, and in the earlier Essay there referred to, yet, for the case of numeral quaternions, the relations (B) between the products of the symbols i, j, k, give results opposed to the commutative formula, namely, the following:

$$ij = -ji, \quad jk = -kj, \quad ki = -ik.$$

In fact, by (149), or by (209), to justify generally this commutative formula of multiplication, as applied to numeral sets of the order n, it would be necessary that the n^3 coefficients of multiplication should be connected with each other by the relations included in the type,

$$n_{r, r', s} = n_{r', r, s}. \qquad (292)$$

Now these relations have, indeed, been established in our theory of numeral couples, since, in the abridged notation of the nineteenth article, and with the values there adopted, we have the equations,

$$f = g; \quad f' = g'; \quad \text{or,} \quad n_{010} = n_{100}; \quad n_{011} = n_{101}; \qquad (293)$$

but they do *not* hold good in our theory of numeral quaternions, since we have been led to adopt values for the coefficients of multiplication, which give, on the contrary,

$$n_{123} = -n_{213}; \quad n_{231} = -n_{321}; \quad n_{312} = -n_{132}. \qquad (294)$$

Thus, if we still adopt the system of values of the coefficients of quaternion multiplication assigned in the twenty-third article, we must *reject* the commutative property; and may establish a formula which is *opposite* in its character to the equation (292), namely, the following:

$$n_{r, r', s} = -n_{r', r, s}, \quad \text{if} \quad r' \gtrless r, \; r > 0, \; r' > 0. \qquad (295)$$

General Division of one numeral Set by another: Combination of the Operations of Division and Multiplication of Quaternions

32. The general *division* of one numeral set by another, if regarded as the operation of *returning to the multiplier*, from the product and the multiplicand, involves no theoretical difficulty, since it depends on the solution, by elimination or otherwise, of a finite system of ordinary equations of the first degree, between the sought numerical constituents of the quotient; and it has been already exemplified, for couples and quaternions, in the nineteenth and twenty-seventh articles. But it is of essential importance to observe that, if division of numeral sets be thus defined by the formula,

$$(q'' \div q) \times q = q'', \tag{296}$$

in which, as in all other cases, we conceive *the symbol of the multiplier to be placed at the left hand*, and which is analogous to (129), we shall then *not have, generally*, for numeral sets, as for numbers, this other usual equation:

$$q \times (q'' \div q) = q''.$$

In fact, if we were to assume, for example, that this latter and usual equation, though true for numbers and for numeral couples, was generally true for numeral quaternions also, we should then, in consequence of the definitional formula (296), which fixes the correlation of the signs $\times$ and $\div$, with respect to numeral sets, be virtually assuming, also, that equation of commutative multiplication, $q'q = qq'$, which, for the case of quaternions at least, we have already seen reason to *reject*. Hence follows the important consequence that, in this case of quaternions, the first member, $q \times (q'' \div q)$, of the lately rejected equation, is the *symbol of a new quaternion, distinct in general from the operand quaternion, q'', which has been first divided and afterwards multiplied by one common operator quaternion, q; these two operations, thus performed, having not generally neutralized each other, on account of the generally noncommutative character of the multiplication of numeral quaternions.* It is, therefore, already an object of interest in this theory, and will be found to be a problem of which the geometrical and physical applications are in a high degree important, *to determine the constituents of that new quaternion*, $q_{\prime\prime}$, distinct from q'', which is thus represented by the symbol $q \times (q'' \div q)$, or which satisfies the equation

$$q \times (q'' \div q) = q_{\prime\prime}. \tag{297}$$

To express the same problem otherwise, with the help of the definition of division, (296), we have now the system of the two equations,

$$q'' = q'q; \quad q_{\prime\prime} = qq'; \tag{298}$$

q'' and $q_{\prime\prime}$ being those two distinct quaternion products which arise from the multiplication of the same two quaternion factors, q and q', with two different arrangements of those factors; and we are to eliminate the four constituents of one of those two quaternion factors, namely, the constituents of the factor q', between the eight separate and ordinary equations into which the two quaternion equations (298) resolve themselves. If we write, for this purpose,

$$\left.\begin{aligned} q &= w + ix + jy + kz, \\ q' &= w' + ix' + jy' + kz', \\ q'' &= w'' + ix'' + jy'' + kz'', \\ q_{\prime\prime} &= w_{\prime\prime} + ix_{\prime\prime} + jy_{\prime\prime} + kz_{\prime\prime}, \end{aligned}\right\} \tag{299}$$

we shall then have the four equations,

$$\left.\begin{aligned} w'' &= w'w - x'x - y'y - z'z; \\ x'' &= w'x + x'w + y'z - z'y; \\ y'' &= w'y - x'z + y'w + z'x; \\ z'' &= w'z + x'y - y'x + z'w; \end{aligned}\right\} \tag{300}$$

together with the four others which result from these by interchanging, in the right-hand members, the accented with the unaccented letters, and by changing in the left-hand members upper to lower accents; namely, the four following:

$$\left.\begin{aligned} w_{\prime\prime} &= ww' - xx' - yy' - zz'; \\ x_{\prime\prime} &= wx' + xw' + yz' - zy'; \\ y_{\prime\prime} &= wy' - xz' + yw' + zx'; \\ z_{\prime\prime} &= wz' + xy' - yx' + zw'. \end{aligned}\right\} \tag{301}$$

It thus appears immediately that $$w_{\prime\prime} = w''; \tag{302}$$

and the elimination, above directed, of the four numbers w', x', y', z', that is, of the constituents of the numeral quaternion q', between the eight equations (300), (301), gives these three other equations, which complete the solution of the problem, so far as it depends on the above-mentioned elimination:

$$\left.\begin{aligned} wx_{\prime\prime} + zy_{\prime\prime} - yz_{\prime\prime} &= wx'' + yz'' - zy''; \\ wy_{\prime\prime} + xz_{\prime\prime} - zx_{\prime\prime} &= wy'' + zx'' - xz''; \\ wz_{\prime\prime} + yx_{\prime\prime} - xy_{\prime\prime} &= wz'' + xy'' - yx''. \end{aligned}\right\} \tag{303}$$

These equations conduct to the relations,

$$xx_{\prime\prime} + yy_{\prime\prime} + zz_{\prime\prime} = xx'' + yy'' + zz'', \tag{304}$$

and

$$x_{\prime\prime}^2 + y_{\prime\prime}^2 + z_{\prime\prime}^2 = x''^2 + y''^2 + z''^2; \tag{305}$$

which, as it is easy to foresee, will be found to have extensive applications, and which may also be easily obtained, by observing that, before the elimination of w', x', y', z', the equations (300), (301) give

$$\left.\begin{aligned} x_{\prime\prime} + x'' &= 2(wx' + w'x); & x_{\prime\prime} - x'' &= 2(yz' - zy'); \\ y_{\prime\prime} + y'' &= 2(wy' + w'y); & y_{\prime\prime} - y'' &= 2(zx' - xz'); \\ z_{\prime\prime} + z'' &= 2(wz' + w'z); & z_{\prime\prime} - z'' &= 2(xy' - yx'). \end{aligned}\right\} \tag{306}$$

33. Although these latter combinations (306), of those equations (300), 301), conduct without difficulty to the equations (303), (304), (305), yet it is still more easy, when once the principles of the present theory have been distinctly comprehended, to deduce the last-mentioned equations, by treating in the following way the problem of the foregoing article.

Instead of *resolving* the numeral quaternion q' into the four separate terms, w', ix', jy', kz', as is done in the second of the four expressions (299), and then eliminating the four constituent numbers w', x', y', z' between the eight ordinary equations into which the two quaternion equations (298) resolve themselves, we may *eliminate the quaternion* q' itself between those two equations (298), and so obtain immediately, without any labour of calculation, this new quaternion equation,

$$q_{\prime\prime}q = qq''; \tag{307}$$

which, by the three remaining expressions (299), and by the equality (302), becomes:

$$(ix_{\prime\prime} + jy_{\prime\prime} + kz_{\prime\prime})(w + ix + jy + kz) = (w + ix + jy + kz)(ix'' + jy'' + kz''). \tag{308}$$

If now we perform the multiplications here indicated, attending to the fundamental expressions (A) (B), for the squares and products of the three symbols, i, j, k, and to the linear independence, already supposed to exist, between the four symbols, i, j, k, and 1, we find that the *one* quaternion formula (308) resolves itself into the *four* equations, (303) and (304). And either from the four equations thus obtained, or by an application of the law of the modulus to the quaternion equation (308), the relation (305) may be obtained. It is worth while observing that we may also write the quaternion formula,

$$(w^2+x^2+y^2+z^2)\,q_{\prime\prime} = (w+ix+jy+kz)\,(w''+ix''+jy''+kz'')\,(w-ix-jy-kz); \qquad (309)$$

or, more fully,

$$(w^2+x^2+y^2+z^2)\,(w_{\prime\prime}-w''+ix_{\prime\prime}+jy_{\prime\prime}+kz_{\prime\prime}) = (w^2-x^2-y^2-z^2)\,(ix''+jy''+kz'')$$
$$+2(xx''+yy''+zz'')\,(ix+jy+kz)+2w\{i(yz''-zy'')+j(zx''-xz'')+(k(xy''-yx'')\}; \qquad (310)$$

by resolving which *one* formula, the same separate values for $w_{\prime\prime}, x_{\prime\prime}, y_{\prime\prime}, z_{\prime\prime}$ may be obtained, as from the system of the *four* ordinary equations (302), (303).

On the Operation of pre-multiplying one numeral Set by another, and on fractional Symbols for Sets

34. Since we have seen that we are not at liberty to assume generally, for *all* numeral sets, that the commutative formula of multiplication holds good, we must (in general) distinguish between *two modes* of combination of two such sets with each other, *as factors*, in some such way as the following. We saw reason, in the twenty-second article, to regard an ordinal set, q, as having been generated by a certain symbolical multiplication, or complex derivation, from a single standard ordinal relation, a, as from an original operand or derivand; the operator, or symbolical multiplier, having been a numeral set, q. If such an ordinal set, q, or $q \times \mathrm{a}$, be again operated on by the new numeral set, q', as by a new symbolical multiplier, the result will be a new ordinal set, $q' \times (q \times \mathrm{a})$, which, in this theory, admits of being denoted also by $(q' \times q) \times \mathrm{a}$; and generally, in the same theory, the conditions of detachment entitle us to write the formula

$$q' \times (q \times \mathrm{q}^{\backprime}) = (q' \times q) \times \mathrm{q}^{\backprime}, \qquad (311)$$

whatever operand set (of the same order) may here be denoted by the symbol $\mathrm{q}^{\backprime}$. Thus, to multiply the numeral set q, as a multiplicand, by the numeral set q', as a multiplier, comes to be regarded as being equivalent to the operations of multiplying some single standard ordinal relations, a, or some ordinal set, $\mathrm{q}^{\backprime}$, *first by the given multiplicand set*, q, and *afterwards by the given multiplier set*, q'; and of then finding that third set, q'', namely, the *product* $q' \times q$, or $q'q$, which, acting *as a single multiplier*, would produce the *same final result*, and would, therefore, serve, by its single operation, to replace this twofold process. In this view of the multiplication of one numeral set by another, the set proposed as a *multiplicand* is itself a *previous* multiplier, and may, therefore, be called a *premultiplicator*, or, more familiarly, a *premultiplier*. And thus, instead of saying that the product $q' \times q$, or $q'q$, is obtained by multiplying q by q', we may be permitted occasionally to say that the same product results from *premultiplying q' by q*; the symbol of the *pre*multiplier being placed towards the *right hand*, as that of the multiplier is placed towards the left.

With this phraseology, and with the definitional formula (296), which easily gives also this other connected formula,

$$(q' \times q) \div q = q', \qquad (312)$$

division and premultiplication are mutually inverse operations; that is to say, a numeral set, q', remains, upon the whole, unchanged, when it is *both* divided and premultiplied, or both premultiplied and divided, by any other numeral set, q (of the same order). We may also agree to express the same results by symbols of *fractional forms*, a *fraction* being defined to be the *quotient* which is obtained when the numerator is divided by the denominator, so that we shall adopt here, as a definition, the formula

$$\frac{q'}{q} = q' \div q; \tag{313}$$

for then we may say that *a fraction gives its numerator as the product, when it is premultiplied by its denominator*; though it does *not* always, at least for the case of quaternions, produce that numerator when it is *multiplied* by that denominator (the order of the factors being then different). In symbols, the equations

$$\frac{q''}{q} q = q'', \quad \frac{q'q}{q} = q', \tag{314}$$

are here regarded as *identical*; whereas these other usual equations,

$$q\frac{q''}{q} = q'', \quad \frac{q'q}{q'} = q,$$

of which the first is only an abridged way of writing a formula already rejected, while the second is connected therewith, are *not generally true* (or, at least, not universally so) *for numeral sets*; because the *order of the factors* in multiplication is, in the present theory of such *sets*, *not generally unimportant to the result*. We have seen, for example, in the foregoing article, that the quaternion which may now be denoted by the symbol $q\frac{q''}{q}$, or by this other symbol, $\frac{qq''}{q}$, or by $qq'' \div q$, instead of being generally equal to the quaternion q'', is equal, in general, to *another quaternion*, $q_{''}$, distinct from the former, though having several simple relations thereto, which will be found to be connected, in their geometrical and physical applications, with questions respecting the transformation of rectangular coordinates in space, and the rotation of a solid body. It may, therefore, be not useless to remark expressly here, that the following usual equations *continue true* in the present theory of numeral sets, as well as in common algebra:

$$q \times \frac{q''}{q} = \frac{qq''}{q}; \quad \frac{q_2 q}{q_1 q} = \frac{q_2}{q_1} = \frac{q_2 q \div q}{q_1 q \div q}; \tag{315}$$

or, in words, that a *fraction* is *multiplied* by a numeral set when its *numerator* is multiplied thereby; and that the *value* of a fraction, regarded as representing a numeral set, remains *unchanged*, or represents the same set as before, when its numerator and its denominator are *both premultiplied*, or *both divided*, *by any common set* (of the same order); both which results depend on the associative property of multiplication, and on the principle that two numeral sets cannot *generally* give equal products, when operating as multipliers on one common multiplicand (different from zero), unless they be themselves equal sets. These general remarks will become more clear by their future applications; meanwhile, we may here agree to use occasionally, for convenience and variety, another form of expression, consistent with the foregoing principles, and to say that, in the product $q'q$, the left-hand factor, q', is multiplied *into* the right-hand factor, q, as the latter has been said to be multiplied *by* the former, and as that former factor again has been said to be *pre*multiplied by the latter.

On the Operations of submultiplying, and of taking the Reciprocal of a numeral Set

35. As it has been found necessary to distinguish, in general, between *two modes of multiplication* of one numeral set by another, with different arrangements of the factors, so is it also necessary in this theory to distinguish generally between *two inverse operations*, namely, between the operation of *division*, and another closely connected operation, which may be called *sub-multiplication*. For if this last-named operation be now defined to be the *returning to the multiplicand*, when the product and the multiplier are given, it will then be evidently distinct, in general, or, at least, for the case of quaternions, from the operation of *division*, which has been already defined to be the *returning to the multiplier*, when the multiplicand and product are given; because these two factors, the multiplier and the multiplicand, when regarded as numeral sets (at least if those sets be quaternions), cannot generally change places with each other, without altering the value of the product. To denote conveniently this new operation of *sub*multiplication, or of returning from the set $q'q$ to the set q, when the set q' is given, we shall now introduce the conception of a *reciprocal set*, which may be denoted by any one of the three symbols,

$$1 \div q = \frac{1}{q} = q^{-1}; \tag{316}$$

and of which the characteristic property is, that it satisfies generally the two reciprocal conditions,

$$q^{-1} \times q\mathrm{q} = \mathrm{q}, \quad q \times q^{-1}\mathrm{q}' = \mathrm{q}', \tag{317}$$

of which the second follows from the first, and which may be more concisely written thus:

$$q^{-1}q = qq^{-1} = 1. \tag{318}$$

Thus, whether a numeral set q be multiplied or premultiplied by its reciprocal set q^{-1}, the product in each case is unity; and when these two reciprocal sets are employed to operate, as successive multipliers, on any ordinal or numeral set as a multiplicand, they *neutralize* the effects of each other. It follows hence, that *to submultiply by any numeral set is equivalent to multiplying by the reciprocal of that set*; so that we may write generally, for such sets, the *formula of submultiplication* (as in ordinary algebra) thus:

$$\frac{1}{q'} . q'q = q'^{-1} . q'q = q. \tag{319}$$

It is evident from what has been said, that the *reciprocal of the reciprocal* of a numeral set is equal to that *set itself*; and that to *divide by* such a set is to *premultiply by* (or to *multiply into*) its *reciprocal*; thus, generally,

$$q'q \div q = q'q \times \frac{1}{q} = q'q . q^{-1} = q'. \tag{320}$$

The *reciprocal of a quaternion* is given by the formula,

$$(w + ix + jy + kz)^{-1} = (w^2 + x^2 + y^2 + z^2)^{-1}(w - ix - jy - kz). \tag{321}$$

In general, the *reciprocal of the product* of any number of sets is equal to the *product of the reciprocals* of those sets, arranged in the *contrary order*: thus we may write,

$$(\dots q_2 q_1 q_0)^{-1} = q_0^{-1} q_1^{-1} q_2^{-1} \dots. \tag{322}$$

On Powers of a Numeral Set, with whole or fractional Exponents; Square and Square Root of a Quaternion; Indeterminate Expressions, by Quaternions, for the Square Roots of Negative Numbers

36. The symbol q^{-1}, for the reciprocal of a numeral set, is only one of a system of symbols of the same sort, which may easily be formed by an adaptation of received algebraic notation. For with the notions given already, respecting multiplication and division of sets, there is no difficulty in interpreting now, in an extended sense, adapted to the present theory, the following usual system of equations,

$$\left.\begin{array}{l} q^0=1, \quad q^1=q, \quad q^2=q\times q^1, \quad q^3=q\times q^2, \ldots \\ q^{-1}=\frac{1}{q}, \quad q^{-2}=\frac{1}{q}\times q^{-1}, \quad q^{-3}=\frac{1}{q}\times q^{-2}, \ldots \end{array}\right\} \tag{323}$$

and then the well-known *equation of the exponential law*,

$$q^s\times q^r=q^r\times q^s=q^{r+s}, \tag{324}$$

will hold good, as in ordinary algebra, the *exponents* r and s being here supposed to denote any two positive or negative whole numbers, or zero.

These two other usual equations,

$$(q^r)^s=q^{sr}, \quad (q^{\frac{t}{s}})^s=q^t, \tag{325}$$

will then also hold good for numeral sets, at least when r, s, t, and $\frac{t}{s}$, denote whole numbers; and the latter of these two formulae may be employed as a *definition* to *interpret* the symbol $q^{\frac{t}{s}}$, when the exponent is a numerical fraction; thus, $q^{\frac{1}{2}}$ will denote that numeral set, *or any one of those numeral sets*, which satisfy, or are roots of, the equation,

$$(q^{\frac{1}{2}})^2=q^1=q. \tag{326}$$

For example, it results from what has been already shown, that if q denote the first numeral quaternion (299), then its *symbolic square*, or *second power*, is another quaternion, q_2, given by the formula

$$q_2=q^2=(w+ix+jy+kz)^2=w_2+ix_2+jy_2+kz_2, \tag{327}$$

where

$$\left.\begin{array}{l} w_2=w^2-x^2-y^2-z^2; \\ x_2=2wx; \quad y_2=2wy; \quad z_2=2wz. \end{array}\right\} \tag{328}$$

And hence, conversely, the symbolic *square root of the quaternion* q_2, or its power with the exponent $\frac{1}{2}$, is to be regarded as being equivalent to this other numeral quaternion,

$$q=q_2^{\frac{1}{2}}=(w_2+ix_2+jy_2+kz_2)^{\frac{1}{2}}=w+ix+jy+kz; \tag{329}$$

where the constituents, w, x, y, z, are *any four numbers* (positive, negative, or zero), *which satisfy the system of the four equations* (328). Those equations give the relation

$$w_2^2+x_2^2+y_2^2+z_2^2=(w^2+x^2+y^2+z^2)^2, \tag{330}$$

which is included in the more general result (251), respecting the multiplication of any two quaternions; therefore, conversely,

$$w^2+x^2+y^2+z^2=\sqrt{(w_2^2+x_2^2+y_2^2+z_2^2)}; \tag{331}$$

and, consequently, by the first of the four equations (328),

$$2w^2=w_2+\sqrt{(w_2^2+x_2^2+y_2^2+z_2^2)}, \tag{332}$$

where the radical in the second member of (331) is to be considered as a positive number: and, therefore, the first constituent, w, of the sought quaternion q, or of the square root of the given quaternion q_2, is itself given, generally, by (332), as either the positive or the negative square root of another given positive number. And after choosing either of these two values (the positive or the negative) for w, the other three constituents, x, y, z, of the sought quaternion q, become, *in general*, entirely determined by the three last equations (328). There are, therefore, *in general, two, and only two, different square roots of any proposed numeral quaternion*; and they differ only in their signs. But there is one very important CASE OF INDETERMINATENESS, in which an *infinite variety* of roots takes the place of that *finite ambiguity*, which has thus been seen to exist generally in the expression for the square root of a quaternion, namely, the *case where the proposed square is equal to a negative number*, presented under the *form* of a quaternion, of which the first constituent is negative, while the three last separately vanish. For, if we suppose the data to be such that

$$w_2 = -r^2, \quad x_2 = 0, \quad y_2 = 0, \quad z_2 = 0, \tag{333}$$

r being some positive or negative number, then the positive radical in (331) becomes

$$\sqrt{(w_2^2 + x_2^2 + y_2^2 + z_2^2)} = r^2 = -w_2, \tag{334}$$

and the equation (332) reduces itself to the following:

$$w = 0. \tag{335}$$

And while the three last of the four equations (328) are then satisfied, independently of the three remaining constituents, x, y, z, the first of those four equations gives this *one* relation, between those three constituents of the sought quaternion q,

$$x^2 + y^2 + z^2 = r^2, \tag{336}$$

which is the *only condition* that they must satisfy. And since we may satisfy this condition by assuming

$$\left.\begin{array}{l} x = \dfrac{lr}{h}, \quad y = \dfrac{mr}{h}, \quad z = \dfrac{nr}{h}, \\ h = \sqrt{(l^2 + m^2 + n^2)}, \end{array}\right\} \tag{337}$$

without any restriction being imposed on the three (positive, or negative, or null) numbers, l, m, n, we see that, in our theory of quaternions, the *square root of a negative number is a partially indeterminate quaternion*, belonging, however, to a certain peculiar *class*, and admitting of being thus denoted:

$$(-r^2)^{\frac{1}{2}} = \frac{(il + jm + kn)r}{\sqrt{(l^2 + m^2 + n^2)}}. \tag{338}$$

In fact, if we square the second member of this last formula, attending to the fundamental expressions, (A), (B), for the squares and products of the three symbols, i, j, k, we find, as the result of this operation, the negative number $-r^2$, which is the square of the first member; for those fundamental expressions give, generally, this very simple and remarkable equation,

$$(ix + jy + kz)^2 = -(x^2 + y^2 + z^2). \tag{339) = (D}$$

For example, in this theory, the square root of -1 itself is represented by a partially indeterminate symbol of the foregoing class, and we may write

$$(-1)^{\frac{1}{2}} = \frac{ix}{r} + \frac{jy}{r} + \frac{kz}{r}, \quad \text{where} \quad r^2 = x^2 + y^2 + z^2. \tag{340}$$

That is to say, whatever three positive, or negative, or null numbers may be denoted by x, y, z, provided that they do not all together vanish, we are allowed in this theory to establish the following *general expression for any one of the infinitely many square roots of negative unity*:

$$(-1)^{\frac{1}{2}} = \frac{ix+jy+kz}{\sqrt{(x^2+y^2+z^2)}}. \tag{341)=(E}$$

Or, with the recent meaning of r, and with a notation which more immediately suggests the conception of a numeral set, we may establish the formula,

$$(-1,0,0,0)^{\frac{1}{2}} = \left(0, \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right). \tag{342}$$

Cubes and Cube Roots of Quaternions; partially indeterminate Expressions by Quaternions for Cube Roots of positive and negative Numbers

37. With the same condition or abridgment, (336), we may write generally, for any numeral quaternion, this expression

$$q = w + (-1)^{\frac{1}{2}} r; \tag{343}$$

or still more briefly and, at the same time, more determinately,

$$q = w + \iota r, \quad \text{where} \quad \iota^2 = -1, \tag{344}$$

and where ι may be conceived to be in general determined when q is determined, since

$$\iota = \frac{ix}{r} + \frac{jy}{r} + \frac{kz}{r}, \quad r = \sqrt{(x^2+y^2+z^2)}. \tag{345}$$

The cube of this expression (344) for q is

$$q^3 = w^3 - 3wr^2 + \iota(3w^2 - r^2)r; \tag{346}$$

and this *cube*, or *third power of a quaternion*, may be equated to a new quaternion, denoted as follows:

$$q_3 = q^3 = w_3 + ix_3 + jy_3 + kz_3 = w_3 + \iota_3 r_3, \tag{347}$$

where

$$r_3^2 = x_3^2 + y_3^2 + z_3^2, \quad \iota_3^2 = -1; \tag{348}$$

provided that we satisfy the two conditions,

$$w_3 = w^3 - 3wr^2, \quad \iota_3 r_3 = \iota r(3w^2 - r^2), \tag{349}$$

of which the second again resolves itself into three others, on account of the mutual linear independence of the three symbols, i, j, k. These last equations give

$$\frac{x_3}{x} = \frac{y_3}{y} = \frac{z_3}{z} = 3w^2 - r^2; \tag{350}$$

and, therefore, it is allowed to write

$$\iota_3 = \iota, \quad r_3 = r(3w^2 - r^2); \tag{351}$$

provided that, if we still choose to consider the radical r as positive, we regard the other radical, r_3, as varying its sign, according to the law

$$r_3 \gtrless 0, \quad \text{according as} \quad 3w^2 \gtrless r^2. \tag{352}$$

If, now, it be required to find conversely the *cube root* q, or the power with exponent $\frac{1}{3}$ of a given quaternion, q_3, we shall have, first, the two equations

$$w^2 + r^2 = (w_3^2 + r_3^2)^{\frac{1}{3}}; \quad \frac{r}{w}\left(3 - \left(\frac{r}{w}\right)^2\right)\left(1 - 3\left(\frac{r}{w}\right)^2\right)^{-1} = \frac{r_3}{w_3}; \tag{353}$$

of which the second may be written more concisely thus:

$$3t-t^3=(1-3t^2)t_3, \quad \text{if} \quad r=tw,\ r_3=t_3w_3; \tag{354}$$

so that

$$t_3^2=w_2^{-2}(x_3^2+y_3^2+z_3^2). \tag{355}$$

The value of this positive number, t_3^2, is known, because the four constituents of the quaternion q_3 are now supposed to be given; hence, three different positive values for t^2 can, in general, be deduced from the square of the first equation (354), which is a well-known cubic; for each such value of t^2, the sign of t_3, and therefore, also (by the same cubic equation), the sign of t may be determined by the condition that r_3 or t_3w_3 is, by (352), to receive the same sign as $3-t^2$; but r is supposed positive, therefore w has the same sign as t; and

$$w^2(1+t^2)=w_3^{\frac{2}{3}}(1+t_3^2)^{\frac{1}{3}}, \tag{356}$$

so that the constituent w is entirely determined: therefore, r (being $=tw$) is known, and then the three remaining constituents, x, y, z, of the sought quaternion, q, are given by (350). Thus, the sought cube root, q, of the proposed numeral quaternion q_3, is, in general, determined; or, at least, is restricted to a finite and *triple variety*, answering to the *three* (real, numerical, and) *unequal roots of the known cubic equation* (354); which roots can always be found by the help of a table of trigonometric tangents. We see, then, by the foregoing process, which will soon be replaced by one more simple and more powerful, that there are, *in general, three, and only three, distinct cube roots of any proposed numeral quaternion.* But when it is required to find, on the same plan, under the form of a quaternion, the cube root of a positive or negative number, w_3, regarded as an abridged expression for the quaternion $(w_3, 0, 0, 0)$, then x_3, y_3, z_3, and r_3, all vanish; and while the ratios of x, y, z remain entirely arbitrary, the numbers w and r are to be determined so as to satisfy the two equations,

$$w_3=w^3-3wr^2; \quad 0=r(3w^2-r^2); \tag{357}$$

which require that we should suppose either

$$r=0, \quad w^3=w_3, \tag{358}$$

or else,

$$r^2=3w^2, \quad w^3=-\tfrac{1}{8}w_3. \tag{359}$$

For example, if we seek the quaternion cube roots of positive unity, regarded as equivalent to the quaternion (1, 0, 0, 0), we find not only unity itself, under the form of the same quaternion, but also this other, and partially indeterminate expression,

$$1^{\frac{1}{3}}=(1,0,0,0)^{\frac{1}{3}}=(-\tfrac{1}{2},x,y,z); \tag{360}$$

where the three positive or negative numbers, x, y, z, are only obliged to satisfy the condition

$$x^2+y^2+z^2=\tfrac{3}{4}. \tag{361}$$

And, in like manner, besides negative unity itself, there are infinitely many quaternion cube roots of negative unity, included in the expression

$$(-1)^{\frac{1}{3}}=(-1,0,0,0)^{\frac{1}{3}}=(+\tfrac{1}{2},x,y,z), \tag{362}$$

under the same condition (361) respecting the sum of the squares of the constituents, x, y, z. The values of this last expression (362), as well as the values of the expression (360), are, therefore, included among those quaternions which are (in this theory) *sixth roots of unity*, or are among the values of the symbol $1^{\frac{1}{6}}$. As one other example, it may be remarked that, by the rule (359), the number negative eight has, for one of its cube roots, the quaternion of which each of the four constituents is equal to positive unity; thus, *one value* of the symbol

$$(-8, 0, 0, 0), \quad \text{is} \quad (1, 1, 1, 1); \tag{363}$$

and, accordingly, we shall find that

$$(1+i+j+k)^3 = -8, \tag{364}$$

if we develop the first member of this last equation, employing the distributive property of multiplication, but *not* the commutative property, and reducing by the values of the symbolic squares and products of i, j, k, which have been already assigned. It may be noted here that, in the more general problem of finding the cube root, q, of a quaternion, q_3, of which the three last constituents, x_3, y_3, z_3, do not all vanish, so that r_3 is different from 0, we might have eliminated r^2 between the first equation (349) and the first equation (353), and so have obtained an ordinary cubic equation in w, which, as well as the equation in t, can be resolved by the trigonometrical tables, namely, the cubic:

$$4w^3 - 3w(w_3^2 + r_3^2)^{\frac{1}{3}} = w_3. \tag{365}$$

Connexion of Quaternions with Couples, and with Quadratic Equations

38. In general, if a numeral quaternion q be required to satisfy any ordinary numerical equation (with real coefficients) of the form

$$0 = a_0 + a_1 q + a_2 q^2 + a_3 q^3 + \&\text{c.}, \tag{366}$$

we may first substitute for q the expression (344), namely, $w + \iota r$, where $\iota^2 = -1$. Then, after finding any one of those systems of values of the two (real) numbers w and r, which satisfy the system of the two equations, obtained by the foregoing substitution, and by equating separately to zero the sums of the terms containing respectively the even and odd powers of ι, namely, the equations

$$\left.\begin{aligned} 0 &= a_0 + a_1 w + a_2(w^2 - r^2) + a_3(w^3 - 3wr^2) + \&\text{c.}, \\ 0 &= a_1 r + a_2(2wr) + a_3(3w^2 r - r^3) + \&\text{c.}; \end{aligned}\right\} \tag{367}$$

we shall only have to change ιr, in the expression for q, to $ix + jy + kz$, and to suppose, as before, that $x^2 + y^2 + z^2 = r^2$. But the process by which the two numbers w and r are thus supposed to be discovered, is precisely the process by which a numeral couple (w, r), of the kind considered in the nineteenth article of this paper, and in the earlier Essay there referred to, would be determined, so as to satisfy the couple-equation,

$$0 = a_0 + a_1(w, r) + a_2(w, r)^2 + \&\text{c.} \tag{368}$$

The calculations required for finding a *couple* (w, r) which shall satisfy this equation (368), are therefore the same as those required for finding a *quaternion* (w, x, y, z), which shall satisfy the equation

$$0 = a_0 + a_1(w, x, y, z) + a_2(w, x, y, z)^2 + \&\text{c.}; \tag{369}$$

provided that we suppose the constituents of these two numeral sets to be connected with each other by the relation already assigned, namely,

$$x^2 + y^2 + z^2 = r^2. \tag{336}$$

Thus, in particular, if it be proposed to satisfy, by a quaternion q, the quadratic equation,

$$0 = a_0 + a_1 q + a_2 q^2, \tag{370}$$

which we may put under the form

$$q^2 - 2aq + b = 0, \tag{371}$$

we may first change q to the couple (w, r), and so obtain the *two* separate equations,

$$w^2 - r^2 - 2aw + b = 0; \quad 2wr - 2ar = 0; \tag{372}$$

of which the latter requires us to suppose, either,

$$\text{1st, } r = 0; \quad \text{or,} \quad \text{2nd, } w = a. \tag{373}$$

The first alternative conducts to a quadratic equation in w, namely,

$$w^2 - 2aw + b = 0, \tag{374}$$

which is precisely the proposed equation (371), with the symbol q of the sought quaternion changed to the symbol w of a sought number; and reciprocally if it be possible to find a real number w, or rather (in general) two such numbers, which shall satisfy the quadratic (374), that is to say, if (the equation have *real roots*, or if) the condition

$$a^2 > b, \quad \text{or} \quad a^2 = b + c^2, \tag{375}$$

be satisfied, where c is a positive or negative number, then the equation (371) will be satisfied by either of the two quaternions which are included in the following expression, and by no other quaternion,

$$q = (w, 0, 0, 0) = (a \pm \sqrt{(a^2 - b)}, 0, 0, 0). \tag{376}$$

The same expression holds good, giving one solution of the equation (371), for the case $a^2 = b$. But in the remaining case, where

$$a^2 < b, \quad a^2 = b - c^2, \tag{377}$$

c being still a positive or negative number, we are to adopt the remaining alternative (373), namely, $w = a$; and instead of supposing $r = 0$, we are now, by the first equation (372), and by (377), to suppose

$$r^2 = w^2 - 2aw + b = b - a^2 = c^2; \tag{378}$$

and the solution of the quadratic equation (371) is now expressed by the partially indeterminate quaternion, connected with the two couple-solutions $(a, \pm c)$,

$$q = (a, x, y, z), \quad \text{where} \quad x^2 + y^2 + z^2 = b - a^2. \tag{379}$$

And thus we may perceive that, if we denote by μ the modulus of the first numeral quaternion (299), which may represent any such quaternion, then *this quaternion, q, is a root of a quadratic equation*, with real coefficients, namely, the following:

$$q^2 - 2wq + \mu^2 = 0. \tag{380}$$

Exponential and Imponential of a numeral Set; general Expression for a Power, when both the Base and the Exponent are such Sets

39. The investigations, in some recent articles, respecting certain powers and roots of a quaternion, may be made at once more simple and more general by the introduction of a well-known exponential series. We shall, therefore, write

$$\mathrm{P}(q) = 1 + \frac{q^1}{1} + \frac{q^2}{1.2} + \frac{q^3}{1.2.3} + \&\text{c.} \tag{381}$$

and shall call this series the *exponential function*, or simply, *the exponential of the numeral set q*, with respect to which the operations are performed; we shall also denote this exponential still more concisely by writing simply $\mathrm{P}q$ instead of $\mathrm{P}(q)$, where no confusion seems likely to arise from this abbreviation. The *inverse function*, which may be conceived to express reciprocally q, by means of Pq, may be called by contrast the *imponential* function, and denoted by the characteristic P^{-1}; thus, we shall suppose $\mathrm{P}^{-1}q$ to be such that

$$\mathrm{PP}^{-1}q = q, \tag{382}$$

or that, more fully,

$$q = 1 + \mathrm{P}^{-1}q + \tfrac{1}{2}(\mathrm{P}^{-1}q)^2 + \frac{1}{2.3}(\mathrm{P}^{-1}q)^3 + \&\mathrm{c}. \tag{383}$$

Then, because the function P is such that

$$\mathrm{P}q' \times \mathrm{P}q = \mathrm{P}(q' + q), \quad \text{if} \quad q'q = qq'; \tag{384}$$

and because, by the *associative* principle of multiplication, *any two whole powers of the same numeral set, q, are commutative as factors*, that is to say, may change their places with each other, without altering the value of the product; we shall have, generally,

$$\mathrm{P}f'(q) \times \mathrm{P}f(q) = \mathrm{P}(f'(q) + f(q)), \tag{385}$$

because we shall have

$$f'(q) \times f(q) = f(q) \times f'(q), \tag{386}$$

if the symbols $f(q)$ and $f'(q)$ denote here any combinations of whole powers of *one common numeral set*, q, and of any given numerical coefficients. For example, if a denote a *number*, we shall have

$$\mathrm{P}a \times \mathrm{P}q = \mathrm{P}(a + q). \tag{387}$$

We may also deduce, from the formula (385), this other important corollary, which is general for numeral sets, and in which the symbol $\mathrm{P}.sq$ represents the same function as $\mathrm{P}(sq)$, while s may, at first, be supposed to denote a whole number:

$$(\mathrm{P}q)^s = \mathrm{P}(sq) = \mathrm{P}.sq. \tag{388}$$

We have, therefore, for *any two whole numbers*, s and t, the relation

$$(\mathrm{P}.sq)^t = (\mathrm{P}.tq)^s; \tag{389}$$

and, therefore, as an equation of which the second member is, at least, *one of the values* of the first, we have

$$(\mathrm{P}.sq)^{\frac{t}{s}} = \mathrm{P}.tq. \tag{390}$$

We are thus led to write, as an equation of the same sort, giving an expression for, at least, one value of any *fractional power of a set*, whenever the *imponential* of that set can be discovered,

$$q^{\frac{t}{s}} = \mathrm{P}\left(\frac{t}{s}\mathrm{P}^{-1}q\right). \tag{391}$$

The simplicity of this equation may now induce us to *extend* it, as we propose to do, by *definition*, to the cases where the *exponent of the power*, instead of being a numerical fraction, is an incommensurable number, *or even a numeral set*. We shall, therefore, write generally

$$q^{q'} = \mathrm{P}(q'\mathrm{P}^{-1}q); \tag{392}$$

and thus we shall have a *general expression for any power of a numeral set*, through the help of the characteristics of the exponential and imponential thereof.

Application to Quaternions; Amplitude and Vector Unit; Coordinates, Radius, and Representative Point

40. On applying these general principles to the case of a quaternion, we have first, by (387),

$$\mathrm{P}q = \mathrm{P}(w + ix + jy + kz) = \mathrm{P}w.\mathrm{P}(ix + jy + kz); \tag{393}$$

and then, if we use the notations (345), and attend to the connexion already established between quaternions and couples, we find that

$$\mathrm{P}(ix+jy+kz)=\mathrm{P}(\iota r)=\cos r+\iota\sin r;\quad \iota^2=-1; \tag{394}$$

where $\cos r$ and $\sin r$ denote, as usual, the cosine and sine of r, so that, in the theory of couples, the following equation holds good:

$$\mathrm{P}(0,r)=(\cos r,\ \sin r). \tag{395}$$

(Compare the earlier Essay, where the functional sign F was used instead of P).* Thus the *exponential of a quaternion* q is expressed generally, with these notations, by the formula,

$$\mathrm{P}q=\mathrm{P}w\,.\,(\cos r+\iota\sin r). \tag{396}$$

Reciprocally the *imponential* $\mathrm{P}^{-1}q'$, of any other quaternion, q', is to be found by comparing this formula (396) with the expression of that quaternion q', when put under the form,

$$q'=w'+\iota'r'=\mu'(\cos\theta'+\iota'\sin\theta'), \tag{397}$$

where

$$\mu'=\sqrt{(w'^2+r'^2)},\quad \tan\theta'=\frac{r'}{w'}. \tag{398}$$

We find, in this manner, that we may suppose

$$q'=\mathrm{P}q,\quad q=\mathrm{P}^{-1}q', \tag{399}$$

provided that we make

$$\mathrm{P}w=\mu';\quad r=\theta'+2n'\pi;\quad \iota=\iota'; \tag{400}$$

where n' is any whole number, and π is, as usual, the least positive root of the numerical equation,

$$\pi^{-1}\sin\pi=0.$$

Hence, the sought *imponential of the quaternion* q' is

$$\mathrm{P}^{-1}q'=\mathrm{P}^{-1}\mu'+\iota'(\theta'+2n'\pi); \tag{401}$$

and, in like manner, by suppressing the accents, the imponential of q is found to be

$$\mathrm{P}^{-1}q=\mathrm{P}^{-1}\mu+\iota(\theta+2n\pi), \tag{402}$$

where θ may be said to be the AMPLITUDE, and μ is what we have already called the MODULUS of q.

41. We may also say that ι is the *imaginary unit*, or perhaps, more expressively, that it is the VECTOR UNIT, of the same quaternion q. For in the applications of this theory to geometrical questions, this imaginary or vector unit ι may be regarded as having in general a given *direction in space* when q is a given quaternion; and if we denote its *direction cosines* by α,β,γ, so that

$$\alpha=\frac{x}{r},\quad \beta=\frac{y}{r},\quad \gamma=\frac{z}{r},\quad \alpha^2+\beta^2+\gamma^2=1, \tag{403}$$

we may write, generally, by (345),

$$\iota=i\alpha+j\beta+k\gamma,\quad \iota^2=-1. \tag{404}$$

This power of representing any DIRECTION IN TRIDIMENSIONAL SPACE, *by one of the quaternion forms of* $\sqrt{(-1)}$, *is one of the chief peculiarities of the present theory*; and will be found to be one of the chief causes of its power, when employed as an instrument in researches of a geometrical kind. If α,β,γ be conceived to be the three rectangular coordinates of a point R upon a spheric surface, with radius unity, described about the origin of coordinates as centre, we may also write, more concisely and, at the same time, not less expressively,

$$\iota=i_{\mathrm{R}};\quad i_{\mathrm{R}}^2=-1. \tag{405}$$

* [See I, p. 89, (117).]

A numeral quaternion q may therefore, in general, be thus expressed:

$$q = \mu(\cos\theta + i_{\mathrm{R}}\sin\theta); \tag{406}$$

where $$\mu = \sqrt{(w^2+x^2+y^2+z^2)}, \quad i_{\mathrm{R}} = i\alpha + j\beta + k\gamma = \sqrt{(-1)}. \tag{407}$$

Its imponential, by (402), will then take the form

$$\mathrm{P}^{-1}q = \log\mu + i_{\mathrm{R}}(\theta + 2n\pi), \tag{408}$$

n denoting here any positive or negative whole number, or zero; and $\log\mu$ denoting the (real and) natural or Napierian logarithm of the positive (or absolute) number μ; or in other words, that determined (real) number, whether positive or negative or null, which satisfies the equation $$\mu = \mathrm{P}(\log\mu). \tag{409}$$

42. Substituting this expression (408) for the imponential of a quaternion in the general expression (392) for a power of a set, we find, for a power of a quaternion q, with another quaternion q' as the exponent of that power, the expression,

$$q^{q'} = \mathrm{P}\{q'\log\mu + q' i_{\mathrm{R}}(\theta + 2n\pi)\}; \tag{410}$$

which, however, it is not *generally* allowed to resolve into the two factors, $\mathrm{P}(q'\log\mu)$ and $\mathrm{P}\{q' i_{\mathrm{R}}(\theta+2n\pi)\}$, because q' and $q' i_{\mathrm{R}}$ are not, in general, *condirectional quaternions*; if this latter name be given to quaternions which have vector units equal or opposite, so that in each case they are commutative with each other, as factors in multiplication. But if we change the exponent q', in (410), to any numerical fraction, $\frac{t}{s}$, where s and t denote whole numbers, then this resolution into factors is allowed, and the formula becomes

$$\begin{aligned} q^{\frac{t}{s}} &= \mathrm{P}\left\{\frac{t}{s}\log\mu + \frac{t}{s} i_{\mathrm{R}}(\theta + 2n\pi)\right\} \\ &= \mathrm{P}\left(\frac{t}{s}\log\mu\right).\ \mathrm{P}\left\{i_{\mathrm{R}}\left(\frac{t\theta}{s} + \frac{2tn\pi}{s}\right)\right\} \\ &= \mu^{\frac{t}{s}}(\cos + i_{\mathrm{R}}\sin)\left(\frac{t\theta}{s} + \frac{2tn\pi}{s}\right); \end{aligned} \tag{411}$$

and thus it will be found that the chief results of the thirty-sixth and thirty-seventh articles, respecting certain powers and roots of a quaternion, are reproduced under a simpler and more general aspect; for instance, the square root of a quaternion is now given under the form

$$q^{\frac{1}{2}} = \mu^{\frac{1}{2}}(\cos + i_{\mathrm{R}}\sin)\left(\frac{\theta}{2} + n\pi\right) = \pm\mu\left(\cos\frac{\theta}{2} + i_{\mathrm{R}}\sin\frac{\theta}{2}\right). \tag{412}$$

But in the particular case where the original quaternion, q, reduces itself to a negative number, $q = w = -\mu$, so that its amplitude, θ, is some odd multiple of π, while the direction of its vector unit is indeterminate or unknown, the formula (412) for a square root becomes simply

$$(-\mu)^{\frac{1}{2}} = \mu^{\frac{1}{2}} i_{\mathrm{R}}; \tag{413}$$

the position of the point R upon the *unit sphere* being now likewise indeterminate or unknown, which agrees with our former results respecting the indeterminate quaternion forms for the square roots of negative numbers. In like manner, the quaternions, distinct from unity itself, which are *cube roots of unity*, are now included in the expression

$$1^{\frac{1}{3}} = \cos\frac{2n\pi}{3} + i_{\mathrm{R}}\sin\frac{2n\pi}{3}; \tag{414}$$

where the *direction* of i_R remains entirely undetermined. But, *in general, the power*, $q^{\frac{t}{s}}$, of *a quaternion, q, admits of s, and only s, distinct quaternion values*, if the exponent, $\frac{t}{s}$, be an arithmetical fraction in its lowest terms, so that the numerator and the denominator of this fractional exponent are whole numbers prime to each other; and if the proposed quaternion q does not reduce itself to a number w, by the three last constituents, x, y, z, all separately vanishing in its expression. As an example of the operation of *raising a quaternion to a power of which the exponent is distinct from all positive and negative numbers, and from zero*, we may remark that the formula (410) gives, generally, for the powers of an imaginary unit, such as i_R $\left(\text{for which we have } \mu=1,\ \theta=\frac{\pi}{2}\right)$, the expression

$$i_R{}^{q'}=\mathrm{P}\left\{q' i_R\left(\frac{\pi}{2}+2n\pi\right)\right\}; \tag{415}$$

making then, in particular, $i_R=i$, and $q'=k$, we find, by (B),

$$i^k=\mathrm{P}\left\{ki\left(\frac{\pi}{2}+2n\pi\right)\right\}=\mathrm{P}\left\{j\left(\frac{\pi}{2}+2n\pi\right)\right\}=\mathrm{P}\left(\frac{j\pi}{2}\right)=j; \tag{416}$$

and by a similar process we find, more generally,

$$i_R{}^{i_{R'}}=i_{R'}i_R, \tag{417}$$

whenever $i_{R'}$ and i_R denote *two rectangular imaginary units*, so that the points R and R′, which mark their directions, are distant from each other by a quadrant on the sphere. We may here introduce a few slight additions to the nomenclature already established in this paper, and may say that, in the general expression $q=w+ix+jy+kz$, the three coefficients, x, y, z, which multiply respectively the three coordinate characteristics, i, j, k, are *the three* COORDINATES *of the quaternion*, and that the square root r of the sum of their squares is the RADIUS of the same quaternion. We shall also say that the point R, on the surface of the unit sphere, which constructs or represents the direction of the vector unit in its expression, is at once the REPRESENTATIVE POINT *of that vector unit*, i_R, and also (in a similar sense) *the representative point of the quaternion q* itself.

On the general Logarithms of a Set, and especially on those of a Quaternion

43. Though we cannot enter here at any length into the theory of *logarithms of sets*, yet it is obvious that if we make

$$q''=q^{q'}, \tag{418}$$

the general expression (392) for a power of a set gives this inverse expression for the exponent q':

$$\log_q q''=q'=\frac{\mathrm{P}^{-1}q''}{\mathrm{P}^{-1}q}; \tag{419}$$

in which expression, however, for a *logarithm of a set*, under the form of a fraction, the numerator and the denominator are to be regarded as *separately subject to that indeterminateness*, whatever it may be, *which arises in the return from the exponential of a set to the set itself*, or in the passage from a set q to its *imponential* $\mathrm{P}^{-1}q$. Thus in the case of quaternions, the *general logarithm of the quaternion q″, to the base q*, may, by (419) and (408), be written thus:

$$\log_q q''=\frac{\log\mu''+i_R(\theta''+2n''\pi)}{\log\mu+i_R(\theta+2n\pi)}. \tag{420}$$

It involves, therefore, *two arbitrary and independent whole numbers*, n'' and n, in its expression. as happens in the theories of John T. Graves, Esq., Professor Ohm, and others,* respecting the general logarithms of ordinary imaginary quantities to ordinary imaginary bases; and also in that theory of the general logarithms of *numeral couples*, with other numeral couples for their bases, which was published by the present author (as part of the Essay already several times cited, on Conjugate Functions and Algebraic Couples, and on Algebra as the Science of Pure Time), in the seventeenth volume of the *Transactions* of this Academy.

Connexion of Quaternions with Spherical Geometry

44. Let $\mathrm{R}, \mathrm{R}', \mathrm{R}'', \ldots, \mathrm{R}^{(n-1)}$ be any n points upon the surface of the unit sphere, so that they may be generally regarded as the corners of a spherical polygon upon that surface; and let them be regarded also as the determining or representative points (in the sense of the forty-second article) of the same number of vector units, i_{R}, $i_{\mathrm{R}'}$, &c. Then the associative property of multiplication will give, on the one hand, the equation

$$i_{\mathrm{R}} i_{\mathrm{R}'} . i_{\mathrm{R}'} i_{\mathrm{R}''} . i_{\mathrm{R}''} i_{\mathrm{R}'''} \ldots i_{\mathrm{R}^{(n-1)}} i_{\mathrm{R}} = (-1)^n; \tag{421}$$

because

$$i_{\mathrm{R}}^2 = i_{\mathrm{R}'}^2 = i_{\mathrm{R}''}^2 = \ldots = -1; \tag{422}$$

and, on the other hand, on substituting the expressions for these vector units, involving their respective direction-cosines and the three fundamental units, i, j, k, which expressions are of the forms

$$i_{\mathrm{R}} = i\alpha + j\beta + k\gamma, \quad i_{\mathrm{R}'} = i\alpha' + j\beta' + k\gamma', \ldots \tag{423}$$

we shall have, for the product of the two first, by the fundamental relations (B), the expression

$$\begin{aligned} i_{\mathrm{R}} i_{\mathrm{R}'} &= (i\alpha + j\beta + k\gamma)(i\alpha' + j\beta' + k\gamma') \\ &= -(\alpha\alpha' + \beta\beta' + \gamma\gamma') + i(\beta\gamma' - \gamma\beta') + j(\gamma\alpha' - \alpha\gamma') + k(\alpha\beta' - \beta\alpha'), \end{aligned} \tag{424}$$

that is,

$$i_{\mathrm{R}} i_{\mathrm{R}'} = -\cos \mathrm{RR}' + i_{\mathrm{P}''} \sin \mathrm{RR}', \tag{425}$$

if RR' denote the arc of rotation in a great circle, round a positive pole P'', from the point R to the point R' upon the sphere, with other similar transformations for the other binary products. By combining these two principles, (421), (425), it is not difficult to see that, *for any spherical polygon*, regarded as having its corners $\mathrm{R}, \mathrm{R}', \ldots$ at the positive poles of the sides of another polygon, the following formula holds good:

$$(\cos \mathrm{R} + i_{\mathrm{R}} \sin \mathrm{R})(\cos \mathrm{R}' + i_{\mathrm{R}'} \sin \mathrm{R}') \ldots (\cos \mathrm{R}^{(n-1)} + i_{\mathrm{R}^{(n-1)}} \sin \mathrm{R}^{(n-1)}) = (-1)^n; \tag{426}$$

in which the symbols $\mathrm{R}, \mathrm{R}', \ldots$ under the characteristics cos and sin, denote the (suitably measured) successive angles at the corners $\mathrm{R}, \mathrm{R}', \ldots$. In particular, for the case of a *spherical triangle*, $\mathrm{RR}'\mathrm{R}''$, the formula (426) gives this less general formula, which, however, may be considered as *including spherical trigonometry*:

$$(\cos \mathrm{R} + i_{\mathrm{R}} \sin \mathrm{R})(\cos \mathrm{R}' + i_{\mathrm{R}'} \sin \mathrm{R}')(\cos \mathrm{R}'' + i_{\mathrm{R}''} \sin \mathrm{R}'') = -1. \tag{427}$$

45. Multiplying both members of this formula (427) into $\cos \mathrm{R}'' = i_{\mathrm{R}''} \sin \mathrm{R}''$, we put it under the less symmetric but sometimes more convenient form,

$$(\cos \mathrm{R} + i_{\mathrm{R}} \sin \mathrm{R})(\cos \mathrm{R}' + i_{\mathrm{R}'} \sin \mathrm{R}') = -\cos \mathrm{R}'' + i_{\mathrm{R}''} \sin \mathrm{R}''. \tag{428}$$

* [See VI, p. 124, second footnote.]

Developing the first member of this last equation, and substituting, for the product of the two vector units, its value (425), we find that it resolves itself into the two following formulae:

$$\cos \mathrm{R} \cos \mathrm{R}' - \cos \mathrm{RR}' \sin \mathrm{R} \sin \mathrm{R}' = -\cos \mathrm{R}''; \tag{429}$$

$$i_{\mathrm{R}} \sin \mathrm{R} \cos \mathrm{R}' + i_{\mathrm{R}'} \sin \mathrm{R}' \cos \mathrm{R} + i_{\mathrm{P}''} \sin \mathrm{R} \sin \mathrm{R}' \sin \mathrm{RR}' = i_{\mathrm{R}''} \sin \mathrm{R}''. \tag{430}$$

Of these two equations, the first agrees with the known expression for the cosine of a side RR′ of a spherical triangle RR′R″, regarded as a function of the three angles R, R′, R″; and the second expresses a theorem, which can easily be verified by known methods, namely, that if a force $= \sin \mathrm{R}''$ be directed from the centre of the sphere to the point R″, that is, to one corner of any such spherical triangle RR′R″, this force is statically equivalent to the system of three other forces, one directed to R, and equal to $\sin \mathrm{R} \cos \mathrm{R}'$; another directed to R′, and equal to $\sin \mathrm{R}' \cos \mathrm{R}$; and the third equal to $\sin \mathrm{R} \sin \mathrm{R}' \sin \mathrm{RR}'$, and directed towards that pole P″ of the arc RR′, which lies at the same side of this arc as does the corner R″.

46. In this, or in other ways, we may be led to establish, as a consequence from the principles which have been already stated, the following general *formula for the multiplication of any two numeral quaternions*:

$$q \times q' = \mu(\cos \mathrm{R} + i_{\mathrm{R}} \sin \mathrm{R}) \times \mu'(\cos \mathrm{R}' + i_{\mathrm{R}'} \sin \mathrm{R}')$$
$$= \mu\mu'\{\cos(\pi - \mathrm{R}'') + i_{\mathrm{R}''} \sin(\pi - \mathrm{R}'')\}; \tag{431}$$

and to interpret it as being equivalent to the system of the three following rules or theorems. First, that (as was seen in the twenty-seventh article), the *modulus* μ'' *of the product* is equal to the *product* $\mu\mu'$ *of the moduli* of the factors. Second, that if a *spherical triangle* RR′R″ be constructed with the representative points of the factors and product for its three corners, the *angles of this triangle* will be respectively equal to the *amplitudes of the two factors*, and to the *supplement of the amplitude of the product*; the amplitude R of the multiplier quaternion q, for example, being equal to the spherical angle at the corner R of the triangle just described. And third, that the *rotation round the product point*, R″, *from the multiplier point*, which is here denoted by R, *to the multiplicand point*, denoted here by R′, is *positive*; or, in other words, this rotation is in the *same direction* (towards the right hand, or towards the left), as the rotation round the positive semiaxis of z or of k $(=ij)$, from that of x or of i, to that of y or of j. The same third rule may also be expressed by saying that the rotation of a great semicircle *round* the multiplier point R, *from* the multiplicand point R′, *towards* the product point R″, is positive; whereas the rotation to the same product point, *from* the multiplier point, *round* the multiplicand point, is, on the contrary, negative. (Compare the remarks in Note A, printed at the end of the present series (see p. 217).)

47. The associative character of multiplication shows that if we assume any three quaternions q, q', q'', and derive two others $q_{,}$, $q_{,,}$ from them, by the equations

$$qq' = q_{,}, \quad q'q'' = q_{,,}, \tag{432}$$

we shall have also the equations

$$q_{,}q'' = qq_{,,} = q''', \tag{433}$$

q''' being a third derivėd quaternion, namely, the ternary product $q\,q'\,q''$. Let R R′ R″ $\mathrm{R}_{,}$ $\mathrm{R}_{,,}$ R‴ be the six representative points of these six quaternions, on the same spheric surface as before; then, by the general construction of a product assigned in the foregoing article, we shall have the following expressions for the six amplitudes of the same six quaternions:

$$\left.\begin{aligned} &\theta = \mathrm{R'RR_{,}} = \mathrm{R_{,,}RR'''}; && \theta_{,} = \mathrm{R''R_{,}R'''} = \pi - \mathrm{RR_{,}R'}; \\ &\theta' = \mathrm{R_{,}R'R} = \mathrm{R''R'R_{,,}}; && \theta_{,,} = \mathrm{R'''R_{,,}R} = \pi - \mathrm{R'R_{,,}R''}; \\ &\theta'' = \mathrm{R_{,,}R''R'} = \mathrm{R'''R''R_{,}}; && \theta''' = \pi - \mathrm{R_{,}R'''R''} = \pi - \mathrm{RR'''R_{,,}}; \end{aligned}\right\} \tag{434}$$

$R'RR_{,}$ being the spherical angle at R, measured from RR' to $RR_{,}$, and similarly in other cases. But these equations between the spherical angles of the figure are precisely those which are requisite, in order that the two points $R_{,}$ and $R_{,,}$ should be the two *foci of a spherical conic inscribed in the spherical quadrilateral* $RR'R''R'''$, or touched by the four great circles of which the arcs RR', $R'R''$, $R''R'''$, $R'''R$, are parts; this geometrical relation between the six representative points $RR'R''R_{,}R_{,,}R'''$ of the six quaternions, q, q', q'', qq', $q'q''$, $qq'q''$, which may conveniently be thus denoted,

$$R_{,}R_{,,}(\dots)RR'R''R''', \tag{435}$$

is, therefore, a consequence, and may be considered as an interpretation of the very simple algebraical formula for associating three quaternion factors,

$$qq'\cdot q'' = q\cdot q'q''.$$

It follows, at the same time, from the theory of cones and conics, that *the two straight lines*, or *radii vectores*, which are drawn from the origin of coordinates to the points $R_{,}$, $R_{,,}$, and *which construct the imaginary parts of the two binary quaternion products*, qq', $q'q''$, *are the two focal lines of a cone of the second degree, inscribed in the tetrahedral angle, which has for four conterminous edges the four radii which construct the imaginary parts of the three quaternion factors* q, q', q'', *and of their continued or ternary product* $qq'q''$.

48. We have also, by the same associative character of multiplication, an analogous formula for the product of any *four* quaternion factors, q, q', q'', q''', namely,

$$q\cdot q'q''q''' = qq'\cdot q''q''' = qq'q''\cdot q''' = q^{\mathrm{IV}}, \tag{436}$$

if we denote this continued product by q^{IV}; and if we make

$$qq' = q_{,},\quad q'q'' = q'_{,},\quad q''q''' = q''_{,},\quad qq'q'' = q'''_{,},\quad q'q''q''' = q^{\mathrm{IV}}_{,}, \tag{437}$$

and observe that whenever E and F are foci of a spherical conic inscribed in a spherical quadrilateral $ABCD$, so that, in the notation recently proposed,

$$EF(\dots)ABCD, \tag{438}$$

then also we may write $\quad FE(\dots)ABCD, \quad$ and $\quad EF(\dots)BCDA, \qquad (439)$

we shall find, without difficulty, by the help of the formula (435), the five following geometrical relations, in which each R is the representative point of the corresponding quaternion q:

$$\left.\begin{aligned}
&R_{,}R'_{,}(\dots)RR'R''R'''_{,};\\
&R'_{,}R''_{,}(\dots)R'R''R'''R^{\mathrm{IV}}_{,};\\
&R''_{,}R'''_{,}(\dots)R''R'''R^{\mathrm{IV}}R_{,};\\
&R'''_{,}R^{\mathrm{IV}}_{,}(\dots)R'''R^{\mathrm{IV}}RR'_{,};\\
&R^{\mathrm{IV}}_{,}R_{,}(\dots)R^{\mathrm{IV}}RR'R''_{,}.
\end{aligned}\right\} \tag{440}$$

These five formulae establish a remarkable *connexion between one spherical pentagon and another* (when constructed according to the foregoing rules), through the medium of *five spherical conics*; of which five conics each touches two sides of one pentagon, and has its foci at two corners of the other. If we suppose, for simplicity, that each of the ten moduli is = 1, the dependence of six quaternions by multiplication on four (as their three binary, two ternary, and one quaternary product, all taken without altering the order of succession of the factors) will give eighteen distinct equations between the ten amplitudes and the twenty polar coordinates of the ten quaternions here considered; it is therefore in general permitted to assume at pleasure twelve of these coordinates, or to choose six of the ten points upon the sphere. Not only, therefore, may we in general take *one of the two pentagons arbitrarily*, but also, at the same time, may *assume one corner of the other pentagon* (subject, of course, to exceptional cases); and, after a suitable choice of the ten amplitudes and four other corners, the five relations (440), between the two pentagons and the five conics, will still hold good.

A very particular (or rather limiting) yet not inelegant case of this theorem is furnished by the consideration of the plane and regular pentagon of elementary geometry, as compared with that other and interior pentagon which is determined by the intersections of its five diagonals. Denoting by $R_{,}$ that corner of the interior pentagon which is nearest to the side RR′ of the exterior one; by $R'_{,}$, that corner which is nearest to R′R″, and so on to $R^{IV}_{,}$; the relations (440) are satisfied, the symbol (...) now denoting that the two points written before it are foci of an ordinary (or plane) ellipse, inscribed in the plane quadrilateral, whose corners are the four points written after it. We may add, that (in this particular case) two points of contact for each of the five quadrilaterals are corners of the interior pentagon; and that the axis major of each of the five inscribed ellipses is equal to a side of the exterior figure.

49. By combining the principles of the forty-seventh with the calculations of the twenty-eighth and thirtieth articles, we see that, with the relations (258), (259), (284), from which the relations (285) have been already seen to follow, we may regard $m_1^{\backprime}, m_2^{\backprime}, m_3^{\backprime}$ as the rectangular coordinates of a point on one focal line, and $m_1^{\backprime\backprime}, m_2^{\backprime\backprime}, m_3^{\backprime\backprime}$ as the rectangular coordinates of a point on the other focal line of a certain cone of the second degree, having its vertex at the origin of those coordinates, and having, on the successive intersections of four of its tangent planes, four points, of which the coordinates are respectively m_1, m_2, m_3; b, c, d; m_1', m_2', m_3'; and m_1'', m_2'', m_3''. Hence, with the same relations between the symbols, the known theory of reciprocal or supplementary cones enables us to infer that the two equations

$$\left.\begin{aligned} xm_1^{\backprime} + ym_2^{\backprime} + zm_3^{\backprime} = 0, \\ xm_1^{\backprime\backprime} + ym_2^{\backprime\backprime} + zm_3^{\backprime\backprime} = 0, \end{aligned}\right\} \tag{441}$$

represent the two cyclic planes of a certain other cone of the second degree, which has its vertex at the origin, and contains upon its surface the four points which are determined by the twelve following rectangular coordinates:

$$\left.\begin{array}{lll} m_2 d - m_3 c, & m_3 b - m_1 d, & m_1 c - m_2 b; \\ cm_3' - dm_2', & dm_1' - bm_3', & bm_2' - cm_1'; \\ m_2' m_3'' - m_3' m_2'', & m_3' m_1'' - m_1' m_3'', & m_1' m_2'' - m_2' m_1''; \\ m_2'' m_3 - m_3'' m_2, & m_3'' m_1 - m_1'' m_3, & m_1'' m_2 - m_2'' m_1. \end{array}\right\} \tag{442}$$

It would have been easy to have given a little more symmetry to these last expressions, if we had not wished to present them in a form in which they might be easily combined with some that had been already investigated, for a different purpose, in this paper.

50. If we denote by the symbol $i_{RR'}$ that vector unit which is directed towards the positive pole of the arc RR′ (*from* the point R *to* the point R′ on the unit sphere), then the general formula (425) for the *product of any two vector units*, i_R and $i_{R'}$, becomes

$$i_R i_{R'} = (\cos + i_{RR'} \sin)(\pi - \widehat{RR'}); \tag{443}$$

and because the positive pole of the arc RR′ is the negative pole of the reversed arc R′R, so that in this reversal the change of sign may be conceived to fall upon the vector unit,

$$i_{R'R} = -i_{RR'}, \tag{444}$$

while the arc itself may thus be regarded as not having changed its sign, but only its pole, we may also write, generally, in this notation, for the *quotient of any two vector units*, the expression

$$i_R i_{R'}^{-1} = -i_R i_{R'} = (\cos + i_{R'R} \sin) . \widehat{R'R}. \tag{445}$$

Hence the associative principle of multiplication gives this other property of any spherical polygon, RR′R″..., which may be regarded as a sort of *polar conjugate* to the property (426), as depending on the consideration of the polar polygon, or *polygon of poles*, namely, the following:

$$(\cos + i_{R'R} \sin) \widehat{R'R} . (\cos + i_{R''R'} \sin) \widehat{R''R'} \ldots (\cos + i_{RR^{(n-1)}} \sin) \widehat{RR^{(n-1)}} = 1. \tag{446}$$

Thus, in particular, for any spherical triangle, of which the three sides may be briefly denoted thus,

$$\widehat{R'R} = \theta''; \quad \widehat{R''R'} = \theta; \quad \widehat{RR''} = \theta'; \tag{447}$$

while the three corresponding vector units, directed to the positive poles of these three arcs, may be thus denoted,

$$\dot{i}_{R'R} = \iota''; \quad \dot{i}_{R''R'} = \iota; \quad \dot{i}_{RR''} = \iota'; \tag{448}$$

the following equation holds good, and may be employed, instead of (427), as a formula for spherical trigonometry:

$$(\cos \theta'' + \iota'' \sin \theta'')(\cos \theta + \iota \sin \theta)(\cos \theta' + \iota' \sin \theta') = 1. \tag{449}$$

Hence also may be derived this other and not less general equation, analogous to (431), and serving in a new way to express the result of the multiplication of any two numeral quaternions, in connexion with a spherical triangle:

$$\mu(\cos \theta + \iota \sin \theta) \times \mu'(\cos \theta' + \iota' \sin \theta') = \mu\mu'(\cos \theta'' - \iota'' \sin \theta''). \tag{450}$$

The sides of the triangle here considered are θ, θ', θ'', that is, they are the amplitudes of the two factors and of the product; and the angles respectively opposite to those three sides are the supplements of the mutual inclinations of the three pairs of vector units, ι', ι''; ι'', ι; ι, ι'; they are therefore, respectively, the inclinations of the two vector units ι' and ι to $-\iota''$, and the supplement of their inclination to each other. But, in the multiplication (450), ι, ι', and $-\iota''$ are respectively the vector units of the multiplier, the multiplicand, and the product; if then we agree to speak of the mutual inclination of the vector units of any two quaternions as being also the mutual *inclination* of those two quaternions themselves, we may enunciate the following Theorem, with which we shall conclude the account of this First Series of Researches: *If, with the amplitudes of any two quaternion factors, and of their product, as sides, a spherical triangle be constructed, the angle of this triangle, which is opposite to the side which represents the amplitude of either factor, will be equal to the inclination of the remaining factor to the product; and the angle opposite to that other side which represents the amplitude of the product, will be equal to the supplement of the inclination of the same two factors to each other.*

NOTE A

Extract from a Letter of Sir William R. Hamilton to John T. Graves, Esq.

'*Observatory of Trinity College, Dublin, 24th October* 1843.

——'The Germans often put i for $\sqrt{-1}$, and therefore denote an ordinary imaginary quantity by $x+iy$. I assume *three* imaginary characteristics or units, i, j, k, such that *each* shall have its square $=-1$, without any one being the equal or the negative of any other;

$$i^2=j^2=k^2=-1. \tag{1}$$

And I assume (for reasons explained in my first letter) the relations

$$ij=k; \quad jk=i; \quad ki=j; \tag{2}$$

$$ji=-k; \quad kj=-i; \quad ik=-j; \tag{3}$$

each imaginary unit being thus the product of the two which precede it in the cyclical order $i\,j\,k$, but the negative of the product of the two which follow it in that order. Such being my fundamental assumptions, which include (as you perceive) the somewhat strange one that *the order of multiplication of quaternions is not, in general, indifferent*, I have at once the theorem that

$$(w+ix+jy+kz)(w'+ix'+jy'+kz')=w''+ix''+jy''+kz'', \tag{4}$$

if the following relations hold good:

$$w''=ww'-xx'-yy'-zz'; \tag{5}$$

$$\left.\begin{aligned} x''&=wx'+xw'+yz'-zy'; \\ y''&=wy'+yw'+zx'-xz'; \\ z''&=wz'+zw'+xy'-yx'; \end{aligned}\right\} \tag{6}$$

and reciprocally that these four relations (5) and (6) are *necessary* (on account of the mutual independence of the three imaginary units, i, j, k, except so far as they are connected by the conditions above assigned), in order that the quaternion $w''+ix''+jy''+kz''$ may result as a product from the multiplication of $w'+ix'+jy'+kz'$, as a multiplicand, by $w+ix+jy+kz$ as a multiplier.

'Making, for abridgment,

$$x''_{\prime}=wx'+xw'; \quad y''_{\prime}=wy'+yw'; \quad z''_{\prime}=wz'+zw'; \tag{7}$$

$$x''_{\prime\prime}=yz'-zy'; \quad y''_{\prime\prime}=zx'-xz'; \quad z''_{\prime\prime}=xy'-yx'; \tag{8}$$

and observing that $\quad xx''_{\prime\prime}+yy''_{\prime\prime}+zz''_{\prime\prime}=0; \quad x'x''_{\prime\prime}+y'y''_{\prime\prime}+z'z''_{\prime\prime}=0; \qquad (9)$

we see easily that $\quad x''_{\prime}x''_{\prime\prime}+y''_{\prime}y''_{\prime\prime}+z''_{\prime}z''_{\prime\prime}=0; \qquad (10)$

therefore, since $\quad x''=x''_{\prime}+x''_{\prime\prime}, \quad y''=y''_{\prime}+y''_{\prime\prime}, \quad z''=z''_{\prime}+z''_{\prime\prime}, \qquad (11)$

we have $\quad x''^2+y''^2+z''^2=x_{\prime}''^2+y_{\prime}''^2+z_{\prime}''^2+x_{\prime\prime}''^2+y_{\prime\prime}''^2+z_{\prime\prime}''^2. \qquad (12)$

Again, $\quad (xx'+yy'+zz')^2+x_{\prime\prime}''^2+y_{\prime\prime}''^2+z_{\prime\prime}''^2=(x^2+y^2+z^2)(x'^2+y'^2+z'^2), \qquad (13)$

$$-2ww'(xx'+yy'+zz')+x_{\prime}''^2+y_{\prime}''^2+z_{\prime}''^2=w^2(x'^2+y'^2+z'^2)+w'^2(x^2+y^2+z^2); \tag{14}$$

therefore, $\quad w''^2+x''^2+y''^2+z''^2=(w^2+x^2+y^2+z^2)(w'^2+x'^2+y'^2+z'^2). \qquad (15)$

Let

$$\left.\begin{array}{llll} w=\mu\cos\theta; & x=\mu\sin\theta\cos\phi; & y=\mu\sin\theta\sin\phi\cos\psi; & z=\mu\sin\theta\sin\phi\sin\psi; \\ w'=\mu'\cos\theta'; & x'=\mu'\sin\theta'\cos\phi'; & y'=\mu'\sin\theta'\sin\phi'\cos\psi'; & z'=\mu'\sin\theta'\sin\phi'\sin\psi'; \\ w''=\mu''\cos\theta''; & x''=\mu''\sin\theta''\cos\phi''; & y''=\mu''\sin\theta''\sin\phi''\cos\psi''; & z''=\mu''\sin\theta''\sin\phi''\sin\psi''; \end{array}\right\} \tag{16}$$

and let μ, $\sin\theta$, and $\sin\phi$, be treated as positive (or, at least, not negative) quantities; we shall then have

$$\mu''=\mu\mu'; \tag{17}$$

which may be enunciated by saying that *the modulus of the product of two quaternions is the product of the moduli of those two factors.*

'At the same time we shall have

$$r=\mu\sin\theta, \quad \text{if we make} \quad r=\sqrt{(x^2+y^2+z^2)}; \tag{18}$$

and may call this quantity, r, the *modulus of the pure imaginary triplet,* $ix+jy+kz$. We may also call it the *radius* of the imaginary part of the quaternion $w+ix+jy+kz$, or even the radius of the quaternion itself; and may speak of the *inclination* of one such radius to another, the cosine of this inclination being

$$\cos.rr'=\cos\phi\cos\phi'+\sin\phi\sin\phi'\cos(\psi'-\psi). \tag{19}$$

The angle ϕ may be called the *colatitude*, and ψ the *longitude*, of the radius, or triplet, or quaternion. And θ may be called the *amplitude* of the quaternion; so that the real part, multiplied by the tangent of the amplitude, produces the radius of the quaternion, or of its imaginary part,

$$w\tan\theta=r. \tag{20}$$

The amplitude, θ, may be supposed to range only from 0 to π. It vanishes for a pure, real, positive quantity, and becomes $=\frac{\pi}{2}$ for a pure imaginary; it is $=\pi$ for a pure real negative.

'The equation (5), combined with (16) and (17), gives

$$\cos\theta''=\cos\theta\cos\theta'-\sin\theta\sin\theta'\{\cos\phi\cos\phi'+\sin\phi\sin\phi'\cos(\psi'-\psi)\}; \tag{21}$$

if, therefore, we construct a spherical triangle, of which one side is the inclination of the factors, while the two adjacent angles are the amplitudes of those factors, the remaining angle will be the supplement of the amplitude of the product.

'Combining (5) with (6), we find that

$$\left.\begin{array}{r} ww''+xx''+yy''+zz''=(w^2+x^2+y^2+z^2)w'; \\ w'w''+x'x''+y'y''+z'z''=(w'^2+x'^2+y'^2+z'^2)w; \end{array}\right\} \tag{22}$$

therefore, by (16) and (17),

$$\left.\begin{array}{r} \cos\theta'=\cos\theta''\cos\theta+\sin\theta''\sin\theta\{\cos\phi''\cos\phi+\sin\phi''\sin\phi\cos(\psi-\psi'')\}; \\ \cos\theta=\cos\theta''\cos\theta'+\sin\theta''\sin\theta'\{\cos\phi''\cos\phi'+\sin\phi''\sin\phi'\cos(\psi'-\psi'')\}; \end{array}\right\} \tag{23}$$

so that in the spherical triangle lately mentioned, the two remaining sides are the inclinations of the two factors to their product. This spherical triangle may, therefore, be constructed by merely joining the points R, R′, R″, where the sphere, with radius unity, and with centre at the origin of x, y, z, is met by the directions of the radii, r, r', r'', of the two factors and the product. The spherical coordinates of these three points are ϕ, ψ; ϕ', ψ'; ϕ'', ψ''; the spherical angles at the same points are $\theta, \theta', \pi-\theta''$. In the solid corner, at the origin, made by the three radii, r, r', r'', whatever the lengths of these radii may be, the three dihedral angles are

$$r''rr' = \theta; \quad rr'r'' = \theta'; \quad r'r''r = \pi - \theta''; \tag{24}$$

that is, they are the amplitudes of the factors, and the supplement of the amplitude of the product.

'Though this theorem of the spherical triangle, R, R′, R″, or solid corner, r, r', r'', when combined with the *law of the moduli* ($\mu'' = \mu\mu'$), reproduces four relations between the four *constituents*, w'', x'', y'', z'', of the quaternion product, and the eight constituents of the two quaternion factors, namely, w, x, y, z, and w', x', y', z', that is to say, the two relations (5) and (15), and the two relations (22); yet it leaves still something undetermined, with respect to the direction of the product, which requires to be more closely considered. In fact, we can thus fix not only the modulus, μ'', and the amplitude, θ'', of the product, but also the inclinations of its radius, r'', to the two radii, r and r'; but the construction, so far, fails to determine *on which side of the plane rr′* of the radii of the factors does the radius of the product lie. In other words, when we deduced the relations (15) and (22), we may be considered as having employed rather the equations (9) and (13), which were derived from (8), than the equations (8) themselves; the three quantities, $x''_{\prime\prime}, y''_{\prime\prime}, z''_{\prime\prime}$, might, therefore, all change signs together, without affecting the law of the moduli, or the theorem of the spherical triangle. And the additional condition, which is to decide between the one and the other set of signs of these three quantities, or between the one and the other set of signs in the expressions

$$x'' = x''_{\prime} \pm x''_{\prime\prime}; \quad y'' = y''_{\prime} \pm y''_{\prime\prime}; \quad z'' = z''_{\prime} \pm z''_{\prime\prime}; \tag{25}$$

is easily seen, on reverting to first principles, to be the choice of the cyclical order ijk, rather than ikj, or the choice of the upper rather than the lower signs in the assumptions

$$ij = -ji = \pm k, \quad jk = -kj = \pm i, \quad ki = -ik = \pm j. \tag{26}$$

This gives a clue, which may be thus pursued. Let

$$\left.\begin{aligned} x''_{\prime} &= r''_{\prime} \cos\phi''_{\prime}, \quad y''_{\prime} = r''_{\prime} \sin\phi''_{\prime} \cos\psi''_{\prime}, \quad z''_{\prime} = r''_{\prime} \sin\phi''_{\prime} \sin\psi''_{\prime}; \\ x''_{\prime\prime} &= r''_{\prime\prime} \cos\phi''_{\prime\prime}, \quad y''_{\prime\prime} = r''_{\prime\prime} \sin\phi''_{\prime\prime} \cos\psi''_{\prime\prime}, \quad z''_{\prime\prime} = r''_{\prime\prime} \sin\phi''_{\prime\prime} \sin\psi''_{\prime\prime}; \end{aligned}\right\} \tag{27}$$

then, by (12) and (16), and by the meaning which we have assigned to r'', we have

$$r''^2 = r''^2_{\prime} + r''^2_{\prime\prime}, \quad \mu''^2 = w''^2 + r''^2. \tag{28}$$

'By (9), $r''_{\prime\prime}$ is perpendicular to the plane of rr'; and therefore, by (10), r'' is in that plane, being, in fact, the projection of r'' thereupon. This projection is entirely fixed by the construction already given; and it remains only to determine the direction of the perpendicular, $r''_{\prime\prime}$, as distinguished from the opposite of that direction. And a rule which shall fix the sign of any one of the coordinates, $x''_{\prime\prime}, y''_{\prime\prime}, z''_{\prime\prime}$, will be sufficient for this purpose. It will be sufficient, therefore, to study any one of the equations (8), for instance the first, namely,

$$x''_{\prime\prime} = yz' - zy',$$

and to draw from it such a rule.

'Substituting for y, z, y', z', their values (16), we find

$$x''_{\prime\prime} = \mu\mu' \sin\theta \sin\theta' \sin\phi \sin\phi' \sin(\psi' - \psi); \tag{29}$$

so that (the other factors having been already supposed positive) $x''_{\prime\prime}$ has the same sign as the sine of the excess of the longitude ψ' of r' over the longitude ψ of r. But these longitudes are determined by the rotation of the plane of xr round the positive semiaxis of x, from the position of xy towards the position of xz, or from the positive semiaxis of y towards that of z; which direction of rotation is here to be considered as the positive one. Consequently, $x''_{\prime\prime}$ is positive or negative, according as the least rotation round $+x$, from r to r', is itself positive or negative; in each case, therefore, the rotation round $x''_{\prime\prime}$, and, consequently, round $r''_{\prime\prime}$, or finally round r'', from r to r', is positive. *The rotation round the product line, from the multiplier to the multiplicand, is constantly right-handed or constantly left-handed,* according as the rotation round $+i$ from $+j$ to $+k$ is itself right-handed or left-handed. Hence, also, to express the same rule otherwise, *the rotation round the multiplier, from the multiplicand to the product,* is (in the same sense) *constantly positive.* In short, the cyclical order is multiplier, multiplicand, product; just as, and precisely because, we took the order $i\,j\,k$ for that in which the rotation round any one, from the next to the one after it, should be accounted positive, and chose that ij should be $=k$, not $=-k$. *The law of the moduli, the theorem of the spherical triangle, and the rule of rotation, suffice to determine entirely the product of any two quaternions.*

'In my former letter I gave a theorem equivalent to that which I have here given as the theorem of the spherical triangle, answering, in fact, very nearly to the polar triangle conjugate therewith, but, as I think, much less geometrically simple, because the three corners had no obvious geometrical meanings, whereas now the corners R, R′, R″ mark the directions of the factors and product respectively. In the new triangle, if we let fall a perpendicular from the extremity R″ of that radius of the sphere which coincides in direction with r'', on the arc RR′, which represents the inclination of the factors to each other, and call the foot of this perpendicular $\mathrm{R}''_{\prime}$, we shall have

$$r''_{\prime} = r'' \cos \mathrm{R}''\mathrm{R}''_{\prime}, \quad r''_{\prime\prime} = r'' \sin \mathrm{R}''\mathrm{R}''_{\prime}; \tag{30}$$

also the spherical coordinates of $\mathrm{R}''_{\prime}$ will be $\phi''_{\prime}$, $\psi''_{\prime}$; and $\phi''_{\prime\prime}$, $\psi''_{\prime\prime}$, in (27), will be the spherical coordinates of a point $\mathrm{R}''_{\prime\prime}$ which will be one pole of the arc RR′, and will be distinguished from the other pole by the rule of rotation already assigned; it might, perhaps, be called the *positive pole* of RR′, though it ought then to be considered as the negative pole of R′R.

'We saw that $r''_{\prime}$ was in the plane of r and r', and this is now constructed by $\mathrm{R}''_{\prime}$ being on the great circle RR′.

'There seem to be some advantages in considering the quaternion

$$w + ix''_{\prime} + jy''_{\prime} + kz''_{\prime} \tag{31}$$

as the *reduced product* of the two factors already often mentioned in this letter; it is the *part* of their complete product (4) which is *independent of their order*; and its radius $r''_{\prime}$, is, as we have seen, the *projection* of the radius r'' of the complete product on the plane of the two factors rr'. We now see that

$$\tan\theta \sin rr''_{\prime} = \tan\theta' \sin r'r''_{\prime} = \tan r''r''_{\prime}; \tag{32}$$

the radius $r''_{\prime}$ of the reduced product divides the angle between the radii r, r', of the factors, into parts, of which the sines are inversely as the tangents of the amplitudes, θ, θ'. Indeed this radius, $r''_{\prime}$, is the statical resultant, or *algebraical sum*, of two lines which coincide in direction with r and r' respectively, if w' and w be positive, but have their lengths equal to the products $w'r$ and wr', or $\mu\mu' \sin\theta \cos\theta'$ and $\mu\mu' \sin\theta' \cos\theta$, or $ww' \tan\theta$ and $ww' \tan\theta'$; as appears (among other ways) from the equations (7). For the same reason, or by a combination of the equations (7), (16), (27), we have

$$r''^{2}_{\prime}\mu^{-2}\mu'^{-2} = \cos\theta^2 \sin\theta'^2 + \cos\theta'^2 \sin\theta^2 + 2\sin\theta\cos\theta\sin\theta'\cos\theta'\cos rr'; \tag{33}$$

and because, by (21), $\cos\theta'' = \cos\theta\cos\theta' - \sin\theta\sin\theta'\cos rr'$, (34)

we arrive at the following pretty simple expression for the radius of the reduced product,

$$r''_{,} = \mu\mu' \sqrt{(\cos\theta^2 + \cos\theta'^2 - 2\cos\theta\cos\theta'\cos\theta'')}. \tag{35}$$

But also, by the general analogy of the present notation, if we denote by $\mu''_{,}$ and $\theta''_{,}$ the modulus and amplitude of the same reduced product (31), we shall have

$$\mu''_{,}\cos\theta''_{,} = w'' = \mu\mu'\cos\theta'', \quad \mu''_{,}\sin\theta''_{,} = r''_{,}; \tag{36}$$

therefore, $$\mu''_{,} = \mu\mu'\sqrt{(\cos\theta^2 + \cos\theta'^2 + \cos\theta''^2 - 2\cos\theta\cos\theta'\cos\theta'')}; \tag{37}$$

and $$\cos\theta''_{,} = \frac{\cos\theta''}{\sqrt{(\cos\theta^2 + \cos\theta'^2 + \cos\theta''^2 - 2\cos\theta\cos\theta'\cos\theta'')}}. \tag{38}$$

Again, by (17), (28), (34), (36), (37),

$$\left.\begin{aligned} r''_{,,} &= \sqrt{(\mu''^2 - \mu''^2_{,})} = \mu\mu'\sqrt{(1 + 2\cos\theta\cos\theta'\cos\theta'' - \cos\theta^2 - \cos\theta'^2 - \cos\theta''^2)} \\ &= \mu\mu'\sin\theta\sin\theta'\sin rr' \end{aligned}\right\} \tag{39}$$

an expression for the radius of the pure imaginary triplet,

$$ix''_{,,} + jy''_{,,} + kz''_{,,}, \tag{40}$$

that is, of the complete product (4) minus the reduced product (31), which agrees with the second equation (30), because, by spherical trigonometry,

$$\sin\theta\sin\theta'\sin rr' = \sin\theta''\sin r''r''_{,}; \tag{41}$$

and which gives $$\mu''_{,} = \mu\mu'\sqrt{(1 - (\sin\theta\sin\theta'\sin rr')^2)}. \tag{42}$$

We might call the triplet (40), (which remains when we subtract the reduced product from the complete product), the *residual triplet*, or simply, the *residual*, of the product of the two proposed quaternions (4). And we see that this *residual is always perpendicular to the reduced product*, when it exists at all; for we shall find that it may sometimes vanish. It is the part of the complete product which changes sign when the order of the factors is changed.

'These remarks on the geometrical construction of the *equations of multiplication* (5) and (6) have, perhaps, been tedious; they certainly are nothing more than deductions from those equations, and, consequently, from the fundamental assumptions (1), (2), (3). Yet it may not be altogether useless, in the way of illustration, to draw some corollaries from them, by the consideration of particular cases.

'*Multiplication of two Reals.*—It is evident from the figure that, as [the two internal angles] θ and θ' tend to 0, [the external angle] θ'' tends to 0 likewise; and that the same thing happens with respect to θ'', when θ and θ' both tend to π. Hence the product of two positive or two negative real quantities is a real positive quantity. But when one of the two amplitudes of the factors, θ or θ', tends to 0, and the other to π, then θ'' also tends to π; the product of two reals is, therefore, real and negative, if one of the two factors is positive and the other negative.

'*Multiplication by a Real.*—If θ tend to 0, θ'' tends to become $=\theta'$, and R$''$ tends to coincide with R$'$; also μ tends to become $= w$. If, therefore, a quaternion be multiplied by a positive real quantity, $w=\mu$, the effect is only to multiply its modulus by that quantity, without changing the amplitude or direction. But if θ tend to π, then μ tends to $-w$; R$''$ tends to become diametrically opposite to R$'$; and θ'' tends to become supplementary to θ'. If a quaternion be multiplied by a real negative, $w = -\mu$, the effect is to multiply the modulus, μ', by the real positive, $-w=\mu$; to change the amplitude θ' to $\pi-\theta'$; the colatitude, ϕ', to $\pi-\phi'$; and the longitude, ψ', to $\pi+\psi'$. Accordingly, by inspection of the second line of the expressions marked (16), we see that these changes are equivalent to multiplying each of the four constituents, w', x', y', z', of the proposed quaternion, by $-\mu$. In each of these two cases of multiplication by a real, the *residual* triplet disappears by (39), because $\sin\theta$ vanishes.

'*Multiplication of a Real by a Quaternion.*—We have only to suppose that θ' tends to 0 or to π. The residual vanishes, and the order of multiplication is indifferent.

'*Multiplication of two pure Imaginaries.*—Here $\theta=\theta'=\frac{\pi}{2}$, $\mu=r$, $\mu'=r'$; R'' coincides with $\mathrm{R}''_{\prime\prime}$, that is, with the positive pole of RR'; the direction of the product is perpendicular to the plane of the factors; and the amplitude of the product is the supplement of the inclination of those two factors to each other. Introducing the consideration of the reduced product and residual, since $\mathrm{R}''\mathrm{R}''_{\prime}=\frac{\pi}{2}$, we have, by (30), $r''_{\prime}=0$, $r''_{\prime\prime}=r''$; the reduced product is a pure real, namely, the real part of the complete product; and the residual is equal to the imaginary part. The amplitude of the reduced product is $=\pi$, or $=0$, according as the inclination of the factors is less or greater than $\frac{\pi}{2}$; such, then, is the condition which decides whether the real part of the product of two pure imaginaries, taken in either order, shall be negative or positive. The real part itself $=\mu\mu'\cos\theta''=-rr'\cos rr'=$ the product of the radii of the factors multiplied by the cosine of the supplement of their mutual inclination. The radius of the residual $=rr'\sin rr'=$ the product of the same radii of the factors multiplied by the sine of their inclination to each other. The product is a pure imaginary, if the factors be mutually rectangular; but a pure real negative, if the factors coincide in direction; and a pure real positive, if their directions be exactly opposite.

'*Squaring of a Quaternion.*—As R' tends to coincide with R, and θ' to become equal to θ, R'' tends to coincide likewise with R, and θ'' to become double of θ, at least if θ be less than $\frac{\pi}{2}$. But if θ be greater than $\frac{\pi}{2}$, then R'' tends to coincide with the point diametrically opposite to R, and θ'' tends to become equal to the double of the supplement of θ. If $\theta=\frac{\pi}{2}$, then R'' tends to become distant by $\frac{\pi}{2}$ from R, but in an indeterminate direction, which is, however, unimportant, because θ'' tends to become $=\pi$, and the square (of a pure imaginary triplet) is thus found to be *a pure real negative*; which agrees with the recent result respecting the product of two pure imaginaries, coincident in direction with each other. In general, *the square of a quaternion may be obtained by squaring the modulus and doubling the amplitude*; that is, the square of

$$\mu\cos\theta+\mu\sin\theta(i\cos\phi+j\sin\phi\cos\psi+k\sin\phi\sin\psi), \tag{43}$$

may *always* be thus expressed:

$$\mu^2\cos 2\theta+\mu^2\sin 2\theta(i\cos\phi+j\sin\phi\cos\psi+k\sin\phi\sin\psi); \tag{44}$$

for instance, $$(i\cos\phi+j\sin\phi\cos\psi+k\sin\phi\sin\psi)^2=-1; \tag{45}$$

although, when $\theta>\frac{\pi}{2}$, $\theta<\pi$, it is supposed, in the *construction*, that we treat $\cos 2\theta$ as $=\cos(2\pi-2\theta)$; $\sin 2\theta\cos\phi$ as $=\sin(2\pi-2\theta)\cos(\pi-\phi)$; $\sin 2\theta\sin\phi\cos\psi$ as $=\sin(2\pi-2\theta)$

$\times \sin(\pi-\phi)\cos(\pi+\psi)$; and $\sin 2\theta \sin\phi \sin\psi$ as $=\sin(2\pi-2\theta)\sin(\pi-\phi)\sin(\pi+\psi)$; all which is evidently allowed.

'*Cubing a Quaternion.*—The cube may always be found by cubing the modulus, and tripling the amplitude.

'*Raising to any whole Power.*—The n^{th} power of the quaternion (43) is the following, if n be a positive whole number:

$$\mu^n \cos n\theta + \mu^n \sin n\theta (i\cos\phi + j\sin\phi\cos\psi + k\sin\phi\sin\psi). \tag{46}$$

'*Extracting a Root.*—The n^{th} root has, in general, n, and only n, values, included under the form

$$\mu^{\frac{1}{n}} \cos\frac{\theta+2p\pi}{n} + \mu^{\frac{1}{n}} \sin\frac{\theta+2p\pi}{n}(i\cos\phi + j\sin\phi\cos\psi + k\sin\phi\sin\psi). \tag{47}$$

'*Roots of Reals.*—If $\theta=0$, so that we have to extract the n^{th} root of a positive real quantity, w, considered as the quaternion

$$w+i0+j0+k0=w, \tag{48}$$

ϕ and ψ remain entirely undetermined, in the formula

$$(\mu+i0+j0+k0)^{\frac{1}{n}} = \mu^{\frac{1}{n}}\cos\frac{2p\pi}{n} + \mu^{\frac{1}{n}}\sin\frac{2p\pi}{n}(i\cos\phi + j\sin\phi\cos\psi + k\sin\phi\sin\psi). \tag{49}$$

For example, unity, considered as $1+i0+j0+k0$, has not only itself as a cube root, but also every possible quaternion which has its modulus $=1$, and its amplitude $=\frac{2\pi}{3}$. (The amplitude $\frac{4\pi}{3}$ corresponds merely to quaternions with directions opposite to those with the amplitude $\frac{2\pi}{3}$, and direction is here indifferent.) But unity has only two square roots, $\pm 1+i0+j0+k0$.

'If $\theta=\pi$, so that we have to extract the n^{th} root of the quaternion (48), when $w=-\mu$, we have still ϕ and ψ left undetermined, but the formula is now

$$(-\mu+i0+j0+k0)^{\frac{1}{n}} = \mu^{\frac{1}{n}}\cos\frac{(2p+1)\pi}{n} + \mu^{\frac{1}{n}}\sin\frac{(2p+1)\pi}{n}(i\cos\phi + j\sin\phi\cos\phi + k\sin\phi\sin\psi). \tag{50}$$

For example, the square root of -1 may have any arbitrary direction, provided that it is a pure imaginary with modulus $=1$;

$$(-1+i0+j0+k0)^{\frac{1}{2}} = i\cos\phi + j\sin\phi\cos\psi + k\sin\phi\sin\psi. \tag{51}$$

'*Exponent any positive quantity.*—The power is

$$\mu^{\frac{m}{n}}\cos\left(\frac{m}{n}\overline{\theta+2p\pi}\right) + \mu^{\frac{m}{n}}\sin\left(\frac{m}{n}\overline{\theta+2p\pi}\right)(i\cos\phi + j\sin\phi\cos\psi + k\sin\phi\sin\psi), \tag{52}$$

if $\frac{m}{n}$ be any positive fraction; and it is natural to define that the power with incommensurable exponent

$$\{\mu\cos\theta + \mu\sin\theta(i\cos\phi + j\sin\phi\cos\psi + k\sin\phi\sin\psi)\}^{\nu} \tag{53}$$

is the limit of the power with exponent $\frac{m}{n}$, if ν be limit of $\frac{m}{n}$; hence, generally, the power (53) is

$$\mu^{\nu}\cos(\nu\theta+2\nu p\pi) + \mu^{\nu}\sin(\nu\theta+2\nu p\pi)(i\cos\phi + j\sin\phi\cos\psi + k\sin\phi\sin\psi); \tag{54}$$

at least, if ν be positive. The reason for this last restriction is, that we have not yet considered *division*, at least in the present letter, which I am aiming to make complete in itself, so far as it goes.

'*Multiplication of codirectional Quaternions.*—If, in fig. 1, we conceive R′ to approach to R, then, in general, R″ will approach either to R or to the point diametrically opposite; and, in the first case, θ'' will tend to become the sum of θ and θ'; but, in the second case, the sum of their supplements. In each case we may treat θ'' as $=\theta+\theta'$, if we treat R″ as coinciding with R, or ϕ'' and ψ'' as equal to ϕ and ψ. Thus, generally,

$$\begin{aligned}&\{\mu\cos\theta+\mu\sin\theta(i\cos\phi+j\sin\phi\cos\psi+k\sin\phi\sin\psi)\}\\&\quad\times\{\mu'\cos\theta'+\mu'\sin\theta'(i\cos\phi+j\sin\phi\cos\psi+k\sin\phi\sin\psi)\}\\&\quad=\mu\mu'\cos(\theta+\theta')+\mu\mu'\sin(\theta+\theta')(i\cos\phi+j\sin\phi\cos\psi+k\sin\phi\sin\psi);\end{aligned}\tag{55}$$

which accordingly agrees with the equations of multiplication (5) and (6), whatever μ, μ', θ, θ', ϕ, and ψ may be. (Indeed, if $\theta'+\theta=\pi$, the position of R″ is undetermined; but this is indifferent, because its amplitude is now $=\pi$, and the product is a pure real negative.) For example, by making $\phi=0$, we fall back on the old and well-known theorem of ordinary imaginaries, that

$$(\mu\cos\theta+i\mu\sin\theta)(\mu'\cos\theta'+i\mu'\sin\theta')=\mu\mu'\cos(\theta+\theta')+i\mu\mu'\sin(\theta+\theta').\tag{56}$$

'*Division* [*Submultiplication*].—By (55),

$$\begin{aligned}&\{\mu\cos\theta+\mu\sin\theta(i\cos\phi+j\sin\phi\cos\psi+k\sin\phi\sin\psi)\}\\&\quad\times\{\mu^{-1}\cos\theta-\mu^{-1}\sin\theta(i\cos\phi+j\sin\phi\cos\psi+k\sin\phi\sin\psi)\}=1.\end{aligned}\tag{57}$$

'The *reciprocal* of a quaternion may be found by changing the modulus to its reciprocal, and then either changing the amplitude to its negative, or else the direction to its opposite; this latter change (of direction, rather than of amplitude), agreeing better than the former with the construction in fig. 1. Accordingly, in that figure or in this, in which R represents the direction of multiplier, and may be called the multiplier-point, R′ multiplicand point, and R″ product point, if we prolong RR′ and RR″ till they meet in R‵, the point diametrically opposite to R; then, in the triangle R‵ R″ R′, the point R′, with amplitude θ', will be equal to the product of R‵ as multiplier, with amplitude θ, and R″ as multiplicand, with amplitude θ'', by the theorems already established. *We may, therefore, return from product to multiplicand, by multiplying by reciprocal of multiplier.* But it is natural to call this return *division* [*submultiplication*]. To *divide* [or rather to *submultiply*] is, therefore, to multiply by the reciprocal of the proposed divisor, if this reciprocal be determined by the rule assigned above. These definitions and theorems respecting division of quaternions lead us to put the equation (4) under the form

$$w'+ix'+jy'+kz'=\ldots=\frac{w-ix-jy-kz}{w^2+x^2+y^2+z^2}(w''+ix''+jy''+kz'');\tag{58}$$

and so conduct us not only to the relation $w'=(w^2+x^2+y^2+z^2)^{-1}(ww''+xx''+yy''+zz'')$, which we had already, but also to these others, which can likewise be deduced easily from the equations of multiplication, (5) and (6),

$$\left.\begin{aligned}x'&=(w^2+x^2+y^2+z^2)^{-1}(wx''-xw''+zy''-yz'');\\y'&=(w^2+x^2+y^2+z^2)^{-1}(wy''-yw''+xz''-zx'');\\z'&=(w^2+x^2+y^2+z^2)^{-1}(wz''-zw''+yx''-xy'').\end{aligned}\right\}\tag{59}$$

The modulus of the quotient is the quotient of the moduli.

$$\left.\begin{aligned}&\frac{\mu''\cos\theta''+\mu''\sin\theta''(i\cos\phi+j\sin\phi\cos\psi+k\sin\phi\sin\psi)}{\mu\cos\theta+\mu\sin\theta(i\cos\phi+j\sin\phi\cos\psi+k\sin\phi\sin\psi)}\\&\quad=\frac{\mu''}{\mu}\cos(\theta''-\theta)+\frac{\mu''}{\mu}\sin(\theta''-\theta)(i\cos\phi+j\sin\phi\cos\psi+k\sin\phi\sin\psi).\end{aligned}\right\}\tag{60}$$

'Codirectional quaternions may be divided by each other, by division of moduli and subtraction of amplitudes; and diametrically opposite quaternions may be treated as codirectional, by changing an amplitude to its negative. A quaternion divided by itself gives unity, under the form $1 + i0 + j0 + k0$.

'*Raising to any real Power.*—The transformation (54) of the ν^{th} power of a quaternion is now seen to hold good, if the exponent ν be any real quantity.

'*Napierian Exponential.*—If $$f(t) = 1 + \frac{t}{1} + \frac{t^2}{1.2} + \&c., \tag{61}$$

then, r being $= \sqrt{(x^2+y^2+z^2)}$, &c.,

$$f(ix+jy+kz) = \cos r + \sin r(i \cos \phi + j \sin \phi \cos \psi + k \sin \phi \sin \psi); \tag{62}$$

the modulus of the function f of a pure imaginary is unity.'

The foregoing is an extract from a letter, hitherto unpublished, which was addressed by the author to his friend, Mr Graves, at the time specified in the date. Two figures have been suppressed, as it was thought that the reader would find no difficulty in constructing them from the indications given. A fractional symbol in the formula (58) has also been suppressed, as not entirely harmonizing, under the circumstances in which it occurs, with a notation subsequently adopted. And the reader is reminded by the words 'submultiplication' and 'submultiply,' inserted within square brackets, that these words have since come to be preferred by the author to the words 'division' and 'divide,' when it is required to mark the return from the product to the multiplicand, in cases when the order of the factors is not indifferent to the result: *division* being (in the text of the present paper) defined to be, in such cases, the return from the product to the multiplier. With these slight changes, it may be interesting to some readers to see how nearly the author's present system, although it has been, since the date of the foregoing letter, in some respects, simplified and extended, besides being applied to a great variety of questions in geometry and physics, agrees with the formulae and constructions for quaternions, which were employed by the writer in October 1843; and were in that month exhibited by this letter to a scientific correspondent, and also soon afterwards to a brother of that gentleman, the Rev. Charles Graves, before the Meeting of the Academy at which the first public communication* on the subject was made, and of which the date (13 November 1843) is prefixed to the present series. As that public communication of November 1843 was in great part oral, and as a considerable interval has since elapsed, the author thinks it may be not irrelevant to mention expressly here that not only were the fundamental formulae (1) (2) (3) of the foregoing letter exhibited to the Academy at the date so prefixed, and a general sketch given of their relation to spherical trigonometry, but also the theorems respecting the connexion established through quaternions between certain spherical quadrilaterals, pentagons, and conics, which form the subject of the forty-seventh and forty-eighth articles of this paper, were then communicated, and illustrated by diagrams. Those theorems have since been printed in the Number† of the 'London, Edinburgh, and Dublin Philosophical Magazine' for March 1845. The fundamental equations between i, j, k received their first *printed* publication in the Number‡ of that Magazine for July 1844; and other articles on Quaternions, by the present writer, which will probably be continued, have appeared in the Numbers of that Magazine for October 1844; July, August, and October 1846; and in that for the present month, June 1847, in which these last sheets of the present paper are now passing

* [See V.] † [See VIII, articles 12–17.] ‡ [See VIII, article 2.]

through the printers' hands.* The articles on Symbolical Geometry, in the 'Cambridge and Dublin Mathematical Journal,' are also designed to have a certain degree of connexion with this subject.†

The 'first letter'‡ to Mr Graves, referred to in the one here printed, was written on 17 October 1843, and has been printed in the Supplementary Number of the same Philosophical Magazine for December 1844. It contained a sketch of the process by which the writer had succeeded in combining, through Quaternions, his old conception of *sets of numbers*, derived from the conception of *sets of moments of time*, with the notion of *tridimensional space*. The former conception had been familiar to him since the year 1834, about the end of which year, and the beginning of the following one, he tried a variety of triplet systems, and obtained several geometrical constructions, but was not sufficiently satisfied with any of them to give them publicity; attaching, perhaps, too much weight to the objection or difficulty, that in every such system of *pure triplets*, the product was found to be liable to vanish, while the factors were still different from zero. It should be here observed that the *triplets* described in the author's two letters of October 1843, are really *imperfect quaternions*; they are, therefore, strictly speaking, *not proper triplets*, such as he had once sought for (and in some degree found); and they cannot be regarded as having at all anticipated the independent discoveries since made by Professor de Morgan, nor those made subsequently by John T. Graves, Esq. and the Rev. Charles Graves, in 1844, respecting certain remarkable systems of *such pure and proper Triplets, with products of a triplet form*, connected with imaginary cube roots of negative or positive unity.

The writer hopes that a very interesting theory of *octaves*, including an extension of Euler's theorem respecting products of sums of squares from four to eight, which was communicated to him as an extension of his quaternions, about the end of 1843 and beginning of 1844, in letters from his friend, Mr John Graves, will yet be published by that gentleman, who has also contributed to the 'Philosophical Magazine' for April 1845, a remarkable paper on Couples.§ Some valuable papers on Couples, Quaternions, and Octaves, have also been communicated to the same magazine, since the commencement of 1845, by Arthur Cayley, Esq., especially an application of quaternions (which appeared in the February of that year) to the representation of the rotation of a solid body.|| That important application of the author's principles had indeed occurred to himself previously; but he was happy to see it handled by one so well versed as Mr Cayley is in the theory of such rotation, and possessing such entire command of the resources of algebra and of geometry. Any further remarks which the writer has to offer on the nature and history of this whole train of inquiry, must be reserved to accompany the account of his Second Series of Researches respecting Quaternions.¶

* [See VIII, up to article 36.]

† [Vols. I (1846) to IV (1849); to appear in volume IV of *Mathematical Papers*.]

‡ [See IV.]

§ [*Phil. Mag.* vol. XXVI (1845), pp. 315–20. See also Appendix 3.]

|| [*Phil. Mag.* vols. XXVI (1845), pp. 141–5; XXVII (1845), pp. 38–40. See also A. Cayley, *Coll. Math. Papers*, I, pp. 28–35, 234–52.]

¶ [See also Appendix 3.]