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BRIEF NOTES

Some finite deformations of transversely isotropic elastic materials

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SOME classes of deformations of transversely isotropic elastic solids are obtained for which the principal axes of the strain are the same as those of the stress. The resulting stress field consists of an uniaxial tension superimposed upon an ambient pressure.

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THE CONSTITUTIVE equation of elastic materials as given in [1] is

$$\mathbf{T} = \mathbf{g}(\mathbf{F}). \tag{1.1}$$

Here, **T** denotes the stress tensor, **F** the deformation gradient at the present time, which is taken relative by to a fixed, arbitrary, local reference configuration, and **g** the response function of the material.

It is known that in an isotropic elastic medium any principal axis of strain in the deformed state is also a principal axis of the stress, but generally that is not true. In the present note, we obtain certain classes of deformations of transversely isotropic elastic solids which satisfy this condition. The tool used is the symmetry principle. We do not appeal to any representation of the constitutive equation.

We recall that an orthogonal tensor $\mathbf{Q}(\mathbf{Q}\mathbf{Q}^T = \mathbf{1})$ belongs to the isotropy group of an elastic material if, and only if,

$$\mathbf{Q}\mathbf{g}(\mathbf{F})\mathbf{Q}^T = \mathbf{g}(\mathbf{Q}\mathbf{F}\mathbf{Q}^T), \tag{1.2}$$

for all invertible tensors **F** [1]. If the local reference configuration is an undistorted state ([1], Sec. 33) and the material considered is a transversely isotropic elastic solid, then the relation (1.2) must hold for all orthogonal tensors of the form:

$$[Q^\alpha_\beta] = \begin{bmatrix} \pm\sin\mu & \pm\cos\mu & 0 \\ \mp\cos\mu & \pm\sin\mu & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}; \quad \alpha, \beta = 1, 2, 3, \mu \in [0, 2\pi]. \tag{1.3}$$

Now, we choose only such deformation gradients as satisfy

$$\mathbf{QFQ}^T = \mathbf{F} \tag{1.4}$$

for all orthogonal tensors of the form (1.3). From (1.2) and (1.4), we infer that

$$\mathbf{Q}\mathbf{g}(\mathbf{F})\mathbf{Q}^T = \mathbf{g}(\mathbf{F}), \tag{1.5}$$

for all orthogonal tensors given by (1.3) and for all deformation gradients which satisfy (1.4). Hence the stress tensor appropriate to this class of deformations is subjected to the conditions

$$\mathbf{QTQ}^T = \mathbf{T} \tag{1.6}$$

for all orthogonal tensors of the form (1.3).

The conditions (1.4) and (1.6) involve, respectively:

$$[F^k_\alpha] = \begin{bmatrix} F^1_1 & F^1_2 & 0 \\ -F^1_2 & F^1_1 & 0 \\ 0 & 0 & F^3_3 \end{bmatrix}, \quad k, \alpha = 1, 2, 3, \tag{1.7}$$

and

$$[\mathbf{T}^{ij}] = \begin{bmatrix} T^{11} & 0 & 0 \\ 0 & T^{11} & 0 \\ 0 & 0 & T^{33} \end{bmatrix}, \tag{1.8}$$

— i.e., deformations whose corresponding deformation gradients have the components given by (1.7) are maintained under uniaxial tension superimposed upon ambient pressure.

R e m a r k 1. It is easily seen that, in this case,

$$\mathbf{TC} = \mathbf{CT}, \tag{1.9}$$

where **C** is the right Cauchy-Green tensor

$$\mathbf{C} = \mathbf{F}^T\mathbf{F}. \tag{1.10}$$

The relation (1.9) indicates that the principal axes of **T** are the same as those of **C**.

R e m a r k 2. In [2], it is proved that the stress on an undistorted state of a transversely isotropic elastic solid must be of the form (1.8). Our result shows that a stress field of the form (1.8) does not characterize the undistorted states of these materials (see also [1], Sec. 50).

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In this section, we deal with some classes of deformations which satisfy the requirement (1.4). We consider only the case of equilibrium subjected to no body force.

Family 1. We use Cartesian coordinates (X, Y, Z) for reference configurations, (x,y,z) for the loaded configuration, and consider the deformation

$$x = f(X, Y), \quad y = g(X, Y), \quad z = h(Z), \tag{2.1}$$

where f, g, h are arbitrary differentiable functions. Taking into account (1.7), we obtain the conditions:

(2.2)
$$\frac{\partial f}{\partial X} = \frac{\partial g}{\partial Y}, \quad \frac{\partial f}{\partial Y} = -\frac{\partial g}{\partial X},$$

which are satisfied if, and only if,

(2.3)
$$f = \frac{\partial \chi}{\partial Y}, \quad g = \frac{\partial \chi}{\partial X}, \quad \Delta\chi = 0.$$

The components of the deformation gradient are given by

(2.4)
$$[F^k{}_\alpha] = \begin{bmatrix} \dfrac{\partial^2\chi}{\partial X \partial Y} & \dfrac{\partial^2\chi}{\partial X^2} & 0 \\ \dfrac{\partial^2\chi}{\partial Y^2} & \dfrac{\partial^2\chi}{\partial X \partial Y} & 0 \\ 0 & 0 & \dfrac{dh}{dZ} \end{bmatrix}, \quad \Delta\chi = 0.$$

The equilibrium equations yield:

(2.5)
$$T^{11} = \psi_1(z), \quad T^{33} = \psi_2(x, y),$$

where ψ_1 and ψ_2 are arbitrary functions.

Family 2. We consider the deformation defined by

(2.6)
$$r = Rf(\Theta), \quad \theta = g(R, \Theta), \quad z = h(Z),$$

where f, g, h are arbitrary functions and where (R, Θ, Z), (r, θ, z) denote, respectively, the cylindrical coordinates of a point in the reference and loaded configurations.

The physical components of the deformation gradient are ([1])

(2.7)
$$F_{\langle 11\rangle} = f(\Theta), \quad F_{\langle 22\rangle} = f(\Theta)\frac{\partial g}{\partial \Theta}, \quad F_{\langle 33\rangle} = \frac{dh}{dZ},$$
$$F_{\langle 12\rangle} = \frac{df}{d\Theta}, \quad F_{\langle 21\rangle} = Rf(\Theta)\frac{\partial g}{\partial R},$$
$$F_{\langle 13\rangle} = F_{\langle 23\rangle} = F_{\langle 31\rangle} = F_{\langle 32\rangle} = 0.$$

The relation (1.7) involves:

(2.8)
$$\frac{\partial g}{\partial \Theta} = 1, \quad \frac{df}{d\Theta} = -Rf(\Theta)\frac{\partial g}{\partial R}.$$

From (2.8), we can infer

(2.9)
$$f = A\exp(C\Theta), \quad g = \Theta - C\log R + B,$$

where A, B, C are arbitrary constants.

([1]) In [3], CARROLL has pointed out that, for a transversely isotropic elastic solid, to physical components of the deformation gradient correspond, by (1.1), physical components of tke stress tensor. Throughtout this paper we denote by $A_{\langle ij\rangle}$ $(i, j = 1, 2, 3)$ the physical components of the second order tensor A.

The deformation field (2.6) becomes:

$$r = AR\exp(C\Theta), \quad \theta = \Theta - C\log R + B, \quad z = h(Z). \tag{2.10}$$

The components of the deformation gradient can be written in the form:

$$[F_{\langle k\alpha\rangle}] = \begin{bmatrix} A\exp(C\Theta) & AC\exp(C\Theta) & 0 \\ -AC\exp(C\Theta) & A\exp(C\Theta) & 0 \\ 0 & 0 & \dfrac{dh}{dZ} \end{bmatrix}. \tag{2.11}$$

The equilibrium equations are satisfied if, and only if,

$$T_{\langle rr\rangle} = \psi_1(z), \quad T_{\langle zz\rangle} = \psi_2(\theta), \tag{2.12}$$

where ψ_1 and ψ_2 are arbitrary functions.

Family 3. Let the point with spherical coordinates (R, Θ, Φ) be deformed into one with spherical coordinates (r, θ, φ), where

$$r = Rf(\Theta), \quad \theta = g(R, \Theta), \quad \varphi = h(\Phi). \tag{2.13}$$

Here f, g, h are arbitrary functions.

By using the same method as for the Family 2, we obtain:

$$r = AR\exp(C\Theta), \quad \theta = \Theta - C\log R + B, \quad \varphi = h(\Phi). \tag{2.14}$$

The components $F_{\langle k\alpha\rangle}$ $(k, \alpha = 1, 2)$ are given by (2.11). We also have

$$\begin{gathered} F_{\langle 33\rangle} = A\exp(C\Theta)\frac{dh}{d\Phi}\frac{\sin(\Theta - C\log R + B)}{\sin\Theta}, \\ F_{\langle 13\rangle} = F_{\langle 23\rangle} = F_{\langle 31\rangle} = F_{\langle 32\rangle} = 0. \end{gathered} \tag{2.15}$$

From the equilibrium equations, we obtain the following system to be solved for $T_{\langle rr\rangle}$ and $T_{\langle\varphi\varphi\rangle}$

$$\begin{gathered} r\frac{\partial T_{\langle rr\rangle}}{\partial r} + T_{\langle rr\rangle} - T_{\langle\varphi\varphi\rangle} = 0, \\ \operatorname{tg}\theta\,\frac{\partial T_{\langle rr\rangle}}{\partial\theta} + T_{\langle rr\rangle} - T_{\langle\varphi\varphi\rangle} = 0, \\ \frac{\partial T_{\langle\varphi\varphi\rangle}}{\partial\varphi} = 0. \end{gathered} \tag{2.16}$$

From the first two equations, we have

$$r\frac{\partial T_{\langle rr\rangle}}{\partial r} - \operatorname{tg}\theta\,\frac{\partial T_{\langle rr\rangle}}{\partial\theta} = 0 \tag{2.17}$$

such that

$$T_{\langle rr\rangle} = \chi(r|\sin\theta|, \varphi), \tag{2.18}$$

where χ is an arbitrary function.

From $(2.16)_1$, we obtain:

$$T_{\langle\varphi\varphi\rangle} = \frac{\partial}{\partial r}(rT_{\langle rr\rangle}) = \frac{\partial}{\partial r}[r\chi(r|\sin\theta|, \varphi)]. \tag{2.19}$$

The Eq. $(2.16)_3$ involves:

$$\frac{\partial^2}{\partial\varphi\partial\varkappa}(\varkappa\chi) = 0, \quad \varkappa \equiv r|\sin\theta|. \tag{2.20}$$

The general solution of this equation is given by

$$\varkappa\chi = \psi_1(\varphi) + \psi_2(\varkappa), \tag{2.21}$$

where ψ_1 and ψ_2 are arbitrary functions.

Finally, we obtain:

$$T_{\langle rr\rangle} = \frac{1}{\varkappa}[\psi_1(\varphi) + \psi_2(\varkappa)], \quad T_{\langle\varphi\varphi\rangle} = \frac{d\psi_2}{d\varkappa}. \tag{2.22}$$

Family 4. The deformation which we now consider carries the particle with Cartesian coordinates (X, Y, Z) to the point with cylindrical coordinates (r, θ, z) as follows:

$$r = f(X, Y), \quad \theta = g(X, Y), \quad z = h(Z). \tag{2.23}$$

Here, as in the above, f, g, h are arbitrary functions.

The relation (1.7) requires, in physical components, that

$$\frac{1}{f}\frac{\partial f}{\partial X} = \frac{\partial g}{\partial Y}, \quad \frac{1}{f}\frac{\partial f}{\partial Y} = -\frac{\partial g}{\partial X}. \tag{2.24}$$

This system can be integrated and we obtain:

$$f = \exp\left(\frac{\partial\chi}{\partial Y}\right), \quad g = \frac{\partial\chi}{\partial X}, \quad \Delta\chi = 0. \tag{2.25}$$

The components of the deformation gradient are

$$[F_{\langle k\alpha\rangle}] = \begin{bmatrix} \exp\left(\frac{\partial\chi}{\partial Y}\right)\frac{\partial^2\chi}{\partial X\partial Y} & \exp\left(\frac{\partial\chi}{\partial Y}\right)\frac{\partial^1\chi}{\partial Y^2} & 0 \\ \exp\left(\frac{\partial\chi}{\partial Y}\right)\frac{\partial^2\chi}{\partial X^2} & \exp\left(\frac{\partial\chi}{\partial Y}\right)\frac{\partial^2\chi}{\partial X\partial Y} & 0 \\ 0 & 0 & \frac{dh}{dZ} \end{bmatrix}, \quad \Delta\chi = 0. \tag{2.26}$$

From the equilibrium equations we infer:

$$T_{\langle rr\rangle} = \psi_1(z), \quad T_{\langle zz\rangle} = \psi_2(r, \theta), \tag{2.27}$$

where ψ_1 and ψ_2 are arbitrary functions.

Remark. The resultant force and the resultant moment which act upon the boundary of the body can be obtained from the known formulae, by means of the two material functions ψ_1 and ψ_2, for deformations of every class (see, for example, [1 and 4]).

References

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