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Axisymmetric self-similar dynamic problem for an elastic half-space with mixed moving boundary conditions

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IN THIS PAPER, we present a method for the determination of the resolving function for the general case of the self-similar mixed problem, the essence of which consists in a reduction of the integral relations for the solving function obtained in [2], to the Riemann boundary value problem. As an example, we consider an axisymmetric problem of propagation with a constant velocity of a crack, on the surface of which there acts a stepwise normal loading of constant intensity, propagated with a constant velocity. Moreover, the solution of the same problem is given where there acts at the origin a concentrated force.

W pracy przedstawiono metodę określenia funkcji rozwiązującej dla ogólnego przypadku samopodobnego zagadnienia mieszanego. Istotą tej metody jest redukcja związków całkowych dla funkcji rozwiązującej, otrzymanych w [2], do zagadnienia brzegowego Riemanna. Jako przykład rozważono osiowosymetryczny problem rozprzestrzeniania się pęknięć ze stałą prędkością na powierzchni, na której działa skokowe obciążenie normalne o stałej intensywności. Uzyskano również rozwiązanie podobnego zagadnienia dla przypadku skupionego obciążenia w początku układu.

В настоящей работе дается регулярный прием нахождения разрешающей функции для общего случая автомодельной смешанной задачи, суть которого состоит в сведении интегральных соотношений, полученных в [2] для разрешающей функции, к краевой задаче Римана. В качестве примера рассмотрена осесимметричная задача о распространении с постоянной скоростью трещины, на поверхности которой действует ступенчатая нормальная нагрузка постоянной интенсивности, разбегающаяся с постоянной скоростью. Кроме того дано решение той же задачи при действии в начале координат сосредоточенный силы.

A REPRESENTATION of the solution of the axisymmetric dynamic problem for an elastic half-space with uniform boundary conditions as a superposition of solutions of plane problems was given by V. I. SMIRNOV and S. L. SOBOLEV [1]. B. V. KOSTROV [2, 3] considered the case of mixed boundary conditions and derived a system of integral equations for the determination of the resolving function of the problem.

Solutions of particular problems in papers [2, 3] were obtained by a very elegant method which, however, in the general case, does not enable us to find the resolving function.

In the present paper, we present a regular method of finding this function for a general case of the axisymmetric self-similar mixed problem. The essence of the method is the reduction of the integral equations for the resolving function, obtained in [2], to the Riemann-Hilbert boundary-value problem.

As an example of the method of application, the problem of a penny-shaped crack propagation with constant velocity is considered. On the crack surface there acts a stepwise uniform normal traction propagated with a constant velocity. The solution of the same problem is extended to the case of a concentrated force at the origin of the coordinate system rather than the moving traction.

1. Statement of the problem and basic equations

LET US consider a homogeneous and isotropic elastic half-space with modulus of rigidity μ and velocities of longitudinal and transverse waves a_1 and a_2, respectively, and refer it to the cylindrical system of coordinates r, φ, z with the origin on the surface of the half-space, the axis z being normal to the surface. The elastic medium occupies the half-space $z > 0$. Assume that the surface of the initially stationary half-space is free from shear stresses

$$\tau_{z\varphi}(r, 0, t) = 0, \quad \tau_{rz}(r, 0, t) = 0, \tag{1.1}$$

one part of the boundary surface is subject to a normal stress $\sigma_z(r, 0, t)$, the other part has a normal velocity $v_z(r, 0, t)$, both independent of φ and constituting homogeneous functions of zero degree. The line of change of the type of the boundary conditions moves with a constant velocity α. Then, obviously, the problem is axisymmetric and self-similar in terms of stresses.

The solution of the axisymmetric self-similar problem can be represented as superposition of solutions of plane problems [1].

Besides the cylindrical introduce Cartesian coordinates x, y, z and by $v_{(k)j}(x, z, t)$, $\sigma_{(k)ij}(x, z, t)$, $(i = x, z)$, $(j = x, z)$ denote a solution of a plane problem independent of y. Now, the solution of the axisymmetric problem can be written in the form

$$\begin{aligned} v_{(k)r}(r, z, t) &= \operatorname{Re} \int_{-\pi}^{\pi} V_{(k)x}(\vartheta_k)\cos\varphi d\varphi, & v_{(k)z}(r, z, t) &= \operatorname{Re} \int_{-\pi}^{\pi} V_{(k)z}(\vartheta_k) d\varphi, \\ \tau_{(k)rz}(r, z, t) &= \operatorname{Re} \int_{-\pi}^{\pi} \sum\nolimits_{(k)xz} (\vartheta_k)\cos\varphi d\varphi, & \sigma_{(k)z}(r, z, t) &= \operatorname{Re} \int_{-\pi}^{\pi} \sum\nolimits_{(k)z} (\vartheta_k) d\varphi, \end{aligned} \tag{1.2}$$

where k is equal to 1 or 2 (longitudinal and transverse components respectively), and ϑ_k is found from the formula

$$\vartheta_k = \frac{a_k tr\cos\varphi + iz\sqrt{a_k^2 t^2 - r^2\cos^2\varphi - z^2}}{a_k(r^2\cos^2\varphi + z^2)}, \quad z > 0. \tag{1.3}$$

In (1.2), we used the possibility of representing the solution of a plane problem as a real (or imaginary) part of an arbitrary function of the complex variable $\tilde{\vartheta}_k$ [4], i.e.

$$\begin{aligned} v_{(k)j}(x, z, t) &= \operatorname{Re} V_{(k)j}(\tilde{\vartheta}_k), \\ \sigma_{(k)ij}(x, z, t) &= \operatorname{Re} \sum\nolimits_{(k)ij} (\tilde{\vartheta}_k), \end{aligned} \tag{1.4}$$

while $\tilde{\vartheta}$ are determined by the equations

$$\tilde{\vartheta}_k = \frac{a_k tx + iz\sqrt{a_k^2 t^2 - x^2 - z^2}}{a_k(x^2 + z^2)}, \quad z > 0.$$

Without loss of generality, the functions $V_{(k)j}(\tilde{\vartheta}_k)$, $\sum_{(k)ij}(\tilde{\vartheta}_k)$ can be continued into the region $z < 0$, $(\operatorname{Im}\tilde{\vartheta}_k < 0)$, namely

$$\overline{V_{(k)j}(\tilde{\vartheta}_k)} = V_{(k)j}(\bar{\tilde{\vartheta}}), \quad \overline{\sum\nolimits_{(k)ij}(\tilde{\vartheta}_k)} = \sum\nolimits_{(k)ij}(\bar{\tilde{\vartheta}}_k). \tag{1.5}$$

For the problem under consideration, in view of (1.1), the solution of a plane problem should satisfy the condition

$$\tau_{xz}(x, 0, z) = 0. \tag{1.6}$$

The conditions that the velocity components labeled 1 and 2 be longitudinal and transverse, respectively, the Hooke law and Eq. (1.6) enable us to express the function in the right side of (1.4) in terms of a single unknown function [2]

$$V_z(\tilde{\vartheta}) = V_{1z}(\tilde{\vartheta}) + V_{2z}(\tilde{\vartheta})$$

as follows:

$$\begin{aligned}
&V'_{1x}(\tilde{\vartheta}) = \frac{\tilde{\vartheta}(1-a_2^2\tilde{\vartheta}^2)}{\sqrt{a_1^{-2}-\tilde{\vartheta}^2}} V'_z(\tilde{\vartheta}), \quad V'_{2x}(\tilde{\vartheta}) = -2a_2^2\tilde{\vartheta}\sqrt{a_2^{-2}-\tilde{\vartheta}^2}\, V'_z(\tilde{\vartheta}),\\
&V'_{1z}(\tilde{\vartheta}) = (1-2a_2^2\tilde{\vartheta}^2)V'_z(\tilde{\vartheta}), \quad V'_{2z}(\tilde{\vartheta}) = 2a_2^2\tilde{\vartheta}^2 V'_z(\tilde{\vartheta}),\\
&\sum\nolimits'_{1xz}(\tilde{\vartheta}) = -\sum\nolimits'_{2xz}(\tilde{\vartheta}) = -2\mu\tilde{\vartheta}(1-2a_2^2\tilde{\vartheta}^2)V'_z(\tilde{\vartheta}),\\
&\sum\nolimits'_{1z}(\tilde{\vartheta}) = -\frac{4\mu a_2^2\left(\tilde{\vartheta}^2-\frac{1}{2}a_2^{-2}\right)^2}{\sqrt{a_1^{-2}-\tilde{\vartheta}^2}} V'_z(\tilde{\vartheta}),\\
&\sum\nolimits'_{2z}(\tilde{\vartheta}) = -4\mu a_2^2\tilde{\vartheta}^2\sqrt{a_2^{-2}-\tilde{\vartheta}^2}\, V'_z(\tilde{\vartheta}),
\end{aligned} \tag{1.7}$$

where prime denotes differentiation with respect to the argument.

In (1.2), (1.7), only the functions appearing in the sequel are written out.

On the plane $z = 0$, (1.7) implies the important equation

$$\sum\nolimits'_{z}(\tilde{\vartheta}) = \sum\nolimits'_{1z}(\tilde{\vartheta}) + \sum\nolimits'_{2z}(\tilde{\vartheta}) = -\frac{4\mu a_2^2 R(\tilde{\vartheta}^2)}{\sqrt{a_1^{-2}-\tilde{\vartheta}^2}} V'_z(\tilde{\vartheta}), \tag{1.8}$$

where $\tilde{\vartheta} = \tilde{\vartheta}_1 = \tilde{\vartheta}_2 = t/x$, and $R(\tilde{\vartheta}^2)$ is the Rayleigh function

$$R(\tilde{\vartheta}^2) = \left(\tilde{\vartheta}^2 - \frac{1}{2}a_2^{-2}\right)^2 + \tilde{\vartheta}^2\sqrt{a_1^{-2}-\tilde{\vartheta}^2}\sqrt{a_2^{-2}-\tilde{\vartheta}^2}.$$

To make use of the boundary conditions, it is necessary to have the time derivatives of $v_z(r, z, t)$ and $\sigma_z(r, z, t)$ on plane $z = 0$; they are obtained from (1.2), namely

$$\dot{v}_z(r, 0, t) = \operatorname{Re}\int_{-\pi}^{\pi} V'_z(\vartheta)\frac{d\varphi}{r\cos\varphi}, \quad \dot{\sigma}_z(r, 0, t) = \operatorname{Re}\int_{-\pi}^{\pi}\sum\nolimits'_{z}(\vartheta)\frac{d\varphi}{r\cos\varphi}, \tag{1.9}$$

where dots denote the derivatives in question and $\vartheta = t/r\cos\varphi$.

Since $V_z(\vartheta)$ is an even function, it is convenient to replace ϑ by $\nu = \vartheta^2$ and to introduce the function

$$F(\nu) = V_z(\vartheta). \tag{1.10}$$

Integrating (1.9) with respect to ν instead of φ, we obtain

$$\frac{r}{2}\dot{\sigma}_z(r,0,t) = \mathrm{Re}\int_{l_\nu}\frac{G'(\nu)}{\sqrt{\nu-\nu_0}}\,d\nu, \quad \frac{r}{2}\dot{v}_z(r,0,t) = \mathrm{Re}\int_{l_\nu}\frac{F'(\nu)}{\nu-\nu_0}\,d\nu. \tag{1.11}$$

Here, the following notations are introduced:

$$G(\nu) = \sum\nolimits_z(\vartheta), \quad \nu_0 = \frac{t^2}{r^2}$$

and, according to (1.8),

$$G'(\nu) = -\frac{4\mu a_2^2}{\sqrt{a_1^{-2}-\nu}}R(\nu)F'(\nu). \tag{1.12}$$

Figure 1 shows the path of integration l_ν. The roots $\sqrt{\nu-\nu_0}$, $\sqrt{a_1^{-2}-\nu}$, and $\sqrt{a_2^{-2}-\nu}$ are uniformized by making cuts for each of them on the plane ν along the positive real

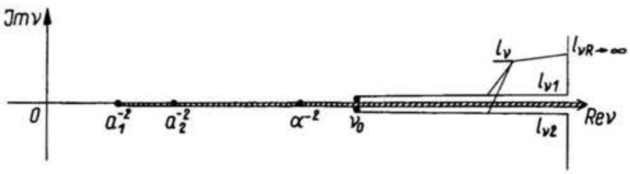

FIG. 1

semi-axis from the points ν_0, a_1^{-2} and a_2^{-2}, respectively, to infinity, and by demanding for $\nu = 0$ that the first root be equal to $i\sqrt{\nu_0}$, while the second and third be positive.

For $v_z(r,0,t)$ and $\sigma_z(r,0,t)$ in view of (1.2), replacing ϑ by ν, we have

$$v_z(r,0,t) = \sqrt{\nu_0}\,\mathrm{Re}\int_{l_\nu}\frac{F(\nu)}{\nu\sqrt{\nu-\nu_0}}\,d\nu, \quad \sigma_z(r,0,t) = \sqrt{\nu_0}\,\mathrm{Re}\int_{l_\nu}\frac{G(\nu)}{\nu\sqrt{\nu-\nu_0}}\,d\nu. \tag{1.13}$$

To satisfy the initial zero conditions, the functions $F(\nu)$ and $G(\nu)$ must be regular for $\mathrm{Re}\,\nu < a_1^{-2}$, and, moreover, the point $\nu = 0$ cannot be a pole. Therefore

$$F(\nu) = \int_0^\nu F'(\nu)\,d\nu, \quad G(\nu) = \int_0^\nu G'(\nu)\,d\nu. \tag{1.14}$$

Integration in (1.14) is to be carried out along the paths running on the same side of the real axis as the point ν.

Eqs. (1.11)–(1.14) obtained in [2] constitute the system of relations for the determination of the resolving function $F'(\nu)$ of the problem. In the next section, a regular method of finding this function for the general case of self-similar mixed boundary problem is described, its essence being the reduction of the integral Eqs. (1.11) to the boundary value Riemann-Hilbert problem for the function $F'(\nu)$.

2. Determination of the resolving function

Assume for certainty that boundary conditions on the plane $z =$ are the following:

$$\sigma_z(r,0,t) = \sigma_z^0(r/t), \quad 0 \leqslant r < \alpha t, \tag{2.1}$$
$$v_z(r,0,t) = v_z^0 r/t, \quad \alpha t < r < \infty,$$

where $\sigma_z^0(r/t)$ and $v_z^0(r/t)$ are known functions. It is convenient to introduce the variable $\nu_0 = t^2/r^2$ and the functions

$$g(\nu_0) = \frac{r}{2}\dot{\sigma}_z^0(r/t), \quad \alpha^{-2} < \nu_0 < \infty, \tag{2.2}$$
$$f(\nu_0) = \frac{r}{2}\dot{v}_z^0(r/t), \quad 0 < \nu_0 < \alpha^{-2}.$$

The first, in view of the boundary conditions (2.1) with the help of (1.11), (1.12), (2.2), may be written in the form

$$g(\nu_0) = -4\mu a_2^2 \operatorname{Re}\left\{ \int\limits_{l_{\nu 1}} \frac{R(\nu)F'(\nu)\,d\nu}{\sqrt{a_1^{-2}-\nu}\,\sqrt{\nu-\nu_0}} + \int\limits_{l_{\nu 2}} \frac{R(\nu)F'(\nu)\,d\nu}{\sqrt{a_1^{-2}-\nu}\,\sqrt{\nu-\nu_0}} + \int\limits_{l_{\nu R\to\infty}} \frac{R(\nu)F'(\nu)\,d\nu}{\sqrt{a_1^{-2}-\nu}\,\sqrt{\nu-\nu_0}} \right\}, \tag{2.3}$$

where the contour l_ν splits into three parts $l_{\nu 1}$, $l_{\nu 2}$ and $l_{\nu R}$, as shown in Fig. 1. The contour $l_{\nu R}$ tends to infinity.

Assume that for $\nu \to \infty$

$$F'(\nu) = o(\nu^{-1}). \tag{2.4}$$

This condition excludes infinite stresses at the origin of the coordinate system which occur in the case of a concentrated force. The latter case will be considered later, in Sec. 4.

Owing to (2.4), the integral along the contour $l_{\nu R}$ in (2.3) vanishes. Denoting by $F'_+(\nu)$ and $F'_-(\nu)$ the upper and lower limits of the function $F'(\nu)$ along the real axis, and taking into account that along the interval under consideration $R_+(\nu) = R_-(\nu) = R(\nu)$, we obtain from (2.3)

$$g(\nu_0) = 4\mu a_2^2 \int\limits_{\nu_0}^{\infty} \frac{R(\nu)[\operatorname{Im}F'_-(\nu) - \operatorname{Im}F'_+(\nu)]}{\sqrt{\nu-a_1^{-2}}\,\sqrt{\nu-\nu_0}}\,d\nu. \tag{2.5}$$

The relation (2.5) may be interpreted as Abel's integral equation. Its solution has the form

$$\frac{4\mu a_2^2 R(\nu)[\operatorname{Im}F'_-(\nu) - \operatorname{Im}F'_+(\nu)]}{\sqrt{\nu-a_1^{-2}}} = -\frac{1}{\pi}\frac{d}{d\nu}\int\limits_{\nu}^{\infty} \frac{g(\nu_0)}{\sqrt{\nu_0-\nu}}\,d\nu_0, \quad \begin{array}{l}\alpha^{-2} < \nu < \infty,\\ \alpha^{-2} < \nu_0 < \infty.\end{array} \tag{2.6}$$

The improper integral in the right side of (2.6) exists, provided the stresses at the origin are finite, i.e. $g(\nu_0) = o(\nu^{-1/2})$.

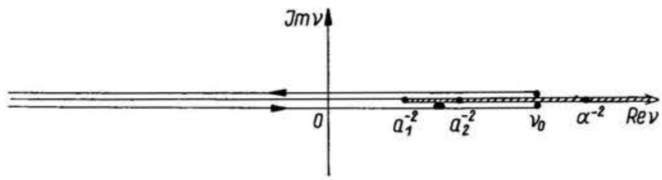

FIG. 2.

Deforming the contour as shown in Fig. 2 and making use of the analyticity of the function $F'(\nu)$, we obtain from the second boundary condition (1.9)

$$(2.7)\qquad f(\nu_0) = \operatorname{Re}\int_{-\infty}^{\nu_0} \frac{F'_-(\nu)\,d\nu}{i\sqrt{\nu_0-\nu}} + \operatorname{Re}\int_{\nu_0}^{-\infty} \frac{F'_+(\nu)\,d\nu}{i\sqrt{\nu_0-\nu}} = \int_{a_1^{-2}}^{\nu_0} \frac{\operatorname{Im}F'_-(\nu)-\operatorname{Im}F'_+(\nu)}{\sqrt{\nu_0-\nu}}\,d\nu .$$

Applying Abel's transformation to (2.7), we have

$$(2.8)\qquad \operatorname{Im}F'_+(\nu)-\operatorname{Im}F'_-(\nu) = -\frac{1}{\pi}\frac{d}{d\nu}\int_{a_1^{-2}}^{\nu} \frac{f(\nu_0)\,d\nu_0}{\sqrt{\nu-\nu_0}}, \qquad \begin{matrix} a_1^{-2} < \nu < \alpha^{-2}, \\ a_1^{-2} < \nu_0 < \alpha^{-2}. \end{matrix}$$

Along the part of the real axis $-\infty < \nu < a_1^{-2}$ the function $F'(\nu)$ is analytic and therefore

$$(2.9)\qquad F'_+(\nu)-F'_-(\nu) = 0, \qquad -\infty < \nu < a_1^{-2}.$$

In view of (1.5) and (1.10), the function $F'(\nu)$ possesses a property

$$(2.10)\qquad \overline{F'(\nu)} = F'(\bar{\nu}).$$

Taking into account (2.10), Eqs. (2.6), (2.8) and (2.9) enable us to formulate the Riemann-Hilbert boundary value problem for the analytical function $F'(\nu)$ as follows:

$$(2.11)\qquad F'_+(\nu)-F'_-(\nu) = \Phi(\nu), \qquad -\infty < \nu < \infty,$$

where the function $\Phi(\nu)$ is determined along the real axis and has the form

$$(2.12)\qquad \Phi(\nu) = \begin{cases} 0, & -\infty < \nu < a_1^{-2}, \\ -\dfrac{i}{\pi}\dfrac{d}{d\nu}\displaystyle\int_{a_1^{-2}}^{\nu} \frac{f(\nu_0)\,d\nu_0}{\sqrt{\nu-\nu_0}}, & a_1^{-2} < \nu < \alpha^{-2}, \\ \dfrac{i\sqrt{\nu-a_1^{-2}}}{4\pi\mu a_2^2 R(\nu)}\dfrac{d}{d\nu}\displaystyle\int_{\nu}^{\infty} \frac{g(\nu_0)}{\sqrt{\nu_0-\nu}}\,d\nu_0, & \alpha^{-2} < \nu < \infty. \end{cases}$$

Equations (2.12) do not describe the behaviour of the function $\Phi(\nu)$ at the points a_1^{-2}, α^{-2} and at infinity. At these points, the function $\Phi(\nu)$ may have singularities of the delta function type which leads to poles in the solution of the Riemann-Hilbert problem (2.11) at the corresponding points. At infinity, according to the condition (2.4), a pole cannot exist. It may be shown that the existence of a pole at the point a_1^{-2} would lead to infinite stresses along the front of the longitudinal wave, which is inadmissible. However, at the point where the type of the boundary conditions changes, a pole should be retained.

Thus a general solution of the problem (2.11) can be written in the form

(2.13)
$$F'(\nu) = \frac{1}{2\pi i}\int\limits_{a_1^{-2}}^{\infty}\frac{\Phi(\tau)d\tau}{\tau-\nu}+\sum_{j=1}^{m}\frac{A_j}{(\nu-\alpha^{-2})^j}.$$

The function $\Phi(\nu)$ may have jump discontinuities or singularities at points $\nu = \nu_s$, $\nu_s \in (a_1^{-2}, \infty)$ of the kind

$$\Phi(\nu) = \frac{\Phi_*(\nu)}{(\nu-\nu_s)^{\gamma}}, \quad \mathrm{Re}\gamma < 1,$$

where $\Phi_*(\nu)$ is a function satisfying Hölder's condition.

The condition (2.4), i.e. the absence of infinite stresses at the origin, ensures the existence of the integral in (2.13). It should be noted that the first term in (2.13), though satisfying Eq. (2.11) where $\Phi(\nu)$ is given by the relations (2.12), does not, however, obey the condition (2.4), for it includes a term of the order of $1/\nu$ at infinity. To cancel this term it is necessary to choose properly the constant A_1. The order of the pole m and remaining constants A_j, $(j = 2, 3, \ldots, m)$ are determined by additional conditions on the line of the change of the type of the boundary conditions.

The case when on one part of the plane $z = 0$, $r < \alpha t$ the velocity $v_z(r, 0, t)$ is given while on the remaining part $\alpha t < r < \infty$ the stress $\sigma_z(r, 0, t)$ prescribed, is similar to the discussed one and results in the Riemann-Hilbert problem of the (2.11) type for the function $G'(\nu)$.

The knowledge of the functions $F'(\nu)$ or $G'(\nu)$ makes it possible to find with the help of (1.2) and (1.5) stresses and velocities at all points of the half-space.

3. Moving load

The case when boundary conditions are given in terms of sufficiently smooth functions may be considered without great difficulties with the help of the Eq. (2.13). As an example, let us consider a more complicated case

(3.1)
$$\begin{aligned} g(\nu_0) &= -\frac{p}{\beta}\delta(\nu_0-\beta^{-2}), \quad && \alpha^{-2} < \nu_0 < \infty,\\ f(\nu_0) &= 0, && \nu_0 < \alpha^{-2}, \end{aligned}$$

where $\delta(\nu_0)$ is the Dirac's delta function.

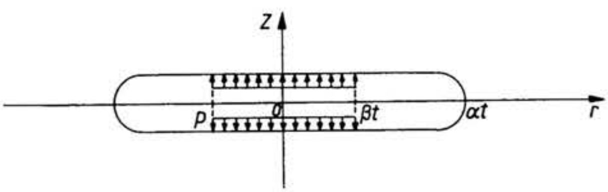

FIG. 3.

The conditions (3.1) correspond to the problem of penny-shaped crack propagated with a constant velocity α, when on the surface of the crack a step-wise normal stress acts with intensity p, moving with a constant velocity β (Fig. 3)

$$\begin{aligned} \sigma_z(r,0,t) &= -pH(\beta t - r), & 0 \leqslant r < \alpha t, \\ v_z(r,0,t) &= 0, & \alpha t < r < \infty. \end{aligned} \tag{3.2}$$

Here $H(\beta t - r)$ is Heaviside's step function.

The initial conditions are assumed to be homogeneous

$$v_j(r,z,0) = 0, \quad \sigma_{ij}(r,z,0) = 0, \quad t \leqslant 0, \quad i = r, \varphi, z, \quad j = r, \varphi, z. \tag{3.3}$$

Besides the initial and boundary conditions it is necessary to introduce an additional condition along the intersection line $r = \alpha t$. Let us assume that stresses and velocities close to the crack edge $r = \alpha t$ have a singularity of the form $\varrho^{-1/2}$, where ϱ is a local radius at the crack edge. This condition corresponds to finite non-zero energy expenditures for crack propagation.

From (3.1) it follows that in the region $-\infty < \nu < \alpha^{-2}$ the function $\Phi(\nu)$ vanishes

$$\Phi(\nu) = 0, \quad -\infty < \nu < \alpha^{-2}, \tag{3.4}$$

while in the region $\alpha^{-2} < \nu < \infty$,

$$\begin{aligned} \Phi(\nu) &= -\frac{ip\sqrt{\nu - a_1^{-2}}}{4\pi\mu\beta a_2^2 R(\nu)} \frac{d}{d\nu} \int_\nu^\infty \frac{\delta(\nu_0 - \beta^{-2})}{\sqrt{\nu_0 - \nu}} d\nu_0 \\ &= -\frac{ip\sqrt{\nu - a_1^{-2}}}{4\pi\mu\beta a_2^2 R(\nu)} \frac{d}{d\nu} \left(\delta(\nu) \ast \frac{H(\beta^{-2} - \nu)}{\sqrt{\beta^{-2} - \nu}} \right) = -\frac{ip\sqrt{\nu - a_1^{-2}}}{4\pi\mu\beta a_2^2 R(\nu)} \frac{d}{d\nu} \frac{H(\beta^{-2} - \nu)}{\sqrt{\beta^{-2} - \nu}}. \end{aligned} \tag{3.5}$$

It is obvious that the function $\Phi(\nu)$ should be considered to be a generalized function [5]. In accordance with the reasoning of Sec. 2, at the point $\nu = \alpha^{-2}$, the function $\Phi(\nu)$ must have the form

$$\Phi(\nu) = 2\pi i \sum_{j=1}^{m} (-1)^j \frac{A_j \delta^{(j-1)}(\nu - \alpha^{-2})}{(j-1)!}. \tag{3.6}$$

The solution of the Riemann-Hilbert problem (2.11) for the jump of the type (3.4), (3.5) exists [6] and formally can be written as

$$F'(\nu) = -\frac{p}{8\pi^2\mu\beta a_2^2} \int_{\alpha^{-2}}^{\beta^{-2}} \frac{\sqrt{\tau - a_1^{-2}}}{R(\tau)(\tau - \nu)} \frac{d}{d\tau} \left(\frac{H(\beta^{-2} - \tau)}{\sqrt{\beta^{-2} - \tau}} \right) d\tau + \sum_{j=1}^{m} \frac{A_j}{(\nu - \alpha^{-2})^j}. \tag{3.7}$$

Applying the rules of operations with generalized functions, the integral in (3.7) can be represented by integrals of ordinary functions

$$\begin{aligned} F'(\nu) &= \frac{p}{8\pi^2\mu\beta a_2^2} \int_{\alpha^{-2}}^{\beta^{-2}} \frac{1}{\sqrt{\beta^{-2} - \tau}\,(\tau - \nu)} \frac{d}{d\tau} \frac{\sqrt{\tau - a_1^{-2}}}{R(\tau)} d\tau \\ &\quad - \frac{p}{8\pi^2\mu\beta a_2^2} \frac{d}{d\nu} \int_{\alpha^{-2}}^{\beta^{-2}} \frac{\sqrt{\tau - a_2^{-2}}\, d\tau}{\sqrt{\beta^{-2} - \tau}\, R(\tau)(\tau - \nu)} + \frac{A_1'}{(\nu - \alpha^{-2})} + \sum_{j=2}^{m} \frac{A_j}{(\nu - \alpha^{-2})^j}. \end{aligned} \tag{3.8}$$

The constant A_1' differs from A_1 in (3.7), due to the non-integral term, which appeared when (3.7) was transformed into (3.8).

To satisfy the additional condition at the crack edge, the function $F'(\nu)$ must have at the point $\nu = \alpha^{-2}$ a pole of the order not greater than two [3]. Therefore it is necessary to take

$$A_j = 0, \quad j = 3, 4, \ldots, m. \tag{3.9}$$

The constant A'_1 is determined from the condition (2.4)

$$A'_1 = \frac{p}{8\pi^2\mu\beta a_2^2} \int\limits_{\alpha^{-2}}^{\beta^{-2}} \frac{1}{\sqrt{\beta^{-2}-\tau}} \frac{d}{d\tau}\left(\frac{\sqrt{\tau-a_1^2}}{R(\tau)}\right) d\tau. \tag{3.10}$$

To find the constant A_2, we first calculate, with the help of (1.11), (1.12), the stress on the plane $z = 0$ for $r < \beta t$ in terms of the function $F'(\nu)$

$$\sigma_z(r, 0, t) = \sqrt{\nu_0} \operatorname{Re} \int\limits_{l_\nu} \frac{G(\nu)\, d\nu}{\nu\sqrt{\nu-\nu_0}} \tag{3.11}$$

$$= -4\mu a_2^2 \sqrt{\nu_0} \operatorname{Re} \int\limits_{l_\nu} \frac{1}{\nu\sqrt{\nu-\nu_0}} \int\limits_0^{\nu} \frac{R(\lambda)}{\sqrt{a_1^{-2}-\lambda}} F'_+(\lambda)\, d\lambda\, d\nu.$$

Figure 4 shows the contours of integration in (3.11). Now (3.11) can be represented in the form:

$$\sigma_z(r, 0, t) = -4\mu a_2^2 \sqrt{\nu_0} \operatorname{Re} \int\limits_{\nu_0}^{\infty} \frac{1}{\nu\sqrt{\nu-\nu_0}} \left[\int\limits_0^{\nu_+} \frac{R_+(\lambda)}{\sqrt{a_1^{-2}-\lambda}} F'_+(\lambda)\, d\lambda + \int\limits_0^{\nu_-} \frac{R_-(\lambda)}{\sqrt{a_1^{-2}-\lambda}} F'_-(\lambda)\, d\lambda \right] d\nu. \tag{3.12}$$

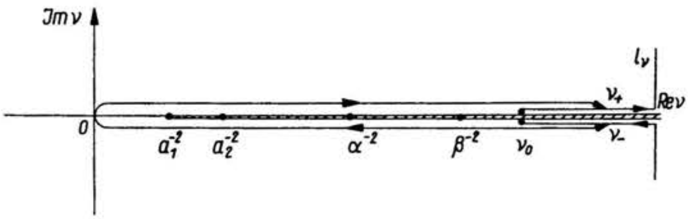

FIG. 4.

Applying the relations (3.4), (3.5), (3.6) for the function $\Phi(\nu) = F'_+(\nu) - F'_-(\nu)$ along the real axis and bearing in mind our convention for the roots, we obtain

$$\sigma_z(r, 0, t) = -4\mu a_2^2 \sqrt{\nu_0} \operatorname{Re}\left\{\int\limits_{\nu_0}^{\infty} \frac{2}{\nu\sqrt{\nu-\nu_0}} \left[\int\limits_0^{a_1^{-2}} \frac{R(\lambda)}{\sqrt{a_1^{-2}-\lambda}} F'(\lambda)\, d\lambda + \int\limits_{a_1^{-2}}^{a_2^{-2}} \lambda\sqrt{\lambda^2 - a_2^{-2}}\, F'(\lambda)\, d\lambda\right] d\nu - \int\limits_{\nu_0}^{\infty} \frac{1}{\nu\sqrt{\nu-\nu_0}} \int\limits_{\alpha^{-2}}^{\beta^{-2}} \frac{R(\lambda)}{\sqrt{a_1^{-2}-\lambda}} \Phi(\lambda)\, d\lambda\, d\nu\right\}. \tag{3.13}$$

Integration in (3.13) with respect to ν yields the expression

(3.14) $$\sigma_z(r,0,t) = p(I_1+I_2+I_3)-A_2(I_4+I_5),$$

where the constants I_1, I_2, I_3, I_4, I_5 are equal to:

$$I_1 = -\frac{p}{\pi\beta}\int\limits_0^{a_1^{-2}} \frac{R(\lambda)}{\sqrt{a_1^{-2}-\lambda}}\Bigg[\int\limits_{\alpha^{-2}}^{\beta^{-2}} \frac{1}{(\tau-\lambda)\sqrt{\beta^{-2}-\tau}}\frac{d}{d\tau}\left(\frac{\sqrt{\tau-a_1^{-2}}}{R(\tau)}\right)d\tau - \frac{d}{d\lambda}\int\limits_{\alpha^{-2}}^{\beta^{-2}} \frac{\sqrt{\tau-a_1^{-2}}}{\sqrt{\beta^{-2}-\tau}}\frac{d\tau}{R(\tau)(\tau-\lambda)} + \frac{1}{\lambda-\alpha^{-2}}\int\limits_{\alpha^{-2}}^{\beta^{-2}} \frac{1}{\sqrt{\beta^{-2}-\tau}}\frac{d}{d\tau}\left(\frac{\sqrt{\tau-a_1^{-2}}}{R(\tau)}\right)d\tau\Bigg]d\lambda,$$

(3.15) $$I_2 = -\frac{p}{\pi\beta}\int\limits_{a_1^{-2}}^{a_2^{-2}} \lambda\sqrt{a_2^{-2}-\lambda}\Bigg[\int\limits_{\alpha^{-2}}^{\beta^{-2}} \frac{1}{\sqrt{\beta^{-2}-\tau}\,(\tau-\lambda)}\frac{d}{d\tau}\left(\frac{\sqrt{\tau-a_1^{-2}}}{R(\tau)}\right)d\tau - \frac{d}{d\lambda}\int\limits_{\alpha^{-2}}^{\beta^{-2}} \frac{\sqrt{\tau-a_1^{-2}}}{\sqrt{\beta^{-2}-\tau}}\frac{d\tau}{R(\tau)(\tau-\lambda)} + \frac{1}{\lambda-\alpha^{-2}}\int\limits_{\alpha^{-2}}^{\beta^{-2}} \frac{1}{\sqrt{\beta^{-2}-\tau}}\frac{d}{d\tau}\left(\frac{\sqrt{\tau-a_1^{-2}}}{R(\tau)}\right)d\tau\Bigg]d\lambda,$$

$$I_3 = -\frac{p}{\beta}\frac{R(\alpha^{-2})}{\sqrt{\alpha^{-2}-a^{-2}}}\int\limits_{\alpha^{-2}}^{\beta^{-2}} \frac{1}{\sqrt{\beta^{-2}-\tau}}\frac{d}{d\tau}\left(\frac{\sqrt{\tau-a_1^{-2}}}{R(\tau)}\right)d\tau,$$

$$I_4 = 8\mu a_2^{-2}\int\limits_0^{a_1^{-2}} \frac{R(\lambda)\,d\lambda}{\sqrt{a_1^{-2}-\lambda}\,(\lambda-\alpha^{-2})^2},\quad I_5 = 8\mu a_2^{-2}\int\limits_{a_1^{-2}}^{a_2^{-2}} \frac{\lambda\sqrt{a_2^{-2}-\lambda}}{(\lambda-\alpha^{-2})^2}d\lambda.$$

The substitution of the boundary condition (3.2) into the left-hand side of (3.13) gives for the constant A_2

(3.16) $$A_2 = p\frac{I_1+I_2+I_3+1}{I_4+I_5}.$$

This completes the solution of the problem.

Note that for $\beta \to \alpha$ the boundary conditions of the problem are identical with those of the paper [3]. In this case, it is seen from Eqs. (3.8), (3.10), (3.15) and (3.16) that the resolving function $F'(\nu)$ is the same as in the above-mentioned paper to within the form of the constants. It can, however, be shown that the constants do actually coincide.

The example discussed here may find applications in the problem of rupture of an oil-bearing layer under the pressure of pumped liquid.

4. Concentrated force

Consider now the case of a concentrated force at the origin of the coordinate system. For simplicity, let the crack be traction free except at the origin, where the normal stresses are defined by a homogeneous function of zero degree containing a delta function

$$\sigma_z(r,0,t) = -P\frac{t}{r}\delta\left(\frac{r}{t}\right), \quad z=0, \quad r<\alpha t. \tag{4.1}$$

It can readily be shown that the integral of (4.1) over any circle on the plane $z = 0$ with centre at the origin has the value

$$\int_0^{2\pi}\int_0^{P} \sigma_z(r,0,t)rdr = -2\pi t^2 P. \tag{4.2}$$

In this case, the condition (2.4) does not hold and the integral along the contour $l_{\nu R}$ in (2.3) cannot be neglected.

Assume that in the problem under consideration, when $\nu \to \infty$, the function $G(\nu)$ behaves as follows:

$$G(\nu) = c\nu + O(1), \tag{4.3}$$

where c is a real constant.

This assumption is based on the same behaviour of the function $G(\nu)$ at infinity in the corresponding Lamb's problem in which the stresses over all surface of the half-space vanish, except at the origin, where they are described by the expression (4.1).

The condition of absence of the stress on the crack surface implies that

$$\operatorname{Re}\int_{l_\nu} \frac{G'(\nu)}{\sqrt{\nu-\nu_0}}d\nu = 0, \quad \alpha^{-2} < \nu_0 < \infty, \tag{4.4}$$

or

$$\int_{\nu_0}^{R} \frac{\operatorname{Re}G'_+(\nu)+\operatorname{Re}G'_-(\nu)}{\sqrt{\nu-\nu_0}}d\nu = -\operatorname{Re}\int_{l_{\nu R}} \frac{G'(\nu)}{\sqrt{\nu-\nu_0}}d\nu, \quad R\to\infty. \tag{4.5}$$

From Eqs. (4.3) and (4.5) it follows that

$$\operatorname{Re}G'_+(\nu)+\operatorname{Re}G'_-(\nu) = 2c, \quad \alpha^{-2}<\nu<\infty \tag{4.6}$$

which results in the following Riemann-Hilbert problem for function $F'(\nu)$:

$$\begin{aligned} &F'_+(\nu)-F'_-(\nu) = 0, && -\infty<\nu<\alpha^{-2},\\ &F'_+(\nu)-F'_-(\nu) = \frac{ic\sqrt{\nu-a_1^{-2}}}{2\mu a_2^2 R(\nu)}, && \alpha^{-2}<\nu<\infty. \end{aligned} \tag{4.7}$$

At the point $\nu = \alpha^{-2}$, the function $F'(\nu)$ due to (3.6) and (3.9) must have a pole of order not greater than two. Hence the solution of this problem has a form

$$F'(\nu) = \frac{c}{4\pi\mu a_2^2}\int_{\alpha^{-2}}^{\infty} \frac{\sqrt{\tau-a_1^{-2}}}{R(\tau)(\tau-\nu)}d\tau + \frac{A_1}{\nu-\alpha^{-2}} + \frac{A_2}{(\nu-\alpha^{-2})^2}. \tag{4.8}$$

This solution contains three constants c, A_1 and A_2 which are to be determined.

Determine first the constant c. To this end, let us calculate the stresses $\sigma_z(r, z, t)$ on a certain small circle $r \leqslant \varrho$, $z = \text{const}$ near the origin, such that $r \ll a_1 t$, $z \ll a_1 t$. The stresses $\sigma_z(r, z, t)$ are given by the last equation of (1.2)

$$\sigma_z(r, z, t) = \operatorname{Re} \int_{-\pi}^{\pi} \left[\sum\nolimits_{1z} (\vartheta_1) + \sum\nolimits_{2z} (\vartheta_2) \right] d\varphi, \tag{4.9}$$

where ϑ_1 and ϑ_2 are determined by Eqs. (1.3)

$$\vartheta_k = \frac{a_k tr\cos\varphi + iz\sqrt{a_k^2 t^2 - r^2\cos^2\varphi - z^2}}{a_k(r^2\cos^2\varphi + z^2)}. \tag{4.10}$$

For small r and φ, we have

$$\vartheta_1 \approx \vartheta_2 \approx \vartheta = \frac{t(r\cos\varphi + iz)}{r^2\cos^2\varphi + z^2}. \tag{4.11}$$

The function (4.8) for $\nu \to \infty$ behaves as follows:

$$F'(\nu) = \frac{c}{2\pi\mu a_2^2(a_1^{-2} - a_2^{-2})}\left(\frac{\pi i}{\sqrt{\nu}} - \frac{2\alpha^{-1}}{\nu}\right) + \frac{A_1}{\nu} + 0\left(\frac{1}{\nu\sqrt{\nu}}\right). \tag{4.12}$$

With the help of (4.11) and (4.12), the stresses (4.9) on the small circle near the origin can be written in the form

$$\sigma_z(r, z, t) \approx c \operatorname{Re} \int_{-\pi}^{\pi} \vartheta^2 d\varphi = ct^2 \int_{-\pi}^{\pi} \frac{r^2\cos^2\varphi - z^2}{(r^2\cos^2\varphi + z^2)^2} d\varphi. \tag{4.13}$$

The calculation of the integral in (4.13) yields

$$\sigma_z(r, z, t) = -\frac{2\pi ct^2 z}{(r^2 + z^2)^{3/2}}. \tag{4.14}$$

Thus, due to the formula

$$\lim_{z \to 0} \frac{z}{(r^2 + z^2)^{3/2}} = \frac{\delta(r)}{r},$$

we obtain

$$\sigma_z(r, 0, t) = -2\pi ct^2 \frac{\delta(r)}{r}. \tag{4.15}$$

The comparison of (4.15) with (4.1) gives the constant

$$c = \frac{1}{2\pi} P. \tag{4.16}$$

The constant A_1 is determined by the condition of absence of the stress on the crack (4.1). According to the expression (4.12), it should be taken equal to

$$A_1 = \frac{P}{2\pi^2 \mu\alpha a_2^2(a_1^{-2} - a_2^{-2})}. \tag{4.17}$$

The constant A_2 is determined as in Sec. 3 and has the form

$$A_2 = P\frac{I_1'+I_2'-\alpha^{-2}}{4\mu a_2^2\int\limits_0^\infty \frac{\left(\nu+\frac{1}{2}a_2^{-2}\right)^2-\nu\sqrt{a_1^{-2}+\nu}\,\sqrt{a_1^{-2}+\nu}}{(\alpha^{-2}+\nu)^2\sqrt{a_1^{-2}+\nu}}\,d\nu}, \tag{4.18}$$

where

$$I_1' = 2\int\limits_0^{a_1^{-2}} \frac{R(\lambda)}{\sqrt{a_1^{-2}-\lambda}}\int\limits_{\alpha^{-2}}^\infty \frac{\sqrt{\tau-a_1^{-2}}}{R(\tau)(\tau-\lambda)}\,d\tau\,d\lambda,$$

$$I_2' = 2\int\limits_{a_1^{-2}}^{a_2^{-2}} \lambda\sqrt{a_2^{-2}-\lambda}\int\limits_{a^{-2}}^\infty \frac{\sqrt{\tau-a_1^{-2}}}{R(\tau)(\tau-\lambda)}\,d\tau\,d\lambda.$$

Finally, it is noteworthy that a similar approach can be suggested in the case of a self-similar problem with displacements as homogeneous functions of zero degree.

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