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The geometrical concept of intermediate configuration and elastic-plastic finite strain

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ELASTIC-PLASTIC finite deformations are usually described by means of an intermediate configuration which leads to the decomposition $\mathbf{F} = \mathbf{F}^{(e)}\mathbf{F}^{(p)}$. In most theories, the intermediate configuration is defined only up to an arbitrary rotation. The kinematics of such materials, as well as the consequences which follow from the arbitrary rotation, are investigated. The invariance conditions are written which must be fulfilled by the constitutive equations. Applications of these ideas are given to show, as an example, how the theories of GREEN and NAGHDI, GORODZOV and LEONOV, and LEE are related to one another.

Skończone deformacje sprężysto-plastyczne są zwykle opisane za pomocą konfiguracji pośredniej, która prowadzi do związku $\mathbf{F} = \mathbf{F}^{(e)}\mathbf{F}^{(p)}$. W wielu teoriach konfiguracja pośrednia jest zdefiniowana z dokładnością do dowolnego obrotu. Zbadano kinematykę takich materiałów oraz konsekwencje wynikające z dowolnego obrotu. Zapisano warunek niezmienniczości, który muszą spełniać równania konstytutywne. Podano zastosowanie tych koncepcji i wykazano dla przykładu związki pomiędzy teoriami GREENA i NAGHDIEGO, GORODZOWA i LEONOWA oraz LEE.

Конечные упруго-пластические деформации описываются обычно при использовании понятия промежуточной конфигурации, что приводит к разложению градиента деформаций по формуле: $\mathbf{F} = \mathbf{F}^{(e)}\mathbf{F}^{(p)}$. В большинстве теорий промежуточная конфигурация определена лишь с точностью до произвольного вращения. Исследованы кинематики таких материалов, а также следствия, вытекающие из произвольности вращений. Выведены условия инвариантности, которым должны удовлетворять определяющие уравнения. Даны приложения этих критериев, показывающие в качестве примера каково соотношение между теориями Грина-Нахди, Городцова-Леонова и Ли.

1. Introduction

IN THE CLASSICAL theory of plasticity, as well as in some simple theories of viscoelasticity, the infinitesimal strain tensor ε_{ij} is decomposed into an elastic part $\varepsilon_{ij}^{(e)}$ and a plastic (or anelastic) part $\varepsilon_{ij}^{(p)}$, these tensors being related by the formula $\varepsilon_{ij} = \varepsilon_{ij}^{(e)} + \varepsilon_{ij}^{(p)}$.

This situation can be extended to large deformations by means of an intermediate state or configuration ([3, 4]).

The elastic deformation is then the deformation of the actual configuration from the intermediate configuration and the plastic deformation is the deformation of the intermediate configuration from the reference configuration. Thus the deformation gradient $\mathbf{F}$ can be decomposed into an elastic part $\mathbf{F}^{(e)}$ and a plastic part $\mathbf{F}^{(p)}$

$$\mathbf{F} = \mathbf{F}^{(e)}\mathbf{F}^{(p)}. \tag{1.1}$$

However, the tensor fields $\mathbf{F}^{(e)}$ and $\mathbf{F}^{(p)}$ are not in general gradient fields and cannot be derived from a displacement function. The intermediate configuration is only locally defined.

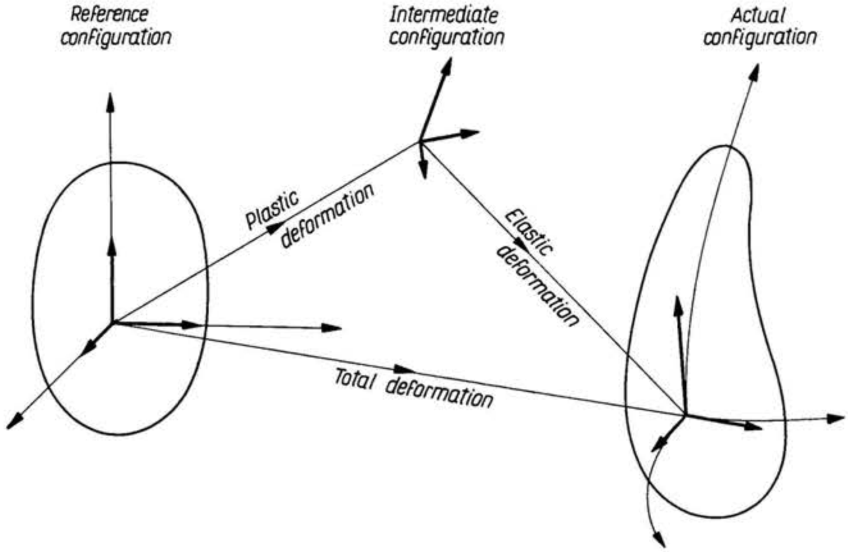

FIG. 1.

In this paper, we first study the kinematics of materials having such an intermediate state and show how some elastic-plastic finite strain theories [5, 7], apparently starting from another concept of plastic deformation, actually fall within this framework. Then we shall state an invariance condition obeyed by most theories, and discuss its implications. A comparison of some finite strain elastic-plastic theories recently proposed will be given as an application.

2. Kinematics

As usual, the rotation tensor **R**, the right and left stretch tensors **U** and **V**, the right and left Cauchy-Green tensors **C** and **B**, the Lagrangian and Eulerian strain tensors **E** and **A**, the velocity gradient **L**, the rate of deformation **D** and the rate of spin **W** can be defined from the deformation gradient **F**:

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}, \quad \mathbf{C} = \mathbf{U}^2 = \mathbf{F}^T\,\mathbf{F}, \quad \mathbf{B} = \mathbf{V}^2 = \mathbf{FF}^T,$$

$$(2.1) \qquad 2\mathbf{E} = \mathbf{C}-\mathbf{1}, \quad 2\mathbf{A} = \mathbf{1}-\mathbf{B}^{-1},$$

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \mathbf{D}+\mathbf{W}, \quad 2\mathbf{D} = \mathbf{L}+\mathbf{L}^T, \quad 2\mathbf{W} = \mathbf{L}-\mathbf{L}^T.$$

From the plastic and elastic deformation gradients $\mathbf{F}^{(p)}$ and $\mathbf{F}^{(e)}$ (which are no longer gradients, but we find it convenient to retain this term), the corresponding plastic and elastic tensors can be defined in the same way:

$$\mathbf{F}^{(p)} = \mathbf{R}^{(p)}\mathbf{U}^{(p)} = \mathbf{V}^{(p)}\mathbf{R}^{(p)}, \quad \mathbf{C}^{(p)} = \mathbf{F}^{(p)T}\mathbf{F}^{(p)}, \quad \mathbf{B}^{(p)} = \mathbf{F}^{(p)}\mathbf{F}^{(p)T},$$

$$(2.2) \qquad 2\mathbf{E}^{(p)} = \mathbf{C}^{(p)}-\mathbf{1}, \quad 2\mathbf{\Delta}^{(p)} = \mathbf{1}-\mathbf{B}^{(p)-1},$$

$$\mathbf{L}^{(p)} = \dot{\mathbf{F}}^{(p)}\mathbf{F}^{(p)-1}, \quad 2\mathbf{D}^{(p)} = \mathbf{L}^{(p)}+\mathbf{L}^{(p)T}, \quad 2\mathbf{W}^{(p)} = \mathbf{L}^{(p)}-\mathbf{L}^{(p)T},$$

$$\begin{aligned}
&\mathbf{F}^{(e)} = \mathbf{R}^{(e)}\mathbf{U}^{(e)} = \mathbf{V}^{(e)}\mathbf{R}^{(e)}, \quad \mathbf{C}^{(e)} = \mathbf{F}^{(e)T}\mathbf{F}^{(e)}, \quad \mathbf{B}^{(e)} = \mathbf{F}^{(e)}\mathbf{F}^{(e)T},\\
&2\boldsymbol{\Delta}^{(e)} = \mathbf{C}^{(e)}-\mathbf{1}, \quad 2\mathbf{A}^{(e)} = \mathbf{1}-\mathbf{B}^{(e)-1},\\
&\mathbf{L}^{(e)} = \dot{\mathbf{F}}^{(e)}\mathbf{F}^{(e)-1}, \quad 2\mathbf{D}^{(e)} = \mathbf{L}^{(e)}+\mathbf{L}^{(e)T}, \quad 2\mathbf{W}^{(e)} = \mathbf{L}^{(e)}-\mathbf{L}^{(e)T}.
\end{aligned} \tag{2.3}$$

We have written $\boldsymbol{\Delta}^{(p)}$ (instead of $\mathbf{A}^{(p)}$) for the Eulerian plastic strain tensor and $\boldsymbol{\Delta}^{(e)}$ (instead of $\mathbf{E}^{(e)}$) for the Lagrangian elastic strain tensor; presently, the tensors $\mathbf{A}^{(p)}$, $\mathbf{E}^{(e)}$ and $\boldsymbol{\Delta}$ are defined by

$$2\mathbf{E}^{(e)} = \mathbf{C}-\mathbf{C}^{(p)}, \quad 2\boldsymbol{\Delta} = \mathbf{C}^{(e)}-\mathbf{B}^{(p)-1}, \quad 2\mathbf{A}^{(p)} = \mathbf{B}^{(e)-1}-\mathbf{B}^{-1} \tag{2.4}$$

and we can write:

$$\begin{aligned}
\mathbf{E} &= \mathbf{F}^{(p)T}\boldsymbol{\Delta}\mathbf{F}^{(p)} = \mathbf{F}^{T}\mathbf{A}\mathbf{F}, & \boldsymbol{\Delta} &= \mathbf{F}^{(e)T}\mathbf{A}\mathbf{F}^{(e)},\\
\mathbf{E}^{(p)} &= \mathbf{F}^{(p)T}\boldsymbol{\Delta}^{(p)}\mathbf{F}^{(p)} = \mathbf{F}^{T}\mathbf{A}^{(p)}\mathbf{F}, & \boldsymbol{\Delta}^{(p)} &= \mathbf{F}^{(e)T}\mathbf{A}^{(p)}\mathbf{F}^{(e)},\\
\mathbf{E}^{(e)} &= \mathbf{F}^{(p)T}\boldsymbol{\Delta}^{(e)}\mathbf{F}^{(p)} = \mathbf{F}^{T}\mathbf{A}^{(e)}\mathbf{F}, & \boldsymbol{\Delta}^{(e)} &= \mathbf{F}^{(e)T}\mathbf{A}^{(e)}\mathbf{F}^{(e)}.
\end{aligned} \tag{2.5}$$

Let $d\mathbf{x}_0$, $d\mathring{\mathbf{x}}$, $d\hat{\mathbf{x}}$ be the line elements, respectively, in the reference, intermediate and actual configurations and ds_0, $d\mathring{s}$, $d\hat{s}$ the corresponding arc length. We can then write:

$$\begin{aligned}
d\hat{s}^2-d\mathring{s}^2 &= 2d\mathbf{x}_0\,\mathbf{E}^{(e)}d\mathbf{x}_0 = 2d\mathring{\mathbf{x}}\boldsymbol{\Delta}^{(e)}d\mathring{\mathbf{x}} = 2d\hat{\mathbf{x}}\mathbf{A}^{(e)}d\hat{\mathbf{x}},\\
d\hat{s}^2-ds_0^2 &= 2d\mathbf{x}_0\mathbf{E}\,d\mathbf{x}_0 = 2d\mathring{\mathbf{x}}\boldsymbol{\Delta}d\mathring{\mathbf{x}} = 2d\hat{\mathbf{x}}\mathbf{A}d\hat{\mathbf{x}},\\
d\mathring{s}^2-ds_0^2 &= 2d\mathbf{x}_0\mathbf{E}^{(p)}d\mathbf{x}_0 = 2d\mathring{\mathbf{x}}\boldsymbol{\Delta}^{(p)}d\mathring{\mathbf{x}} = 2d\hat{\mathbf{x}}\mathbf{A}^{(p)}d\hat{\mathbf{x}}.
\end{aligned} \tag{2.6}$$

And the tensors $\mathbf{E}$, $\mathbf{E}^{(e)}$, $\mathbf{E}^{(p)}$, $\boldsymbol{\Delta}$, $\boldsymbol{\Delta}^{(e)}$, $\boldsymbol{\Delta}^{(p)}$, $\mathbf{A}$, $\mathbf{A}^{(e)}$ and $\mathbf{A}^{(p)}$ are measures of the total, elastic and plastic deformations, respectively, in the reference, intermediate and actual configurations.

The (covariant) convected time derivative of a tensor $\mathbf{T}$ is defined as

$$\mathbf{T}_C = \dot{\mathbf{T}}+\mathbf{L}^T\mathbf{T}+\mathbf{T}\mathbf{L}; \tag{2.7}$$

in the same way, we define the Γ-convected time derivative

$$\mathbf{T}_\Gamma = \dot{\mathbf{T}}+\mathbf{L}^{(p)T}\mathbf{T}+\mathbf{T}\mathbf{L}^{(p)}. \tag{2.8}$$

And it can be shown that

$$\mathbf{A}_C = \mathbf{D}, \quad \boldsymbol{\Delta}^{(p)}_\Gamma = \mathbf{D}^{(p)}, \quad \dot{\mathbf{C}}^{(e)} = 2\mathbf{F}^{(e)T}\mathbf{D}^{(e)}\mathbf{F}^{(e)}. \tag{2.9}$$

The tensors $\dot{\mathbf{E}}$, $\dot{\mathbf{E}}^{(e)}$, $\dot{\mathbf{E}}^{(p)}$, $\boldsymbol{\Delta}_\Gamma$, $\boldsymbol{\Delta}^{(e)}_\Gamma$, $\boldsymbol{\Delta}^{(p)}_\Gamma$, $\mathbf{A}_C$, $\mathbf{A}^{(e)}_C$, $\mathbf{A}^{(p)}_C$ are related by formulae similar to (2.5):

$$\begin{aligned}
\dot{\mathbf{E}} &= \mathbf{F}^{(p)T}\boldsymbol{\Delta}_\Gamma\mathbf{F}^{(p)} = \mathbf{F}^T\mathbf{A}_C\mathbf{F}, & \boldsymbol{\Delta}_\Gamma &= \mathbf{F}^{(e)T}\mathbf{A}_C\mathbf{F}^{(e)},\\
\dot{\mathbf{E}}^{(p)} &= \mathbf{F}^{(p)T}\boldsymbol{\Delta}^{(p)}_\Gamma\mathbf{F}^{(p)} = \mathbf{F}^T\mathbf{A}^{(p)}_C\mathbf{F}, & \boldsymbol{\Delta}^{(p)}_\Gamma &= \mathbf{F}^{(e)T}\mathbf{A}^{(p)}_C\mathbf{F}^{(e)},\\
\dot{\mathbf{E}}^{(e)} &= \mathbf{F}^{(p)T}\boldsymbol{\Delta}^{(e)}_\Gamma\mathbf{F}^{(p)} = \mathbf{F}^T\mathbf{A}^{(e)}_C\mathbf{F}, & \boldsymbol{\Delta}^{(e)}_\Gamma &= \mathbf{F}^{(e)T}\mathbf{A}^{(e)}_C\mathbf{F}^{(e)},
\end{aligned} \tag{2.10}$$

and can be considered as measures of the rate of total, elastic and plastic deformations, respectively, in the reference, intermediate and actual configurations, for we can write:

$$\frac{d}{dt}(d\hat{s}^2-d\mathring{s}^2) = 2d\mathbf{x}_0\dot{\mathbf{E}}^{(e)}d\mathbf{x}_0 = 2d\mathring{\mathbf{x}}\boldsymbol{\Delta}^{(e)}_\Gamma d\mathring{\mathbf{x}} = 2d\hat{\mathbf{x}}\mathbf{A}^{(e)}_C d\hat{\mathbf{x}}, \tag{2.11}$$

$$\frac{d}{dt}(d\hat{s}^2 - ds_0^2) = 2d\mathbf{x}_0\, \dot{\mathbf{E}} d\mathbf{x}_0 = 2d\overset{*}{\mathbf{x}}\, \boldsymbol{\Delta}_\Gamma d\dot{\mathbf{x}} = 2d\hat{\mathbf{x}}\, \mathbf{A}_C\, d\hat{\mathbf{x}}, \tag{2.11 [cont.]}$$

$$\frac{d}{dt}(d\overset{*}{s}{}^2 - ds_0^2) = 2d\mathbf{x}_0\, \dot{\mathbf{E}}^{(p)} d\mathbf{x}_0 = 2d\overset{*}{\mathbf{x}}\, \boldsymbol{\Delta}_\Gamma^{(p)} d\overset{*}{\mathbf{x}} = 2d\hat{\mathbf{x}}\, \mathbf{A}_C^{(p)}\, d\hat{\mathbf{x}}.$$

We also define the tensor $\overline{\mathbf{D}}^{(p)}$ as

$$\overline{\mathbf{D}}^{(p)} = \mathbf{R}^{(e)} \boldsymbol{\Delta}_\Gamma^{(p)} \mathbf{R}^{(e)T} = \mathbf{V}^{(e)} \mathbf{A}_C^{(p)} \mathbf{V}^{(e)}. \tag{2.12}$$

Some elastic-plastic theories [5, 7] use a so-called plastic deformation tensor without referring to (1.1). This tensor, which enters the constitutive equations as a new thermodynamic parameter, is supposed only to be symmetric and invariant in a change of frame. These theories enter the framework described here: We need only to consider this plastic deformation as being our Lagrangian plastic deformation tensor $\mathbf{E}^{(p)}$ [8].

3. Invariance requirements

The intermediate configuration has been defined as a purely kinematical concept. Definition of new concepts in continuum mechanics requires the statement of the invariance condition which they must fulfil. We state —

3.1. Invariance requirement: The intermediate configuration is defined up to an arbitrary rotation

In other words, we define a "material with an intermediate state" as a material, the motion of which is completely described (in addition to the usual deformation function) by a "plastic gradient of deformation" defined up to an arbitrary rotation. From a geometrical point of view, a body is a three-dimensional manifold $\mathscr{B}$, its actual configuration is a (time-dependent) imbedding of $\mathscr{B}$ in the usual Euclidean space E_3, and its intermediate configuration is a (time-dependent) Riemannian structure over $\mathscr{B}$. The curvature of this structure corresponds to the fact that $\mathbf{F}^{(p)}$ cannot be obtained as the gradient of a "plastic displacement function".

This invariance condition, even if not explicitly formulated, is actually underlying most of the theories based on (1.1), but its consequences have not been studied from a general point of view. The intermediate configuration is often defined as the stress-free configuration obtained by some local process of relaxation of the applied stresses. (However, such an interpretation must follow from the constitutive assumptions and cannot be made prior to them). Then this invariance condition follows from the frame indifference principle.

However, it is possible to build up a consistent theory which does not obey this requirement, and in which the intermediate configuration is completely defined [11, 12]. But these theories are essentially different from the others and $\mathbf{F}^{(e)}$ should then be considered as a dipolar displacement (as defined by GREEN and RIVLIN).

In order to satisfy this invariance requirement, we can proceed in two ways.

First point of view. We can assume that all the quantities which depend on $\mathbf{F}^{(p)}$, depend on it only through the tensor $\mathbf{C}^{(p)}$ [5, 6, 7]. This point of view is the more natural one, because it introduces six new parameters, or internal degrees of freedom, defining, instead of an intermediate state, an equivalence class of intermediate states (if two states are said to be equivalent when one can be obtained from the other by a rotation), or equivalently defining the metric associated to the afore-mentioned Riemannian structure.

Second point of view. We can also write first the constitutive equations and then restrict them by writing that they remain invariant in any rotation of the intermediate state, just as they remain invariant in a change of frame from the principle of frame-indifference. Equivalently, we can say that, in a change of frame, the intermediate configuration undergoes an arbitrary rotation and then write the principle of frame-indifference in the following generalized way:

3.2. Property of invariance for the constitutive functions (P.I.F.C.)

The constitutive equations of a material with an intermediate configuration must remain invariant in a Γ-change of frame

$$\mathbf{x}' = \mathbf{c}(t)+\mathbf{Q}_1(t)\mathbf{x}, \qquad \mathbf{F}^{(p)\prime} = \mathbf{Q}_2(x, t)\mathbf{F}^{(p)}, \tag{3.1}$$

where $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are orthogonal tensors.

It must here be emphasized that this should not be considered as a general principle, but rather as a definition of an intermediate configuration.

4. Γ-objective and Γ-invariant tensors

DEFINITION. *Tensors and vectors which remain invariant in a Γ-change of frame will be said to be Γ-invariant. Tensors and vectors which transform like tensors and vectors in the actual configuration will be said to be Γ-objective.*

This is important because, by application of the P.I.F.C. to constitutive equations, we shall obtain for them reduced forms involving only Γ-invariant or Γ-objective quantities.

We now study the behaviour of the kinematic tensors introduced in § 2. From (3.1) it follows that:

$$\mathbf{F}' = \mathbf{Q}_1\mathbf{F}, \quad \mathbf{F}^{(p)\prime} = \mathbf{Q}_2\mathbf{F}^{(p)}, \quad \mathbf{F}^{(e)\prime} = \mathbf{Q}_1\mathbf{F}^{(e)}\mathbf{Q}_2^T. \tag{4.1}$$

Straightforward calculations show that:

$$\mathbf{U}, \mathbf{C}, \mathbf{U}^{(p)}, \mathbf{C}^{(p)}, \mathbf{E}, \mathbf{E}^{(e)}, \mathbf{E}^{(p)}, \dot{\mathbf{E}}, \dot{\mathbf{E}}^{(e)}, \dot{\mathbf{E}}^{(p)} \tag{4.2}$$

are Γ-invariant ($\mathbf{U}' = \mathbf{U}$, etc ...),

$$\mathbf{V}, \mathbf{B}, \mathbf{V}^{(e)}, \mathbf{B}^{(e)}, \mathbf{A}, \mathbf{A}^{(e)}, \mathbf{A}^{(p)}, \mathbf{D}, \mathbf{A}_C^{(e)}, \mathbf{A}_C^{(p)}, \bar{\mathbf{D}}^{(p)} \tag{4.3}$$

are Γ-objective ($\mathbf{V}' = \mathbf{Q}_1\mathbf{V}\mathbf{Q}_1^T$, etc ...),

$$\mathbf{V}^{(p)}, \mathbf{B}^{(p)}, \mathbf{U}^{(e)}, \mathbf{C}^{(e)}, \mathbf{\Delta}, \mathbf{\Delta}^{(e)}, \mathbf{\Delta}^{p}, \mathbf{\Delta}_\Gamma, \mathbf{\Delta}_\Gamma^{(e)}, \mathbf{D}^{(p)} \tag{4.4}$$

behave like tensors in the intermediate configuration ($\mathbf{V}^{(p)\prime} = \mathbf{Q}_2\mathbf{V}^{(p)}\mathbf{Q}_2^T$, etc ...).

The tensors $\mathbf{L}, \mathbf{L}^{(e)}$ and $\mathbf{L}^{(p)}$ transform as:

(4.5)
$$\begin{aligned}
\mathbf{L}' &= \mathbf{Q}_1\mathbf{L}\mathbf{Q}_1^T+\mathbf{\Omega}_1, \qquad (\mathbf{\Omega}_1 = \dot{\mathbf{Q}}_1\mathbf{Q}_1^T),\\
\mathbf{L}^{(p)\prime} &= \mathbf{Q}_2\mathbf{L}^{(p)}\mathbf{Q}_2^T+\mathbf{\Omega}_2, \qquad (\mathbf{\Omega}_2 = \dot{\mathbf{Q}}_2\mathbf{Q}_2^T),\\
\mathbf{L}^{(e)\prime} &= \mathbf{Q}_1\mathbf{L}^{(e)}\mathbf{Q}_1^T+\mathbf{\Omega}_1-\mathbf{Q}_1\mathbf{F}^{(e)}\mathbf{Q}_2^T\mathbf{\Omega}_2\mathbf{Q}_2\mathbf{F}^{(e)-1}\mathbf{Q}_1^T.
\end{aligned}$$

There is one striking point about these equations: while $\mathbf{D}$ and $\mathbf{D}^{(p)}$ are tensors in the final configuration for the deformation, the rate of which they describe ($\mathbf{D}$ is Γ-objective, i.e. a tensor in the actual configuration; $\mathbf{D}^{(p)}$ is a tensor in the intermediate configuration), $\mathbf{D}^{(e)}$ is not and it essentially depends on the orientation of the intermediate configuration. Therefore, $\mathbf{D}^{(e)}$ is not a good measure of the elastic rate of deformation. The elastic rate of deformation will be measured by $\mathbf{A}_C^{(e)}$. For the plastic rate of deformation, we can choose between three measures:

$\mathbf{A}_C^{(p)}$ is Γ-objective and related to $\mathbf{D}$ and $\mathbf{A}_C^{(e)}$ by

(4.6)
$$\mathbf{D} = \mathbf{A}_C^{(e)}+\mathbf{A}_C^{(p)};$$

it also appears naturally when using the first point of view of § 3, [5].

$\mathbf{D}^{(p)}$ is the natural plastic rate of deformation [3] but, being not Γ-objective, it cannot remain in the constitutive equations.

$\overline{\mathbf{D}}^{(p)}$ is Γ-objective and close to $\mathbf{D}^{(p)}$. It corresponds to LEE's choice $\mathbf{R}^{(e)} = \mathbf{1}$ and has the same fundamental invariants as $\mathbf{D}^{(p)}$.

In fact, we shall write constitutive equations involving $\mathbf{D}^{(p)}$ and obtain for them reduced forms involving $\overline{\mathbf{D}}^{(p)}$ or $\mathbf{A}_C^{(p)}$.

5. Constructing a Γ-invariant theory

The general scheme for constructing Γ-invariant constitutive equations is as follows:

1. Write some constitutive assumptions.
2. Use the P.I.F.C. to set the invariance requirements which must be fulfilled.
3. Solve the resulting equations and obtain Γ-invariant reduced forms of the constitutive equations.

As an example, we show how the theory of GREEN and NAGHDI [5] can be obtained from certain general constitutive assumptions (their e'_{KL} and e''_{KL} being our $\mathbf{E}^{(e)}$ and $\mathbf{E}^{(p)}$).

Constitutive assumption 1. *The stress tensor* $\mathbf{T}$ *is a function of* $\mathbf{F}$, $\mathbf{F}^{(p)}$ *and the temperature* θ

(5.1)
$$\mathbf{T} = t(\mathbf{F}, \mathbf{F}^{(p)}, \theta).$$

From the P.I.F.C., this function must be such that

$$t(\mathbf{Q}_1\mathbf{F}, \mathbf{Q}_2\mathbf{F}^{(p)}, \theta) = \mathbf{Q}_1 t(\mathbf{F}, \mathbf{F}^{(p)}, \theta)\mathbf{Q}_1^T;$$

taking $\mathbf{Q}_1 = \mathbf{R}^T$ and $\mathbf{Q}_2 = \mathbf{R}^{(p)T}$, we obtain:

$$\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-1T} = \mathbf{U}^{-1}t(\mathbf{U}, \mathbf{U}^{(p)}, \theta)\mathbf{U}^{-1}$$

or, if $\mathbf{S}$ is the second Piola-Kirchhoff stress tensor:

$$\varrho \mathbf{F}\mathbf{S}\mathbf{F}^T = \varrho_0 \mathbf{T}, \tag{5.2}$$

$$\mathbf{S} = s(\mathbf{E}, \mathbf{E}^{(p)}, \theta). \tag{5.3}$$

Constitutive assumption 1 (*continued*). *The function* (5.3) *can be inverted into*

$$\mathbf{E} = \varepsilon(\mathbf{S}, \mathbf{E}^{(p)}, \theta). \tag{5.4}$$

Constitutive assumption 2. *The yield condition reads*

$$f(\mathbf{F}, \mathbf{F}^{(p)}, \theta) \leqslant \varkappa. \tag{5.5}$$

By writing that this condition is Γ-invariant and using (5.4), we obtain:

$$f(\mathbf{S}, \mathbf{E}^{(p)}, \theta) \leqslant \varkappa. \tag{5.6}$$

Constitutive assumption 3. *If* $\hat{f}$ *is defined as*

$$\hat{f} = \mathrm{tr}\left(\dot{\mathbf{S}}\frac{\partial f}{\partial \mathbf{S}}\right) + \dot{\theta}\frac{\partial f}{\partial \theta}, \tag{5.7}$$

the plastic flow law is:

— *if* $f < \varkappa$ (*elastic range*)

$f = \varkappa$ *and* $\hat{f} \leqslant 0$ (*unloading, neutral loading*),

$$\dot{\varkappa} = 0, \ \mathbf{D}^{(p)} = 0. \tag{5.8}$$

— *if* $f = \varkappa$ *and* $\hat{f} > 0$ (*loading*),

$$\dot{\varkappa} = k(\mathbf{F}, \mathbf{F}^{(p)}, \theta; \mathbf{L}, \dot{\theta}) \tag{5.9}$$

$$\mathbf{D}^{(p)} = d(\mathbf{F}, \mathbf{F}^{(p)}, \theta; \mathbf{L}, \dot{\theta}) \tag{5.10}$$

k *and* d *being linear in* $\mathbf{L}$ *and* $\dot{\theta}$.

From (5.10) we obtain the reduced form:

$$\dot{\mathbf{E}}^{(p)} = e_1(\mathbf{S}, \mathbf{E}^{(p)}, \theta; \dot{\mathbf{E}}, \dot{\theta}). \tag{5.11}$$

Deriving (5.4) with respect to time, we obtain:

$$\dot{\mathbf{E}} = e_2(\mathbf{S}, \mathbf{E}^{(p)}, \theta; \dot{\mathbf{E}}^{(p)}, \dot{\mathbf{S}}, \dot{\theta}). \tag{5.12}$$

Constitutive assumption 3 (*continued*). (5.11) *and* (5.12) *can be solved in*

$$\dot{\mathbf{E}} = e_3(\mathbf{S}, \mathbf{E}^{(p)}, \theta; \dot{\mathbf{S}}, \dot{\theta}), \tag{5.13}$$

$$\dot{\mathbf{E}}^{(p)} = e(\mathbf{S}, \mathbf{E}^{(p)}, \theta; \dot{\mathbf{S}}, \dot{\theta}). \tag{5.14}$$

These functions are linear in $\dot{\mathbf{S}}, \dot{\theta}$. We can write (5.9) as

$$\dot{\varkappa} = \dot{\varkappa}(\mathbf{S}, \mathbf{E}^{(p)}, \theta; \dot{\mathbf{S}}, \dot{\theta}). \tag{5.15}$$

Constitutive assumption 4. *The free energy* A *and the entropy* S *are given by*

$$\begin{aligned} A &= A(\mathbf{F}, \mathbf{F}^{(p)}, \theta), \\ S &= S(\mathbf{F}, \mathbf{F}^{(p)}, \theta). \end{aligned} \tag{5.16}$$

And we obtain the Γ-invariant form:

$$A = A(\mathbf{E}^{(e)}, \mathbf{E}^{(p)}, \theta), \quad S = S(\mathbf{E}^{(e)}, \mathbf{E}^{(p)}, \theta). \tag{5.17}$$

The theory of GREEN and NAGHDI is based on the Eqs. (5.3), (5.4), (5.6), (5.7), (5.8), (5.14), (5.15) and (5.17).

NB1. In constitutive assumption 2, it is essential to have $f(\mathbf{F}, \mathbf{F}^{(p)}, \theta)$. If we had taken $f(\mathbf{T}, \mathbf{F}^{(p)}, \theta)$, then in the reduced form (5.6), f should have been an isotropic function of $\mathbf{S}$.

NB2. If in constitutive assumption 3 we were to define by (5.10) $\mathbf{A}^{(p)}_C$ or $\bar{\mathbf{D}}^{(p)}$ instead of $\mathbf{D}^{(p)}$, we should obtain the same reduced equations. But if we were to define $\mathbf{L}^{(p)}$, then it can easily be seen that we could not meet the P.I.F.C. This is a quite general feature of the theories obeying our invariance requirement: the plastic flow law must give the plastic rate of deformation and not the total time derivative of $\mathbf{F}^{(p)}$. On the contrary, for theories in which $\mathbf{F}^{(p)}$ is entirely defined, then, as in [11], the plastic flow law must give $\mathbf{L}^{(p)}$.

6. An important special case

It is often assumed that the free energy A depends on the deformations only through $\mathbf{F}^{(e)}$

$$A = A(\mathbf{F}^{(e)}, \theta). \tag{6.1}$$

A must be Γ-invariant and we obtain:

$$A(\mathbf{Q}_1 \mathbf{F}^{(e)} \mathbf{Q}_2^T, \theta) = A(\mathbf{F}^{(e)}, \theta), \tag{6.2}$$

and A must be an isotropic function of $\mathbf{C}^{(e)}$, $\mathbf{B}^{(e)}$ or $\mathbf{H}$ ($\mathbf{H} = 1/2\,\mathrm{Log}\,\mathbf{B}^{(e)}$ being the Hencky tensor of the elastic deformations). Using (2.9), we can write:

$$\begin{aligned} \varrho\dot{A} &= -\varrho S\dot{\theta} + \mathrm{tr}\,\mathbf{T}^{(e)}\mathbf{D}^{(e)}, \\ S &= -\frac{\partial A}{\partial \theta}, \\ \mathbf{T}^{(e)} &= 2\varrho\,\mathbf{F}^{(e)}\frac{\partial A}{\partial \mathbf{C}^{(e)}}\mathbf{F}^{(e)T} \end{aligned} \tag{6.3}$$

or, A being an isotropic function of $\mathbf{C}^{(e)}$, $\mathbf{B}^{(e)}$ or $\mathbf{H}$:

$$\mathbf{T}^{(e)} = 2\varrho\mathbf{F}^{(e)}\frac{\partial A}{\partial \mathbf{C}^{(e)}}\mathbf{F}^{(e)T} = 2\varrho\mathbf{B}^{(e)}\frac{\partial A}{\partial \mathbf{B}^{(e)}} = \varrho\frac{\partial A}{\partial \mathbf{H}}. \tag{6.4}$$

The first form is LEE's [3]; the third is that of GORODZOV and LEONOV [9]. The energy balance equation can then be written as:

$$\varrho\dot{U} = \varrho(\dot{A} + \theta\dot{S} + S\dot{\theta}) = \mathrm{tr}(\mathbf{TL}) - \mathrm{div}\,\mathbf{q}, \tag{6.5}$$

$$\varrho\theta\dot{S} = \mathrm{tr}[(\mathbf{T} - \mathbf{T}^{(e)})\mathbf{D}] + \mathrm{tr}[\mathbf{T}^{(e)}(\mathbf{D} - \mathbf{D}^{(e)})] - \mathrm{div}\,\mathbf{q} = \theta\left(\sigma_m + \sigma_t - \mathrm{div}\frac{\mathbf{q}}{\theta}\right), \tag{6.6}$$

$$\begin{aligned} \sigma_t &= -\theta^{-1}\mathbf{q}\cdot\mathrm{grad}\,\theta \\ \theta\sigma_m &= \mathrm{tr}[(\mathbf{T} - \mathbf{T}^{(e)})\mathbf{D}] + \mathrm{tr}[\mathbf{T}^{(e)}(\mathbf{D} - \mathbf{D}^{(e)})], \end{aligned} \tag{6.7}$$

where $\mathbf{q}$ is the heat flux vector.

According to (6.4), $\mathbf{T}^{(e)}$ is Γ-objective, but, as has been previously pointed out, $\mathbf{D}^{(e)}$ is not Γ-objective. However, using the isotropy of A, it is easily shown that $\mathbf{F}^{(e)-1}\mathbf{T}^{(e)}\mathbf{F}^{(e)}$ is symmetric and that

$$\operatorname{tr}[\mathbf{T}^{(e)}(\mathbf{D}-\mathbf{D}^{(e)})] = \operatorname{tr}(\mathbf{T}^{(e)}\mathbf{F}^{(e)}\mathbf{D}^{(p)}\mathbf{F}^{(e)-1}) = \operatorname{tr}\mathbf{T}^{(e)}\mathbf{D}^{(p)}, \tag{6.8}$$

$$\theta\sigma_m = \operatorname{tr}[(\mathbf{T}-\mathbf{T}^{(e)})\mathbf{D}]+\operatorname{tr}(\mathbf{T}^{(e)}\mathbf{D}^{(p)}). \tag{6.9}$$

All the tensors occurring in this decomposition are Γ-objective.

Writing Onsager's relations, we can obtain the viscoelastic materials of GORODZOV and LEONOV. Taking $\mathbf{T} = \mathbf{T}^{(e)}$ and writing a plastic flow law, we can obtain the special case $\gamma = 1$ of LEE's theory. (The general case $\gamma \neq 1$ involves more complicated thermodynamic considerations, which we do not intend do discuss here). The Eq. (6.9) has been obtained by GORODZOV and LEONOV, but our derivation is much more straightforward.

It is worth noting that Lee's decomposition of the rate of expenditure of work into a plastic and an elastic part would not be Γ-invariant without the assumption (6.1).

At the beginning of this paragraph, we showed that if $A = A(\mathbf{F}^{(e)})$, then it must be an isotropic function of $\mathbf{C}^{(e)}$ or $\mathbf{B}^{(e)}$; more generally, if the stress tensor is given by an elastic law from the elastic deformation gradient, $\mathbf{T} = t(\mathbf{F}^{(e)})$, then this elastic law must be isotropic because the P.I.F.C. requires

$$t(\mathbf{Q}_1\mathbf{F}^{(e)}\mathbf{Q}_2^T) = \mathbf{Q}_1 t(\mathbf{F}^{(e)})\mathbf{Q}_1^T,$$

and $\mathbf{T}$ must be given by an isotropic function of $\mathbf{B}^{(e)}$. In order to obtain an anisotropic elastic law, we have to take a more general form than $\mathbf{T} = t(\mathbf{F}^{(e)})$; for example,

$$\mathbf{T} = t(\mathbf{C}^{1/2}\mathbf{C}^{(p)-1}\mathbf{C}^{1/2}).$$

7. Relations between the theories of Green and Naghdi and of Lee

We next show how the constitutive equations of GREEN and NAGHDI can be specialized to obtain LEE's theory.

We first assume the incompressibility of the plastic deformations

$$\det\mathbf{F}^{(p)} = 1, \quad \operatorname{tr}\overline{\mathbf{D}}^{(p)} = 0. \tag{7.1}$$

The free energy A will be taken as an isotropic function of θ and $\mathbf{G}^{(e)}$

$$\mathbf{G}^{(e)} = \mathbf{C}^{1/2}\mathbf{C}^{(p)-1}\mathbf{C}^{1/2} = \mathbf{R}^T\mathbf{B}^{(e)}\mathbf{R}, \tag{7.2}$$

$$A = A(\mathbf{G}^{(e)}, \theta) = A(\mathbf{B}^{(e)}, \theta), \tag{7.3}$$

this function being such that the function

$$\psi(\mathbf{G}^{(e)}) = [\det\mathbf{G}^{(e)}]^{-1/2}\mathbf{G}^{(e)}\frac{\partial A}{\partial\mathbf{G}^{(e)}} \tag{7.4}$$

can be inverted into $\mathbf{G}^{(e)}(\psi)$.

The Eq. (6.11) of [5] leads to

$$\mathbf{S} = 2\varrho_0\mathbf{C}^{1/2}\mathbf{G}^{(e)}\frac{\partial A}{\partial\mathbf{G}^{(e)}}\mathbf{C}^{1/2}, \tag{7.5}$$

$$\mathbf{T} = 2\varrho\mathbf{B}^{(e)}\frac{\partial A}{\partial\mathbf{B}^{(e)}}, \tag{7.6}$$

that is the Eq. (6.4). The inversibility of $\psi(\mathbf{G}^{(e)})$ ensures that $\mathbf{B}^{(e)}$ can be considered as a function of $\mathbf{T}$ and θ.

The yield function f will be taken as an isotropic function of θ and $\overline{\mathbf{S}}$

$$\overline{\mathbf{S}} = \mathbf{C}^{1/2}\mathbf{S}\mathbf{C}^{1/2} = \mathbf{R}^T\mathbf{P}\mathbf{R}, \quad (\varrho\mathbf{P} = \varrho_0\mathbf{T}), \tag{7.7}$$

$$f = f(\overline{\mathbf{S}}, \theta) = f(\mathbf{P}, \theta), \tag{7.8}$$

which, according to (5.4), is a special case of (5.6).

We now choose the β_{KL} and h_{KL} of GREEN and NAGHDI as

$$\boldsymbol{\beta} = \mathbf{C}^{1/2}\mathbf{G}^{(e)-1/2}\frac{\partial f}{\partial \overline{\mathbf{S}}}\mathbf{G}^{(e)-1/2}\mathbf{C}^{1/2}, \tag{7.9}$$

$$\mathbf{h} = \alpha(\theta)\mathbf{C}^{-1/2}\mathbf{G}^{(e)1/2}\overline{\mathbf{S}}\mathbf{G}^{(e)1/2}\mathbf{C}^{-1/2}. \tag{7.10}$$

Straightforward calculations then lead to

$$\overline{\mathbf{D}}^{(p)} = \lambda\dot{f}\frac{\partial f}{\partial \mathbf{P}}, \quad \dot{\varkappa} = \alpha(\theta)\dot{\omega} \tag{7.11}$$

with

$$\lambda = \left[\operatorname{tr}\left(\mathbf{P}\frac{\partial f}{\partial \mathbf{P}}\right)\right]^{-1}, \quad \dot{\omega} = \operatorname{tr}\mathbf{P}\mathbf{E}^{(p)}, \tag{7.12}$$

which are LEE's constitutive equations when $\gamma = 1$.

Thus LEE's theory (when $\gamma = 1$) is included in the theory of GREEN and NAGHDI. Another proof of this fact can be found in [8].

8. Conclusion

The intermediate state as defined here by (1.1) and the invariance condition of § 3 is the fundamental geometrical concept in elastic-plastic finite strain. The first point of view of § 3, ignoring $\mathbf{F}^{(p)}$ and working with $\mathbf{C}^{(p)}$, leads to very general theories. On the other hand, to build up specialized theories, one needs the physical interpretation (1.1). The second point of view of § 3 and the P.I.F.C. enables us to retain the advantages of (1.1) and altogether to meet the requirement of a Γ-invariant theory. We hope it will also lead to a better understanding of the many theories which have been proposed and of the relations which exist between them.

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